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Abstract

A class of mean reverting positive stochastic processes driven by $\alpha-$stable distributions, referred to here as $\alpha-$root processes in analogy to the square root process (Cox-Ingersoll-Ross process), is a subclass of affine processes, in particular continuous state branching processes with immigration (CBI processes). Being affine, they provide semi-analytical results for the implied term structures as well as for the characteristic exponents for their associated distributions. Their use has not been appreciated in the field perhaps due to lack of an efficient numerical algorithm to supplement their semi-analytical results. The present article introduces a convenient formulation of such processes, CBI processes in general, in the form of pure-jump processes of infinite activity. The formulation admits an efficient simulation algorithm that enables an extensive investigation of their properties.

Stochastic processes are the building blocks of modeling discipline. Though Brownian motion has been largely successful in this regard, there are certain areas where more advanced processes could be helpful. This is especially so in mathematical finance wherein alternate processes have been utilized, in particular to provide an explanation to parameter smiles, such as volatility smiles or correlation smiles. Among other approaches, a class of stochastic processes called $\alpha-$stable Lévy processes have been used for this purpose with encouraging results. Because applicable α usually lies between 1 and 2, and the associated stable processes can have negative values, their use has been largely limited to their exponentials as stochastic variables of interest. This makes them analytically intractable for many objects of interest, such as term structures of discount factors in interest rate modeling or survival probabilities in credit risk modeling.

It is known that the Cox-Ingersoll-Ross process, also known as the square-root process, though confined only to the positive real axis, admits analytical results for term structure modeling. It belongs to a class of affine processes, the spot rate in interest rate modeling being related affinely to the short rate. It is driven by Brownian motion which in the language of stable processes has $\alpha = 2$. A natural question then arises as to whether there exist $\alpha-$root processes driven by $\alpha-$stable distributions, and whether they too exhibit the affine property. As it turns out, the answer to this question is pleasantly in the affirmative. $\alpha-$root processes thus provide a natural and appealing approach to affine jump diffusion processes

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by incorporating jumps into the diffusion component itself to turn it into an $\alpha-$root process, rather than extending the process to include a jump component.

The class of affine processes is a well-studied branch of mathematics, and has been characterized in generality by Duffie, Filipovic and Schachermayer [2003]. However, this class being rather large, identification of specific affine processes for their usefulness is important in itself. Being a subclass of affine processes, in particular continuous state branching processes with immigration (CBI processes), and a natural extension of the square-root process, $\alpha-$root processes have caught the attention of researchers in the field. For instance, they are briefly touched upon by Carr and Wu [2004] as an activity process for generating random time. Their use has not been appreciated in the field perhaps due to lack of an efficient numerical algorithm to supplement their semi-analytical results. The present article introduces a convenient formulation of such processes, CBI processes in general, in the form of pure-jump processes of infinite activity. The formulation admits an efficient simulation algorithm that enables an extensive investigation of their properties. The algorithm is also adaptable to the case of standard mean reverting processes (Ornstein-Uhlenbeck-Type processes) driven by $\alpha-$stable processes, or Lévy processes of infinite activity.

Section 1 introduces the $\alpha-$root process, CBI process in general, in the form of a mean-reverting pure-jump process of infinite activity and presents semi-analytical solutions for the implied term structures and the Laplace exponents. Section 2 presents closed form expressions for the Laplace exponents in some special cases. Section 3 presents an efficient Monte Carlo simulation algorithm that enables a numerical investigation of the process. Section 4 discusses the simulation results. Semi-analytical solutions are derived in Appendix A. The results of a numerical investigation are presented in Figures 1-10.

1 Alpha-Root Process

Let us start with the following pure jump process for a positive stochastic variable $r(t)$,

$$dr(t) = [\phi(t) - mr(t)]\, dt + \int_{z=0}^{\infty} h(z/r(t))dM(dz,t). \tag{1}$$

Here $dM(dz,t) = dN(dz,t) - dzdt$ where $N(dz,t)$s are independent Poisson processes. Process $N(dz,t)$ is of intensity dz and is associated with the interval $(z, z+dz)$. If $N(dz,t)$ jumps up by one at time t, $dN(dz,t)$ causes $r(t)$ to jump up by $h(z/r(t_-))$ where t_- is just prior to t. We may refer to $h(x)$ as the jump function. It is taken to be nonnegative, integrable from $x = 0$, going to zero as its argument $x \to \infty$. $M(dz,t)$ is the compensated Poisson process (a Martingale). Parameter m is the mean reversion rate. Drift $\phi(t)$ is assumed to be positive. Note that the total intensity of the Poisson processes is infinite and hence the stochastic process is of infinite activity (however, effective intensity depends on the jump function and is not necessarily infinite).

An attractive feature of the above process is that it is an affine model, just as the well-known square-root process is. Note that process (1) is written in somewhat an unconventional way. It is usual to regard jump h as an independent variable with the Poisson intensity $dz = (dz/dh)dh$ giving rise to an intensity density $|dz/dh|$ called the Lévy density. Working with the jump function $h(x)$ has provided us with a convenient formulation of an affine

process, in particular a CBI process (for constant $\phi(t)$, that could also have square-root diffusion and $r-$independent nonnegative jump component), in the form of a stochastic differential equation that forms the basis of a simulation to be discussed later.

Being an affine model, process (1) admits semi-analytical results for the implied term structures as well as for the characteristic exponents for their associated distributions. The following result is derived in Appendix A,

$$\mathrm{E}_t\left\{\exp\left[-\int_t^T ds u(T-s)r(s)\right]\right\} = \exp\left[-\int_t^T ds\phi(s)B(T-s) - B(T-t)r(t)\right], \qquad (2)$$

where $u(\tau)$ is some deterministic function and $B(\tau)$, satisfying $B(0)=0$, is a solution of

$$\frac{dB(\tau)}{d\tau} + mB(\tau) = u(\tau) + \int_0^\infty dx\,\{1 - h(x)B(\tau) - \exp\left[-h(x)B(\tau)\right]\}. \qquad (3)$$

Result (2) features the affine property, the expression within square brackets being related affinely to $r(t)$. For term structure modeling, one is interested in solving the above equation with $u(\tau)=1$. If interested in the Laplace transform $\mathrm{E}_t\{\exp[-ur(T)]\}$ of the probability density function of $r(T)$, or its negative logarithm known as the Laplace exponent, the equation is solved in the absence of $u(\tau)$, but under the initial condition $B(0)=u$.

The above result is for a general jump function $h(x)$. For $h(x)=ax^{-1/\alpha}$, $1<\alpha<2$, we have $h(z/r)\propto r^{1/\alpha}$ and (1) may be referred to as an $\alpha-$root process. Equation for $B(\tau)$ then becomes

$$\begin{aligned}\frac{dB(\tau)}{d\tau} + mB(\tau) &= u(\tau) - \sigma^\alpha\left[B(\tau)\right]^\alpha, \quad 1<\alpha<2,\\ &= u(\tau) - \sigma B(\tau)\ln B(\tau), \qquad \alpha=1. \end{aligned} \qquad (4)$$

where σ is $a\left[\alpha\Gamma(-\alpha)\right]^{1/\alpha}$ for $1<\alpha<2$ and is a for $\alpha=1$. Equation for $\alpha=1$ is also presented above, though it needs to be treated as a special case. For the Laplace exponent, the above can be solved with $u(\tau)=0$ and $B(0)=u$ to obtain

$$\begin{aligned}B(\tau) &= e^{-m\tau}\left\{u^{-(\alpha-1)} + \frac{\sigma^\alpha}{m}\left[1-e^{-(\alpha-1)m\tau}\right]\right\}^{-1/(\alpha-1)}, \quad 1<\alpha<2,\\ &= \exp\left[e^{-\sigma\tau}\left(\ln u + \frac{m}{\sigma}\right) - \frac{m}{\sigma}\right], \qquad \alpha=1. \end{aligned} \qquad (5)$$

Case $\alpha<1$ turns out to be inconsistent. These results have a limit as $\alpha\to 2$ (given a fixed σ) to correspond to the case of the square-root process. Closed form expressions for the Laplace exponent can be obtained in some special cases as discussed in the next section.

Drift $\phi(t)$ has been assumed to be positive. This ensures that the origin is inaccessible, that the probability density of $r(T)$ as $r(T)\to 0$ goes to zero. This can be examined, as usual in Laplace transforms, by looking at the $u\to\infty$ limit of $u\mathrm{E}_t\{\exp[-ur(T)]\}$. The leading contribution comes from the integral in (2) near $s=T$,

$$u\mathrm{E}_t\{\exp[-ur(T)]\} \sim u\exp\left[-\phi(T)\frac{u^{2-\alpha}}{(2-\alpha)\sigma^\alpha}\right], \qquad \text{as } u\to\infty. \qquad (6)$$

Given $\phi(t) > 0$, this goes to zero as $u \to \infty$. For $\alpha = 2$, one obtains the well-known requirement $\phi(t) > \sigma^2$ (volatility of the square-root process is $\sigma\sqrt{2}$ in our scale convention). As for $\alpha = 1$, $B(\tau) \to \infty$ as $u \to \infty$ for all τ so that the above quantity goes to zero for any $\phi(t) \geq 0$ (in this case, $\phi(t)$ can be zero).

The $\alpha-$root process can be viewed as being driven by an $\alpha-$stable Lévy process. This is analogous to the square root process being driven by the Brownian motion. To see this, consider small $\tau = T - t$ when $B(\tau) \simeq (1 - m\tau)u - \sigma^\alpha \tau u^\alpha$ and the Laplace exponent approximates to

$$[r(t) + (\phi(t) - mr(t))\tau]\, u - \sigma^\alpha r(t)\tau u^\alpha. \tag{7}$$

The u^α term is the Laplace exponent of a stable distribution of index α and skew parameter one (maximally skewed to the right) with zero location, the term linear in u arising from the deterministic part of the $r-$process. Its scale parameter is $\sigma(r(t)\tau)^{1/\alpha}$ (times $[-\cos(\pi\alpha/2)]^{1/\alpha}$ to be exact), as expected with the $\alpha-$root of $r(t)$ attached (similar analysis can be done for $\alpha = 1$). Given the above infinitesimal result, one can indeed recover the full Laplace exponent using the law of iterated expectations. Note that infinitesimally, the $\alpha-$root process can be viewed as being driven by a time-scaled stable process, τ getting effectively scaled by $r(t)$. This is a stochastic scaling of time, scaling by the stochastic process $r(t)$ itself. This gives us an alternate view of process (1) for general $h(x)$ as well, providing a relationship between CBI processes and Lévy processes (known as Lamperti representation).

The expression for term structure in (2) involves convolution of $\phi(s)$ and $B(s)$ (consider $t = 0$). When modeling term structure models, say for interest rates or credit spreads, one approach is to imply the drift $\phi(t)$ from the given data on discount factors or survival probabilities as the case may be. If this deconvolving procedure is not convenient, one may consider the well-known approach in affine modeling of working with a constant ϕ, but with the stochastic variable $r(t)$ related to the variable of interest by a deterministic shift that is implied from the given data (see Brigo and Alfonsi (2005) for such an approach with the square root process).

2 Laplace Exponents

The Laplace exponent of the distribution of $r(T)$ can be obtained given the solution (5) for $B(\tau)$. For constant drift $\phi(t) = \phi$ and for $1 < \alpha < 2$, this gives for the exponent

$$\frac{\nu\phi}{mq^\nu}\int_1^{1+pu^{1/\nu}} dx x^{-\nu}\left(1 + qu^{1/\nu} - x\right)^{\nu-1} + \frac{r(t)e^{-m(T-t)}u}{\left(1 + pu^{1/\nu}\right)^\nu}, \tag{8}$$

where $\nu = 1/(\alpha - 1)$, $q = \sigma^\alpha/m$ and $p = q(1 - e^{-(\alpha-1)m(T-t)})$. The integral can be expressed in terms of incomplete beta functions. For small u, the exponent has the expansion

$$\left[\frac{\phi}{m}(1 - s) + r(t)s\right] u - \left\{\frac{\phi}{m\alpha}\left[q(1 - s) - p\nu s\right] + p\nu r(t)s\right\} u^\alpha, \tag{9}$$

where $s = e^{-m(T-t)}$. This gives the mean, and the scale parameter for the large $r(T)$ behavior (nonanalytic $u^\alpha-$behavior as $u \to 0$ indicates that the $r(T) \to \infty$ tail is similar to that of a

stable distribution of index α). Closed form expression for the exponent can be obtained if $m = 0$, that reads

$$\frac{\phi u^{2-\alpha}}{(2-\alpha)\sigma^\alpha}\left[1-\left(1+pu^{\alpha-1}\right)^{-(2-\alpha)/(\alpha-1)}\right]+\frac{r(t)u}{\left(1+pu^{\alpha-1}\right)^{1/(\alpha-1)}}, \tag{10}$$

where $p = (\alpha-1)\sigma^\alpha(T-t)$. If $m \neq 0$, closed form expressions can be obtained for some special values of α. For the limiting case of $\alpha = 2$, we obtain the well-known result

$$\frac{\phi}{\sigma^2}\ln(1+pu)+\frac{r(t)e^{-m(T-t)}u}{1+pu}, \tag{11}$$

where $p = (\sigma^2/m)(1-e^{-m(T-t)})$. This is the exponent of the non-central chi-square distribution (volatility of the square-root process is $\sigma\sqrt{2}$). For $\alpha = 3/2$, one obtains

$$\frac{2\phi}{mq^2}\left\{\frac{p\sqrt{u}(1+q\sqrt{u})}{1+p\sqrt{u}}-\ln\left(1+p\sqrt{u}\right)\right\}+\frac{r(t)e^{-m(T-t)}u}{\left(1+p\sqrt{u}\right)^2}, \tag{12}$$

where $q = \sigma^{3/2}/m$ and $p = q(1-e^{-m(T-t)/2})$. For $\alpha = 4/3$, the exponent is

$$\frac{3\phi}{mq^3}\left\{\frac{pu^{1/3}(1+qu^{1/3})}{P(u)}\left[\frac{q}{2}u^{1/3}\left(1+\frac{e^{-m(T-t)/3}}{P(u)}\right)-1\right]+\ln\left(P(u)\right)\right\}+\frac{r(t)e^{-m(T-t)}u}{\left(P(u)\right)^3}, \tag{13}$$

where $q = \sigma^{4/3}/m$ and $p = q(1-e^{-m(T-t)/3})$ and $P(u) = 1+pu^{1/3}$. Another integrable case is $\alpha = 5/3$ that gives

$$\frac{3\phi}{mq\sqrt{q}}\left\{\frac{\sqrt{q}u^{1/3}R(u)}{\sqrt{1+pu^{2/3}}}-\mathrm{Sin}^{-1}\left[\frac{\sqrt{q}u^{1/3}R(u)}{1+qu^{2/3}}\right]\right\}+\frac{r(t)e^{-m(T-t)}u}{(1+pu^{2/3})^{3/2}}. \tag{14}$$

Here $q = \sigma^{5/3}/m$, $p = q(1-e^{-2m(T-t)/3})$ and $R(u) = \sqrt{1+pu^{2/3}}-e^{-m(T-t)/3}$. Closed form expressions can be obtained more generally for $\alpha = 1+2/k$ where $k \geq 2$ is an integer.

Closed form expressions for the exponent can also be obtained for certain time-dependent drifts. For instance, consider a time-dependence of the form $\phi(t) = \phi e^{-\kappa m t}$ given some constant κ. The exponent in integral form then reads

$$\frac{\nu\phi e^{-\kappa mT}u^\kappa}{mq^{(1-\kappa)\nu}}\int_1^{1+pu^{1/\nu}} dx x^{-\nu}\left(1+qu^{1/\nu}-x\right)^{(1-\kappa)\nu-1}+\frac{r(t)e^{-m(T-t)}u}{\left(1+pu^{1/\nu}\right)^\nu}, \tag{15}$$

where as before $\nu = 1/(\alpha-1)$, $q = \sigma^\alpha/m$ and $p = q(1-e^{-(\alpha-1)m(T-t)})$. Closed form expression can be obtained for $\kappa = 2-\alpha$,

$$\frac{\phi e^{-(2-\alpha)mT}u^{(2-\alpha)}}{(2-\alpha)\sigma^\alpha}\left[1-\left(1+pu^{\alpha-1}\right)^{-(2-\alpha)/(\alpha-1)}\right]+\frac{r(t)e^{-m(T-t)}u}{\left(1+pu^{\alpha-1}\right)^{1/(\alpha-1)}}. \tag{16}$$

Closed form expressions can also be obtained for some other choices of κ, for instance when $(1-\kappa)\nu$ is a positive integer.

3 Monte Carlo Simulation

Process (1) is of infinite activity as presented. The integral over z needs to be cut off at the higher end to render the total intensity of the Poisson processes finite for simulation purpose. This can be done by forcing $h(x) = 0$ for $x > X$ given a sufficiently large X. Process (1) can now be viewed as being driven by a compound Poisson process of stochastic total intensity $r(t)X$. It can be simulated starting with a more convenient form,

$$d[r(t) - c_X(t)] = -m_X[r(t) - c_X(t)]dt + \int_{z=0}^{r(t)X} h(z/r(t))dN(dz, t). \tag{17}$$

Here $m_X = m + \int_0^X dx h(x)$ and c_X is introduced via $\phi(t) = dc_X(t)/dt + m_X c_X(t)$. Since $\phi(t)$ is taken to be positive, $c_X(t)$ solves to be positive. $c_X(0)$ can be conveniently chosen, say as $r(0)$ or $\phi(0)/m_X$. The algorithm reads as follows.

1. Set $t_o = 0$ and $r = r(0)$.

2. Draw an independent exponentially distributed unit mean random number v. Set t to the next event arrival time $t_o + v/Z$ where $Z = rX$, or the time horizon whichever is earlier.

3. Update r to r_- given by

$$r_- = (r - c_X(t_o))e^{-m_X(t-t_o)} + c_X(t). \tag{18}$$

4. If t is the time-horizon, go to step 6.

5. Draw an independent uniformly distributed random number $w \in [0, 1]$. Update r_- to

$$r = r_- + h(x), \qquad \text{where } x = wZ/r_-. \tag{19}$$

Note that $h(x) = 0$ if $x > X$. Set $t_o = t$ and go to step 2.

6. Collect this sample or value a derivative. For the next scenario, go to step 1.

7. From all the samples thus obtained, determine the distribution, or average the values to obtain a price for the derivative.

An attractive feature of the algorithm is that it does not involve discretization of time. Some improvements are possible to ensure that $Z \geq r(t)X$ in between Poisson events if $c_X(t)$ increases with t and can make r_- larger than r before the next event arrival time. Note that, since jumps are nonnegative, $r(t)$ never goes below $c_X(t)$ (consider $c_X(0) = r(0)$). Hence, because $c_X(t) > c_\infty(t)$ for any finite X (and $t > 0$), to sample $r(t)$ close to its lower bound of $c_\infty(t)$, X will have to be very large. For the $\alpha-$root process, $c_\infty(t)$ is zero and there will always be some region left unsampled near zero for any finite X. This deficiency is corrected in the updated algorithm discussed below.

For $h(x) = ax^{-1/\alpha}$, $1 < \alpha < 2$, there is an issue of convergence. The $x-$integral in (3), limited to $x \leq X$, can be approximated as

$$-\alpha\Gamma(-\alpha)(aB)^\alpha + \frac{\alpha}{2(2-\alpha)}(aB)^2 X^{1-2/\alpha} - \frac{\alpha}{6(3-\alpha)}(aB)^3 X^{1-3/\alpha} + \mathcal{O}\left((aB)^4 X^{1-4/\alpha}\right).$$

Note that, as $\alpha \to 2$, the second term tends to be of the same order as the leading contribution. This makes our Monte Carlo not useful near $\alpha = 2$. Fortunately, there is an interesting solution. Consider extending process (1) to include another set of Poisson processes. If

identical to the first, but with its jump function $h(y) = by^{-1/\omega}$ for some parameters b, ω and cutoff Y, this adds a $y-$integral to (3) that can be approximated as above. Note that the sign of the second term in its expansion can be made negative by choosing $\omega > 2$, or ω large enough to keep $(bB)^\omega$ term farther away. Any such ω could be chosen, in fact, $\omega = \infty$ turns out to be a good choice. For $\omega = \infty$, $h(y) = b$ for $y \leq Y$ and zero otherwise, and the added process is effectively just one Poisson process. Its $y-$integral is $(1 - bB - e^{-bB})Y$ that can be expanded in powers of bB. Parameter b can be chosen so as to cancel the troubling term. The x and $y-$integrals then together get approximated to $-\alpha\Gamma(-\alpha)(aB)^\alpha$.

However, convergence is still not satisfactory, and the issue of the unsampled region near zero remains. Hence, consider extending process (1) with one more Poisson process with its jump function $h(y) = -c$ for $y \leq Y$ and zero otherwise. It is now possible to choose b and c to cancel both the $(aB)^2$ and $(aB)^3$ terms. The equations for b and c turn out to be cubic that can be solved to obtain

$$b = aq(s + d)X^{-1/\alpha}, \quad c = aq(s - d)X^{-1/\alpha}, \tag{20}$$

where

$$s = \sqrt{1/2 - d^2},\ d = \cos((\pi+\theta)/3),\ \theta = \cos^{-1}(p/q^3),\ p = \frac{\alpha X}{(3-\alpha)Y},\ q = \sqrt{\frac{\alpha X}{(2-\alpha)Y}}. \tag{21}$$

As long as $Y/X \leq \alpha(3-\alpha)^2/(2-\alpha)^3$, this gives a solution $b \geq c \geq 0$. To keep the higher order terms introduced by the added processes small, Y should not be too small relative to X. The next correction term is then of $\mathcal{O}(X^{1-4/\alpha})$. The region near zero now gets sampled because of negative jumps introduced. As one gets closer to $r = 0$, the total Poisson intensity becomes small, and hence the likelihood of getting into negative $r-$values is small.

Changes to Monte Carlo are straightforward. There is an additional positive contribution $(b - c)Y$ to m_X. Total Poisson intensity is now $Z_X + 2Z_Y$ where $Z_X = rX$ and $Z_Y = rY$. Further in step 5, the original process is chosen with probability $X/(X + 2Y)$ and the two added processes with probabilities $Y/(X + 2Y)$ each, and an appropriate jump is added to r_- (based on $x = wZ_X/r_-$ or $y = wZ_Y/r_-$). If r does end up negative after adding $-c$ in step 5, it is set to zero (or infinitesimally small). For the present simulation results, Y is chosen to be equal to X. To improve efficiency, Sobol sequences are used to generate each of the independent random numbers.

As $\alpha \to 2$, $a = \sigma[\alpha\Gamma(-\alpha)]^{-1/\alpha}$ tends to zero for a given σ, but b and c tend to a nonzero value ensuring that the square root process is simulated appropriately in the limit as trinomial branching. For processes with generic jump functions, there can be a similar issue of convergence depending on the behavior of $h(x)$ as $x \to \infty$, and a similar approach to convergence can be attempted. The algorithm is also adaptable to the case of standard mean reverting $\alpha-$stable Lévy processes, or more general Lévy processes (Ornstein-Uhlenbeck-Type processes) of infinite activity with $r-$independent jump functions, extended to include negative jumps if desired. The analysis of section 1 can be carried through to obtain the well-known results. The $x-$integral then appears in an equation containing the drift term and, for simulation purpose, the process can be rewritten with a cutoff introducing an appropriately redefined drift term if necessary. In this context, the issue of convergence was addressed in Asmussen and Rosiński (2001) with the addition of a Brownian component that effectively cancels out the $(aB)^2$ term.

4 Simulation Results

Results of the Monte Carlo simulation for constant drift ϕ and a choice of other parameters are presented Figures 1-10 (t is set to zero). Figures 1-5 present the dependence of the probability distribution of $r(T)$ at $T = 5$ on α, σ, m and ϕ. Figure 6 shows the dependence on T itself. As can be seen from Figure 7, X need not be too large. To understand the order of magnitude of X, note that the total intensity of Poisson processes starts off at $3r(0)X$ that is about 10 for $r(0) = 0.03$ and $X = 100$, and corresponds to a time-step of about 0.1. To confirm the accuracy, the Laplace exponent is computed and displayed in Figure 8 for $\alpha = 3/2, 5/3$ and 2 for which closed form expressions are available from section 2.

An usual approach to understanding the distribution of a positive random variable is to compare it to a lognormal one. This can be done by computing the implied Black-Scholes volatility for a call or a put option on $r(T)$ at various strikes, ignoring discounting and setting the underlying to $\mathrm{E}_0(r(T))$. The resulting volatility smile is plotted in Figure 9 for different values of α. Figure 10 shows its dependence on T. The smile features are encouraging and further study is needed to confirm their applicability.

A Semi-Analytics

Because affine processes have been well-studied, analytics of an $\alpha-$root process can be written down as a special case. However, for our purpose, it is simpler and more illuminating to derive the same starting with the pure-jump process

$$dr(t) = [\phi(t) - m_X r(t)]\, dt + \int_{z=0}^{\infty} h_X(z/r(t)) dN(dz, t). \tag{22}$$

Here $h_X(x) = 0$ for $x > X$ given a large X and $h_X(x) = h(x)$ for $x \leq X$. This effectively cuts off the integral over z at the higher end ensuring that the total intensity of the Poisson processes is finite. The object of interest is the following expectation value

$$F_T(r(t), t) \equiv \mathrm{E}_t \left\{ \exp\left[-\int_t^T ds u_T(s) r(s) \right] \right\}. \tag{23}$$

Its differential can be written down using Ito's calculus leading to

$$\frac{\partial F_T}{\partial t} + (\phi - m_X r)\frac{\partial F_T}{\partial r} - u_T r F_T + r \int_0^{\infty} dx \left[F_T(r + h_X(x), t) - F_T(r, t) \right] = 0. \tag{24}$$

Integration variable z is scaled to $x = z/r(t)$. The above can be solved with the ansatz

$$F_T(r(t), t) = \exp\left[-A_T(t) - B_T(t) r(t) \right]. \tag{25}$$

Equating coefficients of F_T independent of r and those linear in r separately gives

$$\begin{aligned} \frac{dA_T(t)}{dt} + \phi(t) B_T(t) &= 0, \\ \frac{dB_T(t)}{dt} - m_X B_T(t) + u_T(t) + \int_0^X dx \left\{ 1 - \exp\left[-h(x) B_T(t) \right] \right\} &= 0. \end{aligned} \tag{26}$$

Consider now $u_T(t) = u(\tau)$ as a function of $\tau = T - t$ only. Then $B_T(t) = B(\tau)$ is also a function of τ only, satisfying $B(0) = 0$ and the differential equation

$$\frac{dB(\tau)}{d\tau} + m_X B(\tau) = u(\tau) + \int_0^X dx \left\{1 - \exp\left[-h(x)B(\tau)\right]\right\}. \tag{27}$$

With $h(x)$ assumed to go to zero as $x \to \infty$, the integrand above goes to zero as $h(x)B(\tau)$, so that the above equation tends to be independent of X for large X if $dm_X/dX = h(X)$. A choice for m_X with such a large X behavior is $m_X = m + \int_0^X dxh(x)$ (assuming $h(x)$ is integrable from $x = 0$). Equation for $B(\tau)$ then reads as in (3) in the limit $X \to \infty$. Given a solution for $B(\tau)$ satisfying $B(0) = 0$, and $A_T(t)$ expressed as an integral of $B(\tau)$, solution for $F_T(r(t), t)$ is as given in equation (2).

If $u(\tau) = u\delta(\tau - 0_+)$ where $\delta(\tau - 0_+)$ is a Dirac-delta function sitting just above $\tau = 0$, one obtains the Laplace transform $\mathrm{E}_t\left\{\exp\left[-ur(T)\right]\right\}$ of the probability density function of $r(T)$, or its negative logarithm known as the Laplace exponent. For this, equation (3) is solved for $B(\tau)$ in the absence of $u(\tau)$, but under the initial condition $B(0) = u$.

For $h(x) = ax^{-1/\alpha}$, $1 < \alpha < 2$, equation for $B(\tau)$ reads as in (4). The $x-$integral in (3) is $-\alpha\Gamma(-\alpha)a^\alpha\left[B(\tau)\right]^\alpha$. Note that $\int_0^X dxh(x) = a\alpha X^{(\alpha-1)/\alpha}/(\alpha - 1)$ diverges as $X \to \infty$, but gets absorbed into m_X. The $\alpha = 1$ case is special. The $x-$integral in (27) is $-aB(\tau)\ln B(\tau)$ up to terms linear in $B(\tau)$ that are taken care of by $m_X = m + a\ln(X/a) + a(1 - \gamma)$ where γ is the Euler's constant.

One may wonder whether an $\alpha-$root process can be defined for $\alpha < 1$ as well. After all, the $x-$integral in (27) is then finite as $X \to \infty$ and is $-\alpha\Gamma(-\alpha)a^\alpha\left[B(\tau)\right]^\alpha$. However, the integral dominates the $m_X B(\tau)$ term as $B(\tau) \to 0$, and solving for the Laplace exponent with $u(\tau) = 0$ and $B(0) = u$ yields a $B(\tau)$ that does not go to zero as $u \to 0$.

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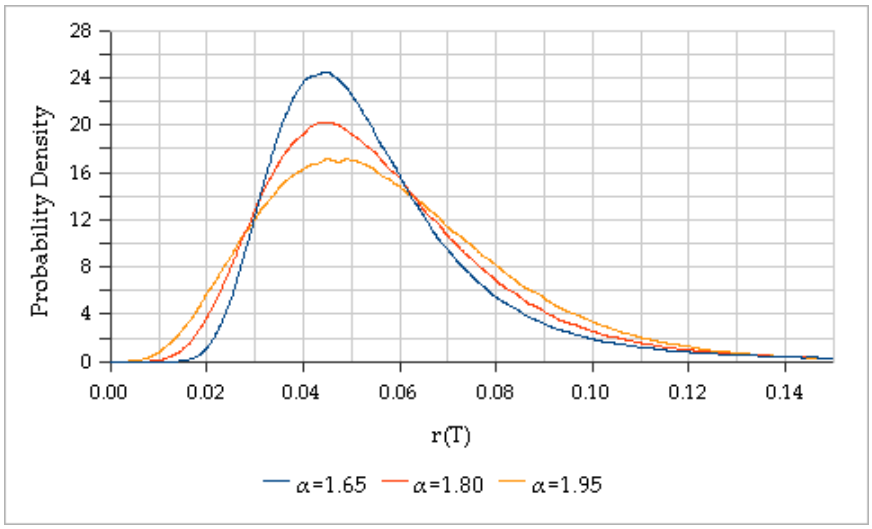

Figure 1: Plots of the probability density functions for $\alpha = 1.65, 1.80$ and 1.95. Other parameters chosen are $T = 5, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is one million and cutoff X is 100.

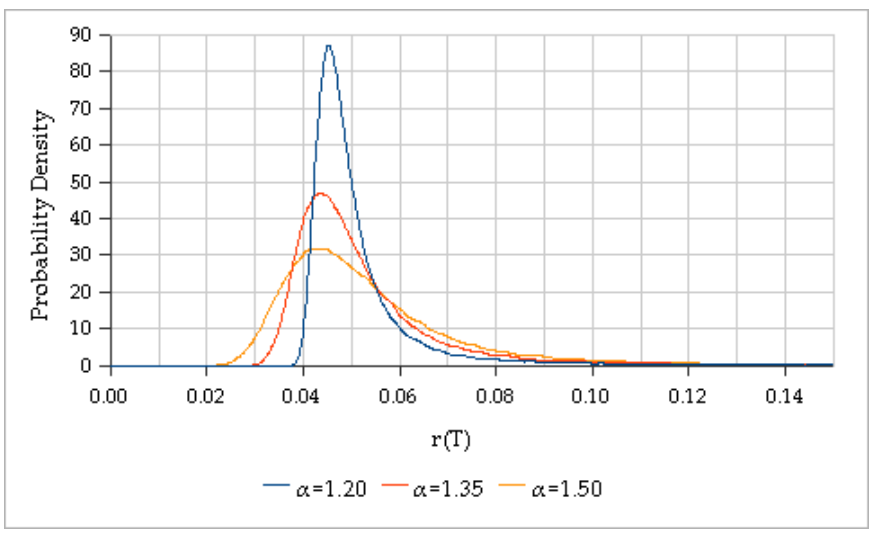

Figure 2: Plots of the probability density functions for $\alpha = 1.20, 1.35$ and 1.50. Other parameters chosen are $T = 5, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

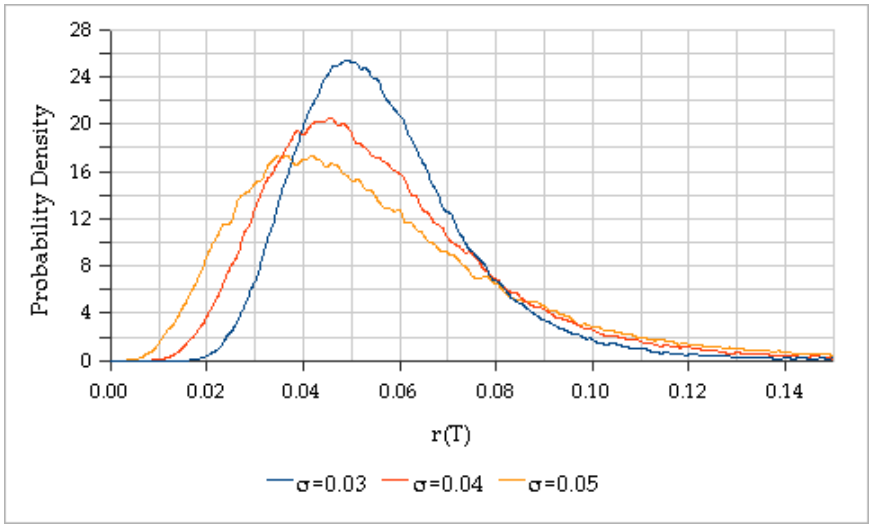

Figure 3: Plots of the probability density functions for $\sigma = 0.03, 0.04$ and 0.05. Other parameters chosen are $T = 5, \alpha = 1.80, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

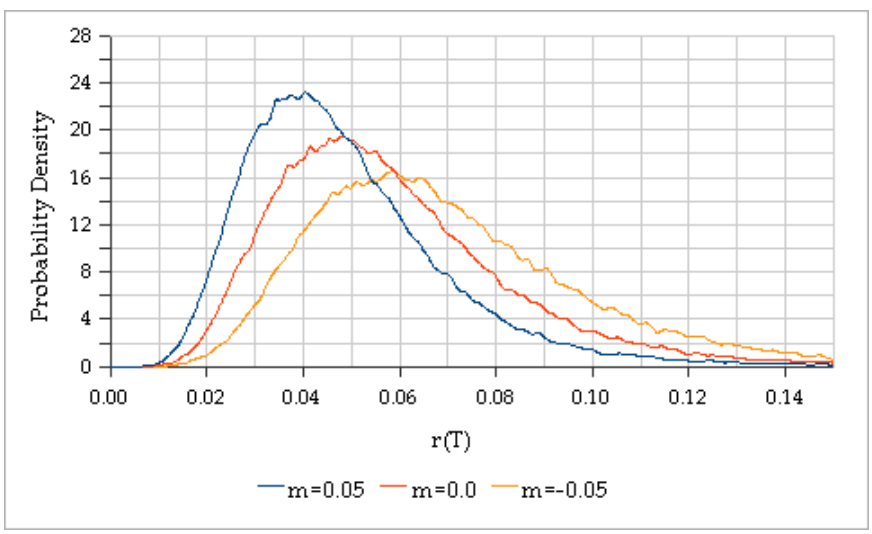

Figure 4: Plots of the probability density functions for $m = 0.05, 0.0$ and -0.05. Other parameters chosen are $T = 5, \alpha = 1.80, \sigma = 0.04, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

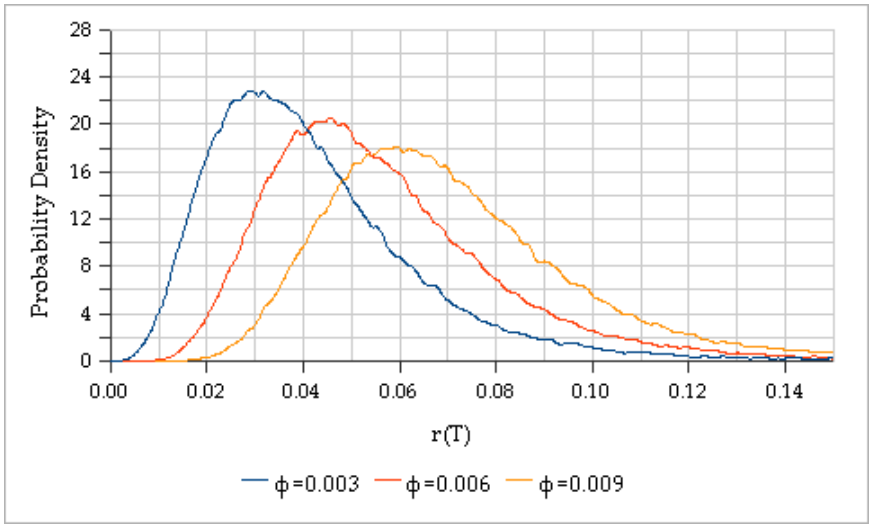

Figure 5: Plots of the probability density functions for $\phi = 0.003, 0.006$ and 0.009. Other parameters chosen are $T = 5, \alpha = 1.80, \sigma = 0.04, m = 0.01$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

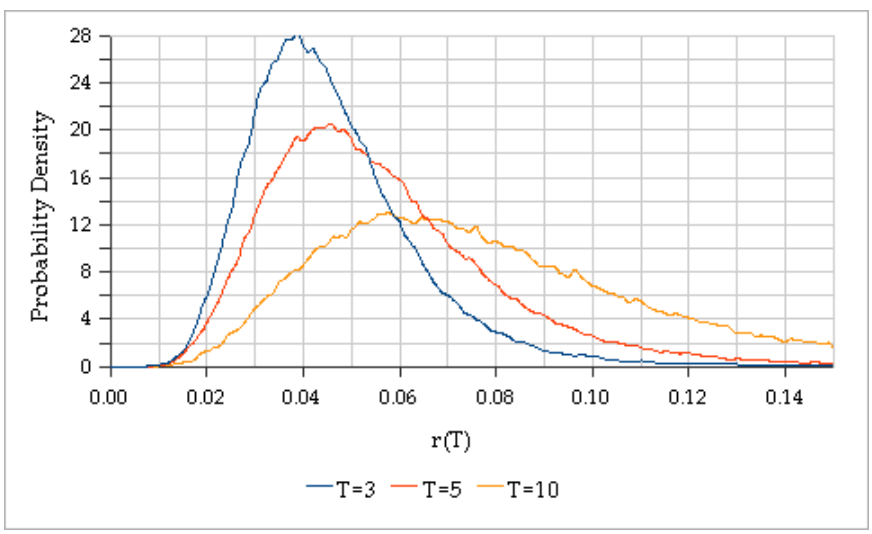

Figure 6: Plots of the probability density functions for $T = 3, 5$ and 10. Other parameters chosen are $\alpha = 1.80, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

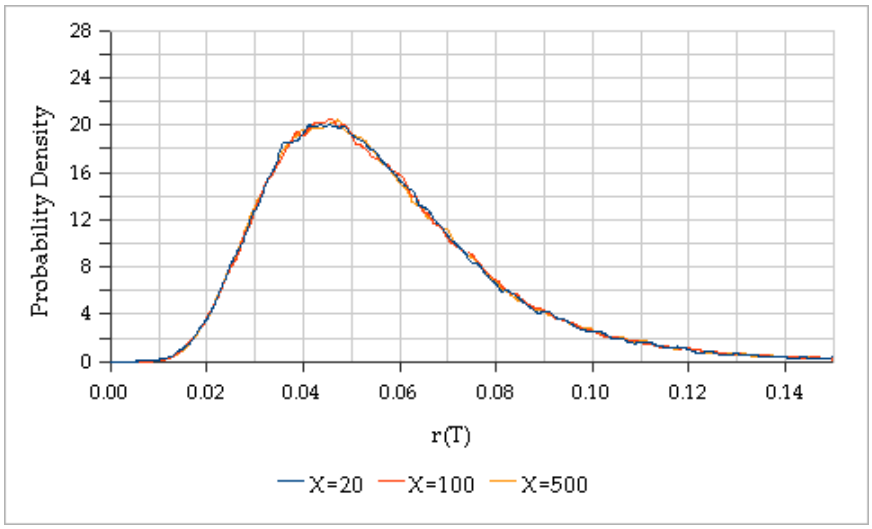

Figure 7: Plots of the probability density functions for cutoff $X = 20, 100$ and 500. Other parameters chosen are $T = 5, \alpha = 1.80, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000.

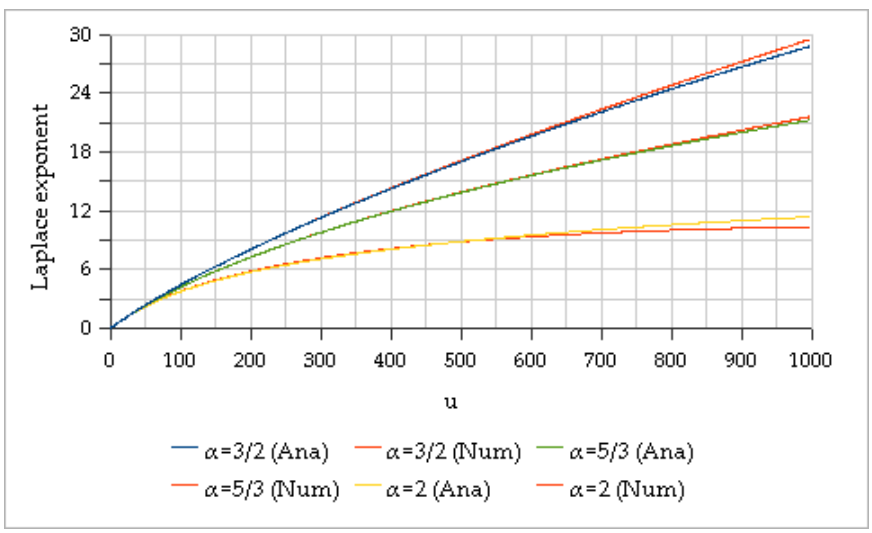

Figure 8: Plots of the Laplace exponents computed analytically and numerically for $\alpha = 3/2, 5/3$ and 2. Other parameters chosen are $T = 5, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

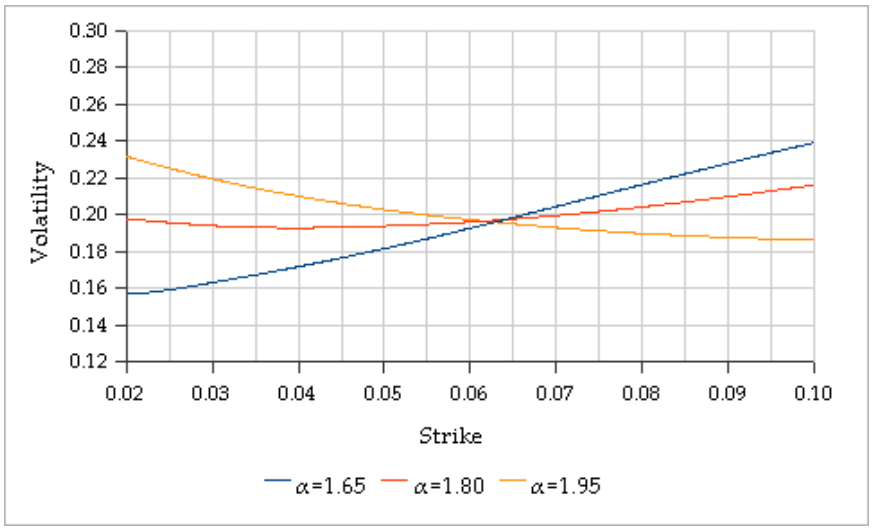

Figure 9: Plots of the volatility smiles for $\alpha = 1.65, 1.80$ and 1.95. Other parameters chosen are $T = 5, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff $X = 100$.

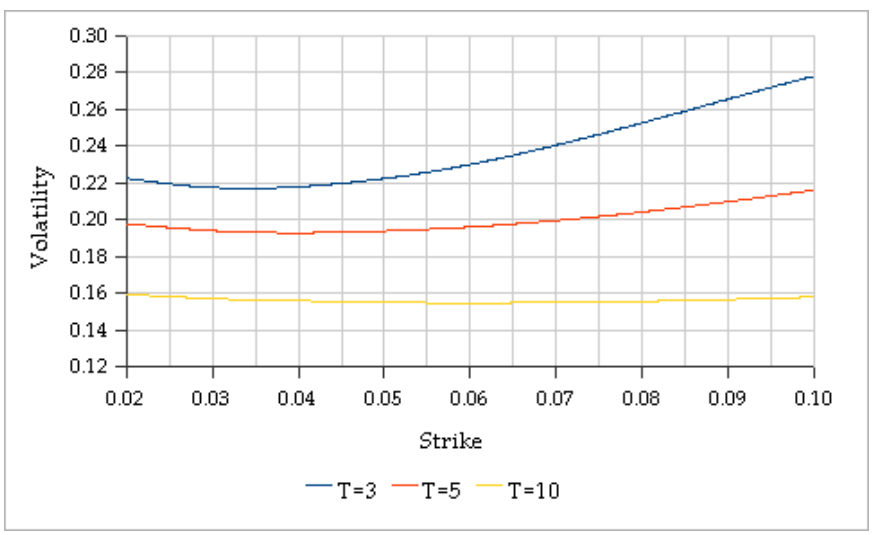

Figure 10: Plots of the volatility smiles for $T = 3, 5$ and 10. Other parameters chosen are $\alpha = 1.80, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff $X = 100$.