Equity-linked insurances and guaranteed annuity options

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We consider here term and whole-life cases of the equity-linked life insurance (ELLI), and the guaranteed annuity option (GAO). For the introduction for ELLI and GAO see, e.g., [13]. We present a financial instrument which is a combination of ELLI and GAO in a stochastic interest rate framework.

The equity-linked life insurance can be defined as an agreement between a buyer and a seller, where the buyer is under an obligation to pay, usually every year until the time to maturity of the contract or the death of the buyer (it depends what comes first) the periodic premium to the seller. At death of the buyer or maturity of the contract, the issuer is obliged to provide a cash benefit, according to the made agreement. The benefit equals greater of the values:

1. a function depending on the periodic premium and on the history of the spot price of an index or mutual fund from the date when the arrangement was defined to the expiration date of the contract,
2. a deterministic value depending on the periodic premium.

The equity-linked life insurance has been studied in [10], [11], [12], where the price of the mutual fund was assumed to be a geometric Brownian motion and the interest rate was constant. [1] and [19] took also ELLI into consideration but they assumed the deterministic interest rate. [2], [3] and [4] assumed that the interest rate is deterministic and modelled by a geometric Brownian motion. In 1993 they extended the model to the case, when the short-term interest rate is described by an Ornstein-Uhlenbeck process. In [4] these both events were widened and Monte Carlo simulations were used. The relatively new papers, which are related to ELLI are [17] and [5]. The results presented here are besad on the former. Because of the specific nature of ELLI, computer simulations must be employed. [15] described the pricing of ELLI without using the simulation techniques since they claimed an analytical formula they found. Unfortunately, their formula is wrong, which was proved in [16]. [5] presented another way to price ELLI. The author shows that founding the analytical formula is possible but the final formula overestimates the benefit. From [5] it is clear that any pricing of ELLI should employ simulations, which agrees with [17]. Another important problem, which was considered in the context of ELLI, was the uniqueness of the fair premium. This issue was the main subject of [18].

The guaranteed annuity option is an arrangement under which the writer delivers at policy maturity a cash benefit which can be converted to an annuity with the minimum guaranteed or greater market rate. This contract gives the holder the right to receive at retirement the life annuity obtained by converting the cash benefit at the guaranteed or market rate. [6], [7] and [8] provide information about this instrument.

Following [17] we price ELLI tied to the term life insurance. We also price ELLI tied to the whole life insurance. We calculate the fair premium. Moreover, we introduce corrections to the original ideas, which make the simulations more reliable.

The main presented result is the price of the ELLI contract combined with GAO. The obtained formulas take into account different practical cases, for example, when the policyholder changes the mutual fund or when one wants to invest additional money in the fund.

In Section 2 we introduce some definitions, notations and facts, which are needed in the further part of the Chapter. In Section 3 we deal with calculations, which are important in pricing of the equity-linked life insurance. Section 4 is devoted to pricing. We apply the methods from [17] to price ELLI in the term case. Then, we propose a formula tied to the whole life insurance case. Finally, we price ELLI combined with GAO. Our idea lies in the combination of the instrument which guarantees a person the maximum of the guaranteed amount and the amount collected in the mutual fund, and the one which gives option the policyholder reaching the retirement age to receive an annuity with a guaranteed interest rate. Section 5 shows the numerical results.

1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probabilistic space. Following [17] we use the following notation.

- K is the periodic premium paid by the insured,
- k is a share of the periodic premium, such that $k = a \cdot K$, where $a \in [0; 1]$,
- $g(K)$ is the guaranteed, deterministic amount,
- t_i is a premium payment date, where $i = 0, 1, \ldots, n-1$ and $t_0 = 0$,
- t_n is maturity date, where $t_n = T$,
- $D(t, t')$ is the price of a zero coupon bond at time t with time to maturity $t' \geq t$,
- $S(t)$ is the price of a mutual fund at time t,
- $\pi(t)dt$ is the probability that the buyer will die in the time interval $(t, t+dt]$
- W_1, W_2 are the standard Wiener processes under the P measure.

We assume that $S(t)$ and $D(t, t')$ satisfy the following stochastic differential equations:

$$dS(t) \stackrel{\text{def}}{=} \mu S(t)dt + \sigma_1 S(t)dW_1(t) + \sigma_2 S(t)dW_2(t)$$

$$dD(t, t') \stackrel{\text{def}}{=} \mu(t, t')D(t, t')dt + \sigma(t, t')dW_1(t), \quad \sigma(t, t) = 0, \quad D(t, t) = 1.$$

Therefore $D(t,t')$ is the geometric Brownian motion. The condition $D(t,t)=1$ is needed because $D(t,t')$ is the price of the zero coupon bond. The factor $\sigma(t,t')$ is responsible for a volatility of $D(t,t')$ process, thus $\sigma(t,t)=0$.

Let us introduce a definition which will be useful in pricing of ELLI and GAO: the periodic premium is fair if the discounted value of the future benefits at time 0 is equal to the discounted value of the future premiums.

We now recall Itô's lemma which will be applied further in the text. Let $f = f(X_1, X_2, t)$ where X_1 and X_2 are two processes of the following form

$$dX_1 = \mu_1(X_1, X_2, t)dt + \sigma_1(X_1, X_2, t)dB_1,\ dX_2 = \mu_2(X_1, X_2, t)dt + \sigma_2(X_1, X_2, t)dB_2$$

with correlation $dX_1 dX_2 = \rho dt$, where B_1 and B_2 are the standard Brownian motions. Then

$$\begin{aligned} df &= \left(\mu_1 f_{X_1} + \mu_2 f_{X_2} + f_t + \frac{1}{2}\sigma_1^2 f_{X_1X_1} + \rho\sigma_1\sigma_2 f_{X_1X_2} + \frac{1}{2}\sigma_2^2 f_{X_2X_2}\right) dt \\ &+ \sigma_1 f_{X_1} dB_1 + \sigma_2 f_{X_2} dB_2. \end{aligned}$$

Now, we give the definition of equivalence of two measures. Let $(\Omega, \mathcal{F}, \mathrm{P}, \mathrm{Q})$ be the probability space with measures P and Q. Two measures P and Q are **equivalent** iff $\forall A \in \mathcal{F}$

$$\mathrm{P}(A) > 0 \Longleftrightarrow \mathrm{Q}(A) > 0.$$

The function $\frac{\mathrm{dQ}}{\mathrm{dP}}(x) > 0$ such that $\forall A \in \mathcal{F}$

$$\mathrm{Q}(A) = \int_A \frac{\mathrm{dQ}}{\mathrm{dP}}(x)\mathrm{P}(dx)$$

is called **Radon-Nikodym derivative**.

The solution of the stochastic differential equation

$$dX(t) = aX(t)dt + b_1 X(t)dW_1(t) + b_2 X(t)dW_2(t)$$

is

$$X(t) = X(0)\exp\left[\left\{a - \frac{1}{2}\left(b_1^2 + b_2^2\right)\right\}t + b_1 W_1(t) + b_2 W_2(t)\right].$$

2 Pricing of the ELLI contract

The value of the investments in the equity fund at time T is equal to $k\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}$, thus we want to evaluate $\frac{S(T)}{S(t_i)}$. Using some results obtained in [17] instead of changing the probability twice we set

$$\lambda_1(t) \stackrel{\text{def}}{=} \frac{\mu(t,t') - \sigma(t,T)}{\sigma(t,t')}, \quad \lambda_2(t) \stackrel{\text{def}}{=} \frac{\mu - \sigma(t,T)}{\sigma_2} - \frac{\sigma_1}{\sigma_2}\cdot\frac{\mu(t,t') - \sigma(t,T)}{\sigma(t,t')},$$

what gives the same outcome as in [17] (see [17] for more details). Let P^T be a probability measure equivalent to P such that

$$\begin{aligned}\frac{d\mathrm{P}^T}{d\mathrm{P}} &= \exp\left[-\int_{t_0}^{T}\{\lambda_1-\sigma(t,T)\}\,dW_1-\int_{t_0}^{T}\lambda_2 dW_2\right.\\ &- \left.\frac{1}{2}\int_{t_0}^{T}\left\{(\lambda_1-\sigma(t,T))^2+\lambda_2^2\right\}dt\right]. \qquad (1)\end{aligned}$$

If we use for W_1^T and W_2^T the Radon-Nikodym derivative and the Girsanov theorem with (1) we get

$$(dW_1^T,dW_2^T)=(dW_1+(\lambda_1(t)-\sigma(t,T))dt,dW_2+\lambda_2(t)dt). \qquad (2)$$

Now we will present some equations, which will be useful hereafter. From (2) and the form of λ_1 and λ_2 we get

$$\begin{aligned}dS(t) &\stackrel{\text{def}}{=} \mu S(t)dt+\sigma_1 S(t)dW_1(t)+\sigma_2 S(t)dW_2(t)\\ &= \mu S(t)dt+\sigma_1 S(t)dW_1^T-\sigma_1 S(t)\frac{\mu(t,t')-\sigma(t,T)}{\sigma(t,t')}dt+\sigma_1 S(t)\sigma(t,T)dt\\ &+ \sigma_2 S(t)dW_2^T-\mu S(t)dt+\sigma(t,T)S(t)dt+\sigma_1 S(t)\frac{\mu(t,t')-\sigma(t,T)}{\sigma(t,t')}dt.\end{aligned}$$

And simplifying:

$$dS(t)=(\sigma_1\sigma(t,T)+\sigma(t,T))S(t)dt+\sigma_1 S(t)dW_1^T(t)+\sigma_2 S(t)dW_2^T(t). \qquad (3)$$

Similarly, we obtain

$$\begin{aligned}dD(t,t') &\stackrel{\text{def}}{=} \mu(t,t')D(t,t')dt+\sigma(t,t')D(t,t')dW_1=\mu(t,t')D(t,t')dt\\ &= \mu(t,t')D(t,t')dt+\sigma(t,t')D(t,t')dW_1^T-\mu(t,t')D(t,t')dt\\ &+ \sigma(t,T)D(t,t')dt+\sigma(t,t')\sigma(t,T)D(t,t')dt.\end{aligned}$$

Thus

$$\begin{aligned}dD(t,t') &= \sigma(t,T)D(t,t')dt+\sigma(t,t')\sigma(t,T)D(t,t')dt\\ &+ \sigma(t,t')D(t,t')dW_1^T(t)\end{aligned}$$

and

$$dD(t,T)=\left(\sigma(t,T)+\sigma^2(t,T)\right)D(t,T)dt+\sigma(t,T)D(t,T)dW_1^T(t).$$

Applying Itô's lemma to $\frac{D(t,t')}{D(t,T)}$ we get

$$\begin{aligned}d\left(\frac{D(t,t')}{D(t,T)}\right) &= \left[\{\sigma(t,T)+\sigma(t,t')\sigma(t,T)\}\frac{D(t,t')}{D(t,T)}-\left\{\sigma(t,T)+\sigma^2(t,T)\right\}\right.\\ &\cdot D(t,T)\frac{D(t,t')}{D^2(t,T)}-\sigma(t,t')D(t,t')\sigma(t,T)D(t,T)\frac{1}{D^2(t,T)}\\ &+ \left.\frac{1}{2}\sigma^2(t,T)D^2(t,T)\frac{2D(t,t')}{D^3(t,T)}\right]dt+\sigma(t,t')D(t,t')\frac{1}{D(t,T)}\\ &\cdot dW_1^T(t)-\sigma(t,T)D(t,T)\frac{D(t,t')}{D^2(t,T)}dW_1^T(t).\end{aligned}$$

Hence

$$d\left(\frac{D(t,t')}{D(t,T)}\right) = (\sigma(t,t') - \sigma(t,T))\,\frac{D(t,t')}{D(t,T)} dW_1^T(t). \tag{4}$$

From 2-dimensional version of Itô's formula we get

$$\begin{aligned} d\left(\frac{S(t)}{D(t,T)}\right) &= \Bigg[\{\sigma(t,T) + \sigma_1\sigma(t,T)\}\, S(t)\frac{1}{D(t,T)} - \{\sigma(t,T) + \sigma^2(t,T)\} \\ &\cdot \quad D(t,T)\frac{S(t)}{D^2(t,T)} - \sigma_1 S(t)\sigma(t,T)D(t,T)\frac{1}{D^2(t,T)} + \frac{1}{2}\sigma^2(t,T) \\ &\cdot \quad D^2(t,T)\frac{2S(t)}{D^3(t,T)}\Bigg] dt + \sigma_1 S(t)\frac{1}{D(t,T)} dW_1^T + \sigma_2 S(t) \\ &\cdot \quad \frac{1}{D(t,T)} dW_2^T - \sigma(t,T)D(t,T)\frac{S(t)}{D^2(t,T)} dW_1^T. \end{aligned}$$

Therefore

$$d\left(\frac{S(t)}{D(t,T)}\right) = \{\sigma_1 - \sigma(t,T)\}\,\frac{D(t,T)}{S(t)} dW_1^T(t) + \sigma_2\frac{D(t,T)}{S(t)} dW_2^T(t). \tag{5}$$

A big advantage of the equations (4) and (5) is the lack of the drift. From [9] we know that $\frac{S(T)}{D(t,T)}$ and $\frac{D(t,t')}{D(t,T)}$ are martingales under the P^T measure. This feature is needed in order to have no arbitrage in the market. The second important feature, which can be sometimes troublesome to proved, is completeness of the market. However, from [9] we know that our model guarantees completeness of the market. Solving (5) we get

$$\begin{aligned} \frac{S(T)}{D(T,T)} &= \frac{S(t)}{D(t,T)} \exp\Bigg[-\frac{1}{2}\int_t^T \left\{(\sigma_1 - \sigma(u,T))^2 + \sigma_2^2\right\} du \\ &+ \int_t^T (\sigma_1 - \sigma(u,T)) dW_1^T(u) + \int_t^T \sigma_2 dW_2^T(u)\Bigg]. \end{aligned}$$

Thus

$$\begin{aligned} \frac{S(T)}{S(t)} &= \frac{1}{D(t,T)} \exp\Bigg[-\frac{1}{2}\int_t^T \left\{(\sigma_1 - \sigma(u,T))^2 + \sigma_2^2\right\} du \\ &+ \int_t^T \{\sigma_1 - \sigma(u,T)\}\, dW_1^T(u) + \int_t^T \sigma_2 dW_2^T(u)\Bigg]. \end{aligned} \tag{6}$$

Now our aim is to get rid of $D(t,T)$, which complicates calculation of $\frac{S(T)}{S(t)}$ in (6). When we solve (4) we obtain

$$\begin{aligned} \frac{D(t,t')}{D(t,T)} &= \frac{D(t_0,t')}{D(t_0,T)} \exp\Bigg[\int_{t_0}^t \{\sigma(u,t') - \sigma(u,T)\}\, dW_1^T(u) \\ &- \frac{1}{2}\int_{t_0}^t \{\sigma(u,t') - \sigma(u,T)\}^2 du\Bigg]. \end{aligned} \tag{7}$$

Therefore for $t' := t$ we have

$$
\begin{aligned}
D(t,T) &= \frac{D(t_0,T)}{D(t_0,t)} \cdot \exp\Bigg[- \int_{t_0}^{t} \{\sigma(u,t) - \sigma(u,T)\}\, dW_1^T(u) \\
&+ \frac{1}{2}\int_{t_0}^{t} \{\sigma(u,t) - \sigma(u,T)\}^2\, du \Bigg]. \qquad (8)
\end{aligned}
$$

Substituting (8) in (6) we obtain

$$
\begin{aligned}
\frac{S(T)}{S(t)} &= \frac{D(t_0,t)}{D(t_0,T)} \exp\Bigg[\int_{t_0}^{t} \{\sigma(u,t) - \sigma(u,T)\}\, dW_1^T du \\
&- \frac{1}{2}\int_{t}^{T} \{\sigma(u,t) - \sigma(u,T)\}^2\, du \\
&- \frac{1}{2}\int_{t}^{T} \left\{(\sigma_1 - \sigma(u,T))^2 + \sigma_2^2\right\} du \\
&+ \int_{t}^{T} \{\sigma_1 - \sigma(u,T)\}\, dW_1^T(u) + \int_{t}^{T} \sigma_2 dW_2^T(u) \Bigg]. \qquad (9)
\end{aligned}
$$

Thus $\frac{S(t)}{S(t_i)}$ is lognormally distributed. Hence, it is impossible to find distribution of $\sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)}$. This forces us to use Monte Carlo simulations in pricing ELLI. The simulation technique is well known and widely applied when the other ideas fail. But first let us find $\sigma(t,t')$. It is very important to choose such $\sigma(t,t')$ that makes the model more feasible. Nielsen and Sandmann (see [17]) in order to make the model computationally feasible set $\sigma(t,t') = \sigma(t'-t)$. This choice of σ transforms the equation (9) to

$$
\begin{aligned}
\frac{S(T)}{S(t)} &= \frac{D(t_0,t)}{D(t_0,T)} \exp\Bigg[-\frac{1}{2}(T-t)^2\sigma^2 t - \frac{1}{2}\int_{t}^{T} \left\{(\sigma_1 - (T-u)\sigma)^2 + \sigma_2^2\right\} du \\
&- (T-t)\sigma W_1^T(t) + \int_{t}^{T} \{\sigma_1 - (T-u)\sigma\}\, dW_1^T(u) + \int_{t}^{T} \sigma_2 dW_2^T(u) \Bigg].
\end{aligned}
$$

In this parametrization

$$
\mathrm{E}\frac{S(T)}{S(t)} = \frac{D(t_0,t)}{D(t_0,T)},
$$

$$
\begin{aligned}
\mathrm{Var}\frac{S(T)}{S(t)} &= \frac{D^2(t_0,t)}{D^2(t_0,T)} \Bigg[\exp\Big\{ (T-t)^2\sigma^2 t + \left(\sigma_1^2 + \sigma_2^2\right)(T-t) \\
&- \sigma\sigma_1(T-t) + \frac{1}{3}\sigma^2(T-t)^3 \Big\} - 1 \Bigg].
\end{aligned}
$$

It can be shown that $\mathrm{Var}\frac{S(T)}{S(t)}$ grows rapidly with T. This is a very undesirable feature because it makes the results more variable. The second reason why it is worth to change parametrization is that $D(t,t')$ is a bond process, so it should be less risky than mutual fund $S(t)$. For constants σ, σ_1 and σ_2 used in [17], $\sigma(t,t')$ being the factor of $D(t,t')$ can be large for a long maturity time. Thus, we set $\sigma(t,t') = \sigma(1-e^{-(t'-t)})$. We must keep in mind that $\sigma(t,t)$ must be equal to 0 but for the new form of σ this condition is satisfied. It is worth noting that $1-e^{-(t'-t)} \approx t'-t$ when the right side is small. Now (9) can be symbolically expressed as

$$\frac{S(T)}{S(t)} = \frac{D(t_0,t)}{D(t_0,T)} \exp\left(-\frac{1}{2}A + B\right),$$

where A and B stand for a deterministic and stochastic part, respectively.

The stochastic part

$$\begin{aligned} B &= \int_0^t \sigma\left(e^{u-T} - e^{u-t}\right) dW_1^T(u) + \int_t^T \left\{\sigma_1 - \sigma(1-e^{u-T})\right\} dW_1^T(u) \\ &+ \int_t^T \sigma_2 dW_2^T(u) = \sigma e^{-T} \int_0^T e^u dW_1^T(u) + (\sigma_1 - \sigma)\left\{W_1^T(T) - W_1^T(t)\right\} \\ &- \sigma e^{-t} \int_0^t e^u dW_1(u) + \sigma_2 \left\{W_2^T(T) - W_2^T(t)\right\}, \end{aligned}$$

while deterministic part

$$A = \left\{(\sigma_1-\sigma)^2 + \sigma_2^2\right\}(T-t) - \frac{1}{2}\sigma^2\left(e^{-t} - e^{-T}\right)^2 + \left(2\sigma\sigma_1 - \sigma^2\right)\left(1 - e^{-(T-t)}\right).$$

Hence, we finally get the following equality

$$\begin{aligned} \frac{S(T)}{S(t)} &= \frac{D(t_0,t)}{D(t_0,T)} \exp\Bigg[-\frac{1}{2}\left\{(\sigma_1-\sigma)^2 + \sigma_2^2\right\}(T-t) + \frac{1}{4}\sigma^2\left(e^{-t}-e^{-T}\right)^2 \\ &- \frac{1}{2}\left(2\sigma\sigma_1 - \sigma^2\right)\left\{1 - e^{-(T-t)}\right\} + \sigma e^{-T}\int_0^T e^u dW_1^T(u) \\ &+ (\sigma_1-\sigma)\left\{W_1^T(T) - W_1^T(t)\right\} - \sigma e^{-t}\int_0^t e^u dW_1^T(u) \\ &+ \sigma_2\left\{W_2^T(T) - W_2^T(t)\right\} \Bigg]. \end{aligned}$$

Similarly as it was in the previous parametrization, we calculate $\mathrm{E}\frac{S(T}{S(t)}$ and $\mathrm{Var}\frac{S(T)}{S(t)}$. We obtain that

$$\mathrm{E}\frac{S(T)}{S(t)} = \frac{D(t_0,t)}{D(t_0,T)},$$

and

$$\begin{aligned} \mathrm{Var}\frac{S(T)}{S(t)} &= \frac{D^2(t_0,t)}{D^2(t_0,T)}\Bigg[\exp\Big[\left\{(\sigma_1-\sigma)^2+\sigma_2^2\right\}(T-t) - \frac{1}{2}\sigma^2\left(e^{-t}-e^{-T}\right)^2 \\ &+ \left(2\sigma\sigma_1 - \sigma^2\right)\left\{1-e^{-(T-t)}\right\}\Big] - 1\Bigg]. \end{aligned}$$

Now we will analyze the order of the variance for two choices of σ. The order of the variance for the old parametrization is

$$R_1 = \exp\left\{(T-t)^2\sigma^2 t + \left(\sigma_1^2+\sigma_2^2\right)(T-t) - \sigma\sigma_1(T-t) + \frac{1}{3}\sigma^2(T-t)^3\right\} - 1$$

whereas for the new parametrization the order is

$$\begin{aligned} R_2 &= \exp\left[\left\{(\sigma_1-\sigma)^2+\sigma_2^2\right\}(T-t) - \frac{1}{2}\sigma^2\left(e^{-t}-e^{-T}\right)^2 + \left(2\sigma\sigma_1-\sigma^2\right)\right. \\ &\quad \cdot \left.\left\{1-e^{-(T-t)}\right\}\right] - 1. \end{aligned}$$

Now we compare R_1 and R_2 with respect to different T's and t's. Let $\sigma = 0.08$, $\sigma_1 = 0.1$ and $\sigma_2 = 0.15$.
In Table 1 we can observe that the alternation of σ makes sense.

Table 1: The order of the variance with respect to old and new σ

T	t	Var_{old}	Var_{new}
10	0	0.4758	0.2654
10	5	2.2845	0.1320
30	0	$e^{7.6575}$	1.0005
30	15	$e^{29.208}$	0.4233
50	0	$e^{33.9458}$	2.1626
50	25	$e^{133.2146}$	0.7898

For $\sigma = \sigma\left(1-e^{-(t'-t)}\right)$ the variance is bigger for $t = 0$ than for $t > 0$. For $\sigma(t,t') = \sigma(t'-t)$ the situation is quite different. However, we see that the new σ is much better, because the variance is always smaller. The differences between variances under the same t and T parameters are visible: for new σ it is not greater than 2.2 and for old σ it can reach e^{133}. Therefore the new choice of σ is justified.

2.1 The term case

First, we recall the definition of ELLI. It is an agreement under which the maximum of the money collected in the investment fund and the guaranteed amount is paid. The benefit for the insured from the insurance contract received at maturity time T can be written in the following way (see [17]).

$$g(K) + V(T,T) = g(K) + \left(k\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)} - g(K), 0\right)_+,$$

where $(x)_+ = \max(x,0)$. Please notice that the benefit is de facto a call option with the exercise price equal to the guaranteed amount. We can also see the necessity of calculating the expression $\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}$ and finding its easy form in the previous section. We know from the definition of the fair periodic premium that in order to evaluate this premium we must take into account the expected discounted benefit, which is equal to the expected discounted cost of the contract. It is evident that the cost which must bear the policy holder is equal to

$$K\sum_{i=0}^{n-1} D(t_0,t_i)\left\{1-\int_{t_0}^{t_i}\pi(t)dt\right\}.$$

The expected discounted benefit is of more complicated form, because it consists of the guaranteed part

$$g(K)\int_{t_0}^{T} D(t_0,t)\pi(t)dt + g(K)D(t_0,T)\left\{1-\int_{t_0}^{T}\pi(t)dt\right\}$$

and the bonus part

$$\begin{array}{rl} \int_{t_0}^{T} & D(t_0,t)\mathrm{E}^t\left[\left\{k\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}-g(K)\right\}_+\right]\pi(t)dt \\ + & D(t_0,T)\mathrm{E}^T\left\{k\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}-g(K)\right\}_+\left\{1-\int_{t_0}^{T}\pi(t)dt\right\}, \end{array}$$

where $n^*(t)=\min\{i: t_i>t\}$. Observe that we incorporate the mortality case, and $\pi(t)dt$ stands for the probability that the policy holder will die in the time interval $(t,t+dt]$. Denoting

$$B(t_0,t)=D(t_0,t)\mathrm{E}^t\left[\left\{k\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}-g(K)\right\}_+\right]$$

and

$$B(t_0)=\int_{t_0}^{T} B(t_0,t)\pi(t)dt + B(t_0,T)\left\{1-\int_{t_0}^{T}\pi(t)dt\right\}$$

the pricing formula for ELLI can be reduced to the following expression (see [17])

$$\begin{array}{rl} B(t_0) & +g(K)\int_{t_0}^{T} D(t_0,t)\pi(t)dt + g(K)D(t_0,T)\left\{1-\int_{t_0}^{T}\pi(t)dt\right\} \\ = & K\sum_{t=0}^{n-1} D(t_0,t_i)\left\{1-\int_{t_0}^{t_i}\pi(t)dt\right\}. \end{array} \quad (10)$$

This equation enables us to calculate the fair premium K. But instead of it we will use a slightly simpler pricing formula. The formula

$$\begin{array}{rl} \int_{t_0}^{T} & \mathrm{E}^t\left[\left\{g(K)-aK\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}\right\}_+\right]D(t_0,t_i)\pi(t)dt \\ + & \mathrm{E}^T\left[\left\{g(K)-aK\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}\right\}_+\right]D(t_0,T)\left\{1-\int_{t_0}^{T}\pi(t)dt\right\} \\ = & (1-a)K\sum_{i=0}^{n-1} D(t_0,t_i)\left\{(1-\int_{t_0}^{t_i}\pi(t)dt\right\} \end{array} \quad (11)$$

is equivalent to (10).

Left side of the above equation can be interpreted as the cost, which must bear the policy seller, when he guarantees to give the policyholder the amount $g(K)$. The right side of the equation is the amount of money, which must pay the buyer in order to have right to get $g(K)$.

Equation (11) concerns the term case, thus it can be used when we want to find a fair periodic premium in the term life insurance.

2.2 The whole life case

In the case when we consider the whole life insurance equation (11) reduces to

$$\begin{aligned} &\int_{t_0}^{T} \mathrm{E}^t \left[\left\{ g(K) - aK \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)} \right\}_+ \right] D(t_0, t_i)\pi(t)dt \\ = \quad & (1-a)K \sum_{i=0}^{n-1} D(t_0, t_i) \left\{ 1 - \int_{t_0}^{t_i} \pi(t)dt \right\}. \qquad (12) \end{aligned}$$

Remembering, that $(1-a)K = K - k$ we can rewrite (12) in the form

$$\int_{t_0}^{T} \mathrm{E}^t \left[\left\{ g(K) - aK \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)} \right\}_+ \right] D(t_0, t_i)\pi(t)dt \qquad (13)$$

$$= \quad (K-k) \sum_{i=0}^{n-1} D(t_0, t_i) \left\{ 1 - \int_{t_0}^{t_i} \pi(t)dt \right\}. \qquad (14)$$

This equation (like (10)) has some advantages. It enables us to calculate $K - k$ when the mutual fund has been changed, when the buyer wants to adjust $g(K)$ or pay in some addition money into the fund.

3 Pricing of the combined instrument

In this section we price whole-life and term equity-linked life insurance linked with the guaranteed annuity option. This idea comes from [16].

3.1 ELLI and GAO

Our financial instrument can be characterized as follows: when the insured dies before the age of 65 the seller pays the amount of money due to ELLI and when the buyer survives 65 he has a option to receive an annuity with the guaranteed or the market interest rate. Thus this instrument can be described as the life insurance with the investment fund changing into pension at age of 65. Let v be the discounting factor under the guaranteed interest rate and $D(t, t')$ is defined as in Section 3 as the price of a zero coupon bond. The general formula, which describes the actuarial value V of annual payments can be written in the following way: $V = S\sum_{i=0}^{\infty} v^i {}_ip_x$, where v is the discounting factor. In our case V can be expressed as

$$V = S\sum_{i=1}^{\infty} D(T, T+i)_i p_T \tag{15}$$

which is actuarial value of the annuity under the market rate calculated for a retirement age T of the policyholder. Analogically

$$V = S'\sum_{i=1}^{\infty} v^i {}_ip_T \tag{16}$$

is the actuarial value of the annuity with the guaranteed interest rate for a person aged T. On the other hand we are aware of the fact that V is also the amount of money collected in the mutual fund, because it is the amount of money guaranteeing us getting the annuity. Therefore $V = \max\left\{g(K), \sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)}\right\}$. The loss of the issuer, connected with variable rates is $\sum_{i=1}^{\infty} D(T, T+i)_i p_T \max(S' - S, 0)$. Equations (15) and (16) imply that

$$S = \frac{V}{\sum_{i=1}^{\infty} D(T, T+i)_i p_T} \text{ and } S' = \frac{V}{\sum_{i=1}^{\infty} v^i {}_ip_T}.$$

Thus the expected loss of the seller can be written as

$$\begin{aligned} \mathrm{E}^T & \left[\sum_{i=1}^{\infty} D(T, T+i)_i p_T \max\left\{\frac{V}{\sum_{i=1}^{\infty} v^i {}_ip_T} - \frac{V}{\sum_{i=1}^{\infty} D(T, T+i)_i p_T}, 0\right\}\right] \\ \cdot \quad & D(t_0, T)\left\{1 - \int_{t_0}^{T} \pi(t)\right\} = \mathrm{E}^T\left[\sum_{i=1}^{\infty} D(T, T+i)_i p_T V\right. \\ \cdot \quad & \left.\left\{\frac{1}{\sum_{i=1}^{\infty} v^i {}_ip_T} - \frac{1}{\sum_{i=1}^{\infty} D(T, T+i)_i p_T}\right\}_+\right] D(t_0, T)\left\{1 - \int_{t_0}^{T} \pi(t)dt\right\}. \end{aligned} \tag{17}$$

Similarly (17) can be represented as

$$\begin{aligned} \mathrm{E}^T & \left[\sum_{i=1}^{\infty} D(T, T+i)_i p_T \max\left(g(K), \sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)}\right)\right. \\ \cdot \quad & \left.\left(\frac{1}{\sum_{i=1}^{\infty} v^i {}_ip_T} - \frac{1}{\sum_{i=1}^{\infty} D(T, T+i)_i p_T}\right)_+\right] D(t_0, T)\left(1 - \int_{t_0}^{T} \pi(t)dt\right). \end{aligned} \tag{18}$$

Now we use (11) presenting the cost of the guaranteed part of the ELLI-contract. To price ELLI mixed with GAO we should take into consideration (18), because this equation concerns the loss (so costs) connected with different interest rates, under which the annuity is calculated. Thus all costs bearing by the issuer of the option must be included in our pricing equation. Finally, we present the following equation which summarizes (11) and (18):

$$
\begin{aligned}
\int_{t_0}^{T} & \quad \mathrm{E}^t\left[\left\{g(K)-aK\sum_{i=0}^{n^*(t)-1}\frac{S(t)}{S(t_i)}\right\}_+\right]D(t_0,t_i)\pi(t)dt \\
+ & \quad \mathrm{E}^T\left[\sum_{i=1}^{\infty}D(T,T+i)_i p_T \max\left\{g(K),\sum_{i=0}^{n-1}\frac{S(T)}{S(t_i)}\right\}\right. \\
\cdot & \quad \left.\left\{\frac{1}{\sum_{i=1}^{\infty}v^i{}_ip_T}-\frac{1}{\sum_{i=1}^{\infty}D(T,T+i)_ip_T}\right\}_+\right]D(t_0,T)\left\{1-\int_{t_0}^{T}\pi(t)dt\right\} \\
= & \quad (1-a)K\sum_{i=0}^{n-1}D(t_0,t_i)\left(1-\int_{t_0}^{t_i}\pi(t)dt\right). \qquad (19)
\end{aligned}
$$

4 Numerical results

In this section we will show some results obtained via simulations and we will analyze them. We assume that mortality follows Makeham,s law, namely the survivor function l_x can be described as

$$l_x = bs^x g^{\ c^x},$$

where $b>0$, $s>0$, $g>0$, $c>0$. The first step is to estimate the parameters b, s, g and c using Polish life tables for the year 2005. We obtain that
$b = 99704.1832$,
$s = 0.99961478$,
$g = 0.99991201$,
$c = 1.11595563$.
Now (see [17]) we can compute the density function of the random variable describing the future lifetime for a person aged x:

$$\pi_x'(t) = \frac{(l_{x+t})'}{l_x}.$$

We study $K-k$. This value depends on $g(K)$, because this is the amount of money, which the policyholder must pay in order to receive the guaranteed amount $g(K)$. We make the antithetic Monte Carlo simulations with 1000 paths (i.e. using 2000 paths, which stems from the definition of the antithetic method). We set the flat initial term structure $D(t_0,t_i)=(1.06)^{-t_i}$, what implies that the annual interest rate is equal to 6%.

4.1 Results for ELLI whole life case

In all simulations we have considered a person aged 30. Now we show how $K-k$ changes with $g(K)$ for ELLI. Let $k=100$.

First we analyze the variance, which changes, of course, but is small and varies only little. It is also obvious that the bigger $g(K)$ the larger $K-k$ because when we want to be offered a higher protection we must bear the cost connected with

Table 2: $K-k$ with respect to different $g(K)$

$g(K)$	$K-k$	$\mathrm{Var}(K-k)$
5 000	3.409146	0.018879562
10 000	12.30417	0.074464537
15 000	25.01689	0.116564028
20 000	40.29600	0.133931721
25 000	57.33679	0.134654032
30 000	75.62904	0.119531645
35 000	94.84122	0.109018809
40 000	114.7477	0.123134971
45 000	135.1923	0.155640645
50 000	156.0627	0.194149183

this privilege, thus the probability that the amount collected in the mutual fund will exceed $g(K)$ becomes small. In Table 2 we see that $K-k>100=k$ for $g(K)\geq 40000$ which means that over half of the periodic premium guarantees the amount $g(K)$. For $g(K)<20000$ $K-k$ is less than the third part of k. So little $K-k$ with respect to the guaranteed amount can mean that the mutual fund will exceed the guaranteed amount. It can also give evidence that the probability of death of 30-year-old person, until the mutual fund will exceed $g(K)$, is not too big. Therefore the part guaranteeing getting of $g(K)$ is relatively small.

In Figure 1 quantile lines of the mutual fund shares are depicted.

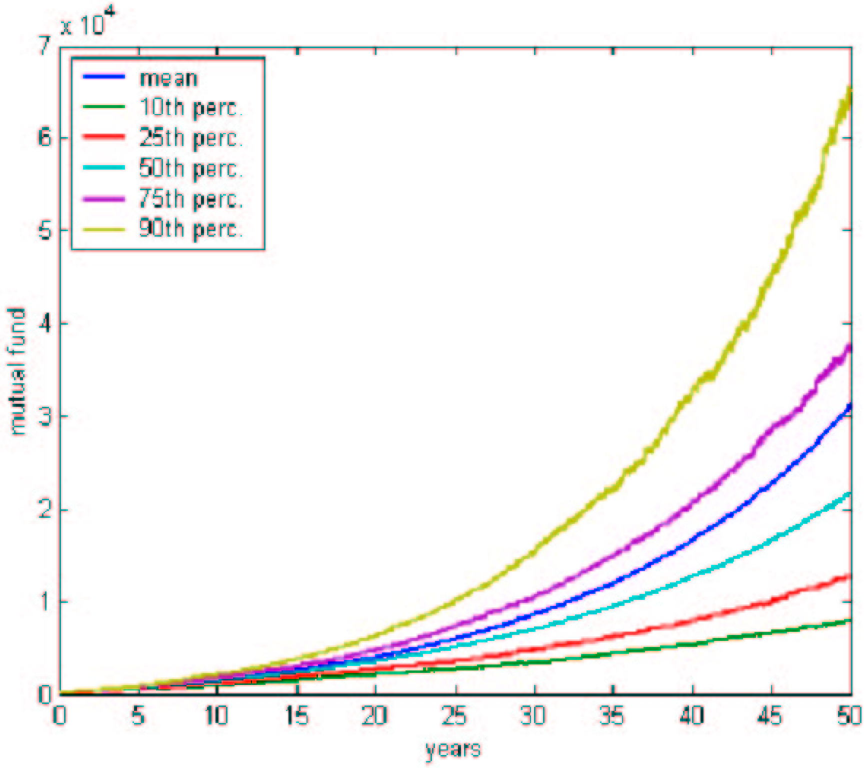

Figure 1: Quantile lines of the mutual fund shares

In Figure 1 we can observe how the amount of money collected in the fund can vary. It results from the pessimistic and optimistic cases of the sum $\sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)}$. The yellow line is the most optimistic trajectory, so the mutual fund can "earn" more than 65 000 over 50 years. Below we can see the 75th percentile, so only 25% of trajectories provide better value for the policyholder. The dark blue line stands for the mean of the collected mutual fund shares while the light blue refers to the median. We note that the median and mean differ. The median of the trajectories is below the mean. The best for the issuer and the worst for the buyer are red and dark green lines. They present 25th and 10th percentiles of trajectories, respectively. For these trajectories $\sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)}$ is about 10 000. Thus, they can be called the most pessimistic from the point of view of the policyholder.

If we look at the Table 3 we observe that the more optimistic scenario for the buyer the greater the bonus part. It seems to be sensible, because if we take 10th percentile of the bonuses and the 90th percentile of the bonuses then the bonus part is larger for the second one. Besides the bonus begins to appear at different times with different scenarios. For the 90th percentile it appears for 35 years of the contract's duration. For the most pessimistic case even 55 years of having the contract are not enough to give the bonus part. For the 50th percentile the bonus part is less than for the mean trajectories. That means that there are fewer optimistic trajectories than pessimistic trajectories but their character is "more aggressive".

Table 3: The bonus part with respect to different trajectories

	Percentiles of the bonus parts					
Years	mean	10%	25%	50%	75%	90%
15	0	0	0	0	0	0
20	0	0	0	0	0	0
25	19.1519	0	0	0	0	0
30	289.459	0	0	0	0	0
35	1248.88	0	0	0	0	2194.56
40	3577.61	0	0	0	743.053	12408.6
45	7743.85	0	0	0	7817.69	27062.1
50	15171.4	0	0	1602.64	18064.9	43468.4
55	25839.8	0	0	8259.41	30801.6	71697.8

4.2 Results for ELLI mixed with GAO

We present here simulation results obtained for ELLI combined with GAO (see (19)). In Table 4 we present the changeability of two parts: the first part assures $g(K)$ and the second assures the annuity with the guaranteed interest rate i. Both $g(K)$'s-part assurance and interest's part assurance are calculated from (19). This equation consists of the parts coming from ELLI and GAO. The first one equals

$$\begin{aligned} &\Big[\int_{t_0}^{T} \mathrm{E}^t \Big[\Big\{ g(K) - aK \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)} \Big\}_+ \Big] D(t_0,t_i)\pi(t)dt \Big] \\ &\Big/ \sum_{i=0}^{n-1} D(t_0,t_i) \Big\{ 1 - \int_{t_0}^{t_i} \pi(t)dt \Big\} \end{aligned} \tag{20}$$

and is the factor connected with ELLI. The second one is equal to

$$\begin{aligned} &\Big[\mathrm{E}^T \Big[\sum_{i=1}^{\infty} D(T,T+i)_i p_T \max\Big\{ g(K), \sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)} \Big\} \\ &\cdot \Big\{ \frac{1}{\sum_{i=1}^{\infty} v^i{}_i p_T} - \frac{1}{\sum_{i=1}^{\infty} D(T,T+i)_i p_T} \Big\}_+ \Big] D(t_0,T) \\ &\cdot \Big\{ 1 - \int_{t_0}^{T} \pi(t) \Big\} \Big] \Big/ \sum_{i=0}^{n-1} D(t_0,t_i) \Big\{ 1 - \int_{t_0}^{t_i} \pi(t)dt \Big\} \end{aligned} \tag{21}$$

and comes from the GAO's part in (19).

First we show how much the policyholder must pay for the guaranteed amount $g(K)$ and guaranteed interest rate i. We assume that the buyer pays $k = 100$ in the mutual fund.

Table 4: The costs of the guaranteed amount (20) and the guaranteed interest rate (21).

		Interests's part					
g(K)	$g(K)$'s	2%	3%	4%	5%	6%	7%
5000	4.91914	7.83983	12.0134	17.4918	24.6413	33.3009	43.3178
10000	25.9181	8.67955	13.3056	19.3954	27.3387	36.9763	48.1514
15000	59.7834	10.3754	15.9068	23.2113	32.7299	44.3011	57.7739
20000	100.087	12.6114	19.3201	28.2065	39.7844	53.8749	70.3186
25000	143.868	15.1473	23.1924	33.8627	47.7576	64.6789	84.4522
30000	189.509	17.8326	27.3075	39.8825	56.2581	76.2005	99.5161
35000	236.187	20.6222	31.5881	46.1398	65.0832	88.1521	115.13
40000	283.378	23.4524	35.9323	52.4884	74.0379	100.279	130.972
45000	330.773	26.3142	40.3197	58.8945	83.0678	112.501	146.93
50000	378.354	29.1917	44.7276	65.3303	92.1434	124.787	162.973

If we look at Table 4 we can observe that the costs grow with i, what is obvious, if we want to have greater interest rate we must bear the costs connected with this privilege. Therefore we pay only 7.83983 for the guarantee of the annuity with 1% interest rate and 43.3178 for the annuity with 7% interest rate. We must also pay in the greater periodic premium when we want to have greater $g(K)$. If we want to be sure that we will get $g(K)$=5 000 we pay only 4.91914 but the guarantee of $g(K)$=50000 is bigger and it costs 378.354. Therefore the growth of the price of i's and $g(K)$'s assurance part is evident. It is worth noting that the assurance of $g(K)$ is more expensive than assurance of the guarantted interest rate i. But we cannot say that the assurance of $g(K)$ is always more expensive because we cannot be sure what it is happening in case when i is greater than 0.07. One thing we can be sure of is that the average cost of the growth of $g(K)$ for $i = 0.07$ is smaller than the same growth of $g(K)$-part assurance (although for $i = 0.07$ and g(K)=5 000 the cost is equal to 43.3178 and for the same $g(K)$ the cost of the guaranteeing part equals only 4.91914, then for $g(K) = 50000$ these costs are 162.973 and 378.354 for i and the guaranteeing part, respectively). We pay attention to i once more. Observe that for the last 3 rows the values of the part guaranteeing the annuity are almost proportional to the $g(K)$. This behavior can be explained in the following way. As we remember $g(K)$ and the amount gathered on the mutual fund decide how much the policyholder will get as the annuity. The mutual fund is the dominating factor until we decide the guaranteed amount to be too big. For the last table's rows the guaranteed amount is probably so big that the mutual fund does not influence the annuity but $g(K)$ does. To make this consideration clearer we show some example: we want to pay in 100 in the mutual fund and to have the guaranteed amount equals 45 000. This guarantee costs 330.773, so over 3 times more than k. The additional privilege of i is 26.3142 for 1% and 146.93 for 7%, thus it is not so expensive as the bearing the costs of $g(K)$. Of course it depends on us what we want to have - greater $g(K)$ or i.

Now it will be shown, how much money the policyholder gets as the annuity. We assume that the guaranteed interest rate $i = 0.06$ and $k = 100$.

Table 5: The annuity with respect to different guaranteed amounts $k = 100$

		Percentiles of the actuarial value of annuities					
g(K)	K	mean	5th	25th	50th	75th	95th
5000	140.936	1417.25	*570.761*	*744.151*	1095.09	1691.62	*2626.82*
10000	164.603	1618.93	*1023.32*	*1039.53*	1297.73	1728.71	*2626.82*
15000	205.154	2000.38	**1534.98**	**1534.98**	1709.49	2119	*2751.53*
20000	254.797	2472.95	**2046.64**	**2046.64**	2171.02	2659.54	3168.47
25000	308.834	2991.74	**2558.3**	**2558.3**	2656.12	3228.69	3765.03
30000	365.237	3534.17	**3069.96**	**3069.96**	3154.27	3811.97	4414.23
35000	422.798	4089.18	**3581.62**	**3581.62**	3650.49	4419.07	**5119.00**
40000	481.241	4651.83	**4093.29**	**4093.29**	4156.75	5020.63	**5824.39**
45000	540.201	5218.47	**4604.95**	**4604.95**	4668.28	5618.65	**6512.78**
50000	599.396	5787.47	**5116.61**	**5116.61**	5180.14	6222.7	**7195.22**

Two factors influence the annuity: v (which corresponds to i) and $g(K)$. We prove this fact by the analysis of Table 5. The right part of the table shows the influence of $S(t)$ and $g(K)$ on the top right and down, respectively (see ephasized parts of the table). On the down right the annuities are proportional to $g(K)$, when on the top right are not. But comparing the down parts (see text in bold face) we note the big differences between left and right side. On the left the annuity is 4093.29 for $g(K) = 40000$ and the 5th percentile of the annuities, and it is 5824.39 for the same $g(K)$ and the 95th percentile. The question is, where such a big difference comes from. The reason why the annuities are greater in the right block is one factor $-D(t, t')$, which dominates for greater percentiles. The influence of v becomes weak with the percents. The left side on the top shows the beginning of the $S(t)$'s activity but this behavior is not so evident as it was on the right. The bold part on the left represents the interesting event. Columns of 5th and 25th percentile are the same. We note the same outcomes, what testifies to the domination of $g(K)$ over $S(t)$ and v greater than $D(t, t')$. $S(t)$ determines the annuity on the top and $g(K)$ determines the annuity on the lower part of the table from 2001. Summarizing, for the little guaranteed amount the mutual fund and v or $D(t, t')$ influence the annuity, which gets the policyholder. The existence of the guaranteed interest rate is noticable for the worst scenarios of the annuities and the zero-coupon bond is important for the higher percents of the annuities' results.

In Table 6 we show what happens, when the guaranteed interest rate v changes, $g(K)$=20 000 and the trajectories changes from pessimistic trough median to optimistic. The interest rate i is the guaranteed interest rate resulting from the discounting factor v.

Table 6: The annuity with respect to different guaranteed interest rates. $g(K) = 20000$

		Percentiles of the annuities					
i	K	mean	5%	25%	50%	75%	95%
0.01	208.501	2257.96	1392.12	1735.71	2124.37	2595.55	3125.32
0.02	213.564	2274.86	1440.01	1738.21	2126.54	2595.63	3125.32
0.03	220.513	2301.29	1584.02	1748.53	2133.62	2596.19	3127.44
0.04	229.753	2341.14	1733.42	1756.28	2136.85	2603.82	3129.76
0.05	241.478	2397.71	**1887.77**	**1887.77**	2159.46	2620.85	3140.82
0.06	255.792	2473.83	**2046.64**	**2046.64**	2178.24	2636.83	3164.68
0.07	272.552	2570.7	**2209.63**	**2209.63**	**2209.63**	2664.7	3199.62
0.08	291.478	2687.82	**2376.36**	**2376.36**	**2376.36**	2701.46	3278.89

Considering Table 6 we can find some interesting elements. Observing the first and the last column of the percentiles we see some differences inside each of them. In the first one we can note the numbers from 1392.12 to 2376.36, and in the last one from 3125.32 to 3278.89. So in the last column the numbers are almost the same, when in the first are not. We can say, that in contrast to the first column, the last one does not depend on i, because it does not change so much with i. Therefore for the higher percentiles of the annuities the zero-coupon bond, not the guaranteed interest rate, determines the value of the annuity. We can find out that the influence of $D(t,t')$ grows with the growth of the percentiles of the annuities. The bold part of the table implies the next interesting event. The annuities are the same in the rows for 5th, 25th and 50th percentile. This fact gives evidence that only the guaranteed interest rate plays a large part in changeability of the annuity for little percentiles. By the analyze of table 6 we can also make clear that the instrument priced by us protects against the little annuity, when i grows. It is obvious, if we look at the fair periodic premium K, the mean annuity and the 5th percentile. The growth of i makes the growth of K from 208.501 to 291.478 (so it is about half more than at the beginning). This growth of K does not put the medium annuity up so much, which evaluates from 2257.96 to 2687.82, but it puts the pessimistic annuity up. Hence the choice of the greater guaranteed interest rate protects the policyholder against the pessimistic scenario of the interest rate and then it puts the annuity up.

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