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Abstract This paper analyses the dynamics of a double Cobb-Douglas economy with overlapping generations and public health investments that affect the supply of efficient labour of the old-aged. It is shown that the positive steady state of the economy is unique. Moreover, we provide necessary and sufficient conditions for the emergence of endogenous deterministic complex cycles when individuals are perfect foresighted. Interestingly, the equilibrium dynamics shows rather complicated phenomena such as a multiplicity of period-bubbling.

Keywords OLG model; Productivity; Perfect foresight; Public health expenditure

JEL Classification C62; I18; O41

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1. Introduction

The macroeconomic literature has devoted a long lasting attention to the understanding of the fundamental causes that generate fluctuations in the economic activity (i.e., business cycles). Starting from the Keynesian tradition, business cycles are generated from changes in aggregate demand conditions through the sluggish adjustment of prices and wages (e.g., Hicks, 1937). While such strand of literature was focused on short-run issues in explaining business cycles, the emergence of the real business cycle literature (e.g., Long and Plosser, 1983), which instead aims to explain cycles in macroeconomic variables through the propagation of temporary fundamental shocks within the neoclassical growth model, renewed the interest in the issue of long-run growth and cycle. However, such literature is grounded on a stochastic origin of economic cycles.

In contrast with this, another strand of literature argued that exogenous shocks are not necessary for the emergence of fluctuations. Instead, non-monotonicities in the dynamic behaviour of economic variables can generate periodic as well as aperiodic, but deterministic, orbits that resemble random ones (e.g., Goodwin, 1967; Grandmont, 1985; Farmer, 1986; Reichlin, 1986). While the first author has explained the occurrence of growth and cycles in a Marxian labour market context, which has been criticised for the lacking of sound micro-foundations, the other three authors emphasized the occurrence of business cycles in the neoclassical OLG model with rational individuals, showing that endogenous deterministic, rather than exogenous stochastic, fluctuations may arise also in very orthodox models.

However, regular as well complex cycles may emerge in the framework of the neoclassical OLG growth model with production à la Diamond (1965), which thus conjugates more clearly growth and cycle, only either assuming not fully rational individuals (e.g. myopic or adaptive expectations) (Benhabib and Day, 1982; Michel and de la Croix, 2000; de la Croix and Michel, 2002) or when such a framework is extended, for instance, with the assumptions of endogenous labour supply (Medio and Negroni, 1996), production externality (Cazzavillan, 1996), market imperfections (Aloi et al., 2000), PAYG pensions depending on previous earnings (Wagener, 2003), or taking the accumulation of government debt into consideration (Yokoo, 2000), which result in higher dimensional systems than the one-dimensional Diamond's model and thus capable to show more complicated dynamical events such as the Hopf-Neimark bifurcations.[1]

In fact, the literature has definitely shown, by resorting to various mathematical tools, that the Diamond's model with rational individuals (perfect foresight) can never possess an unstable equilibrium, and thus business cycle is prevented in that case (Galor and Ryder, 1989; Longo and Valori, 2001; Wendner, 2003; Chen et al., 2008).

Therefore, extensions of the Diamond's model that preserve the feature of the one-dimensional map may hardly transform the OLG growth model with production in a model suited to explain fluctuations in macroeconomic variables.

An exception is Bhattacharya and Qiao (2007) who endogenised adult mortality into an otherwise standard overlapping generations model with capital. In particular, they considered individual longevity as being increased by private health investments accompanied by complementary tax-financed public health programs, and showed that such an input complementary may expose the economy to aggregate endogenous fluctuations and even chaotic motions. However, in their model savings are independent from the future interest rates and, hence, the hypothesis of perfect foresight of agents is irrelevant.

An evident stylised fact regards the widespread rise in health spending, which is mainly publicly provided in several countries in the world, especially in Europe. In fact, as noted by Leung and Wang (2010, p. 11) "With few exceptions, we have observed consistent and steady rises over time of health-care expenditure, both in absolute terms and as percentages of GDP, in almost all

[1] Note that even in higher dimensional systems the emergence of business cycles in OLG models with perfect foresighted individuals is an exception rather than the rule. For instance, also in Lines (2001), where the assumption that retirees are not endowed with the entire capital stock results in a two-dimensional system, the modified OLG model is able to describe the initial phase of accumulation but unable to generate any type of cycles.

countries in the world. The total health-care expenditures among European Economic Monetary Union countries, for example, reached an average 9% of GDP in the 1990s".

This paper aims to investigate the properties of the equilibrium dynamics in the conventional Diamond's (1965) OLG growth model under two slight but realistic assumptions. The first one, rather realistic especially in front of the lengthening of life, is that individuals supply labour also in their second period of life in a measure tuned by an efficiency parameter, as in de la Croix and Michel (2007). The second assumption is that the supply of efficient labour of the old-aged, whose value results from various combined effects of age experience, sick days, disabilities, retirement periods and so on, depends on the individual health status,[2] which is, in turn, augmented by the public investments. In particular, it is assumed that health spending is transformed into better health according to a S-shaped relationship, following a recent literature (e.g. Blackburn and Cipriani, 2002; Blackburn and Issa, 2002; de la Croix and Ponthiere. 2009). This shape may capture the realistic fact that in many cases health spending has a more intense effect in promoting individual health only once a certain threshold is approached (e.g. programs of vaccines, immunization and so on) while becoming scarcely effective when the individual health status is close to its saturating value. In turn, the level of health determines the level of efficient labour supply (e.g. the number of healthy days) in the second period of life.

Our main results are the following. First, we characterise the "bi-modal" shape of the phase map and show the existence and uniqueness of the positive equilibrium. Second, we analyse the local dynamics of the system and consider local stability and the emergence of a local bifurcation. We find necessary and sufficient conditions for the emergence of oscillations around the unique equilibrium as well as necessary and sufficient conditions for such an equilibrium to be non-stationary, while numerical simulations also reveal that for an *intermediate-sized* provision of health

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[2] The link between health status and labour productivity has been early recognised by the pioneering Grossman (1972) who argues that: "… the level of ill-health measured by the rates of mortality and morbidity, influences the amount and productivity of labour supplied to an economy" (p. xiii).

care services by the government, chaotic cycles emerge and, in particular, multiple "period-bubbling" phenomena may occur. We find that endogenous fluctuations are more likely if the degree of "parsimony" of agents is larger, i.e. when individuals prefer to smooth consumption over the second period of life.

In conclusion, it is worth noting that the equilibrium dynamics in our model (*i*) may be cyclical, and most importantly, regular and chaotic business cycles appear to be the rule rather than the exception, and (*ii*) shows a strong complexity – i.e., a multiplicity of period-bubbling.

This paper contributes to two strands of literature centred on the issues of: (*i*) endogenous, as an alternative to stochastically driven, business cycles, and (*ii*) endogenous individual health. The value added of the paper is twofold. First, it is shown that deterministic endogenous complex cycles[3] are generated in a model where otherwise would be prevented, and the equilibrium dynamics would therefore be monotonic towards a globally stable steady state, as is usual in the economic literature.[4] Second, while the effects of the introduction of endogenous health as the main determinant of the individual length of life has recently been investigated with either a privately organised health system (Leung and Wang, 2010) or publicly provided (Chakraborty, 2004) or both (Bhattacharya and Qiao, 2007), less attention has been paid to the link between endogenous health and labour productivity, and thus this paper wanted to fill this gap in a simple OLG context.

The remainder of the paper is organised as follows. In Section 2 we present the model. In Section 3 we analyse the equilibrium dynamics of the economy and give necessary and sufficient conditions

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[3] This result is of interest because it shows that the equilibrium dynamics may generate business cycles without the need of any exogenous shock. In fact, as Bhattacharya and Qiao (2007, p. 2528) claimed: "These fluctuations are interesting to economists because they represent stylized business cycles that are generated purely from within an economic system and not from exogenous stochastic shocks."

[4] In fact, we recall that in the absence of our two assumptions the equilibrium dynamics in the model would be globally stable.

for the emergence of endogenous fluctuations. In Section 4 numerical simulations of the emergence of deterministic chaos are shown. Section 5 concludes.

2. The model

Consider a general equilibrium overlapping generations (OLG) closed economy populated by identical individuals, identical firms and a government that finances a public health programme through a proportional wage income tax.

2.1. Individuals

Each generation is composed by a continuum of agents of measure one. Population is stationary. The typical agent lives for three periods. An agent born at time $t-1$ draws utility from consumption when middle-aged ($c_{1,t}$) and old-aged ($c_{2,t+1}$):

$$U_t(c_{1,t}, c_{2,t+1}), \tag{1}$$

We assume that $U(.)$ is increasing in its arguments and concave; it is homogeneous of degree one (homothetic preferences) and satisfies the Inada conditions.

Workers inelastically supply one unit of efficient labour when middle-aged (young), while receiving a unitary wage income at the rate w_t, and d_{t+1} units of efficient labour when old, and receive earnings equal to $d_{t+1} w^e{}_{t+1}$ in that case, where $w^e{}_{t+1}$ is the wage individuals expect to earn at time $t+1$.

The coefficient d defines the endowment of efficient labour when old (e.g., the behaviour of the productivity over life), and thus the slope of earnings over the life cycle positively depends on: (*i*) the individual health level when old, which, in turn, determines the endowment of efficient labour of the old-aged (of course, also an early or late retirement age, which affects the labour supply of workers in the second period of life, is strictly related with the measure of individual health); (*ii*) the effect of both age and experience on productivity.[5] Therefore, a priori the supply of efficient labour when old may be larger or smaller than that when young, and this crucially depends on two counterbalancing forces: the negative effect of the reduction in health when old, and the positive effect due to the experience. In particular, we assume that the efficient labour supply of the old-aged at $t+1$, d_{t+1}, depends only on the individual health status[6] at the same time which, in turn, positively depends on health expenditure at t, h_t (in a nutshell, the higher the public health spending, the higher the health status and thus the higher the efficiency of labour when old). In particular, we assume the relationship between the old age endowment of efficient labour and health expenditure is described by the following *non-decreasing* – though bounded – function (see Blackburn and Cipriani, 2002; Blackburn and Issa, 2002):[7]

$$d_{t+1} = d(h_t) = \frac{d_0 + d_1 \Delta(h_t)^{\delta}}{1 + \Delta(h_t)^{\delta}}, \tag{2}$$

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[5] de la Croix and Michel (2007, p. 511): "The parameter [d] defines the income growth ability over life, and is determined by different factors: health when old, determining the old-age endowment in efficient labor; retirement age; effect of experience on human capital."

[6] The age profile of the productivity due to the experience effects, whose behaviour is empirically rather controversial, is left exogenous, in that we concentrate on the health effect of the productivity.

[7] Although Blackburn and Cipriani (2002) assume the dependent variable as the rate of longevity rate instead of the coefficient of labour efficiency when old, and the independent variable is human capital instead of public health capital, the line of reasoning to justify this formulation may be the same. Realistically, in many cases health investments have a more intense effect in promoting individual health (and thus labour efficiency) when old when a certain threshold level of public health expenditure is reached, while becoming scarcely effective when efficient labour is close to its saturating value (e.g., the functional relationship between health investment and individual health (i.e. labour efficiency) when old may be S-shaped).

where $\delta, \Delta > 0$, $0 \le d_0 < d_1$ $d(0) = d_0$, $\lim_{h \to \infty} d(h) = d_1$, $d'_h(h) = \dfrac{\delta \Delta h^{\delta - 1}(d_1 - d_0)}{(1 + \Delta h^{\delta})^2} > 0$, $d''_h(h) < 0$ if

$\delta \le 1$ and $d''_{hh}(h) \underset{<}{\overset{>}{-}} 0$ for any $h \underset{>}{\overset{<}{-}} h_T := \left[\dfrac{\delta - 1}{(1 + \delta)\Delta}\right]^{\frac{1}{\delta}}$ if $\delta > 1$.

Now, some clarification on Eq. (2) are in order. We define d_0 as the value of the "natural" labour efficiency when old, that is when any public health spending is absent. The parameter d_1, instead, captures the intensity of the efficiency of public health investments on the supply of efficient labour when old.[8] Finally, from Eq. (2) it can readily be seen that a rise in δ (Δ) reduces (increases) the effectiveness of the public health spending as an inducement to higher efficient labour of the old-aged for different values of the health investment, *ceteris paribus* as regards the parameters of the health technology. In other words, it measures how the old-aged productivity reacts to a change in h_t. In particular, when $\delta \le 1$ ($\delta > 1$) threshold effects of health investments on the old-aged endowment of efficient labour are absent (exist) and, hence, the function $d(h_t)$ is concave (S-shaped), so that efficient labour increases less than proportionally (more [less] than proportionally until [once] the turning point h_T is achieved) from the starting point d_0 to the saturating value d_1 as h rises.

The government levies a wage tax at the constant tax rate $0 < \tau < 1$ only on the young workers' wage. Therefore, the budget constraint of an individual born at $t - 1$ simply reads as:

$$c_{1,t} + s_t = w_t(1 - \tau), \tag{3.1}$$

i.e. wage income – net of contributions paid to finance the public provision of health expenditure – is divided into material consumption when young, c_t, and savings, s_t.

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[8] A rise in d_1 may be interpreted as exogenous medical advances due, for instance, to scientific research (i.e., antibiotics, vaccination programmes, innovations in medical technologies and so on).

When old, individuals work with either a larger or smaller efficiency than young workers do depending on their health status, and live with the amount of resources given by savings when young plus the expected interest accrued from time t to time $t+1$ at the rate $r^{e}{}_{t+1}$, and also by the labour income they receive in that period. Hence, the budget constraint of an old individual started working at t can be expressed as

$$c_{2,t+1} = \left(1 + r^{e}{}_{t+1}\right)s_t + d_{t+1} w^{e}{}_{t+1}, \tag{3.2}$$

where $c_{2,t+1}$ is old-aged consumption.

The representative individual entering the working period at t must choose how much to save out of her disposable income to maximise the lifetime utility Eq. (1), which we assume, for simplicity, to be of the logarithmic type, that is

$$U_t = \ln\left(c_{1,t}\right) + \beta_t \ln\left(c_{2,t+1}\right), \tag{4}$$

subject to Eqs. (3), where $0 < \beta_t < 1$ is the subjective discount factor. The constrained maximisation of Eq. (4), where, as usual, actual and expected factor prices and the future supply of efficient labour are assumed as exogenously given, gives the following saving rate:

$$s_t = \frac{\beta_t w_t (1-\tau)}{1+\beta_t} - \frac{d_{t+1} w^{e}{}_{t+1}}{(1+\beta_t)\left(1 + r^{e}{}_{t+1}\right)}. \tag{5}$$

Although the subjective discount factor might depend on health when old (for instance, as a "proxy" of the rate of longevity, see Chakraborty, 2004), we assume $\beta_t = \beta$ constant for any t, because this paper is essentially focused on the relationship between health and labour efficiency when old, rather than between health and longevity.

2.2. Firms

At time t firms produce a homogeneous good, Y_t, combining capital and labour, K_t and L_t, respectively, through the constant returns to scale Cobb-Douglas technology $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where

$A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. Labour supply is $L_t = \bar{L}(1 + d_t)$, where $\bar{L}$ is the constant number of workers in each cohort (young and old); then, without loss of generality, we may assume $\bar{L} = 1$. Therefore, output per efficient worker (y_t) as a function of capital per efficient worker (k_t) is

$$y_t = Ak_t^{\alpha}, \tag{6}$$

where $y_t := Y_t / L_t$ and $k_t := K_t / L_t$.

Firms maximise profits[9] and perfect competition guarantees that factor inputs are paid their marginal products, that is

$$r_t = \alpha A k_t^{\alpha-1} - 1, \tag{7}$$

$$w_t = (1-\alpha) A k_t^{\alpha}. \tag{8}$$

2.3. Government

The government invests in public health (e.g. hospitals, vaccination programmes, new medical health care services, scientific research and so on), by collecting a constant wage tax at the rate $0 < \tau < 1$ on young workers' labour income (see Chakraborty, 2004). Therefore, the per capita budget constraint faced by the government at t reads as

$$h_t = \tau w_t, \tag{9}$$

the left-hand side being the health expenditure and the right-hand side the tax receipt.

2.4. Equilibrium

[9] Without loss of generality, we assume the price of final output is normalised to unity and capital totally depreciates at the end of each period.

Market-clearing in goods and capital market leads to the equilibrium condition:

$$k_{t+1}(1+d_{t+1})=s_t\,. \tag{10}$$

More in detail, using Eq. (5) to substitute out for s_t into Eq. (10), equilibrium implies:

$$k_{t+1}=\frac{\beta}{(1+\beta)[1+d(k_t)]}w_t(1-\tau)-\frac{d(k_t)}{(1+\beta)[1+d(k_t)]}\frac{w^e{}_{t+1}}{1+r^e{}_{t+1}}\,. \tag{11}$$

where the relationship between the productivity of the old-aged and the capital stock, $d(k_t)$, is obtained by Eq. (2) upon substitution of h_t from Eq. (9) and w_t from Eq. (8).

As known, it is usual in the dynamical analyses of OLG models (see e.g., de la Croix and Michel, 2002) to investigate how the path of capital accumulation evolves depending on whether individuals have either perfect or myopic expectations about factor prices. In the next section we deal with this argument and go on studying the dynamics under perfect foresight for the interesting dynamical features, as regards, in particular, the possibility of the emergence of non-monotonic behaviours and deterministic endogenous fluctuations that the model generates in that case.

3. Dynamics under perfect foresight

With perfect foresight, the expected interest and wage rates depend on the future value of the stock of capital per efficient worker, that is

$$\begin{cases}1+r^e{}_{t+1}=\alpha A k_{t+1}{}^{\alpha-1}\\ w^e{}_{t+1}=(1-\alpha)A k_{t+1}{}^{\alpha}\end{cases}. \tag{12}$$

Therefore, combining Eqs. (8), (11), (12) and rearranging terms, the dynamic equilibrium sequence of capital can be written as

$$k_{t+1}=\frac{H k_t^{\alpha}}{\alpha(1+\beta)+(1+\alpha\beta)d(k_t)}, \tag{13}$$

where $H = \beta\alpha(1-\alpha)A(1-\tau)$ is a positive constant used to simplify notation. Substituting out for $d(k_t)$ from Eqs. (2), (8) and (9) into (13), the law of motion for capital can alternatively be expressed as:

$$k_{t+1} = \frac{Hk_t^{\alpha}\left(1+Bk_t^{\alpha\delta}\right)}{G+EBk_t^{\alpha\delta}}, \tag{14}$$

where $B := \Delta[\tau(1-\alpha)A]^{\delta} > 0$, $E := \alpha(1+\beta)+d_1(1+\alpha\beta) > 0$, $G := \alpha(1+\beta)+d_0(1+\alpha\beta) > 0$ and $E > G$ because $d_1 > d_0 \geq 0$.

Steady states of the time map Eq. (14) are determined as $k_{t+1} = k_t = k^*$. The following propositions show (*i*) the existence of a unique non-trivial time-invariant solution of Eq. (14), and (*ii*) despite the seemingly simplistic form implied by such a dynamic system, there exists a decreasing relationship between capital stocks at two successive dates (i.e. the law of motion in Eq. 14 may be non-monotonic) that might also generate complex cycles. The latter result depends on mutual relationship between the intensity of the reaction of the supply of efficient labour when old to a change in the health spending, δ, and the health tax rate, τ.

Analysis of Eq. (14), therefore, gives the following propositions.

Proposition 1. *(Existence and uniqueness of the steady state). (1) The zero equilibrium of the dynamic system described by Eq. (14) is unstable. (2) In addition, a unique non-trivial steady state* $k^* > 0$ *exists.*

Proof. Define the right-hand side of Eq. (14) as $J(k)$. Differentiating $J(k)$ with respect to k gives:

$$J'_k(k) = \frac{\alpha H\left(Mk^{2\alpha\delta}+Fk^{\alpha\delta}+G\right)}{k^{1-\alpha}\left(G+EBk^{\alpha\delta}\right)^2}, \tag{15}$$

where $M := EB^2 > 0$, $F := B[E+G-\delta(E-G)]$ and $E-G = (1+\alpha\beta)(d_1-d_0) > 0$. Moreover, $F > 0$ ($F < 0$) if and only if $\delta < \overline{\delta}$ ($\delta > \overline{\delta}$), where

$$\bar{\delta} := \frac{E+G}{E-G} > 1\,. \tag{16}$$

Since $J(0)=0$ and

$$\lim_{k\to 0^+} J'_k(k) = \alpha H \lim_{k\to 0^+} \frac{Mk^{2\alpha\delta} + Fk^{\alpha\delta} + G}{k^{1-\alpha}\left(G + EBk^{\alpha\delta}\right)^2} = +\infty\,, \tag{17}$$

then Point (1) of Proposition 1 holds.

Now, fixed points of Eq. (14) are determined as solutions to $k = J(k)$, that can also be rearranged as $Z_1(k) = Z_2(k)$, where $Z_1(k) := k^{1-\alpha}$ and $Z_2(k) := \dfrac{H\left(1 + Bk^{\alpha\delta}\right)}{G + EBk^{\alpha\delta}}$.

Therefore, since:

(*i*) $Z_1(0)=0$, $Z'_{1,k}(k) = (1-\alpha)k^{-\alpha} > 0$ for any $k>0$ and $\lim_{k\to+\infty} Z_1(k) = +\infty$, and

(*ii*) $Z_2(0) = \dfrac{H}{G} > 0$, $Z'_{2,k}(k) = \dfrac{-\alpha\delta HB(E-G)k^{\alpha\delta-1}}{\left(G+EBk^{\alpha\delta}\right)^2} < 0$ for any $k>0$ and $\lim_{k\to+\infty} Z_2(k) = \dfrac{H\left(\dfrac{1}{k^{\alpha\delta}} + B\right)}{\dfrac{G}{k^{\alpha\delta}} + EB} = \dfrac{H}{E} > 0$, with $\dfrac{H}{E} < \dfrac{H}{G}$, then

for any $k>0$ $Z_1(k) = Z_2(k)$ only once at k^*. This proves Point (2) of Proposition 1. **Q.E.D.**

Proposition 2. *(Stability and non-monotonic behaviour). (1) (Stability). Let* $0 < \delta \le \delta_2$ *hold. Then, the phase map* $J(k)$ *monotonically increases for any* $k>0$ *and the unique non-trivial steady state* k^* *is locally asymptotically stable. (2) (Non monotonic behaviour). If* $\delta > \delta_2$ *and* $\tau_1(k^*) < \tau < \tau_2(k^*)$*, then the law of motion in Eq. (14) is non-monotonic, where*

$$\delta_2 := \frac{E+G+2\sqrt{EG}}{E-G} > \bar{\delta}\,, \tag{18}$$

$$\tau_1(k^*) := \left\{ \frac{-[E+G-\delta(E-G)]-\sqrt{\Lambda}}{2(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}} \right\}^{\frac{1}{\delta}}, \tag{19}$$

$$\tau_2(k^*) := \left\{ \frac{-[E+G-\delta(E-G)]+\sqrt{\Lambda}}{2(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}} \right\}^{\frac{1}{\delta}}, \tag{20}$$

and $\Lambda := (E-G)^2\delta^2 - 2(E-G)(E+G)\delta + (E-G)^2 > 0$ *for any* $\delta > \delta_2$.

Proof. Let first the proof of Point (1) be outlined. From Eq. (15) we find that:

$$\lim_{k\to+\infty} J_k'(k) = \lim_{k\to+\infty} \frac{\alpha H\left(Mk^{2\alpha\delta} + Fk^{\alpha\delta} + G\right)}{k^{1-\alpha}\left(G+EBk^{\alpha\delta}\right)^2} = \alpha H \lim_{k\to+\infty} \frac{M + \frac{F}{k^{\alpha\delta}} + \frac{G}{k^{2\alpha\delta}}}{k^{1-\alpha}\left(\frac{G^2}{k^{2\alpha\delta}} + \frac{2GEB}{k^{\alpha\delta}} + E^2B^2\right)} = 0. \tag{21}$$

Now, define $k^{\alpha\delta} := x$ as a new supporting variable. Then, Eq. (15) can be rearranged as

$$j(k,x) = \frac{\alpha H\left(Mx^2 + Fx + G\right)}{k^{1-\alpha}(G+EBx)^2}. \tag{22}$$

Solving Eq. (22) for x gives

$$\hat{x}_1 = \frac{-[E+G-\delta(E-G)]-\sqrt{\Lambda}}{2BE}, \tag{23}$$

$$\hat{x}_2 = \frac{-[E+G-\delta(E-G)]+\sqrt{\Lambda}}{2BE}. \tag{24}$$

Then, it is easy to verify that $\Lambda > 0$ if and only if either $0 < \delta < \delta_1$ or $\delta > \delta_2$ holds, where $\delta_1 := \frac{E+G-2\sqrt{EG}}{E-G} < 1$ and $\delta_2 > \bar{\delta}$ is defined by Eq. (18) above. Since $\delta_1 < 1$, it can be ruled out because $F < 0$ if and only if $\delta > \bar{\delta} > 1$ (see Eq. 16).

Now, using $k^{\alpha\delta} := x$, from (21) and (22) we obtain

$$\hat{k}_1 = \left\{ \frac{-[E+G-\delta(E-G)]-\sqrt{\Lambda}}{2BE} \right\}^{\frac{1}{\alpha\delta}}, \tag{23}$$

$$\hat{k}_2 = \left\{ \frac{-[E+G-\delta(E-G)]+\sqrt{\Lambda}}{2BE} \right\}^{\frac{1}{\alpha\delta}}, \tag{24}$$

which represent the two solutions of Eq. (15) for k. Therefore, if $0<\delta<\overline{\delta}$, then $F>0$ and, hence, no positive real roots of Eq. (20) can exist for x. If $\overline{\delta}<\delta<\delta_2$, then $F<0$ and $\Lambda<0$, and no positive real roots of Eq. (20) can exist for x. Therefore, through Eq. (15), for any $0<\delta<\delta_2$, no positive real roots can exist for k and $J'_k(k)>0$ for any $k>0$. Hence, since Proposition 1 holds, $k^*>0$ is the unique locally asymptotically stable steady state of the dynamic system Eq. (14) in that case. This proves Point (1).

We now proceed to prove Point (2). If $\delta>\delta_2$, then it is straightforward to verify that $F<0$, $\Lambda>0$ and, hence, two positive real roots of Eq. (20) exist in that case, namely $\hat{x}_1>0$ and $\hat{x}_2>0$ (see Eqs. 21 and 22). Therefore, through Eq. (15), two positive real roots exist for k as well, namely $\hat{k}_1>0$ and $\hat{k}_2>0$ (see Eqs. 23 and 24). Then $J'_k(k^*)>0$ for any $0<k^*<\hat{k}_1$, $J'_k(k^*)<0$ for any $\hat{k}_1<k^*<\hat{k}_2$, and $J'_k(k^*)>0$ for any $k^*>\hat{k}_2$, where $\hat{k}_1>0$ and $\hat{k}_2>0$ represent the maximal and minimal points of the time map $J(k)$ for any $k>0$, respectively. Hence, $\delta>\delta_2$ is a necessary condition for the law of motion in Eq. (14) to be oscillatory around the steady state k^*. Moreover, from the numerator of Eq. (15), $J'_k(k^*)\leq 0$ if and only if

$$T_1(k^*)\tau^{2\delta}+T_2(k^*)\tau^{\delta}+G\leq 0, \tag{25}$$

where $T_1(k^*):=(k^*)^{2\alpha\delta}\Delta^2[(1-\alpha)A]^{2\delta}E>0$ and $T_2(k^*):=(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}[E+G-\delta(E-G)]<0$ for any $\delta>\delta_2$. Defining now $\tau^{\delta}:=p$ as a new supporting variable, (25) can be rearranged as

$$T_1(k^*)p^2+T_2(k^*)p+G\leq 0, \tag{26}$$

Since $T_1(k^*)>0$, $T_2(k^*)<0$ for any $\delta>\delta_2$ and $G>0$, then applying the Descartes' rule of sign we find the two positive real solutions of (26) for p when equality holds, that is:

$$p_1(k^*) := \frac{-[E+G-\delta(E-G)]-\sqrt{\Lambda}}{2(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}}, \tag{27}$$

$$p_2(k^*) := \frac{-[E+G-\delta(E-G)]+\sqrt{\Lambda}}{2(k^*)^{\alpha\delta}\Delta[(1-\alpha)A]^{\delta}}, \tag{28}$$

where $p_2(k^*) > p_1(k^*)$ and $\Lambda > 0$ for any $\delta > \delta_2$. Therefore, inequality (26) is verified if and only if, for any $\delta > \delta_2$, $p_1(k^*) < p < p_2(k^*)$. Now, applying $\tau^{\delta} := p$, we get the two positive real solutions of (25) for τ when equality holds, namely Eqs. (19) and (20), where $\tau_2(k^*) > \tau_1(k^*)$. Therefore, inequality (25) is verified if and only if, for any $\delta > \delta_2$, $\tau_1(k^*) < \tau < \tau_2(k^*)$, which is sufficient to have $J_k'(k^*) < 0$ for any $\hat{k}_1 < k^* < \hat{k}_2$. This proves Point (2). **Q.E.D.**

Although Eq. (14) is *not* a unimodal function, Proposition 1 showed the existence and uniqueness of the steady state. Moreover, Proposition 2 provided: (*i*) necessary and sufficient conditions for the unique steady state k^* (a closed-form solution for k^*, however, is prevented) to be locally asymptotically stable; this is the case when thresholds effects of public health investments on the level of the health and thus on the supply of efficient labour when old are either absent or relatively scarce, and (*ii*) necessary and sufficient conditions for the law of motion in Eq. (14) to be oscillatory, this is the case when thresholds effects exists and the size of the public health spending is neither too small nor too large (i.e. *intermediate-sized*). The set of conditions stated in Point (2) of Proposition 2 is indeed *necessary* to generate endogenous fluctuations around the steady state.

Therefore, we are now in a position to contribute to bring to light the *sufficient* conditions for the equilibrium to be non-stationary and, hence, for deterministic endogenous fluctuations to be generated. The following proposition deals with this argument and characterises, for the case when the law of motion in Eq. (14) is oscillatory, all the possible outcomes as regards stability and instability of the unique stationary state of real activity of the economy.

Proposition 3. *(Endogenous fluctuations). If Point (2) of Proposition 2 holds,* $\delta_2 < \delta < \delta(\tau, k^*)$ *and,*

(1) if $\tau_1(k^*) < \tau < \underline{\tau}(\tau, k^*)$, *then the law of motion in Eq. (14) is non-monotonic and convergent to* k^**;*

(2) if $\tau = \underline{\tau}(\tau, k^*)$, *then a flip bifurcation generically occurs;*

(3) if $\underline{\tau}(\tau, k^*) < \tau < \overline{\tau}(\tau, k^*)$, *then the law of motion in Eq. (12) is non-monotonic and divergent from* k^**;*

(4) if $\tau = \overline{\tau}(\tau, k^*)$, *then a reverse flip bifurcation generically occurs;*[10]

(5) if $\overline{\tau}(\tau, k^*) < \tau < \tau_2(k^*)$, *then the law of motion in Eq. (12) is non-monotonic and convergent to* k^**, where*

$$\delta(\tau, k^*) := \frac{\alpha H \Lambda}{4EG(E-G)(k^*)^{1-\alpha}}, \tag{29}$$

$$\underline{\tau}(\tau, k^*) := \left[\frac{-T_4(\tau, k^*) - \sqrt{\Omega}}{2T_3(\tau, k^*)}\right]^{\frac{1}{\delta}}, \tag{30}$$

$$\overline{\tau}(\tau, k^*) := \left[\frac{-T_4(\tau, k^*) + \sqrt{\Omega}}{2T_3(\tau, k^*)}\right]^{\frac{1}{\delta}}, \tag{31}$$

$T_3(\tau, k^*) := (k^*)^{2\alpha\delta} \Delta^2 [(1-\alpha)A]^{2\delta} E\left[\alpha H + (k^*)^{1-\alpha} E\right] > 0$,

$T_4(\tau, k^*) := (k^*)^{\alpha\delta} \Delta [(1-\alpha)A]^{\delta} \left\{\alpha H [E + G - \delta(E-G)] + (k^*)^{1-\alpha} 2EG\right\}$,

$\Omega := \alpha^2 H^2 \Lambda - 4\alpha HEG(E-G)(k^*)^{1-\alpha} \delta > 0$ *for any* $\delta_2 < \delta < \delta(\tau, k^*)$ *and, we recall* $H = H(\tau) := \beta\alpha(1-\alpha)A(1-\tau)$.

[10] Numerically, we may show (see the next section) that both the flip bifurcation and reverse flip bifurcation are super-critical and, hence, attractive (i.e., the bifurcation points are symmetrical and stable).

Proof. The proof of Proposition 3 is articulated as follows. From Eq. (15) we find that $J_k'(k^*)\leq -1$ if and only if

$$T_3(\tau,k^*)\tau^{2\delta}+T_4(\tau,k^*)\tau^{\delta}+T_5(\tau,k^*)\leq 0\,. \tag{32}$$

where $T_5(\tau,k^*):=G\lfloor \alpha H+(k^*)^{1-\alpha}G\rfloor>0$. Using the definition $\tau^{\delta}:=p$, (32) can be rearranged as

$$T_3(\tau,k^*)p^2+T_4(\tau,k^*)p+T_5(\tau,k^*)\leq 0\,, \tag{33}$$

Since $T_3(\tau,k^*)>0$, $T_5(\tau,k^*)>0$ and assuming $T_4(\tau,k^*)<0$ for any $\delta_2<\delta<\delta(\tau,k^*)$, then applying the Descartes' rule of sign there exists the two positive real solutions of (33) for p when equality holds, as given by:

$$\underline{p}(\tau,k^*):=\frac{-T_4(\tau,k^*)-\sqrt{\Omega}}{2T_3(\tau,k^*)}\,, \tag{34}$$

$$\overline{p}(\tau,k^*):=\frac{-T_4(\tau,k^*)+\sqrt{\Omega}}{2T_3(\tau,k^*)}\,, \tag{35}$$

where $\overline{p}(\tau,k^*)>\underline{p}(\tau,k^*)$. Therefore, inequality (33) is verified if and only if, for any $\delta_2<\delta<\delta(\tau,k^*)$, $p_1(k^*)<p<p_2(k^*)$. Now, applying $\tau^{\delta}:=p$, we get the two positive real solutions of (32) for τ when equality holds, namely Eqs. (30) and (31), where $\overline{\tau}(\tau,k^*)>\underline{\tau}(\tau,k^*)$. Therefore, for any $\delta_2<\delta<\delta(\tau,k^*)$ we get, (*i*) $-1<J_k'(k)<0$ if $\tau_1(k^*)<\tau<\underline{\tau}(\tau,k^*)$, i.e. the steady state is stationary through oscillations, (*ii*) $J_k'(k^*)=-1$ if $\tau=\underline{\tau}(\tau,k^*)$, i.e. a flip bifurcation generically occurs, (*iii*) $J_k'(k^*)<-1$ if $\underline{\tau}(\tau,k^*)<\tau<\overline{\tau}(\tau,k^*)$, i.e. the steady state is non-stationary and deterministic endogenous fluctuations occurs, (*iv*) $J_k'(k^*)=-1$ if $\tau=\overline{\tau}(\tau,k^*)$, i.e. a reverse flip bifurcation generally occurs, and (*v*) $-1<J_k'(k^*)<0$ if $\overline{\tau}(\tau,k^*)<\tau<\tau_2(k^*)$, i.e. the steady state is stationary through oscillations. **Q.E.D.**

The phase map Eq. (14) is a first order non-linear difference equation in k and the law of motion of capital accumulation may be oscillatory, i.e. it may become a "backward-bending" capital accumulation function. In particular, this happens when: (*i*) threshold effects of public health investments exist as an inducement to higher efficient labour when old, and (*ii*) the provision of public health care services, as measured by the health tax rate τ, is *intermediate-sized* (see Point 2 of Proposition 2). In fact, when the public health expenditure is either small or large enough the law of motion is monotonic and the steady state is locally asymptotically stable regardless of the size of δ (see Point 1 of Proposition 2).

Indeed, two different (negative) effects emerge when the health tax rate raises, and it is neither too small nor too large. First, it reduces the disposable income of the young and, through this channel, it negatively affects savings and capital accumulation. Second, it increases health care services and, hence, the individual health status which, in turn, implies a rise in the units of efficient labour supplied when old; the rise in d, therefore, reducing the need of savings for sustaining the old-age consumption contributes to depress capital accumulation further on at the steady state.

The final result of this two negative forces, therefore, may be such that, when $\delta > \delta_2$, the phase map $J(k)$ is negatively sloped before the unique steady state is achieved, that is $k^* > \hat{k}_1$, at least for an intermediate-sized provision of health care services that affects the efficient labour supply of the old-aged.

Figure 1 shows in a stylised way the behaviour of the capital accumulation function Eq. (14) and the evolution of the unique steady state k^* when for different sizes of the public health system, in the case of the existence of threshold effects of public health investments on the supply of efficient labour when old (i.e. $\delta > \delta_2$). It is easy to see that the phase map Eq. (14) may be either monotonically increasing or bimodal, and the unique equilibrium may be locally stable or unstable. In the next section it is illustrated a complete description of the impact of different levels of the public health system on the steady state.

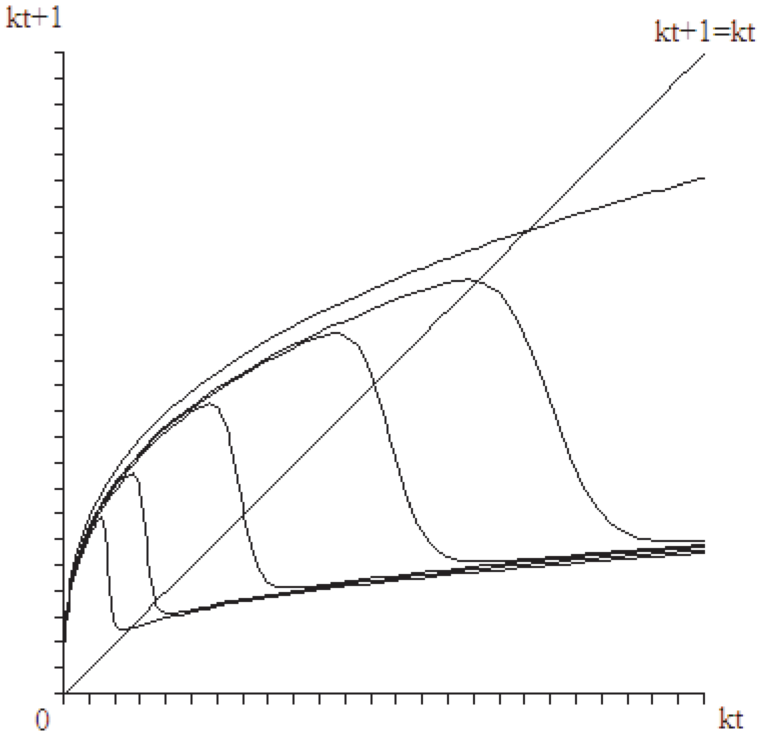

Figure 1. A pictorial view of the evolution of both the capital accumulation function Eq. (14) and the steady state for different sizes of the public health system.

3.1. Steady state effects of changes in the health tax rate

In order to elucidate how the main steady-state macroeconomic variables react to a change in the health tax rate, τ, as well as to grasp the economic intuition behind the results, we now perform a sensitivity analysis by taking the following configuration of parameters (exclusively chosen for illustrative purposes): $\alpha = 0.45$ (which is an average value between the values usually referred to developed countries, i.e. $\alpha = 0.36$, see e.g., Kehoe and Perri, 2002, and values usually used for developing countries, i.e. $\alpha = 0.50$, see Purdue University's Global Trade Analysis Project 2005 database – GTAP), $\beta = 0.20$, $d_0 = 0$, $d_1 = 25$. This parameter set generates $\delta_2 = 1.3239$. Then we choose $A = 22$ and $\delta = 14$[11], while also assuming $\Delta = 1$ without loss of generality, given the purely technical (and not economically interpretable) nature of such a parameter.

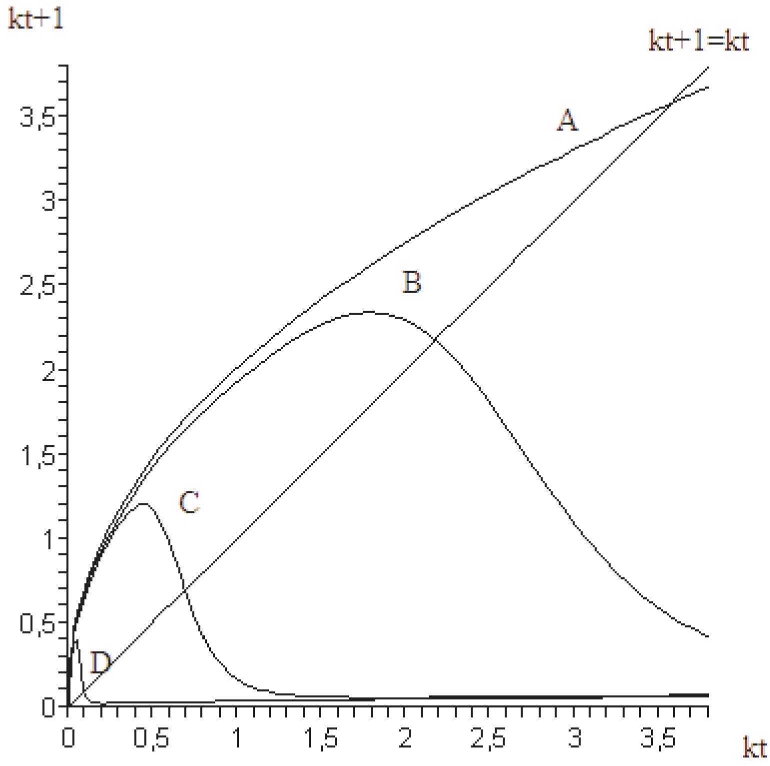

Figure 2. The phase map Eq. (14) with the corresponding steady state when τ varies.

Table 1 below summarises Figure 2 from two point of views. The former is concerned with the evolution of the steady state stock of capital as well as the macroeconomic variables of interest when τ increases. The latter regards the slope of the phase map evaluated at different equilibrium points (to this purpose Table 1 reports the values of the first derivative of Eq. 14 with respect to k_t evaluated at the steady state, namely $J'_k(k^*)$). As regards the former point, Table 1 illustrates the variations of k^* along with the per capita health expenditure, h^*, the supply of efficient labour when old, d^*, the level of per capita GDP, $Y^* = A(k^*)^{\alpha}(1 + d^*)$, and the ratio of per capita health

[11] Note that we used a value of δ close to that adopted by de la Croix and Ponthiere (2009), who assumed $\delta = 10$.

spending to per capita GDP, h^*/Y^*, as well as the steady state lifetime utility index of the representative individual, U^*.[12]

Table 1. Macroeconomic variables at the steady state when τ varies.

τ	0 (A)	0.04 (B)	0.075 (C)	0.20 (D)	0.25	0.30	0.40
k^*	3.57	2.17	0.69	0.098	0.062	0.042	0.023
$J'_k(k^*)$	0.45	-0.85	-2.99	-4.26	-4.30	-4.29	-4.23
h^*	0	0.68	0.77	0.85	0.86	0.87	0.892
d^*	0	0.129	0.633	2.37	2.95	3.46	4.23
Y^*	39.05	35.27	30.5	26.08	24.94	23.77	21.24
h^*/Y^*	0	0.019	0.025	0.032	0.034	0.036	0.042
U^*	3.45	3.22	2.72	1.73	1.46	1.22	0.78

τ	0.60	0.80	0.92	0.94	0.96	0.98	0.99
k^*	0.0096	0.0047	0.003	0.0027	0.0023	0.0018	0.0008
$J'_k(k^*)$	-4.18	-4.30	-3.96	-3.64	-2.9	-0.98	0.43
h^*	0.899	0.87	0.81	0.79	0.76	0.69	0.49
d^*	4.63	3.28	1.44	1.04	0.60	0.14	0.001
Y^*	15.35	8.50	3.95	3.15	2.34	1.47	0.90
h^*/Y^*	0.058	0.102	0.207	0.253	0.32	0.46	0.54
U^*	-0.08	-1.21	-2.52	-2.92	-3.48	-4.47	-5.67

[12] Note that in Table 1 are reported the different values of the variables of interest for τ corresponding to Case A-Case D in Figure 2 (i.e., $0 < \tau < 0.2$) as well as for other several values of τ, which are not reported in the figure for reasons of clarity of the pictorial view.

From Table 1, therefore, the following results hold.[13]

Result 1. *The steady state stock of capital per efficient worker,* k^*, *monotonically reduces when* τ *raises.*

Result 2. *The steady state health expenditure per capita,* h^*, *and the supply of efficient labour when old,* d^*, *are inverted U-Shaped functions of the health tax rate* τ.

Result 3. *The per capita GDP,* Y^*, *monotonically reduces when* τ *raises.*

Result 4. *The steady state lifetime utility index of the representative individual,* U^*, *monotonically reduces when* τ *raises.*

Result 1 is not entirely surprising, since a rise in the health tax rate negatively acts on k^* through two channels. First, the rise in τ reduces the disposable income of the young and, hence, the saving shrinks. Second, higher health tax rates mean, for values of τ relatively low, an increasing health expenditure, which is transformed into a higher supply of efficient labour when old. This, in turn, determines a higher working income when old, so that the saving rate shrinks because of the reduced need to support consumption in the second period of life.[14] This effect eventually contributes to depress capital accumulation further on.

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[13] Note that these results are robust to different parametric configurations, as extensive numerical simulations confirmed.

[14] Of course, as mentioned in the introduction, the public health spending may also increase the rate of longevity of people (see Chakraborty, 2004) and, through this channel, the saving rate. The analysis of this additional case is certainly interesting and is left for further research.

As regards Result 2, it can be noted that the health expenditure is a humped function of the tax rate because of the existence of a sort of "Laffer curve" when τ raises. In fact, for increasing values of such a tax, the tax base (i.e. the wage rate) shrinks (see Result 1), meaning that the total health spending is reduced even if the tax rate is increased. As a consequence, the supply of efficient labour of the old, which solely depends on the government health spending, shrinks as well when the tax rate is high enough. This causes a positive effect on capital accumulation which, nevertheless, is clearly of second order relative to the direct negative effect on k^* of the rise in τ. Therefore, as a final effect, we observe a strong reduction in the steady state stock of capital.

As regards Result 3, even if the effects of higher health taxes on per capita income may be a priori ambiguous, given the negative monotonic effect on capital accumulation and the possible increase in the supply of efficient labour when old, numerical simulations revealed that the negative effect of the reduced capital accumulation always prevails on the possible increase in the efficient labour productivity when old and, hence, per capita GDP shrinks.[15] Finally, as a consequence of Result 3, also the lifetime utility index at the steady state always shrinks as well.

This paper, however, is mainly concerned with the analysis of the effects of the introduction of a public health care system which, in turn, influences the efficient labour supply of the old, on the stability issue along with the possible emergence of deterministic endogenous fluctuations in a simple OLG model of neoclassical growth with production under rational expectations which otherwise would be unable to generate endogenous fluctuations. Therefore, we do not proceed further with a deeper investigation of the steady state effects of the health tax rate τ on per capita GDP and welfare, although interesting, and concentrate on studying the possible emergence of complex cycles under rational dynamics, which is precisely the object of the next section.

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[15] The impact of health expenditure on per capita GDP as reported in Table 1 (i.e. the ratio h^* / Y^*) is consistent with many actual values. For instance, for $0 < \tau < 0.80$ such a ratio varies from 2 per cent when $\tau = 0.04$ to 10.2 per cent when $\tau = 0.80$, which is consistent with the actual value in several countries in the world, as observed in many empirical works (see, e.g., Leung and Wang, 2010).

As regards the stability of the equilibrium, before starting with the numerical analysis of the possible emergence of deterministic chaos, it is interesting to note that Table 1 makes also clear the reason why stable or unstable non-monotonic behaviours occur only within a range of values of the health tax rate (see Point 2 of Proposition 2 and Proposition 3). In fact, for relatively small values of τ, the negative effect on savings and capital accumulation due to the reduction in the disposable income of the young is relatively small and, hence, even if the supply of efficient labour when old increases in that case (which, in turn, contributes to reduce capital accumulation further on), the final negative effect of a rise in τ (for τ small enough) on the shape of the phase map Eq. (14) is not strong enough to generate a decreasing relationship between two different values of the capital stock per efficient worker at two successive dates. In contrast, for relatively high values of τ, even if the negative effect on savings and capital accumulation due to the strong reduction in the disposable income when young, it is mitigated by the reduction experienced in the health expenditure and, hence, in the supply of efficient labour when old (which, in turn, causes a positive effect on capital accumulation). As a consequence, the reduction caused by a rise in τ when the health tax rate is already high is not strong enough to continue to produce unstable oscillations around the steady state, i.e. the law of motion in Eq. (14) returns to be stable either through oscillations or monotonically.

4. A numerical example of chaotic motions

In this section we wish to check for the possibility of chaotic motions that the phase map Eq. (14) may generate under the hypotheses of Proposition 3 above. To this purposes, an application of the theorem of Li and Yorke (1975) could for example be used.[16] However, given the economical rather

[16] Mitra (2001) also provided sufficient conditions to check for topological chaos when the Li–Yorke condition cannot be satisfied. However, since Proposition 2.3 by Mitra (2001, p. 143) holds only for unimodal maps, we cannot apply the

than mathematical motivation of the paper, we only use a graphical tool (e.g., bifurcation diagrams) for a pictorial view of possible chaotic dynamic behaviours, thus without embarking in more sophisticated analyses for the detection of chaos (e.g. Lyapunov's exponents).

Below, we resort again to numerical simulations to give an example of the possible chaotic behaviour generated by the financing of public health care services when threshold effects of public health investments on the individual health status, which in turn affects the old-aged endowment of efficient labour, exist (i.e. $\delta > \delta_2$).

Figure 3-5 below (where Figures 4 and 5 represent an enlarged view of Figure 2) represent the bifurcation diagrams and show, for the same parameter values used in the previous Section 3.1 (moreover, now we assume $k_0 = 1$ as the initial value of the stock of capital per efficient worker), the emergence of deterministic chaos depending on the values of τ. On the vertical axis we show the limit points of the equilibrium sequence of capital, and on the horizontal axis the value of τ ($0 < \tau < 1$). Figure 2 shoes two-period bubbling phenomena which are linked by a stable two-period cycle (note that the scale of the figure prevents the vision of the second period bubbling phenomenon when τ is close to unity; such a case is in fact represented in Figure 5 for reasons of clarity). When the health tax is relatively low (i.e. $0 < \tau < 0.042$) a unique limit point exists. Then a flip bifurcation emerges followed by a period-doubling bifurcations cascade leading to chaotic behaviours (see Figure 4): chaos occurs for the parametric interval $0.055 < \tau < 0.0985$, which is interwoven, as usually occurs, by small parametric windows (for instance, $0.091 < \tau < 0.915$) in which fluctuations become regular. At $\tau = 0.0986$ a reverse period-doubling bifurcations cascade occurs, leading, for the large interval $0.125 < \tau < 0.924$, to a stable two-period cycle. Subsequently, the same qualitative change in the dynamics described above again holds (see Figure 5 in

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result by Mitra in this model to ensure that the endogenous fluctuations generated in our model are chaotic, since the phase map Eq. (14) is bimodal.

comparison with Figure 4): for $0.925 < \tau < 0.98$ a period-doubling route to chaos followed by a reverse period-doubling route to stability occurs.

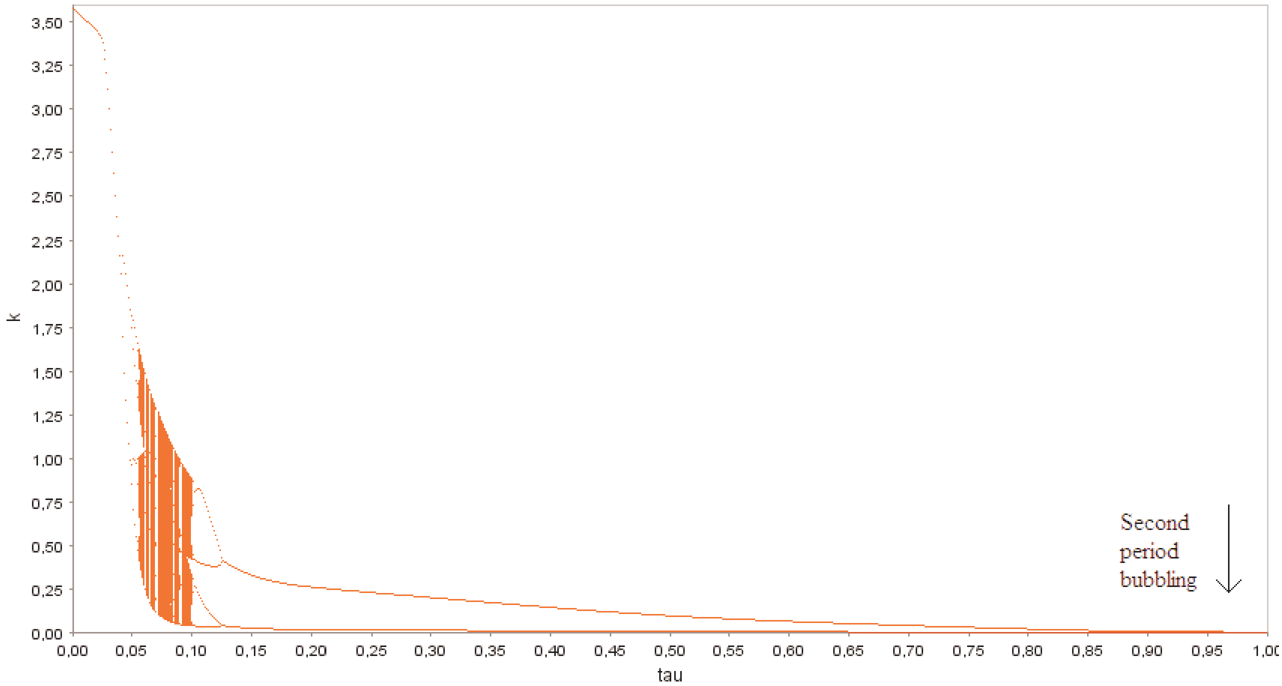

Figure 3. Bifurcation diagram for τ when $\beta = 0.2$.

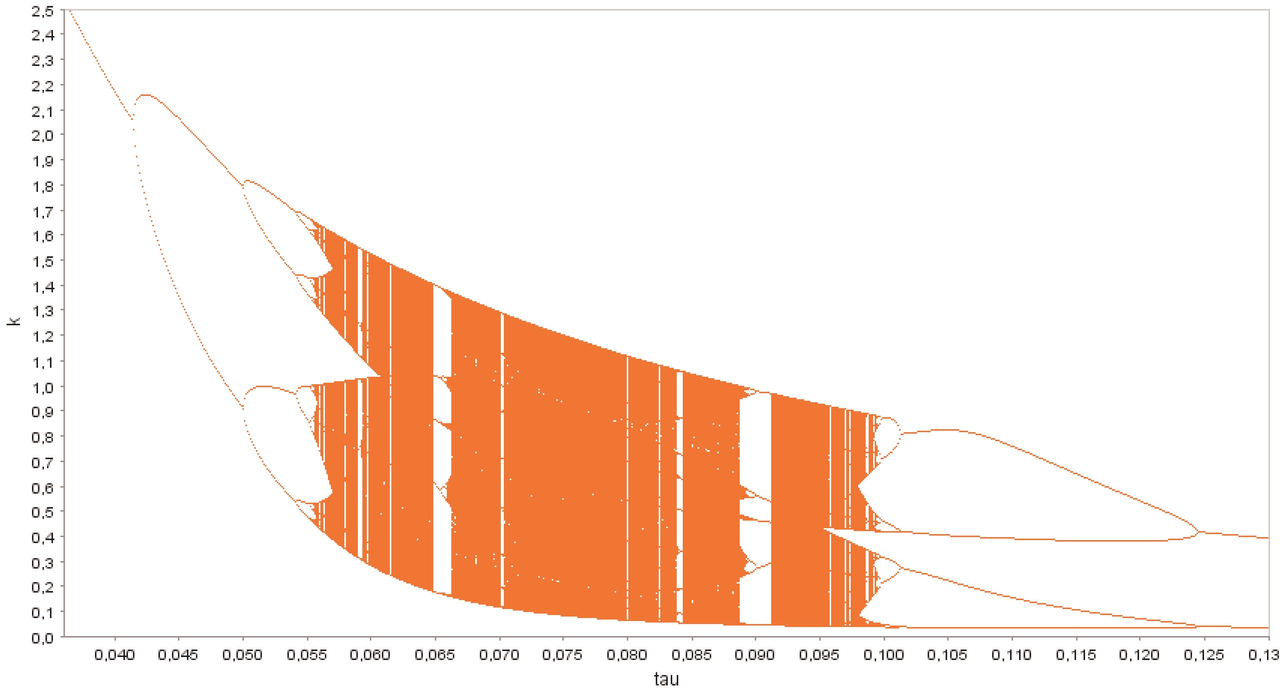

Figure 4. An enlarged view of the bifurcation diagram for τ when $\beta = 0.2$. (window for $0.036 < \tau < 0.13$ and $0 < k < 2.5$).

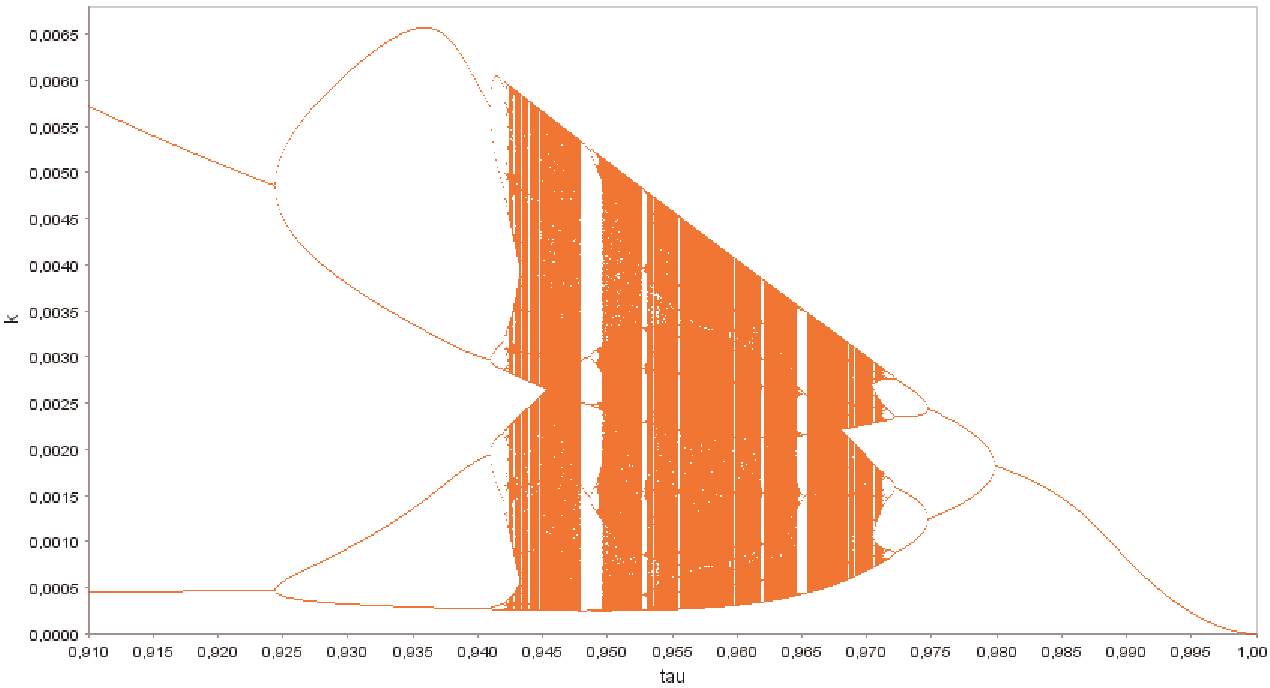

Figure 5. An enlarged view of the bifurcation diagram for τ when $\beta = 0.2$. (window for $0.91 < \tau < 1$ and $0 < k < 0.0068$).

Therefore, a schematic resume of the qualitative dynamics of this economy is the following. When the size of the public health system increases, *ceteris paribus* with respect to the other economic parameters, the map Eq. (14) exhibits: (*i*) for low values of τ the classic period-doubling route to chaos; (*ii*) for a further increase in τ period-bubbling.[17] As regards the period-bubbling phenomenon, the regular bifurcation pattern reverses itself, undergoes period halving via flip bifurcations, and eventually returns to a unique steady state for larger parameter values. However, as regards the outcome of the period-bubbling, in this model we observe a return to a stable two-period cycle instead of a return to a steady state, followed, for further increases in τ, by a second

[17] Pioneering discussions about this dynamical feature are Bier and Bountis (1984) and Stone (1993).

period-doubling route to chaos as well as a second period-bubbling in which this time the bifurcation pattern definitely returns to a stable steady state.

In order to check for the robustness of the qualitative dynamics of the model, we now vary the preference parameter β from 0.2 to 0.6.[18]

By using as the criterion to evaluate the stability of the unique positive equilibrium of the model the measure of the parametric interval of τ for which the equilibrium is unstable, we may conclude that a rise in the subjective discount factor reduces stability. In fact, while when $\beta = 0.2$ the economy is unstable for $0.042 < \tau < 0.98$, when $\beta = 0.6$ the parametric window of instability is enlarged to $0.021 < \tau < 0.9912$ (see Figure 6).[19] Moreover, and most important, with this new value of β the map Eq. (14) exhibits even three period-doubling routes to chaos followed by three period-bubbling phenomena, each of which is separated by two windows (for $0.058 < \tau < 0.217$ and $0.853 < \tau < 0.9695$) of two-period cycles (see Figures 7-9).

The above increase in the subjective discount factor implies that: (*i*) the portion of the entire domain of the health tax rate that guarantees stability of the steady state shrinks from 6.2 to 3 per cent; (*ii*) the portion of the domain of τ in which 2-period cycles occurs, reduces from 82.4 to

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[18] In the economic literature several examples exists as regards different values of the subjective discount factor β used in numerical simulations. For instance, Strulik (2008) used $\beta = 0.09$, Strulik (2004) $\beta = 0.2$, de la Croix and Michel (2002) $\beta = 0.3$, while Blackburn and Cipriani (2002), Žamac (2007) and Leung and Wang (2010) considered $\beta = 0.6$. In this paper, therefore, we have chosen two typical values among those widely adopted in the economic literature.

[19] Similar to Figure 2, Figure 6 shows period-bubbling when τ is close to unity even for the case $\beta = 0.6$ (see Figure 9 for a clear pictorial view of it), thus giving rise to a triple, rather than double (as in the case of $\beta = 0.2$), period-bubbling phenomenon. Interestingly, we note that when $\alpha = 0.33$, $\beta = 0.2$ and $\delta = 30$ (with the other parameters being unchanged) the bifurcation diagram for τ allows to observe even quadruple period-bubbling. For the sake of brevity, however, we do not report such a case in the paper.

27.55 per cent, and finally (*iii*) the higher periodicity or chaotic regime occurs for the 69.5 of the values of τ when $\beta = 0.6$ versus the 7.02 per cent when $\beta = 0.2$.

Therefore, it follows that when the individuals are more "parsimonious" and prefer to smooth consumption over their second period of life (i.e., β is high), the occurrence of business cycles is more likely while also becoming more irregular.

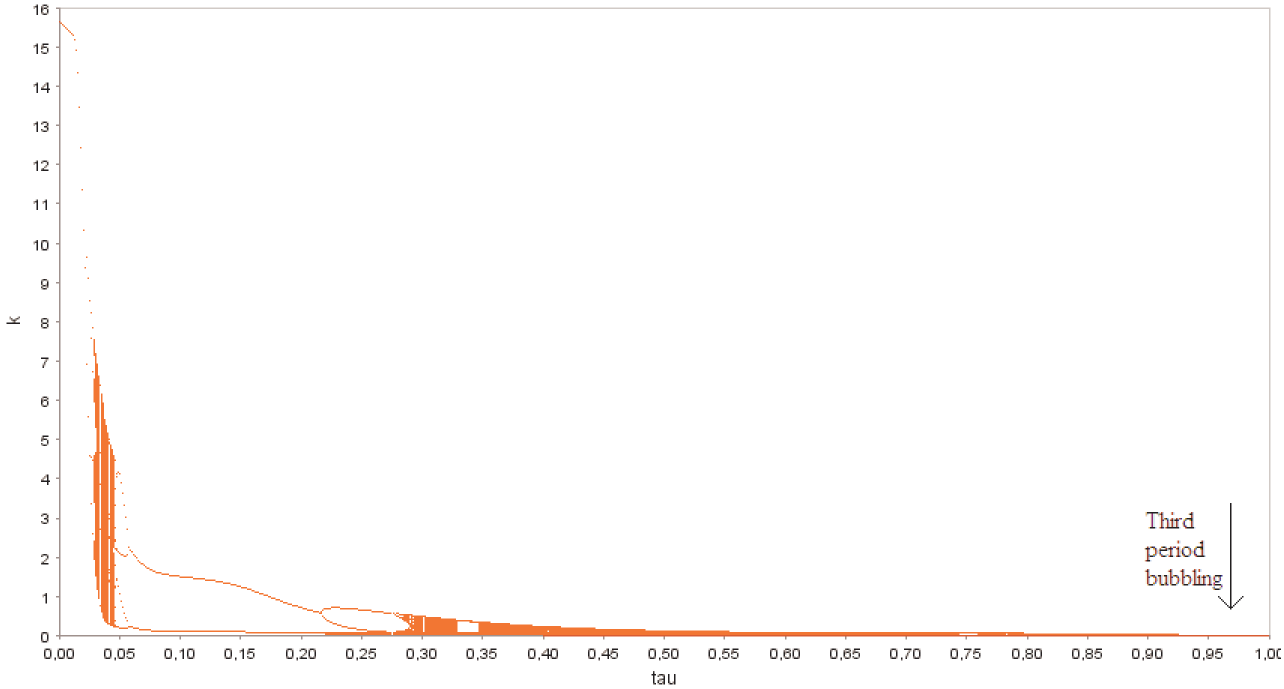

Figure 6. Bifurcation diagram for τ when $\beta = 0.6$.

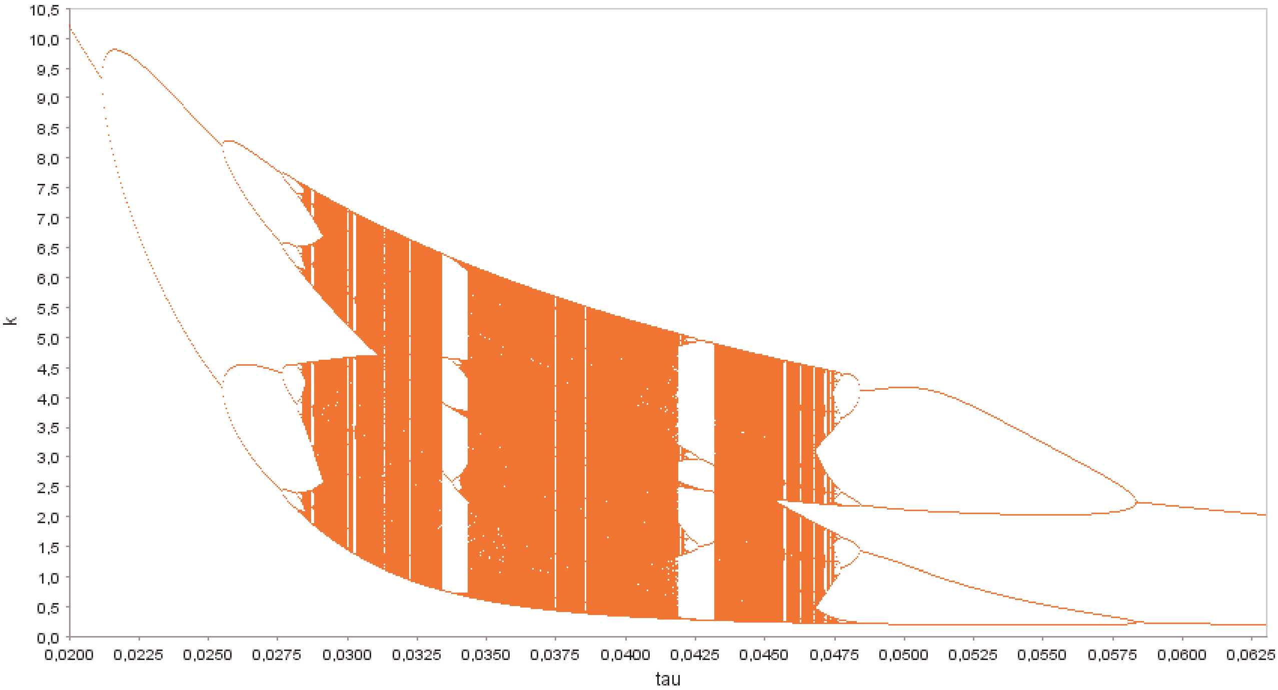

Figure 7. An enlarged view of the bifurcation diagram for τ when $\beta = 0.6$. (window for $0.02 < \tau < 0.063$ and $0 < k < 10.5$).

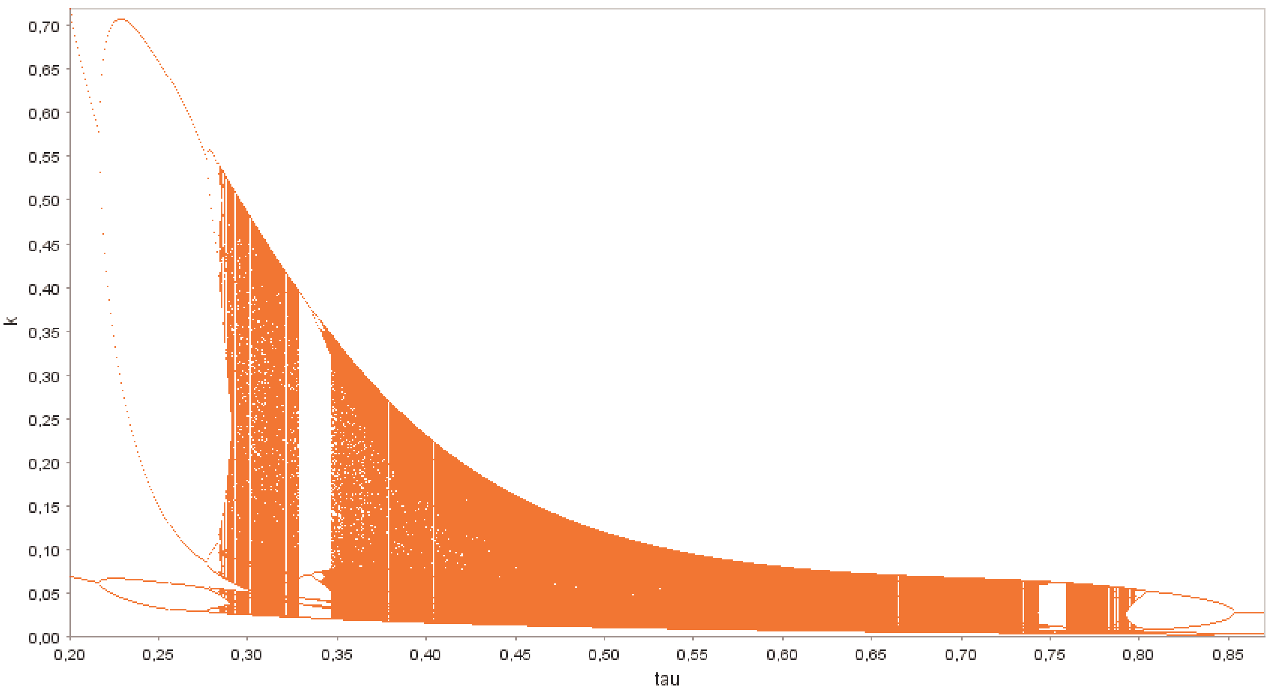

Figure 8. An enlarged view of the bifurcation diagram for τ when $\beta = 0.6$. (window for $0.2 < \tau < 0.87$ and $0 < k < 0.72$).

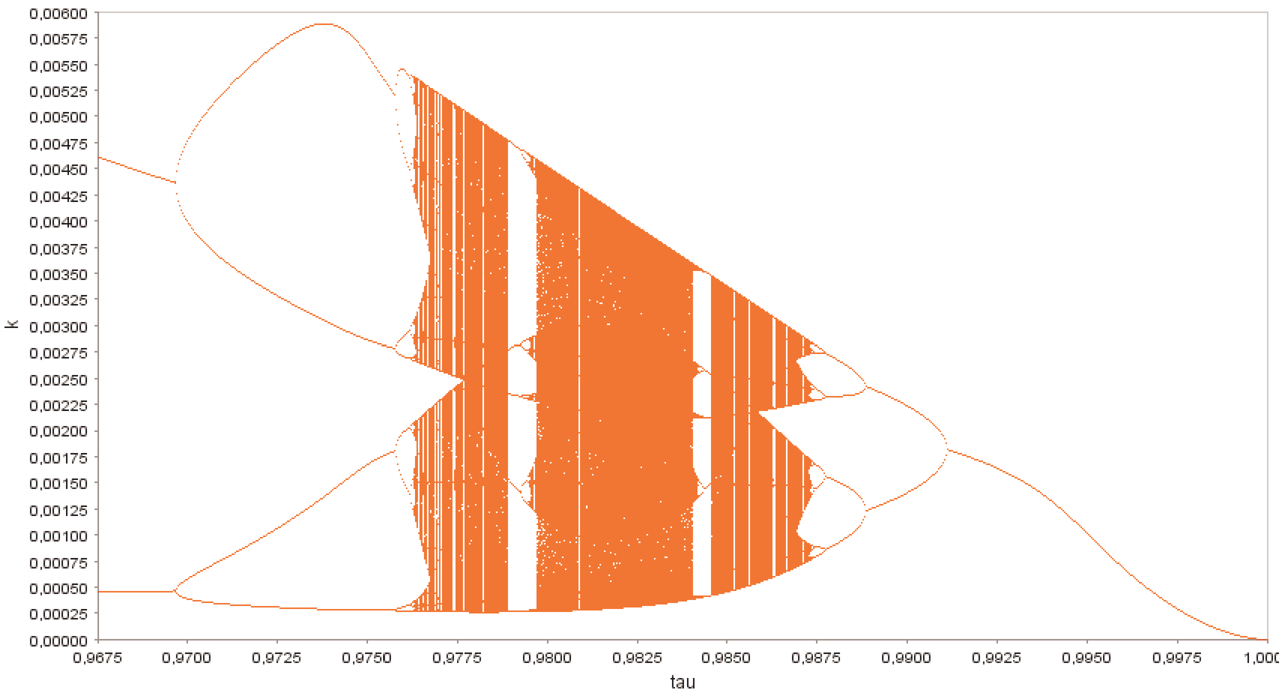

Figure 9. An enlarged view of the bifurcation diagram for τ when $\beta = 0.6$. (window for $0.9675 < \tau < 1$ and $0 < k < 0.006$).

Therefore, we may conclude that the presence of either a pair or triple period-bubbling phenomena linked each other by regular (i.e., two-period) cycles or irregular cycles instead of a stable steady state, seems to be the major and robust dynamical feature of this simple economy.

It is worth noting that with plausible parameter sets we obtain a very remarkable result: regular as well as chaotic business cycles seems to be the rule rather than the exception in an OLG growth model à la Diamond (1965) extended with both a public health system financed at a balanced budget and old-age working individuals.

Moreover, the complexity of the dynamics above described – i.e., the multiplicity of period-bubbling – is notable for one-dimensional map.[20]

[20] In fact, in an influential paper, Stone (1993) claimed: "The period-doubling route to chaos is a well known feature of a range of simple, nonlinear difference equations routinely used in modelling biological populations. It is not generally understood, however, that the process may easily break down and suddenly reverse, giving rise to distinctive period-halving bifurcations." (p. 617).

Finally, we attempt to provide a "heuristic" interpretation of the complicated equilibrium dynamics of this economy. Although there is no simple closed-form solutions for the steady-state of Eq. (14), the above analysis has shown that the steady state equation has a unique interior fixed point and, as illustrated by Figure 1, the map (14) is "bimodal" when $\delta > \delta_2$. This means that, as it may be easily ascertained drawing the phase diagram, even if the equilibrium occurs when the map is negatively sloped and, thus, it is locally unstable, the spiralling trajectories outward from such an unstable equilibrium remain entrapped thanks to the "bimodality" feature of the map: in particular, given such a shape of the map, the more the equilibrium appears in the intermediate portion of the negatively inclined map – despite the higher negative inclination and thus the larger local instability of such an equilibrium–, the more likely the oscillations displayed by the phase diagram are regular (conversely, the more the equilibrium appears in either the high or low portion of the negatively inclined map – despite the map at the equilibrium is less inclined and, thus, less locally unstable –, the more likely we observe irregular fluctuations).

This intuitively explains (see the case jointly portrayed in Figure 2 – i.e., changes in the shape of the phase map when τ varies –, and Figures 3-5 – i.e., the bifurcation diagrams when τ varies) because if τ increases, determining a downward shift in the time map and, hence, at the equilibrium it is initially less inclined, then it becomes more inclined and, finally, again less inclined, we observe initially irregular fluctuation, then regular fluctuations (e.g., a two-period cycle) and finally again irregular fluctuations, followed by a definitive stabilisation for large enough values of the health tax rate τ.[21]

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[21] As an example to illustrate this line of reasoning, we observe that, although the slope of the phase map Eq. (14) at the equilibrium point when $\tau = 0.82$ is, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} \cong -4.31$ – which corresponds to the maximal local instability – only a two-period cycle exists, while when $\tau = 0.945$, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} \cong -3.50$ – and thus the slope of the phase map is lower, i.e., the local instability is smaller than in the previous case – a chaotic behaviour exists.

5. Conclusions

In this paper we studied the equilibrium dynamics of the Diamond's (1965) model with perfect foresight extended with endogenous health and old-age efficient labour.

We show that a unique positive equilibrium exists as Diamond (1965). However, while with the equilibrium dynamics in the Diamond's model the unique equilibrium cannot ever display either temporary or permanent oscillations, in this paper it is shown that the introduction of the link between health and labour productivity of the old-aged may be responsible of the appearance of robust endogenous fluctuations, which may be either regular or chaotic. In particular, a rise in the public provision of health care services, as represented by a rise in the health tax rate collected on the young's labour income, reduces capital per efficient worker, per capita GDP and welfare, because by raising the health status, and thus the productivity of the mature workers, savings shrinks because of the reduced need to save to sustain old-age consumption.

Moreover, and more important, a rather low level of the health tax may trigger deterministic endogenous chaotic business cycles. The qualitative features of the equilibrium dynamics, despite the simplicity of the one-dimensional map and the assumption of rational expectations are very rich: indeed, multiplicity of period-bubbling phenomena is shown.[22]

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[22] We note that the presence of period-doubling reversals may be economically interesting since they have the potential to curb dangerous chaotic fluctuations in macroeconomic variables.

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