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Geometrical Considerations on Heston's Market Model

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Abstract

We propose to discuss a new technique to derive an good approximated solution for the price of a European call and put options, in a market model with stochastic volatility. In particular, the model that we have considered is the Heston's model. This allows arbitrary correlation between volatility and spot asset returns. We are able to write the price of European call and put, in the same form in which one can see in the Black-Scholes model. The solution technique is based upon coordinate transformations that reduce the initial PDE in a straightforward one-dimensional heat equation.

1 Introduction

The Heston's model is versatile to describe stock options, bond options and currency options, as the same S. Heston [1993] shows in his work. In particular it links the biases to the dynamics of the spot price and the distribution of spot returns. All option models with the same volatility are equivalent for at-the-money options. Since options are usually traded near-the-money, this explains some of the empirical support for the Black-Scholes model. Correlation between volatility and the spot price is necessary to generate skewness, and this in the distribution spot returns affects the pricing of in-the-money options relative to out-of-the-money options. It is worth noting that without this correlation, stochastic volatility only changes the Kurtosis. Kurtosis affects the pricing of near-the-money versus far-from-the-money. These are the principal characteristics of Heston's model. Several authors use, Heston's model for its features, in order to make pricing and to make hedging strategy in Finance, as one can see in the references section.

By means our paper we want to illustrate a new technique, by which we are able to transform original PDE, in a straightforward one-dimensional heat equation. This results is important, because exists and it is known its analytical solution. In order to obtain the above coordinate transformations, we discuss briefly the stochastic models. In a continuos-time framework, the random volatility σ_t is usually assumed to obey a diffusion-type process. Let the stock price S_t be given as

$$dS_t = \mu(S_t, t)dt + \sigma_t S_t dW_t^{(1)} \tag{1}$$

with the stochastic volatility σ_t (also known as the instantaneous volatility or the spot volatility) satisfying

$$d\sigma_t = a(\sigma_t, t)dt + b(\sigma_t, t)dW_t^{(2)} \tag{2}$$

where $W^{(1)}$ and $W^{(2)}$are standard one-dimensional Brownian motions defined on some filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, with the cross-variation satisfying $dW_t^{(1)} dW_t^{(2)} = \rho dt$ for some constant $\rho \in [-1, 1]$. Recall that the Brownian motions $W^{(1)}$ and $W^{(2)}$ are mutually independent if and only if they are uncorrelated, that is, when $\rho = 0$. More generally, we may assume that ρ_t is a stochastic process adapted to the filtration $\mathbb{F}$generated by $W^{(1)}$ and $W^{(2)}$. For a fixed horizon date T, a martingale measure $\mathbb{Q}$ for the process $S_{t,(Q)} = S_t/B_t$ is defined as probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathbb{F}_{\mathbb{T}})$ such that $S_{(Q)}$is a local martingale under $\mathbb{Q}$. Under any martingale measure $\mathbb{Q}$, we have

$$dS_t = rS_t, dt + \sigma_t S_t dW_{t,(Q)}^{(1)} \tag{3}$$

with the spot volatility σ satisfying

$$d\sigma_t = \overline{a}(\sigma_t, t)dt + b(\sigma_t, t)dW_{t,(Q)}^{(2)} \tag{4}$$

for some drift coefficient $\overline{a}_t$. We shall adopt a commonly standard convention that

$$\overline{a}_t(\sigma_t, t) = a(\sigma_t, t) + \lambda(\sigma_t, t)b(\sigma_t, t) \tag{5}$$

for some (sufficiently regular) function $\lambda(\sigma_t, t)$. The presence of the additional term in the drift of the stochastic spot volatility σ under an equivalent martingale measure is an immediate consequence of Girsanov's theorem. A specific form of this term, as given in (5), is a matter of convenience and its choice is motivated by practical considerations. Under suitable regularity conditions, a unique solution (S_t, σ_t) to (non-linear) stochastic differential equations (3), (4) is known to follow a two dimensional diffusion process; results concerning the existence and uniqueness of the SDEs can be found, e.g., in Ikeda and Watanabe (1981) or Karatzas and Shreve (1998). The existence of an equivalent probability measure under which the process $S_{t,(Q)} = S_t/B_t$ is a martingale (as opposed to a local martingale) is a non trivial issue, however, and thus it needs to be examined on a case-by-case for each particular stochastic volatility model. Stochastic volatility models of the stock price are also supported by empirical studies of stock returns. Early studies of market stock prices (reported in Mandelbrot (1983), Fama (1985), Praetz (1972), and Blattberg and Gonedes (1974)) concluded that the lognormal law is an inadequate desriptor of stock returns. More recent studies (see, for instance, Hsu. (1974) and Kon (1984)) have found that the mixture of Gaussian distributions. Ball and Torous (1985) have empirically estimated models of returns as mixtures of a continuos and jump processes. Empirical studies of Black (1976), Schmalensee and Trippi (1978), and Christie (1982) uncoverd an invese correlation between stock returns and changes in volatility. This peculiar feature of stock returns supports the conjecture that the stock price volatility should be modelled by means of an autonomous stochastic process, rather then as a function of the underlying asset price.

2 PDE Approach

Generally speaking, stochastic volatility models are not complete, and thus a typical contingent claim (such as a european option) cannot be priced by arbitrage. In Other words, the standard replication arguments cannot longer be applied to most contingent claims. For this reason, the issue of valuation of derivative securities under market incompleteness has attracted considerable attention in recent years, and various alternative approaches to this problem were subsequently developed. Seen form a different perspective, the incompleteness of a generic stochastic volatility model is reflected by the fact that the class of all martingale measure for the process $S_{t,(Q)} = S_t/B_t$ comprises more than on probability measure, and thus the necessity of specifying a single pricing probability arises.

Since under (3), (4) we deal with a two-dimensional diffusion process, it is possible to derive, under mild additional assumptions, the partial differential equation satisfied by the

value function of a European contingent claim. For this purpose, one needs first to specify the market price of volatility risk $\lambda(\sigma, t)$. Mathematically speaking, the market price for the risk is associated with the Girsanov transformation of the underlying probability measure leading to a particular martingale measure. Let us observe that pricing of contingent claims using the market price of volatility risk is not preferences-free, in general (typically, one assumes that the representative investor is risk-averse and has a constant relative risk-aversion utility function).
to illustrate the PDE approach mentioned above, assume that the dynamics of two dimensional diffusion process (S, σ) under a martingale measure are given by (3), (4), with Brownian motions $W^{(1)}_{(Q)}$ and $W^{(2)}_{(Q)}$ such that $dW^{(1)}_{(Q)} dW^{(2)}_{(Q)} = \rho dt$ for some constant $\rho \in [-1, 1]$. Suppose also that both processes S and σ, are nonnegative. Then the price function $f = f(s, \sigma, t)$ of a European contingent claim is well known to satisfy a specific PDE (see for instance, German (1976) or Hull and White (1976)).

Proposition Consider a European contingent claim $Y = g(S_T)$ that settles at time T. Assume that the price of Y is given by the risk-neutral valuation formula under $\mathbb{Q}$ for the process $S_{t,(Q)} = S_t/B_t$. Then the pricing function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ solves the PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \rho\sigma s b(\sigma, t)\frac{\partial^2 f}{\partial s \partial t} + \frac{1}{2} b(\sigma, t)^2 \frac{\partial^2 f}{\partial \sigma^2} + rs\frac{\partial f}{\partial s} + [a(\sigma, t) + \lambda(\sigma, t) b(\sigma, t)]\frac{\partial f}{\partial \sigma} - rf = 0, \quad (6)$$

with the terminal condition $\phi(s) = f(s, \sigma, T)$ for every $s \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+$.

Let us stress once again that we do not claim here that $\mathbb{Q}$ is a unique martingale measure for a given model. Hence unless volatility-based derivatives are assumed to be among primary assets, the market price of volatility risk needs to be exogenously specified. For some specifications of stochastic volatility dynamics and the market price of volatility risk, a closed-form expression for the option's price is available. In other cases, suitable numerical procedures need to be employed. Since we deal based on the discretization of the partial differential equation satisfied by the pricing function appear excessively time-consuming. An alternative Monte Carlo approach for stochastic volatility models was examined by Fournié (1997).

3 Heston's Model and Pricing Options

A widely popular stochastic volatility model, proposed by Heston (1993), assumes that the asset price S satisfies

$$dS_t = S_t(\mu_t dt + \sqrt{\nu}_t)dW_t^{(1)} \qquad S \in [0, \infty) \tag{7}$$

with the instantaneous variance ν governed by the SDE

$$d\nu_t = k(\theta - \nu_t)dt + \alpha\sqrt{\nu_t}dW_t^{(2)}, \qquad \nu \in (0, \infty); \quad k, \theta, \alpha \in \mathbb{R} \tag{8}$$

where $W^{(1)}$ and $W^{(2)}$ are standard one-dimensional Brownian motions defined on filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, which the cross-variation $\langle W^{(1)}, W^{(2)}\rangle = \rho t$ for some constant $\rho \in [-1, 1]$. In this case, it is more convenient to express the pricing function f and the market price of volatility risk λ in terms of variables (S, ν, t), rather than (S, σ, t). We now make a judicious choice of the market price of volatility risk; specifically, we set $\lambda(\nu_t, t) = \lambda\sqrt{\nu_t}$ for some constant λ such that $\lambda\alpha \neq k$. Hence, under a martingale measure $\mathbb{Q}$, equations (3.16)-(3.17) became

$$dS_t = S_t(rdt + \sqrt{\nu_t})dW_{t,(Q)}^{(1)} \tag{9}$$

and

$$d\nu_t = \kappa(\Theta - \nu_t)dt + \alpha\sqrt{\nu_t}dW_{t,(Q)}^{(2)} \tag{10}$$

where we set

$$\kappa = (k - \lambda\alpha), \qquad \Theta = \theta k(k - \lambda\alpha)^{-1}, \tag{11}$$

and where $W_{(Q)}^{(1)}$ and $W_{(Q)}^{(2)}$ are standard one-dimensional Brownian motions such that $\langle dW^{(1)}, dW^{(2)}\rangle = \rho dt$. It is now easy see that the pricing PDE for European derivatives in Heston model, by Itô's lemma, has the following form:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 f}{\partial S^2} + \rho\nu\alpha S\frac{\partial^2 f}{\partial S\partial\nu} + \frac{1}{2}\nu\alpha^2\frac{\partial^2 f}{\partial \nu^2} + \kappa(\Theta - \nu)\frac{\partial f}{\partial \nu} + rS\frac{\partial f}{\partial S} - rf = 0 \tag{12}$$

with the terminal condition $f(S, \nu, T) = \phi(S)$ for every $S \in \mathbb{R}_+$, $\nu \in \mathbb{R}_+$ and $t \in [0, T]$. We take here for granted the existence and uniqueness of (nonnegative) solutions S and ν to Heston's SDE. It is common to assume $2K\Theta/\alpha^2 > 1$, so that, the solution ν is strictly positive if $\nu_0 > 0$.

4 Numerical methods for Option Valuation

For the Heston model, the closed form solution does not exist, but we can obtain its solution by numerical techniques, like:

• Finite Difference method (Crank Nicolson);

• Monte-Carlo simulation method combined with a variance reduction technique:

• Fourier transform technique.

• Geometrical approximation.

Here, we want to highlight some important aspects. The PDE method is a flexible method which can be used for many pay-offs: European Options or certain path dependent derivatives; in this case, the drawback is that we have to approximate the option prices on a grid. Accurate pricing requires a substantial amount of grid points. The PDE method is somewhat expensive.

The Monte-Carlo method is the most general, but it has long computation times.

The Fourier transformation technique has been used to evaluate the model option prices. This method is both fast and accurate. Its major technical difficulty lies in the derivation of a characteristic function, i.e., the Fourier transform of the risk-neutral density function. See Carr and Madan for further details. The Fourier transformation technique can take advantage of a very numerical algorithm called the Fast Fourier Transform (FFT) technique, which drastically improves the numerical efficiency of the calibration.

For more details about above methods, see the references. Now, we focus on proposed method, that we have called "Geometrical Approximation", it is based only on considerations about the pay-off function. For suitable values of ρ, ν, α, where $\epsilon = \frac{\rho\nu}{\alpha} << 1$, we have a closed form solution of the exact PDE, but with modified Cauchy condition, in which we consider the following pay-off function $(S_T e^{-\frac{\rho\nu}{\alpha}} - E)^+$, instead of, the standard pay-off function $(S_T - E)^+$. It is clear that the former goes to the latter for ϵ that goes to zero.

$$e^{-\epsilon} \simeq (1-\epsilon)$$

thus

$$f(T,S,\nu) = \left(S_T e^{-\epsilon} - E\right)^{+} \simeq \left(S_T\left(1-\epsilon\right) - E\right)^{+}$$

$$\lim_{\epsilon \to 0} \left(S_T\left(1-\epsilon\right) - E\right)^{+} = \left(S_T - E\right)^{+}$$

In order to evaluate a European call option, first we simplify the PDE (12) at hand. To this end, let us introduce a new variable x and a new function f_1:

$$S = e^{x}, \quad \nu = \tilde{\nu}\alpha, \qquad x \in (-\infty,\infty), \qquad \nu \in [0,\infty), \qquad t \in [0,T]]$$
$$f(t,S,\nu) = e^{-r(T-t)} f_1(t,x,\tilde{\nu})$$

so that we have a new PDE

$$\frac{\partial f_1}{\partial t} + \frac{1}{2}\tilde{\nu}\alpha\left(\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x \partial\tilde{\nu}} + \frac{\partial^2 f_1}{\partial\tilde{\nu}^2}\right) + \frac{\kappa}{\alpha}\left(\Theta - \tilde{\nu}\alpha\right)\frac{\partial f_1}{\partial\tilde{\nu}} + \left(r - \frac{1}{2}\tilde{\nu}\alpha\right)\frac{\partial f_1}{\partial x} = 0, \tag{13}$$

now we consider only the terms that have derivatives of the second order and after that, we try a new set of coordinates that transform the PDE in its canonical form. It is important remember that our PDE, is of parabolic kind and its canonical form is the heat equation, and we want to transform the above PDE in a heat equation. First step, we write the characteristic equation associated to the second order terms of our PDE (13) , thus we compute its roots:

$$\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x \partial\tilde{\nu}} + \frac{\partial^2 f_1}{\partial\tilde{\nu}^2} = 0.$$

The characteristic equation results to be

$$\left(\frac{dx}{d\tilde{\nu}}\right)^2 - 2\rho\left(\frac{dx}{d\tilde{\nu}}\right) + 1 = 0,$$

$$\Delta = 4^2(1-\rho^2) \leq 0, \qquad \rho \in (-1,1)$$

so that the squared term is of elliptic kind, and the roots belong at the set of complex numbers

$$\left(\frac{dx}{d\tilde{\nu}}\right)_{1/2} = \rho \pm \imath\sqrt{1-\rho^2}.$$

At this point we can define the characteristic lines (or remembering what said in the chapter 1, these are also defined like geodesics) as follows

$$
\begin{aligned}
x - (\rho + \imath\sqrt{1-\rho^2})\tilde{\nu} &= z \\
x - (\rho - \imath\sqrt{1-\rho^2})\tilde{\nu} &= w.
\end{aligned}
$$

Through another change of variable, that we show hereafter, we obtain a linear system easy to solve

$$z = \xi + \imath\eta; \qquad w = \xi - \imath\eta;$$

so that results $w = \overline{z}$

$$
\begin{aligned}
\tilde{\nu} = -\frac{\eta}{\sqrt{1-\rho^2}} \qquad & x = \frac{\xi\sqrt{1-\rho^2} - \rho\eta}{\sqrt{1-\rho^2}} \\
\eta = -\tilde{\nu}\sqrt{1-\rho^2} \qquad & \xi = x - \rho\tilde{\nu}
\end{aligned} \tag{14}
$$

where $\eta \in (-\infty, 0)$ and $\xi \in (-\infty, \infty)$ and it is clear that our function f_1 must be transformed in another, that we call f_2.

In order to understand our method, is useful make the following geometrical consideration. We have defined a new system of coordinates, where $\vec{e}_\eta, \vec{e}_\xi, \vec{e}_t$, are ortogonal directions; we can think x, ν as vectors, whose projections on the axes are respectively given by

$$\vec{x} = (0)\vec{e}_\eta + (x)\vec{e}_\xi \qquad \vec{\tilde{\nu}} = (\tilde{\nu}\cos\theta_\rho)\vec{e}_\eta + (\tilde{\nu}\sin\theta_\rho)\vec{e}_\xi$$

where, we have supposed $\rho = \sin\theta_\rho$ and $\sqrt{1-\rho^2} = \cos\theta_\rho$, $\theta_\rho \in (-\pi/2, \pi/2)$. Now we can define a new vector, that we call $\vec{V}$, whose projections are

$$\vec{V} \equiv (V_\eta, V_\xi) \qquad V_\eta = -\tilde{\nu}\cos\theta_\rho \qquad V_\xi = x - \tilde{\nu}\sin\theta_\rho$$

where $\theta_\rho \in (-\pi/2, \pi/2)$, and by which, we can show the vectorial relation that exists between the variables $(x, \tilde{\nu})$. Now, from the Cauchy's condition, we are able to write the new function f_2, like function of variables t and $V_\xi(x, \tilde{\nu})$, because, the function f depends, at the time T, only from the projection terms upon the axis ξ,

$$f(T, S, \nu) = (S_T - E)^+ = \left(e^{x'} - E\right)^+ = \left(S' e^{-\frac{\rho\nu'}{\alpha}} - E\right)^+$$

(where with the apex ($'$) we indicate the variables at the time $t = T$), therefore, because of the continuity properties of the Feynman-Kač formula, we can suppose that is true at any time t.

$$f_1(t, x, \tilde{\nu}) = f_2(t, V_\xi(x, \tilde{\nu})); \qquad t \in [0, T]$$

now we may substitute them in the old squared term

$$\frac{\partial^2 f_1}{\partial x^2} + 2\rho \frac{\partial^2 f_1}{\partial x \partial \tilde{\nu}} + \frac{\partial^2 f_1}{\partial \tilde{\nu}^2} = (1 - \rho^2)\nabla^2_{V_\xi} f_2(t, V_\xi(x, \tilde{\nu})).$$

Thus, the new Black-Sholes PDE of Hesten's model is become

$$\frac{\partial f_2}{\partial t} - \frac{\alpha V_\eta}{\sqrt{1-\rho^2}} \left[\frac{(1-\rho^2)}{2} \frac{\partial^2 f_2}{\partial V_\xi^2} + \left(\frac{1}{2} - \frac{\kappa}{\alpha}\rho\theta \right) \frac{\partial f_2}{\partial V_\xi} \right] + \left(r - \frac{\kappa}{\alpha}\rho\theta \right) \frac{\partial f_2}{\partial V_\xi} = 0 \qquad (15)$$

where we have changed the final condition $(S_T - E)^+$, in

$$\left(S' e^{-\frac{\rho\nu'}{\alpha}} - E \right)^+ = \left(e^{V'_\xi} - E \right)^+$$

Now, we can compute the solution of PDE (15) in closed form, that is an approximation of the original problem for $\frac{\rho\nu}{\alpha} << 1$.

It is worth noting that is sufficient another change of coordinates to simplify last PDE. We may define a new transformation of coordinates; and the new function f_3, as follows

$$\gamma = V_\xi + \left(r - \frac{k}{\alpha}\rho\theta \right)(T - t), \qquad \gamma \in (-\infty, \infty);$$
$$\tau = -\int_t^T ds \frac{\alpha V_\eta}{\sqrt{1-\rho^2}} = \int_t^T ds \nu(s), \qquad \tau \in \left[0, \int_0^T ds \nu(s) \right];$$
$$f_2(t, V_\xi) = f_3(\tau(t, V_\eta), \gamma(t, V_\xi));$$

for $t = T$ we have

$$f_3(0, \gamma') = \left(e^{\gamma'} - E \right)^+.$$

Substituting what we have just found, in the previous equation, we have finally a very easy partial differential equation

$$\frac{\partial f_3}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla_\gamma^2 f_3 + \left(\frac{1}{2} - \frac{\kappa\rho}{\alpha}\right)\frac{\partial f_3}{\partial \gamma}$$
$$\gamma \in (-\infty, \infty), \quad \tau \in \left[0, \int_0^T ds\nu(s)\right]; \tag{16}$$

Now we can rewrite the function f_3 as follows, in order to obtain the one-dimensional heat equation:

$$f_3(\tau,\gamma) = e^{\lambda\tau+\beta\gamma} f_4(\tau,\gamma);$$

where

$$\lambda = -\frac{(1/2 - \kappa\rho/\alpha)^2}{2(1-\rho^2)}, \qquad \beta = -\frac{(1/2 - \kappa\rho/\alpha)}{(1-\rho^2)};$$

so that substituting, we have

$$\frac{\partial f_4}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla_\gamma^2 f_4$$

At this point our final value problem is became another problem, more easy than before, indeed we have

$$\frac{\partial f_4}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla_\gamma^2 f_4 \qquad \gamma \in (-\infty, +\infty), \tau \in \left[0, \int_0^T ds\nu(s)\right]$$
$$f_4(0,\gamma') = \left(e^{\gamma'} - E\right)^+$$

Now, we are able to write the solution, that is

$$f_4(\tau,\gamma) = \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}}\int_{-\infty}^{+\infty} d\gamma' f_4(0,\gamma') \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$
$$= \int_{-\infty}^{\infty} d\gamma' f_4(0,\gamma') G(\gamma', 0|\gamma,\tau) \tag{17}$$

where

$$G(\gamma', 0|\gamma,\tau) = \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}} \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

where

$$f(t,S,\nu) = e^{-r(T-t)+\lambda\tau+\beta\gamma} f_4(\tau,\gamma)$$

$$f(T,S,\nu) = e^{\beta\gamma'} f_4(0,\gamma')$$

$$f_4(0,\gamma') = e^{-\beta\gamma'}\left(e^{\gamma'} - E\right)^+$$

At this point we have

$$f_4(\tau,\gamma) = \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}} \int_{-\infty}^{+\infty} d\gamma' e^{-\beta\gamma'}\left(e^{\gamma'} - E\right)^+ \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}} \int_{\ln E}^{+\infty} d\gamma' e^{-\beta\gamma'}\left(e^{\gamma'} - E\right) \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

Thus we can write the price of a European Call option in Heston's market model as follows

$$f(t,S,\nu) = \frac{e^{-r(T-t)+\lambda\tau+\beta\gamma}}{\sqrt{2\pi(1-\rho^2)\tau}} \int_{\ln E}^{+\infty} d\gamma' e^{-\beta\gamma'}\left(e^{\gamma'} - E\right) \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$
$$= \left(S_t e^{-\frac{\rho\nu}{\alpha}}\right) e^{\delta_1^\rho}\mathbf{N}(d_1^\rho) - E e^{\delta_2^\rho}\mathbf{N}(d_2^\rho) \quad (18)$$

where

$$\delta_1^\rho = -\left[\frac{\kappa}{\alpha}\rho\Theta - \left(\lambda + \frac{(1-\beta)^2}{2}(1-\rho^2)\overline{\nu}\right)\right](T-t);$$
$$\delta_2^\rho = -\left[r - \left(\lambda + \frac{\beta^2}{2}(1-\rho^2)\right)\overline{\nu}\right](T-t);$$
$$\overline{\nu}_\rho = \frac{1}{(T-t)}\int_t^T ds(1-\rho^2)\nu(s)$$
$$\overline{\nu}_{\rho=0} = \overline{\nu} = \frac{1}{(T-t)}\int_t^T ds\nu(s)$$

$$d_1^\rho = \frac{\ln(S/E) - \frac{\rho}{\alpha}\nu + \left[\left(r - \frac{\kappa}{\alpha}\rho\Theta\right) + (1-\beta)\overline{\nu}_\rho\right](T-t)}{\sqrt{\overline{\nu}_\rho(T-t)}}$$
$$d_2^\rho = \frac{\ln(S/E) - \frac{\rho}{\alpha}\nu + \left[\left(r - \frac{\kappa}{\alpha}\rho\Theta\right) - \beta\overline{\nu}_\rho\right](T-t)}{\sqrt{\overline{\nu}_\rho(T-t)}}$$
$$d_2^\rho = d_1^\rho - \sqrt{\overline{\nu}_\rho(T-t)}$$

Thus for $\epsilon = \frac{\rho\nu}{\alpha} << 1$, the final value of Call option is given by:

$$C_{\rho,\alpha,\Theta,\kappa}(t, S_t, \nu_t) = S_t\,(1-\epsilon)\,e^{\delta_1^\rho}\mathbf{N}(d_1^\rho) - Ee^{\delta_2^\rho}\mathbf{N}(d_2^\rho), \tag{19}$$

and for a Put, the final value is

$$P_{\rho,\alpha,\Theta,\kappa}(t, S_t, \nu_t) = Ee^{\delta_2^\rho}\mathbf{N}(-d_2^\rho) - S_t\,(1-\epsilon)\,e^{\delta_1^\rho}\mathbf{N}(-d_1^\rho); \tag{20}$$

5 Numerical Test

Now, we can compare options prices calculated according to techniques described above, with our approximation method. The Monte-Carlo algorithm was implemented in $C++$ code, while others algorithm are implemented in $MatLab$ code. For $\rho = 0$, we obtain the Black-Scholes solution with averaged volatility, as in Hull-White formula. This proof that our approach, also if it is an approximation, it is correct. For values of $\rho = -1, +1$ we

Table 1: Black-Scholes price $S(0) = 100$, $E = 100$

σ_t	r	T	Value
0.1	0.03	0.5	3.6065
0.1	0.05	0.5	4.1923
0.3	0.03	1	13.2833
0.5	0.05	1	21.7926
0.5	0.05	5	49.5965

have two degenerate cases, and them are not interesting. In order to have any idea of the derivatives price, we compute Vanilla Call Option value in Balck-Scholes market model; and after that, one can see the price of Vanilla Call Option for Heston market model. Here, we have compared our method, G.A., with others obtained by Heston and Lipton, Fourier transform method, and by finite difference method, f.d.m.(Crank Nicolson). Our results are suitable, and this prove in analytical way, the goodness of method proposed. It is worth noting that our prices go to heston prices, by increasing maturity date T, unlike that for f.d. method. We compare also our results with those obtained by Monte Carlo method, for different values of parameters. As our tables show, we can be satisfied,

Table 2: Heston price $S(0) = 100$, $E = 100$, $Err = \|(Heston)_{value} - (G.A.)_{value}\|$

r	ρ	κ	α	Θ	ν_t	T	G.A. Value	H. Value	f.d.m. Value	Err
0.03	0.1	1.0	0.2	0.01	0.01	0.5	3.2992	3.4386	3.4376	0.1394
0.03	0.1	1.0	0.2	0.01	0.01	1	5.2461	5.2953	5.2840	0.0492
0.03	0.1	1.0	0.2	0.01	0.01	2	8.4954	8.4583	8.5943	0.0371
0.05	0.1	1.0	0.2	0.01	0.01	1	6.4339	6.5025	6.5223	0.0686
0.05	0.1	1.0	0.2	0.01	0.01	2	10.9954	11.0196	11.2186	0.0242
0.03	0.4	1.0	0.6	0.01	0.01	2	7.4459	7.3439	7.7829	0.1020
0.03	0.1	1.0	0.2	0.01	0.01	5	16.5891	16.5906	24.7230	0.001

Table 3: Heston price $S(0) = 100$, $E = 50$ $Err = \|(Heston)_{value} - (G.A.)_{value}\|$

r	ρ	κ	α	Θ	ν_t	T	G.A. Value	H. Value	f.d.m. Value	Err
0.03	0.1	1.0	0.7	0.04	0.01	0.5	50.7421	50.7341	50.8215	0.0080
0.03	0.2	1.0	0.5	0.0225	0.01	0.5	50.1853	50.7336	50.7756	0.5483
0.03	0.1	1.0	0.5	0.0225	0.01	1	50.7597	51.4585	51.8893	0.6988
0.05	0.1	1.0	0.5	0.0225	0.01	2	53.7232	54.6672	55.9912	0.9940
0.03	0.1	1.0	0.5	0.0225	0.01	0.5	50.6919	50.7352	51.0340	0.0433
0.03	0.1	1.0	0.5	0.0225	0.01	1	51.5730	51.5830	56.3770	0.0100
0.03	0.1	1.0	0.5	0.0225	0.01	3	56.3680	56.8155	59.4993	0.4475

the Geomtrical Approximation method does work. It is clear that must be verified the following condition:

$$\left(S_t e^{-\frac{\rho\nu_T}{\alpha}} - E\right)^+ \simeq (S_T - E)\,,^+ \qquad t \in [0, T]; \tag{21}$$

where

$$\|1 - e^{-\frac{\rho\nu_T}{\alpha}}\| \sim \|10^{-2}\|. \tag{22}$$

Table 4: Heston price for a Call with $S(0) = 100$, $E = 100$, $Err = \|(M.C._{value} - (G.A.)_{value}\|$ for Monte Carlo method we used day pass $(1/250)$ and 10^6 trajectories

r	ρ	κ	α	Θ	ν_t	T	G.A. Value	M.C. Value	S.S.E.	Err
0.03	0.1	1.0	0.2	0.01	0.01	0.5	3.2992	3.4591	0.0022	0.1599
0.03	0.1	1.0	0.2	0.01	0.01	1	5.2461	5.3417	0.0031	0.0956
0.03	0.1	1.0	0.2	0.01	0.01	2	8.4954	8.5857	0.0042	0.0903
0.05	0.1	1.0	0.2	0.01	0.01	5	22.9333	23.4234	0.0039	0.4901

So that, first to use G.A. method is necessary estimate the value of volatility at maturity date T. If our condition (22) is satisfied, thus we are able to write the Vanilla Option price in very accurate way.

5.1 Hedging and Put-Call-Parity

In order to find the better hedging strategy from the market risk, we use a replicant portfolio. So that we need to know the value of the first and second, derivative of the price, with respect to S_t, that we call them as Δ and Γ strategies respectively for a European call option and European put option, where $\epsilon << 1$:

$$\Delta_{call} = \frac{\partial C_{\rho,\alpha,\Theta,\kappa}}{\partial S} = (1-\epsilon)\, e^{\delta_1^\rho} N(d_1^\rho)$$

$$\Gamma_{call} = \frac{E e^{\delta_1^\rho - (d_1^\rho)^2/2}}{S\sqrt{2\pi\overline{\nu}_\rho (T-t)}} \tag{23}$$

and

$$\Delta_{put} = \frac{\partial P_{\rho,\alpha,\Theta,\kappa}}{\partial S} = -\,(1-\epsilon)\, e^{\delta_1^\rho} N(-d_1^\rho)$$

$$\Gamma_{put} = \frac{E e^{\delta_1^\rho - (d_1^\rho)^2/2}}{S\sqrt{2\pi\overline{\nu}_\rho (T-t)}} \tag{24}$$

Thus we have

$$\Gamma_{put} = \Gamma_{call}$$

It is worth noting that, in the Heston's model, the Put-Call-Parity condition is verified, and this proof that we are in a free arbitrage market.

6 Conclusions

The proposed method gives an approximation value of the vanilla option price. As one can see in the previous tables. Our method is more efficent, than f.d.m method, when the maturity date of the option, is more long than three years. Another important consideration is following: the 'Geometrical Approximation' method is more sensible at the volatility, i.e., the options price is sensible, to the volatility increase or decrease. It is important highlight that we solve the exact PDE of Heston but with different Cauchy's condition, or also called pay-off function, in which there is the volatility in explicit way.

$$f(T, S, \nu) = \left(S_T e^{-\frac{\rho\nu_T}{\alpha}} - E\right)^+$$

We think that would be more correct to use our pay-off function, when one want use a stochastic volatility market model. Because, in this way is clear how the volatility contributes at the option price; and for suitable values of ρ, α and ν, so that, the argument of exponential function $\epsilon = \frac{\rho\nu}{\alpha}$, goes to zero we would have standard solutions, as if we had considered the pay-off function $f(T, S, \nu) = (S_T - E)^+$.

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