The Evolution of Population, Technology and Output

Galindev, Ragchaasuren

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Ragchaasuren Galindev†
Queen's University Management School

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Abstract

This paper extends Galor and Weil's (2000) unified growth model on the evolution of population, technology and output by replacing the parental utility function in which consumption and children are unrelated, with a more general specification in which some commodities are unrelated with children while the others are substitutes. Considering some leisure goods as the substitutes for children, it aims to explain the demographic transition from high to low fertility with the observed increase in the relative price of children to that of leisure goods along with Galor and Weil's quality-quantity mechanism based on the observed increase in the educational attainments. This modification leads to a conclusion that the demographic transition is a natural phenomenon in this environment when children become relatively more expensive than leisure goods, even for a given level of education and a given price of leisure goods. In addition, an increase in education and a decrease in the price of leisure goods contribute to the demographic transition.

Key words: Malthusian Regime, Modern Growth, Demographic Transition

JEL: J13, O11, O33, O40

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†Contact address: 25 University Square, Queens University Belfast, Belfast, BT7 1NN, United Kingdom; E-mail: g.ragchaasuren@qub.ac.uk, Phone: +44 (0) 28 9097 3290.

1 Introduction

Economic history of Western European countries brings up an important challenge for theories of development. These countries have undergone three different economic regimes in terms of the evolution of population, output and technology during the process of economic development. In the early stage of development, prior to 1800, these countries were trapped in the Malthusian regime where agricultural technology was dominant and technological progress was more or less totally absorbed by the number of inhabitants and hence income per capita remained roughly constant. Countries then moved into the Post-Malthusian regime where both income per capita and population growth increased. While mortality was falling, fertility was increasing until the second half of the nineteenth century and reached its highest level around the 1870s. Through the demographic transition from high to low fertility, countries finally entered the Modern Growth regime where the relationship between income per capita and the growth rate of population becomes negative.[1]

It is widely accepted that the transition from the Malthusian regime to the Post-Malthusian regime is due to the industrial revolution which took place during the late eighteenth century and the first half of the nineteenth century whereas the challenge has been to explore the mechanisms responsible for the demographic transition from high to low fertility.[2] Faster technological progress and an increase in the educational attainments observed in those countries in the late nineteenth century is understood to be the main driving force behind the demographic transition.[3] The

[1]For relevant empirical evidence in each regime, see Galor and Weil (2000) and Galor (2005).

[2]There is a substantial body of literature on the relationship between long-run growth and population dynamics. Galor and Weil (1999, 2000), Fernandez-Villaverde (2001), Jones (2001), Kögel and Prskawets (2001), Galor and Moav (2002), Greenwood and Seshadri (2002), Hansen and Prescott (2002), Tamura (2002) and Doepke (2004) develop models which generate a transition from the Malthusian stagnation to modern growth, accompanied by a demographic transition from high to low fertility.

[3]The other theories of the fertility decline include the following. Becker (1981) argues that fertility

quality-quantity tradeoff model, pioneered by Becker (1960), is the foundation of theoretical contributions that promote this mechanism (e.g., Galor and Weil, 2000; Fernandez-Villaverde, 2001; Doepke, 2004). Although different authors take different approaches,[4] the main feature of the mechanism is that parents care about the quantity (number) and quality (e.g., human capital) of their children and a rise in returns to investment in human capital induces them to choose a fewer but highly educated children.[5]

This paper analyzes a quality-quantity tradeoff model in which, however, the increase in educational attainments is combined with an additional force in explaining the demographic transition. The additional force works through the substitutability between children and some commodities in the parental utility function. We claim that certain types of leisure goods can be substituted for children in enhancing parental welfare. In the literature, children (quantity augmented with quality) are treated as a durable consumption good which produce parental satisfaction. Substantial amount of such satisfaction is produced when parents engage in activities such as playing with and talking to their children throughout their lifetime. Francis and Ramey (2008) consider these as high enjoyment activities and classify them as leisure

decreases as the substitution effect of an increase in wages dominates the income effect, implying that the opportunity cost of children increases. Becker et al. (1990) explain that fertility decreases as the aggregate level of human capital increases. Galor and Weil (1996) explain the fertility decline in relation to an increase in the gender wage gap which is the result of women-specific technological progress. Boldrin and Jones (2002) take the old-age security hypothesis and explain the fertility decline as a result of a decrease in infant mortality rate. See Galor (2005) for an extensive literature review on the empirical relevance of the theories of the demographic transition.

[4] We will shortly see Galor and Weil's (2000) mechanism generating the rise in returns to investment in education. In Doepke (2004), faster technological progress in industrial technology increases the returns to education by increasing the gap between skilled and unskilled wages. Fernández-Villaverde (2001) considers the role of capital-skill complementarity for explaining the rise in returns to investment in education.

[5] However, empirical evidence on the rise in returns to education during the nineteenth century is scarce and hence quite controversial. Clark (2003) argues that it did not increase noticeably. In response, Galor (2005) explains that an increase in the demand for skilled labor could meet an increase in supply, leaving the skill premium roughly constant.

while other childcare activities are classified as non-leisure. In that sense, children can be considered as a leisure good in the economic analysis of fertility. Under such circumstances, one could think of some commodities that may be better substitutes for children than the others. For example, playing sports, clubbing, developing hobbies and so forth can be substituted for the satisfactions from having children at certain degrees. In addition, increased consumption of such activities may be responsible for the fertility decline. Francis and Ramey (2008) finds that the average leisure per week did not increase dramatically during the twentieth century for most of the population in the United States. However, the consumption of leisure goods increased. According to Lebergott (1996), the consumption share of leisure goods increased from 3 percent in 1900 to just over 8 percent in 2001.[6] Another compelling piece of evidence is that leisure goods have become relatively cheaper than other consumption goods. According to Kopecky (2005), the relative price of leisure goods to that of consumption goods declined by 26 percent between 1900 and 1950. Our paper is the first contribution in the literature that claims children are a type of leisure good and tries to attribute the increased consumption of other leisure goods (that can be substituted for children) to the fertility decline. The idea is that when leisure goods are expensive in the early stages of economic development, parents raise children to gain utility from leisure activities during their lifetime. In the literature, the biggest cost of raising children is time so that the market wage is often used as the price of children. The observed increase in the market wage and the observed decline in the price of leisure goods increase the relative price of children which induces parents to substitute leisure goods for children.

The quality-quantity tradeoff model that we choose to account for this additional

[6] We have not found similar evidence in the case of Western European countries. However, given the experience of the U.S. and Western European countries over the last century in terms of economic development, we project a similar pattern.

force on the demographic transition, is that of Galor and Weil (2000) (henceforth, GW). GW's model has the following features. Parents are assumed to care about both the number of children they want and the human capital of each child alongside a consumption set. The human capital of each child depends positively on the level of education per child but negatively on the rate of technological progress (i.e., the erosion effect). Technological progress is endogenous and depends positively on both the size of population and the existing level of education. Raising children requires time and its opportunity cost is measured by the level of potential income that can be earned in the labor market. In this environment, the optimal level of education (quality) per child depends positively on the rate of technological progress while the optimal number of children depends positively on parental potential income but negatively on education per child. In the Malthusian regime, population size is small and technological progress is slow so that parents have no incentive to educate their children. Population growth absorbs technological progress fully hence income per capita remains low and roughly constant. Over time, an increase in population size raises technological progress and economies enter the Post-Malthusian regime. Faster technological progress produces the opposite effect on population growth. On the one hand, it allows parents to spend more resources on children by raising their income. On the other hand, it induces parents to invest in their children's education which tends to decrease the number of children they want. Overall, the positive effect dominates the negative effect hence population growth increases, but more slowly than the rate of technological progress, consequently, income per capita increases. When income per capita reaches a sufficiently high level, the positive effect of technological progress on fertility vanishes hence its negative effect through education leads to the demographic transition. The economies enter the Modern Growth regime. As population growth declines, income per capita increases even faster. GW obtain these results qualitatively while Lagerlof (2006) examines the model quantitatively and finds that the simulation results are consistent with those in GW.

We choose GW's model for the following reason. Since Becker (1960), it has been a common practise in the economic analysis of fertility to consider that families derive utility from a single aggregate commodity (consumption) along with the quantity and quality of children rather than the quantities of individual commodities. The reason for this is that there are no good or close substitutes for children according to Becker (1960). However, Becker admits that there may be many poor substitutes for children. GW assume a Cobb-Douglas utility function which implies no substitutes for children - i.e., the elasticity of substitution between children and consumption is unity. We replace this assumption with a more general one that some goods are unrelated with children while the others are substitutes, at certain degrees. Considering substitutes for children is not new in the literature. Moreover, models that assume substitutes for children generate the demographic transition from high to low fertility (e.g., Jones, 2001; Kögel and Prskawets, 2001). However, these authors emphasize the property of the utility function without identifying potential substitutes for children so that they do not have any empirical relevance. In particular, the whole consumption set is a substitute for children in Jones (2001) while all manufacturing goods are substitutes for children in Kögel and Prskawets (2001).[7]

We disaggregate the consumption set in GW into two broad categories: consumption good, an index of commodities that are not substitutes for or unrelated with children and leisure good, an index of leisure goods and services that are substitutes for children. All individuals are assumed to produce both goods using different technologies. They use the same technology in GW to produce the consumption good

[7] Kögel and Prskawets (2001) also consider a single agricultural good which is assumed to be unrelated to children and represents subsistence consumption.

while use a fraction of the consumption good as an input to manufacture the leisure good. The production of the leisure good is subject to an exogenous technological change which captures decreases in the price of leisure goods. Except for this modification, the structure of the present model is the same as that in GW.

In this model, the decision rule governing the optimal level education per child is the same as that in GW and depends only on the rate of endogenous technological change expected to occur in the sector producing the consumption good. The decision rule for the optimal number of children, on the other hand, shows a hump-shaped relationship with parental potential income, for a given level of education per child and a given price of the leisure good as in Jones (2001) and Kögel and Prskawets (2001), rather than a positive relationship in GW. The reason is that the price of the leisure good is important when choosing child quantity as they are substitutes. The price of children is measured by parental potential income which tends to increase with the endogenous technological progress while the price of the leisure good is an inverse of exogenous technological change in this sub-sector. In fact, the model can generate the hump-shaped relationship between fertility and income per capita for any growth mechanism as long as the relative price of children to that of the leisure good increases over time.

We initially analyze the dynamical system of the model for a constant, yet very high price of the leisure good. Under such circumstances, the model behaves exactly the same as that in GW in both the Malthusian and Post-Malthusian regimes in terms of the evolution of population, technology and output. As in GW, there is a frontier, called the Malthusian Frontier, which separates the Modern Growth regime from the other two. In this model, fertility starts falling as soon as the economy crosses over the Malthusian Frontier into the Modern Growth region while it would become constant in GW, for a given level of education. The reason is that the Malthusian

Frontier separates one region where children are a relatively cheaper commodity than the leisure good from the other region where the opposite is true. In that sense, the demographic transition is a natural phenomenon in the current model. While only further increases in education generates the demographic transition in GW, it is not necessary in this model in spite of its contribution.

A decrease in the price of the leisure good is another mechanism that contributes to the demographic transition as it increases the relative price of children. For a given level of education, an increase in the relative price of children to that of the leisure good contracts the boundary of the Malthusian region and hence the economy can cross over the Malthusian Frontier earlier than otherwise.

The remainder of the paper is organized as follows. Section 2 discusses the model and solves it. Section 3 analyzes the evolution of the dynamical system of the model and conclusions are given in Section 4.

2 The Model

We consider GW's overlapping-generations economy in which there are many identical individuals who live for two periods. As children in the first period of life, individuals are economically inactive and consume a fraction of their parents' time. As adults in the second period of life, they decide on the amount of consumption, the quantity (number) and quality (education) of their children and the labor market participation. A consumption set in GW is now disaggregated into two types of goods: consumption good, an index of goods that are unrelated with children in parental utility and leisure good, an index of leisure goods that, to some extent, are substitutes for children. All adults work to produce the two goods using different technologies. The consumption good is produced using land and efficiency units of labor as inputs as in GW in which the land is exogenous and fixed over time and the quantity of efficiency units of labor is endogenously determined from households' optimization problem in the previous period. The production of the leisure good, on the other hand, uses a fraction of the consumption good as an input. The demands for both goods together with the quantity and quality of children is determined by the households' decisions in each period.

2.1 Technology

Each adult produces z_t unit of the consumption good for each unit of time in period t in accordance with the following constant-returns-to-scale technology:

$$z_t = h_t^{\alpha} x_t^{1-\alpha} = h_t^{\alpha} \left(\frac{A_t X}{L_t} \right)^{1-\alpha} \tag{1}$$

where h_t is the amount of efficiency units of labor or human capital per adult, L_t is the size of the working age population, X is total (exogenous and constant) land, A_t is the level of land augmenting technology and $\alpha \in (0, 1)$ is the labor income share. The term $A_t X$ is total effective resources hence $x_t = A_t X / L_t$ is effective resources per adult.[8]

All working age individuals manufacture the leisure good by combining a fraction of the consumption good and the current state of technology in this sub-sector. Specifically, we adopt a technology similar to that used by Vandenbroucke (2009) for his leisure good production and assume the following production function so as to simplify the analysis:

$$q_t = B_t m_t, \tag{2}$$

[8]Multiplying (1) by L_t yields the aggregate production function, $Y_t = H_t^{\alpha}(A_t X)^{1-\alpha}$ where H_t is the aggregate amount of efficiency units of labor - i.e., $H_t = h_t L_t$.

where q_t is the units of output of the leisure good, B_t is a productivity parameter, and m_t is the fraction of the consumption good. The maximization problem is given by:

$$\max_{m_t} \{p_t B_t m_t - m_t\}$$

where p_t is the unit price of the leisure good relative to that of the consumption good which is normalized to one. The optimal decision rule for this problem is $p_t = 1/B_t$.

2.2 Preferences and Budget Constraints

Each adult member of generation t is assumed to derive utility from the amount of the consumption good, c_t, in excess of its subsistence level $\tilde{c} > 0$, child quantity, n_t, augmented with human capital of each child, h_{t+1}, and the amount of the leisure good, d_t, in accordance with the following function:[9]

$$u = \gamma \log(c_t - \tilde{c}) + (1-\gamma) \log \left[\mu \left(h_{t+1} n_t\right)^{\sigma} + (1-\mu) d_t^{\sigma}\right]^{\frac{1}{\sigma}} \qquad (3)$$

An important feature of the utility function in (3) is that the satisfaction from the consumption good is logarithmically separated from those generated from both children and the leisure good. The implication is that the consumption good is neither a complement nor a substitute for both children and the leisure good - i.e., the elasticity of substitution between both $c_t - \tilde{c}$ and $h_{t+1} n_t$ and $c_t - \tilde{c}$ and d_t is unity. Another, yet the most important feature of (3) is that the leisure good and children enter the utility function through a CES (Constant Elasticity of Substitution) function where

[9] GW consider potential income of each child, $w_{t+1} h_{t+1}$, rather than human capital per child, h_{t+1}, in the parental utility function. Since they consider a Cobb-Douglas utility function which yields the same results as the log-separated function, w_{t+1} does not play any role as individuals take it as given. In the current model, however, considering $w_{t+1} h_{t+1}$ would greatly complicate the analysis. For that reason, we follow Galor (2005) and consider only h_{t+1} in the utility function.

μ and $1-\mu$ denote their respective utility weights. The value of the parameter, σ, determines whether children and the leisure good are substitutes, complements or independent for parental utility. The elasticity of substitution is then $1/(1-\sigma)$. When $0 < \sigma \leq 1$ ($\sigma < 0$), they are substitutes (complements), implying that the marginal utility of children decreases (increases) with an increase in the amount of the leisure good. If $\sigma = 0$, the expression converges to a Cobb-Douglas function, implying that the marginal utilities of children and the leisure good are independent of each other in the utility function after being logged. The following analysis considers the case with $0 < \sigma \leq 1$ to be consistent with the idea that the leisure good is a better substitute for children than the consumption good. Setting either $\sigma = 0$ or $\mu = 1$, we can derive a log-seperated utility function as in Galor (2005) which imply no substitutes for children in the economy as in GW.

As in GW, the adult is endowed with one unit of time which can be allocated between two mutually independent activities: working and raising children. Given her endowment of efficiency units of labor, h_t, the adult would earn potential income equal to z_t if she spent her entire time endowment in the labor market. Raising children requires only time as an input. Let τ^q be the fraction of adult's time associated with producing and raising one child regardless of quality and τ^e be the fraction of the adult's time connected with each level of education (quality) for each child, e_{t+1}. Since time devoted to raising children can be exchanged for the consumption good in the market, the opportunity cost of this activity is a fraction of her potential income: $(\tau^q n_t + \tau^e e_{t+1} n_t) z_t$ which can also be understood as the total spending on the purchase of children (both quality and quantity). Her actual income is then determined by $(1 - \tau^q n_t - \tau^e e_{t+1} n_t) z_t$. Since the adult allocates her actual income between purchasing the consumption and leisure goods, her budget constraint in GW is generalized to

$$c_t + p_t d_t = (1 - \tau^q n_t - \tau^e e_{t+1} n_t) z_t \tag{4}$$

where p_t is the relative price of the leisure good to that of the consumption good which is normalized to unity.

Unlike GW, we assume a constraint that governs the minimum number of children that parents want as in Jones (2001) and Kögel and Prskawets (2001), that is

$$n_t \geq \bar{n} > 0. \tag{5}$$

According to (5), the minimum number of children constraint binds if the optimal number of children derived from the optimization problem is smaller than it.

Following GW, we assume that human capital of each child, h_{t+1}, depends on the expected rate of technological progress between periods t and $t+1$ in the sector producing the consumption good, $g_{t+1} \equiv (A_{t+1} - A_t)/A_t$ and education per child, e_{t+1} in an implicit fashion:

$$h_{t+1} \equiv h(g_{t+1}, e_{t+1}) \tag{6}$$

where $h(\cdot) > 0$, $h_g(\cdot) < 0$, $h_{gg}(\cdot) > 0$, $h_e(\cdot) > 0$, $h_{ee}(\cdot) < 0$, and $h_{eg}(\cdot) > 0$ $\forall$ $(e_{t+1}, g_{t+1}) \geq 0$. The interpretation of these conditions is that the rate of technological progress has a negative effect on human capital and this "erosion effect" declines as g_{t+1} increases while education has a positive effect on human capital and its effect declines as e_{t+1} increases. The last property implies that technological progress increases the return to investments in education or education reduces the adverse effect of technological progress.[10]

[10] Lagerlof (2006) considers an explicit functional form for $h(g_{t+1}, e_{t+1})$, that is $h_{t+1} = \frac{e_{t+1}+\rho\tau}{e_{t+1}+\rho\tau+g_{t+1}}$ where $\rho \in (0,1)$ is an exogenous part of the total fixed time cost of raising children, τ, that contributes towards building human capital - i.e., children acquire some knowledge while being raised (public good) which is, however, not as effective as formal education. Therefore

2.3 Utility Maximization

Given z_t, p_t and g_{t+1}, the adults choose c_t, d_t, n_t and e_{t+1} to maximize their utility in (3) subject to the budget constraint in (4) and the minimum number of children constraint in (5). The first order conditions with respect to d_t, n_t and e_{t+1} are given by

$$\frac{(1-\gamma)p_t}{c_t - \tilde{c}} = \frac{\gamma(1-\mu)d_t^{\sigma-1}}{\mu\left(h_{t+1}n_t\right)^{\sigma} + (1-\mu)d_t^{\sigma}}, \tag{7}$$

$$\frac{(1-\gamma)(\tau^q + \tau^e e_{t+1})z_t}{c_t - \tilde{c}} = \frac{\gamma\mu h_{t+1}^{\sigma} n_t^{\sigma-1}}{\mu\left(h_{t+1}n_t\right)^{\sigma} + (1-\mu)d_t^{\sigma}}, \tag{8}$$

$$\frac{(1-\gamma)\tau^e n_t z_t}{c_t - \tilde{c}} = \frac{\gamma\mu h_{t+1}^{\sigma-1} n_t^{\sigma} h_e(\cdot)}{\mu\left(h_{t+1}n_t\right)^{\sigma} + (1-\mu)d_t^{\sigma}}. \tag{9}$$

In (7), the expression on the left-hand side represents the utility cost generated from purchasing one unit of the leisure good measured by the forgone consumption good while the utility gain is on the right-hand side. In (8), the expression on the right-hand side shows the utility gain of having one child while its utility cost is on the left-hand side as it decreases the amount of the consumption good by decreasing the adult's labor market participation. The price the adult pays for each child quantity is $(\tau + \tau^e e_{t+1})z_t$ which is increasing in z_t as well as in e_{t+1} as the same level of education has to apply to each child. The expression in (9) shows the utility gain of purchasing an extra unit of education for each child on the right-hand side while the utility cost is on the left-hand side which is measured by the forgone consumption good through a decrease in the adult's time available for generating income. The price paid for the extra unit of education is $\tau^e n_t z_t$ which is an increasing function of z_t as well as of n_t because an additional unit of education must apply to more units. The presence of n_t in the price of quality and that of e_{t+1} in the price of quantity will ensure the classic interaction between the quality and quantity of children when the other things such

$e_{t+1} + \rho\tau$ is effective education.

as parental income change (e.g., Becker and Lewis, 1973).

We divide (8) by (9) and obtain the same expression as in GW:

$$G(e_{t+1}, g_{t+1}) \begin{cases} \leq 0 & if \quad e_{t+1} = 0 \\ = 0 & if \quad e_{t+1} > 0 \end{cases} \tag{10}$$

where $G(e_{t+1}, g_{t+1}) = (\tau^q + \tau^e e_{t+1}) h_e(e_{t+1}, g_{t+1}) - \tau^e h(e_{t+1}, g_{t+1})$. Assuming $G(0, 0) < 0$, GW derive the following decision rule for the optimal level of education per child:

$$e_{t+1} = e(g_{t+1}) \begin{cases} = 0 & if \quad g_{t+1} \leq \hat{g} \\ > 0 & if \quad g_{t+1} > \hat{g} \end{cases} \tag{11}$$

where $\hat{g} > 0$ and $e'(g_{t+1}) > 0$ for any $g_{t+1} > \hat{g}$. According to (11), the optimal level of education per child is 0 when the rate of technological progress is sufficiently slow, but positive and increases with g_{t+1} for sufficiently fast technological progress. According to GW, a decrease in the level of human capital due to the erosion effect of technological progress is reduced by an increase in education for $g_{t+1} > \hat{g}$. The implication is that the overall effect of technological progress on human capital is still negative - i.e., $h_g\left(e(g_{t+1}), g_{t+1}\right) < 0$.[11] It is also true in Lagerlof (2006) for his choice of the functional form for $h(e_{t+1}, g_{t+1})$. The optimal level of education per child hence their human capital are, however, independent of parental potential income, z_t. The reason can be explained using (8) and (9). Other things being equal, an increase in z_t tends to increase the demand for both e_{t+1} and n_t by decreasing the marginal utility of the consumption good, $(1 - \gamma)/(c_t - \tilde{c})$, implying that they are normal goods - i.e., it generates the wealth effect. At the same time, it produces

[11] This assumption simplifies the following analysis greatly. In the other cases, an increase in education would either fully recover or more than fully recover the loss in human capital - i.e., $h_g\left(e(g_{t+1}), g_{t+1}\right) \geq 0$. While the former would not affect the qualitative results, the latter would lead to more general results.

a substitution effect on the demand for both commodities by increasing their prices $\tau^e n_t z_t$ and $(\tau^q + \tau^e e_{t+1})z_t$: directly as well as indirectly through n_t for the former and through e_{t+1} for the latter. This interaction between the quality and quantity of children leaves the level of education per child unaffected to the changes in z_t. In other words, the income and substitution effects are cancelled out.

After some manipulations, one may obtain the following expressions for the optimal quantities of children and the leisure good:

$$n_t = \max\left\{\bar{n}, \frac{\gamma\left(1 - \tilde{c}/z_t\right)}{(\tau^q + \tau^e e_{t+1})(1 + \Omega_t)} \equiv n(p_t, g_{t+1}, z_t)\right\}, \tag{12}$$

$$d_t = M h_{t+1}^{\frac{\sigma}{\sigma-1}} \left[\frac{z_t(\tau^q + \tau^e e_{t+1})}{p_t}\right]^{\frac{1}{1-\sigma}} n_t \equiv d(p_t, g_{t+1}, z_t) \tag{13}$$

where $M = ((1-\mu)/\mu)^{\frac{1}{1-\sigma}}$ and $\Omega_t = M\left[\frac{z_t(\tau^q + \tau^e e_{t+1})}{h_{t+1}p_t}\right]^{\frac{\sigma}{1-\sigma}}$. Let $n(\cdot)$ and $d(\cdot)$ be shorthand notations for $n(p_t, g_{t+1}, z_t)$ and $d(p_t, g_{t+1}, z_t)$ respectively where we use $e_{t+1} = e(g_{t+1})$ and $h_{t+1} = h(e(g_{t+1}), g_{t+1}) = h(g_{t+1})$. The functional properties of $n(\cdot)$ and $d(\cdot)$ are summarized in the following proposition.[12]

Proposition 1 *Other things being equal*

(a) A decrease in the price of the leisure good, p_t, leads to a decrease in n_t, but an increase in d_t - i.e.,

$$n_p(\cdot) > 0 \quad \text{and} \quad d_p(\cdot) < 0.$$

(b) Technological progress expected to occur between time t and $t+1$ in the sector producing the consumption good, g_{t+1}, has a negative effect on n_t but an ambiguous

[12]GW assume that the total derivative $\partial z(h_t, x_t)/\partial g_t > 0$ (holding A_{t-1} constant) although the partial derivative is negative (holding x_t and thus A_t constant). This assumption is maintained in the current model.

effect on d_t for all g_{t+1} - i.e.,

$$n_g(\cdot) < 0 \qquad and \qquad d_g(\cdot) \lesseqgtr 0.$$

(c) There exists a time varying critical value $\tilde{z}_t \equiv \tilde{z}(p_t, g_{t+1})$ so that an increase in the parental potential income, z_t, has (c.1) a positive effect on both n_t and d_t if $z_t < \tilde{z}_t$ and (c.2) a negative effect on n_t but a positive effect on d_t if $z_t > \tilde{z}_t$ - i.e.,

$$\begin{array}{lllllll} (c.1) & n_z(\cdot) & \geq & 0 & and & d_z(\cdot) > 0 & if\ z_t \leq \tilde{z}_t, \\ (c.2) & n_z(\cdot) & \leq & 0 & and & d_z(\cdot) > 0 & if\ z_t \geq \tilde{z}_t. \end{array}$$

where $\tilde{z}_t > \frac{\tilde{c}}{\sigma} > \tilde{c}$ for $\sigma \in (0,1)$. In addition, $\tilde{z}_g(\cdot) < 0$ and $\tilde{z}_p(\cdot) > 0$.

Proof. The results in parts (a) and (b) follow directly from differentiating n_t in (12) and d_t in (13) with respect to p_t and g_{t+1} after substituting (11) and (6) into (12) and (13). The sign of $n_z(\cdot)$ in part (c) is determined by:

$$sgn\{n_z(\cdot)\} \equiv sgn\{\tilde{c}(1-\sigma) - \Omega_t(\sigma z_t - \tilde{c})\}.$$

It is clear that $n_z(\cdot) > 0$ for $0 \leq z_t \leq \frac{\tilde{c}}{\sigma}$, but $n_z(\cdot) \gtreqless 0$ for $z_t > \frac{\tilde{c}}{\sigma}$. If $\sigma = 0$, it is true that $n_z(\cdot) > 0$. If $\sigma = 1$, the reverse is true. In the ideal case with $\sigma \in (0,1)$, the second term, $\Omega_t(\sigma z_t - \tilde{c})$, is a monotonically increasing function of z_t. Thus there exists a unique $\tilde{z}_t \equiv \tilde{z}(p_t, g_{t+1})$ such that $n_z(z_t = \tilde{z}_t) = 0$. Given (11), (6) and $h_g(g_{t+1}) < 0$, the implicit function theorem suggests that $\tilde{z}_g(\cdot) < 0$ and $\tilde{z}_p(\cdot) > 0$. ■

The intuitions of the results in Proposition 1 are the following. Define first the relative price of child quantity to that of the leisure good, r_t, as $r_t \equiv \frac{z_t(\tau^q + \tau^e e_{t+1})}{p_t}$: an increase in both z_t and e_{t+1} but a decrease in p_t cause an increase in r_t. For $g_{t+1} \leq \hat{g}$, the relative price is $r_t \equiv \frac{\tau^q z_t}{p_t}$. The results in part (a) can be explained on the basis of the substitutability between children and the leisure good: the lower the price of the leisure good, the lower (higher) the demand for children (the leisure good). Since child quality (e_{t+1}) does not depend on p_t, child quantity decreases but the leisure good increases directly due to an increase in r_t.

An increase in g_{t+1} has three negative effects on n_t. GW's quality-quantity tradeoff is one of them which makes n_t more expensive relative to e_{t+1} for $g_{t+1} > \hat{g}$. In general, even a small initial increase in e_{t+1} due to an increase in g_{t+1} can lead to a large decrease (increase) in n_t (e_{t+1}) if the interaction between quality and quantity is strong - i.e., close substitutes (e.g., Becker and Lewis, 1973). The second negative effect makes child quantity more expensive relative to the leisure good by increasing r_t for $g_{t+1} > \hat{g}$. In that sense, the current setup strengthens the usual interaction between e_{t+1} and n_t. If there are no substitutes for children - i.e., σ converges to zero, this effect will diminish to zero. The last effect of g_{t+1} on n_t arises through a decrease in h_{t+1}. As parents derive utility from the level of human capital of each child, a decrease in h_{t+1} decreases the marginal utility of child quantity according to (8) for $\sigma \in (0, 1)$. Again, if there are no substitutes for children ($\sigma = 0$), this effect will vanish. The increase in g_{t+1}, on the other hand, has an ambiguous effect on d_t. A positive effect works through an increase in r_t and a decrease in h_{t+1} while a negative effect works through a decrease in n_t.[13]

An increase in z_t has opposite effects on n_t: the income (positive) and substitution (negative) effects. In a simple model with $\tilde{c} = 0$ and $\sigma = 0$, these effects are exactly offset, leaving n_t unchanged. Introducing $\tilde{c} > 0$ strengthens the income effect hence n_t increases. The strength of this additional income effect, however, becomes weaker

[13] For the functional forms used by Lagerlof (2006), we find that the positive effect dominates and the amount of the leisure good increases with g_{t+1}.

as z_t increases. In the current setup with $\sigma > 0$, the substitution effect is also stronger than usual due to $r_t > 0$. At $\tilde{z}_t$, the sum of both the usual and additional income effects is exactly offset by the stronger substitution effect. The case with $z_t < \tilde{z}_t$ implies that the positive effects dominate the substitution effect hence the number of children increases with z_t. In other words, children are relatively cheaper than the leisure good in this region. In the case with $z_t < \tilde{z}_t$, however, children are more expensive than the leisure good (or the substitution effect dominates the income effect) hence child quantity decreases with z_t. Alternatively, the model generates a hump-shaped relation between n_t and z_t for given p_t and g_{t+1}. A decrease in p_t and an increase in g_{t+1} make the transition from $n_z(\cdot) > 0$ to $n_z(\cdot) < 0$ easier by decreasing $\tilde{z}_t$. The optimal amount of the leisure good increases with z_t for two reasons. Firstly, the leisure good is a normal good. Secondly, it is a substitute for children whose price increases with z_t, leading to an increase in r_t.

2.4 Technological Progress

The rate of technological progress that occurs between time t and time $t+1$ in the sector producing the consumption good, g_{t+1}, is the same as that in GW, that is, an implicit function of education at time t, e_t, and the size of working age population at time t, L_t:

$$g_{t+1} \equiv \frac{A_{t+1} - A_t}{A_t} = g(e_t, L_t) \tag{14}$$

where for $L_t \gg 0$ and $e_t \geq 0$, $g(0, L_t) > 0$, $g_i(\cdot) > 0$, and $g_{ii}(\cdot) < 0$, $i = e_t, L_t$. The rate of technological progress is an increasing concave function of each determinant for a sufficiently large population size. Moreover, there is positive technological progress even if education is zero.[14] GW assume $g_L(0, L_t) = 0$ for a sufficiently small

[14] Lagerlof (2006) considers the explicit form for $g(e_t, L_t)$, that is, $g_{t+1} = (e_t + \rho\tau) \min\{\theta L_t, a^*\}$ where $\theta > 0$ measures the "scale" effect of L_t while $a^* > 0$ corresponds to $\lim_{L \to \infty} g(\cdot) = a^*$ for

population size to ensure that early stages of development take place in a Malthusian steady state. This assumption is kept in our analysis.

The rate of technological change in the production of the leisure good, g^B_{t+1}, is assumed to follow an exogenous process,

$$g^B_{t+1} \equiv \frac{B_{t+1} - B_t}{B_t} \geq 0, \tag{15}$$

so as to simplify the analysis. In the following analysis, the dynamical system of the economy is initially studied under the assumption that $g^B_{t+1} = 0$ - i.e., the level of technology in this sector hence the price of the leisure good is constant. We will, however, analyze the effect of $g^B_{t+1} > 0$ on the evolution of the dynamical system.

2.5 Population, Technology and Effective Resources

The evolution of the size of working population, L_t, technology, A_t, and effective resources per worker, x_t, is the same as those in GW and is governed by the following three difference equations:

$$L_{t+1} = n_t L_t, \tag{16}$$

$$A_{t+1} = (1 + g_{t+1})A_t, \tag{17}$$

$$x_{t+1} = \frac{1 + g_{t+1}}{n_t} x_t \tag{18}$$

where their initial levels are historically given at L_0, A_0 and $x_0 = (A_0 X)/L_0$ respectively. The number of children per person, n_t, and the rate of technological progress, g_{t+1}, are determined by the expressions in (12) and (14) respectively.

Using (15), the evolution of the relative price of the leisure good, $p_t = 1/B_t$, can

given e_t. Thus population increases technological progress linearly for $L_t \leq a^*/\theta$ and then has no effect.

be written as

$$p_{t+1} = \frac{1}{1+g^B_{t+1}} p_t \tag{19}$$

where p_0 is historically given. $g^B_{t+1} > 0$ implies that the leisure good becomes more affordable over time.

3 The Dynamical System

This section analyzes the dynamical system of the economy which determines its development through the evolution of population, income per capita, technology levels in the production of both the consumption and leisure goods, education per worker, human capital per worker and effective resources per worker. The sequence that determines the development of the economy in GW, $\{e_t, g_t, x_t, L_t\}_{t=0}^{\infty}$, is now extended to $\{e_t, g_t, x_t, L_t, p_t\}_{t=0}^{\infty}$ in the current analysis that satisfies (14)-(19).

Since we do not follow GW in solving for the household's optimization problem, the dynamical system is characterized by one regime rather than two.[15] For a given size of population L, and a given price of the leisure good p, the development of the economy is determined by the following three-dimensional nonlinear system of

[15] The utility maximization problem in GW is rather unconventional in the sense that they do not equate the marginal benefit and cost of children when choosing the optimal number of children. If they did, the decision rule for n_t would be $n_t = \frac{\gamma(1-\tilde{c}/z_t)}{\tau^q+\tau^e e_{t+1}}$ according to which n_t is an increasing concave function of z_t and converges to $\frac{\gamma}{\tau^q+\tau^e e_{t+1}}$ smoothly as z_t converges to infinity for given e_{t+1}. This rule can be derived as a special case of the one in the current model by setting either $\mu = 1$ or $p_t = \infty$ in (12) as $\Omega_t = 0$. The decision rule in GW is more or less an approximation to this rule in the sense that the same convergence is not smooth and takes place immediately when $z_t = \tilde{z}$. However, this particular approximation simplifies the dynamical system in GW greatly as the system is divided into two regions by $\tilde{z}$. Otherwise, the entire space (x_t, e_t), in which the system is analyzed, would be the Malthusian region where z_t has always a positive effect on n_t as there is no $\tilde{z}$. As can be seen from Proposition 1, the current model generates $\tilde{z}_t$ without following the optimization in GW.

difference equations:

$$\begin{cases} e_{t+1} = e(g(e_t); L) \\ g_{t+1} = g(e_t; L) \\ x_{t+1} = \phi(e_t, g_t, x_t; L, p)x_t \end{cases} \tag{20}$$

where $\phi(e_t, g_t, x_t; L, p) \equiv (1 + g_{t+1})/n_t$ and the initial values e_0, g_0 and x_0 are historically given.

The evolution of e_t and g_t is independent of the x_t. Therefore, the analysis of the joint dynamics of education and technology is exactly the same as those in GW. Broadly speaking, this dynamical subsystem is characterized by three different configurations in the (e_t, g_t) space, depending on the size of population. For a small population size, there is a unique globally stable steady-state equilibrium $(\bar{e}, \bar{g}) = (0, g^l)$ characterizing the dynamical subsystem (see Figure 3 in GW). For a moderate population size, the dynamical subsystem is characterized by multiple steady-state equilibria: $(\bar{e}, \bar{g}) = (e^u, g^u)$ is unstable and lies between $(\bar{e}, \bar{g}) = (0, g^l)$ and $(\bar{e}, \bar{g}) = (e^h, g^h)$ which are stable. This is depicted in Figure 4 in GW. Figure 5 in GW shows the dynamical subsystem for a large population size which is characterized by a globally stable steady-state equilibrium $(\bar{e}, \bar{g}) = (e^h, g^h)$.

3.1 Global Dynamics

We analyze the evolution of the dynamical system of the economy in (20) using a series of phase diagrams in the (e_t, x_t) space, as described in GW. Each phase diagram, shown in Figures 2-4, has three components: the Malthusian Frontier which separates one regime where parental potential income has a positive effect on the chosen number of children from the other where the effect is negative, the XX locus along which the effective resources per worker is constant and the EE locus along which the level of education per worker is constant. There is one similarity and two differences in the phase diagrams between ours and GW's. They are the same in terms of the EE locus. The first difference lies on the Malthusian Frontier which in GW separates the one regime where the correlation between parental income and child quantity is positive from the other where the correlation is zero. The second difference is on the shape of the XX locus as the current model is more general.

The Conditional Malthusian Frontier According to Proposition 1, the economy switches the regime from the one where individuals' income has a positive effect on their chosen number of children to the other where the effect becomes negative when potential income z_t exceeds the time varying critical level $\tilde{z}_t$.

For the dynamical system in (20), the Conditional Malthusian Frontier, $MM_{|g_t}$, is the set of all pairs (e_t, x_t) conditional on given g_t, and $z_t = \tilde{z}_t \equiv \tilde{z}(g_{t+1}; p)$. More formally, $MM_{|g_t}$ is written as

$$MM_{|g_t} \equiv \left\{(e_t, x_t) : x_t^{1-\alpha} h(e_t, g_t)^{\alpha} = \tilde{z}(g(e_t)) \mid g_t\right\}.$$

Lemma 1 *If $(e_t, x_t) \in MM_{|g_t}$, x_t is a monotonically decreasing function of e_t. Moreover, a decrease in x_t along $MM_{|g_t}$ is larger than that in the case where $\tilde{z}_t$ is constant for an equal increase in e_t. Furthermore, the critical $\tilde{z}_t$ decreases along $MM_{|g_t}$ as e_t increases.*

Proof. Given the result in part (c) of Proposition 1 that $\tilde{z}_t$ is a decreasing function of g_{t+1}, an increase in e_t has a negative effect on $\tilde{z}_t$ through an increase in g_{t+1}. Since h_t is an increasing function of e_t, x_t must decrease in response to an increase in e_t along $MM_{|g_t}$. ■

The Conditional Malthusian Frontier is similar to that in GW in the sense that it is a downward sloping curve, intersects the x_t axis and asymptotically approaches to the e_t axis as x_t approaches to zero. As the functional forms are implicit, however, we cannot predict the second order property of the frontier while it is a strictly convex function in GW. Without loss of generality, the Conditional Malthusian Frontier $MM_{|g_t}$ is depicted as a downward sloping convex curve in Figures 2-4.[16]

The frontier is affected by the evolution of population, L, and the price of the leisure good, p.

Lemma 2 *(a) An increase in L and a decrease in p lead $MM_{|g_t}$ to shift downward and leftward.*

Proof. According to (14), an increase in L leads to an increase in g_{t+1}. Given the result in part (c) of Proposition 1, both an increase g_{t+1} and a decrease in p have a negative effect on $\tilde{z}_t$ so that x_t must decrease for given e_t. ■

By setting $p_t = \infty$ and following GW's utility maximization approach, we can derive their frontier as $\Omega_t = 0$ in (12). The intuition of the results in Lemma 2 is that the boundary of the Malthusian region where potential income has a positive effect on child quantity will shrink as the leisure good becomes cheaper and population size increases.

The XX Locus According to (18), the effective resources per worker, x_t, is constant if growth rates of working population and technology are equal. The conditional XX locus is the set of all pairs (e_t, x_t) for given g_t, such that x_t is in a steady state. More formally,

$$XX_{|g_t} \equiv \{(e_t, x_t) : x_{t+1} = x_t \mid g_t\}.$$

[16] The second order property of the Conditional Malthusian Frontier would not affect the qualitative analysis.

Lemma 3 *There exists a unique value* $0 < \hat{e} < e^h$ *such that (a) for each* $0 \leq e_t < \hat{e}$*, there are two values for* $x_t \in XX_{|g_t}$*:* $x_t^h > x_t^l$ *such that* $z(e_t, x_t^h) > \tilde{z}_t$ *and* $z(e_t, x_t^l) < \tilde{z}_t$*, and a unique value* $x_t^l < \tilde{x}_t < x_t^h$ *such that* $z(e_t, \tilde{x}_t) = \tilde{z}_t$*; (b) for* $e_t = \hat{e}$*, there is a unique* $\hat{x} \in XX_{|g_t}$ *such that* $z(\hat{e}, \hat{x}) = \tilde{z}_t$*; and (c) for* $\hat{e} < e_t \leq e^h$*, there is no* $x_t \in XX_{|g_t}$*. Moreover, for* $z_t \geq \tilde{z}_t$*,*

$$x_{t+1} - x_t \begin{cases} > 0 & if\ [(e_t, x_t) > (e_t, x_t^h(e_t))\ and\ 0 \leq e_t < \hat{e}],\ [(\hat{e}, x_t) > (\hat{e}, \hat{x})]\ or\ [e_t > \hat{e}] \\ = 0 & if\ [(e_t, x_t) = (e_t, x_t^h(e_t))\ and\ 0 \leq e_t < \hat{e}]\ or\ [(e_t, x_t) = (\hat{e}, \hat{x})] \\ < 0 & if\ (e_t, x_t) < (e_t, x_t^h)\ and\ 0 \leq e_t < \hat{e}. \end{cases}$$

For $z_t \leq \tilde{z}_t$*,*

$$x_{t+1} - x_t \begin{cases} < 0 & if\ (e_t, x_t) > (e_t, x_t^l(e_t))\ and\ 0 \leq e_t < \hat{e} \\ = 0 & if\ [(e_t, x_t) = (e_t, x_t^l(e_t))\ and\ 0 \leq e_t < \hat{e}]\ or\ [(e_t, x_t) = (\hat{e}, \hat{x})] \\ > 0 & if\ [(e_t, x_t) < (e_t, x_t^l(e_t))\ and\ 0 \leq e_t < \hat{e}],\ [(\hat{e}, x_t) < (\hat{e}, \hat{x})]\ or\ [e_t > \hat{e}]. \end{cases}$$

Proof. Rewrite (12) as follows:

$$n_t = \frac{\gamma\left(1 - \tilde{c}/z_t\right)}{\left(\tau^q + \tau^e e_{t+1}\right)\left(1 + \Psi_t z_t^{\frac{\sigma}{1-\sigma}}\right)} \equiv n(e_t, x_t) \tag{21}$$

where $\Omega_t = \Psi_t z_t^{\frac{\sigma}{1-\sigma}}$, $\Psi_t = M\left(\frac{\tau^q + \tau^e e_{t+1}}{h_{t+1} p}\right)^{\frac{\sigma}{1-\sigma}}$, $e_{t+1} = e(g_{t+1})$, $h_{t+1} = h(g_{t+1})$, $g_{t+1} = g(e_t; L)$ and $z_t = x_t^{1-\alpha} h(e_t, g_t)^\alpha$. We find that $n_x(e_t, x_t) = \tilde{c}(1-\sigma) - F(x_t, e_t)$ where $F(x_t, e_t) = \Psi_t z_t^{\frac{\sigma}{1-\sigma}}(\sigma z_t - \tilde{c})$ which is monotonically increasing in x_t for each e_t - i.e., $F_x(x_t, e_t) > 0$.[17] Since $\tilde{c}(1-\sigma)$ is independent of x_t, there exists a unique $\tilde{x}_t$ for given e_t, such that $\tilde{c}(1-\sigma) = F(x_t, e_t)$ and $\tilde{z}_t = \tilde{x}_t^{1-\alpha} h(e_t, g_t)^\alpha$ - i.e., the pair $(\tilde{x}_t, e_t)$ is on $MM_{|g_t}$. Both $\tilde{c}(1-\sigma)$ and $F(x_t, e_t)$ are depicted in panel (a) of Figure 1 where

[17] This is essentially the same expression found to prove the result in part (c) of Proposition 1.

$\ddot{x}(e_t) = (\tilde{c}/(\sigma h_t^{\alpha}))^{1/(1-\alpha)}$ such that $\sigma z_t = \tilde{c}$.[18] For $x_t < \tilde{x}_t$, $\tilde{c}(1-\sigma) > F(x_t, e_t)$ hence $n_x(x_t, e_t) > 0$. For $x_t > \tilde{x}_t$, $\tilde{c}(1-\sigma) < F(x_t, e_t)$ hence $n_x(x_t, e_t) < 0$. Thus there is a hump-shaped relationship between n_t and x_t for each e_t. Hence we can obtain $\tilde{n}_t = n(\tilde{x}_t, e_t)$ such that $n_x(\tilde{x}_t, e_t) = 0$. Now depict both $1 + g(e_t)$ and $n(x_t, e_t)$ in the (n_t, x_t) space which is given in panel (b) of Figure 1 where $\bar{x}(e_t)$ can be determined from (12) such that $n_t = \bar{n}$. Since $1+g(e_t)$ is independent of x_t, it is a horizontal line. The intersection of $1 + g(e_t)$ and $n(x_t, e_t)$ yields two steady-state values, $x^l(e_t)$ and $x^h(e_t)$, such that $x^l(e_t) < \tilde{x}(e_t) < x^h(e_t)$ if $\tilde{n}_t > 1 + g(e_t)$. Furthermore, $x_{t+1} - x_t > 0$ for both $x_t < x^l(e_t)$ and $x_t > x^h(e_t)$ as $n_t < 1 + g(e_t) < \tilde{n}_t$. However, $x_{t+1} - x_t < 0$ for $x^l(e_t) < x_t < x^h(e_t)$ as $1 + g(e_t) < n_t \leq \tilde{n}_t$.

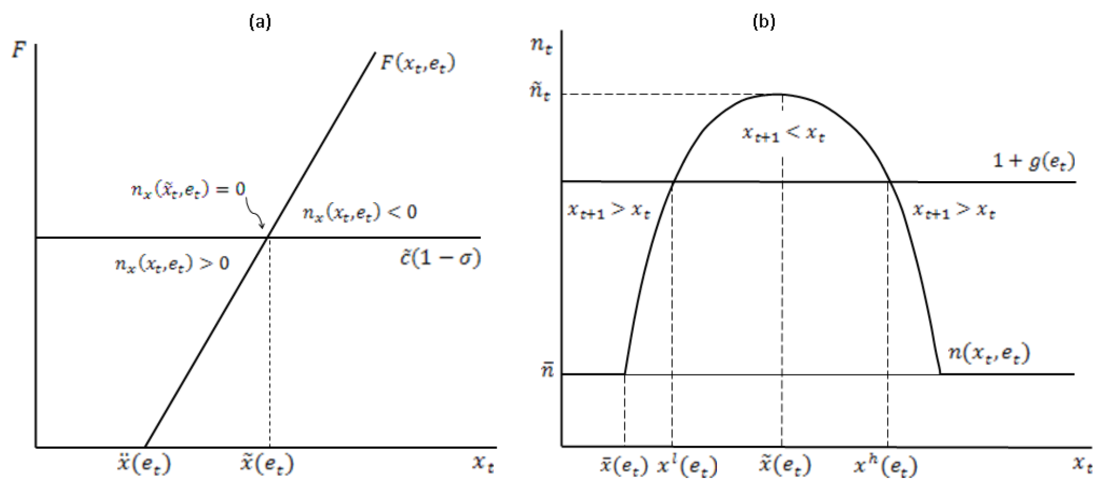

Figure 1. Derivation of the XX locus

Let us now analyze the situation where e_t increases. For each x_t, $F(x_t, e_t)$ increases as both z_t and Ψ_t increase. For the new e_t and Ψ_t, a decrease in $\tilde{x}_t$ must be sufficient to restore $\tilde{c}(1-\sigma) = \Psi_t \tilde{z}_t^{\frac{\sigma}{1-\sigma}}(\sigma\tilde{z}_t - \tilde{c})$. In particular, the new $\tilde{x}_t$ must be smaller than

[18] The second order property of $F(x_t, e_t)$ is ambiguous depending the values of α and σ. However, the qualitative analysis is not affected by this. Hence without loss of generality, $F(x_t, e_t)$ is depicted as a straight line in Figure 1.

the old level such that the new $\tilde{z}_t$ is smaller than the old level. Thus $\tilde{n}_t$ is smaller than its old level. It implies that the point, $(\tilde{n}_t, \tilde{x}_t)$, shifts leftward and downward in the (n_t, x_t) space. In other words, $\tilde{n}_t$ decreases as e_t increases. Since $1 + g(e_t)$ increases as e_t increases, there exists a unique $\hat{e}$ such that $\tilde{n}_t = 1 + g(\hat{e})$. Thus $\tilde{x}_t$ for $\tilde{n}_t = 1+g(\hat{e})$ is $\hat{x}$ and the pair $(\hat{e}, \hat{x})$ is on $MM_{|g_t}$. Since $\tilde{n}_t < 1+g(e_t)$ for $\hat{e} < e_t \leq e^h$, there is no $x_t \in XX_{|g_t}$. ■

The locus $XX_{|g_t}$ is strictly below the curve $MM_{|g_t}$ for $e_t < \hat{e}$ and $x_t < \hat{x}$, but strictly above the curve $MM_{|g_t}$ for $e_t < \hat{e}$ and $x_t > \hat{x}$. At $(\hat{e}, \hat{x})$, the curve $MM_{|g_t}$ and the locus $XX_{|g_t}$ coincide. Since the curve $n(x_t, e_t)$ shifts leftward and downward as e_t increases, x_t^l may either increase or decrease as in GW while x_t^h decreases unambiguously. Hence without loss of generality, the part of the locus $XX_{|g_t}$ below the curve $MM_{|g_t}$ is depicted in Figures 2-4 as an upward sloping curve as in GW.

The locus $XX_{|g_t}$ is also affected by the evolution of population and the price of the leisure good.

Lemma 4 *An increase in L and a decrease in p lead $XX_{|g_t}$ under $MM_{|g_t}$ to shift up, but above $MM_{|g_t}$ to shift down. Furthermore, the critical $\hat{e}$ decreases as L increases and p decreases.*

Proof. An increase in L leads to $\phi(\cdot) = (1 + g_{t+1})/n_t > 1$ through its direct positive effect on g_{t+1} and an indirect negative effect on n_t which works through an increase in e_{t+1}, a decrease h_{t+1} and hence an increase in Ψ_t. Thus n_t must increase to restore the steady-state condition for x_t, $\phi(\cdot) = 1$. It must be achieved as x_t^l increases (i.e., z_t increases) and x_t^h decreases (i.e., z_t decreases) for each e_t. The increase in Ψ_t leads to a decrease in the corresponding $\tilde{x}_t$ by shifting $F(x_t, e_t)$ leftward which is consistent with the result in Lemma 2. For given e_t, a decrease in $\tilde{x}_t$ leads to a decrease in $\tilde{z}_t$ so that $\tilde{n}_t$ decreases. Thus $\hat{e}$ decreases. A decrease in p has the same effect which works through a decrease in n_t for $\phi(\cdot)$. ■

The intuition of the result in Lemma 4 is that the economy may cross over the Malthusian Frontier for a given level of education.[19] The locus $XX_{|g_t}$ in GW can be derived when $p_t = \infty$ as the part of the locus above the Malthusian Frontier becomes vertical at $\hat{e}$ if one follows GW's utility maximization.[20] In that case, the locus $XX_{|g_t}$ shifts up when the size of population increases as $\Psi_t = 0$.

The EE locus The conditional EE locus is exactly the same as that in GW, that is a set all pairs (e_t, x_t) conditional on given g_t such that education per worker e_t is in a steady state:

$$EE_{|g_t} \equiv \{(e_t, x_t) : e_{t+1} = e_t \mid g_t\}.$$

GW shows that the steady-state values of e_t are independent of g_t and x_t, for a given size of population. Therefore the locus EE is a vertical line in the (e_t, x_t) space and shifts rightward as population size increases. The location of the locus EE identifies one of three phases of economic development in terms of the evolution of education and technology. In the early stage of development, the locus EE is vertical at $e = 0$ representing the globally stable temporary steady-state equilibrium, $(\bar{e}, \bar{g}) = (0, g^l)$ and

$$e_{t+1} - e_t \begin{cases} = 0 & if\ e_t = 0 \\ < 0 & if\ e_t > 0. \end{cases} \tag{22}$$

[19]As far as the consequence of an increase in the population size is concerned, this result is not consistent with the underlying assumption of the model that education per person increases with the rate of technological progress which is an increasing function of the population size. However it is a theoretical possibility which can also be predicted by GW.

[20]Under such circumstances, one must maintain the assumption (A4) in GW to ensure that the XX locus is nonempty for $z_t \geq \tilde{z}$, that is, $\hat{g} < (\gamma/\tau^q) - 1 < g(e^h(L_0), L_0)$.

In the intermediate stage of development, characterized by the multiple locally stable temporary steady-state equilibria, $(0, g^l)$, (e^u, g^u) and (e^h, g^h), the locus EE is vertical at $e_t = 0$, $e_t = e^u$ and $e_t = e^h$. The locus EE at $e_t = e^u$ and $e_t = e^h$ shift rightward as population size increases. The global dynamics of e_t are given by

$$e_{t+1} - e_t \begin{cases} = 0 & if\ e_t \in \left\{0, e^u, e^h\right\} \\ > 0 & if\ e^u < e_t < e^h \\ < 0 & if\ 0 < e_t < e^u \text{ or } e_t > e^h. \end{cases} \tag{23}$$

In the advanced stage of development, the locus EE at $e_t = e^h$ represents a globally stable steady-state equilibrium, (e^h, g^h). It shifts rightward as population size increases. The global dynamics of e_t in this case is given by

$$e_{t+1} - e_t \begin{cases} = 0 & if\ e_t = e^h \\ > 0 & if\ 0 \le e_t < e^h \\ < 0 & if\ e_t > e^h. \end{cases} \tag{24}$$

3.2 Conditional Steady-State Equilibria

The dynamical system in the early stage of economic development with small population sizes is characterized by two conditional steady-state equilibria which are given by the intersection between the XX locus and the EE locus in the (e_t, x_t) space, as shown in Figure 2. Both equilibria are conditional on the rate of technological progress, the size of population and the price of the leisure good. Since the conditional steady-state equilibrium $(\bar{e}, \bar{x}) = (0, x^h)$ is unstable, the locally stable conditional steady-state equilibrium $(\bar{e}, \bar{x}) = (0, x^l)$ is the Malthusian steady-state. Another reason why the unstable steady-state equilibrium is not the Malthusian steady-state is that an increase in potential income has a negative effect on child quantity which is opposite to the assumption on which the Malthusian model is built.

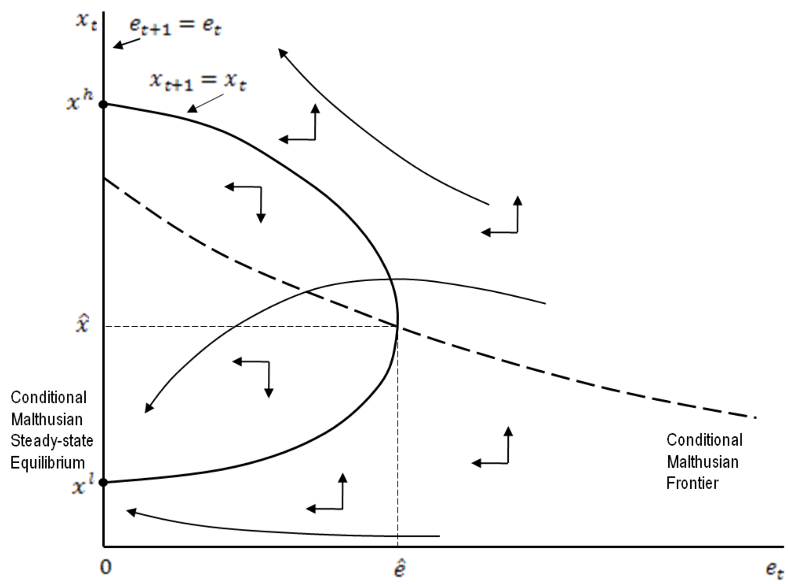

Figure 2. The Conditional Dynamical System in the Early Stage

The dynamical system in the intermediate stage of development with moderate population sizes, depicted in Figure 3, is similar to that in GW in the sense that the Malthusian conditional steady-state is locally stable and the conditional steady-state equilibrium $(e^u, x^l(e^u))$ is a saddlepoint. In addition to those in GW, there are two conditional unstable steady-state equilibria: $(0, x^h(0))$ and $(e^u, x^h(e^u))$. If the level of education is above e^u, the dynamical system converges to an equilibrium with a level of education e^h and possibly a steady-state growth rate of x_t, given the population size and the price of the leisure good. In the advanced stage of development with large population sizes, the dynamical system is, as depicted in Figure 4, converges globally to an education level e^h and possibly a steady-state growth rate of x_t, given the population size and the price of the leisure good.

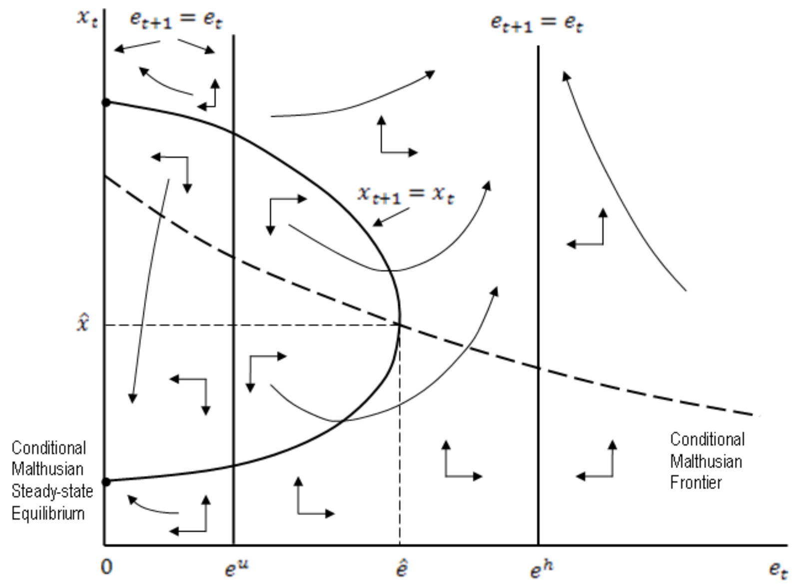

Figure 3. The Conditional Dynamical System in the Intermediate Stage

3.3 Analysis

Collecting the results obtained so far, we can now discuss the process of economic development from a Malthusian regime to a Modern Growth regime and a demographic transition through a Post-Malthusian regime. Consider that the economy is in the early stage of development where population size is sufficiently small, the rate of technological progress is so slow that parents find it inefficient to invest in their children's education. In addition, the price of the leisure good is extremely high as the technology of producing these goods is so primitive that its consumption/production is small and children are very cheap relative to the leisure good. In Figure 2, the situation is represented by the temporary, conditional and locally stable Malthusian steady-state equilibrium where both the level of education and effective resources per worker are constant, for a given rate of technological change. Consequently, output per capita is constant from (1) and (11). Thus population is growing slowly, at the same rate as technological progress. For an extremely high price of the leisure good, the unstable conditional steady-state equilibrium, $(0, x^h)$, may not exist so that the Malthusian steady-state equilibrium can be globally stable as the XX locus for $z_t > \tilde{z}_t$ (i.e., above the Malthusian Frontier) may become vertical at $\hat{e}$ as in GW. Temporary shocks to technology and population will be adjusted towards the Malthusian steady-state.

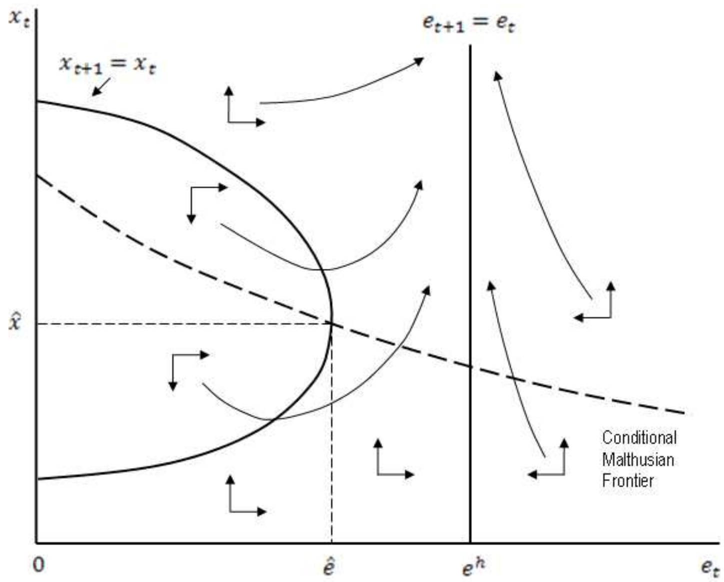

Figure 4. The Conditional Dynamical System in the Advanced Stage

Over time, the price of the leisure good may decrease and create the unstable conditional steady-state equilibrium (i.e., the XX locus for $z_t > \tilde{z}_t$ is no longer vertical at $\hat{e}$). If the effective resources per worker jump above the XX locus for $z_t > \tilde{z}_t$ due to temporary shocks to population and technology, the system will converge to zero education level and possibly a steady-state growth rate of x_t, given the population size, the rate of technological progress and the price of the leisure good. Under such circumstances, the number of children per person will decrease until the minimum quantity of children constraint binds as children are expensive relative to the leisure good in this region. In other words, it will generate a demographic transition. However, we do not consider this possibility. Instead, we assume that the economy stays around the Malthusian steady-state at this early stage of development.

When the size of the population reaches a sufficiently high level, the dynamical system will be characterized by multiple steady-state equilibria as in GW: the Malthusian steady-state with constant income per capita, slow technological progress and zero education, and the Modern Growth steady-state with fast technological progress, a high level of education and increasing income per capita. As depicted in Figure 3, the convergence towards these steady states is history dependent. In addition, the economy may jump over the boundary and evolve accordingly due to shocks to technology and education. Like GW, however, we are interested in the economy starting out in the Malthusian steady-state so that it stays there at this intermediate stage.

Figure 5 in GW shows that the evolution of education and technology is monotonic and converges to a unique globally stable steady-state with fast technological progress and a high level of education when the size of population reaches a sufficiently high level. Simultaneously, the Malthusian steady-state disappears. Exactly the same mechanism is at work in the current analysis. As in GW, technological progress has opposite effects on the evolution of population growth in the Malthusian region of Figure 4. As shown in Proposition 1, technological progress has a negative effect on the number of children per adult through an increase in education per child and an overall decrease in human capital per child, for a given level of potential income and a given price of the leisure good. At the same time, it increases parental potential income which has a positive effect on the number of children per person. If the price of the leisure good is decreasing, child quantity will tend to decrease. Initially, the positive effect dominates all the negative effects hence the rate of population growth will increase, reflecting the characteristics of the Post-Malthusian regime.

If the positive effect of technological change continues to dominate the negative effects on the number of children per person, the rate of population growth will increase continuously until the economy crosses over the Malthusian Frontier. As soon as the economy enters the Modern Growth region, the rate of population growth will decrease unambiguously as the growth in parental potential income produces a negative effect on the number of children per person. Income per capita continues to increase while the rate of population growth continues to decrease even without any further improvements in technology (i.e., both education per person and human capital per person are constant). The source of growth for income per capita will be the growth of effective resources per capita as the rate of technological change is faster than population growth in this region. This is the main difference between the current analysis and that in GW. Under the same circumstances, GW would predict a constant growth rate of population but growing income per capita as parental potential income has no effect on fertility. In fact, only faster technological change will lead to a decrease in population growth by raising education. In the current model, however, rapid changes in technology will produce an even faster decrease in population growth than that in GW as it will be decreasing already due to the negative effect of growing income per capita.

If the price of the leisure good decreases, it will be easier for the economy to move into the Modern Growth region as the Malthusian Frontier shifts down and the critical level of education decreases. Suppose that the highest level of education falls below its critical level and stays unchanged - i.e., $e_t = e^h < \hat{e}$. If it was GW's economy, it would stay in the globally stable steady-state characterized by the intersection of the XX and EE locuses. In the current model, however, a continuous decrease in the price of the leisure good reduces the critical level of education below the existing level of education. Consequently, the economy moves into the Modern Growth regime. In other words, it will require a lower level of education hence a lower rate of technological progress for the economy to experience the characteristics of the Modern Growth regime as children are more expensive relative to the leisure good. In that sense, the current model offers an alternative mechanism for the transition out of the Malthusian stagnation.

The decrease in population growth is bounded from below as the minimum quantity of children constraint binds. If the minimum quantity of children is one per adult, population growth will be zero in the Modern Growth regime. Under such circumstances, the economy converges to a global steady state in which both education and the rate of technological progress are constant as population size is constant. If the minimum number of children above (below) one, the size of population will increase (decrease) and hence the evolution of education and technology will move accordingly.

4 Conclusion

The paper has extended the unified growth model of GW by generalizing their utility function in which consumption and children are independent of each other, with one in which some commodities independent of children while the others are substitutes for children. We consider some leisure goods as substitutes for children. The model is then used to account for the historical evolution of population, technology and output over the course of economic development of Western European countries. The performance of the model is the same as that of GW in the Malthusian and Post-Malthusian regimes. In GW, an increase in education generates the demographic transition from high to low fertility through which the economy enters the Modern Growth regime. In the current model, however, the demographic transition happens naturally when children becomes relatively more expensive than their substitutes, for a given level of education and a given price of leisure goods. A decrease in the price of leisure goods and an increase in education reinforce the underlying mechanism.

The model is theoretical and the results are qualitative. In the future work, we will examine the model quantitatively as Lagerlof (2006) does GW's model.

References

[1] Becker, G., 1960. "An Economic Analysis of Fertility" in Demographic and Economic Change in Developed Countries. Universities-NBER Conference Series 11, pp. 209-231.

[2] Becker, G., 1981. A Treatise on the Family. Cambridge: Harvard Press.

[3] Becker, G., Lewis, H, G., 1973. On the Interaction between Quantity and Quality of Children. Journal of Political Economy, 81, pp. S279-S288.

[4] Becker, G., Murphy, K. M., Tamura, R., 1990. Human Capital, Fertility and Economic Growth. Journal of Political Economy, 98, pp. 12-37.

[5] Boldrin, M., Jones, L., 2002. Mortality, Fertility and Saving in a Malthusian Economy. Review of Economic Dynamics, 5, pp. 775-814.

[6] Clark, G., 2003. The Condition of the Working-Class in England, 1200-2000: Marna Carta to Tony Blair. University of California, Davis.

[7] Doepke, M., 2004. Accounting for Fertility During the Transition to Growth. Journal of Economic Growth, 9 (3), pp. 347-383.

[8] Fernandez-Villaverde, J., 2001. Was Malthus Right? Economic Growth and Population Dynamics. University of Pennsylvania.

[9] Francis, N., Ramey, A. V., 2008. A Century of Work and Leisure. forthcoming in American Economic Journal: Macroeconomics.

[10] Galor, O., 2005. From Stagnation to Growth: Unified Growth Theory. Handbook of Economic Growth, 171-293.

[11] Galor, O., Moav, O., 2002. Natural Selection and the Origin of Economic Growth. Quarterly Journal of Economics, 117, pp. 1133-1192.

[12] Galor, O., Weil, D., 1996. The Gender Gap, Fertility, and Growth. American Economic Review, 86, pp. 374-387.

[13] Galor, O., Weil, D., 1999. From Malthusian Stagnation to Modern Growth. American Economic Review, 89, pp. 150-154.

[14] Galor, O., Weil, D., 2000. Population, Technology and Growth: From the Malthusian Regime to the Demographic Transition. American Economic Review, 110, pp. 806-828.

[15] Greenwood, J., Seshadri, A., 2002. The U.S. Demographic Transition. American Economic Review, 92, pp. 153-159.

[16] Jones, C., 2001. Was an Industrial Revolution Inevitable? Economic Growth Over the Very Long Run. Advances in Macroeconomics, Volume 1, Issue 2, Article 1.

[17] Hansen, G., Prescott, E., 2002. Malthus to Solow. American Economic Review, 92(4), pp. 1205-1217.

[18] Kögel, T., Prskawets, A., 2001. Agricultural Productivity Growth and Escape from the Malthusian Trap. Journal of Economic Growth, 6(4), pp. 337-357.

[19] Kopecky, K., 2005. The Trends in Retirement. Research Report no. 12, Economie d'avant garde, Department of Economics, University of Rochester.

[20] Lagerlof. N., 2006. The Galor-Weil Model Revisited: A Quantitative Exploration. Review of Economic Dynamics 9, pp. 116-142.

[21] Lebergott, S., 1996. Consumer Expenditures: New Measures and Old Motives. Princeton University Press, Princeton, NJ.

[22] Tamura, R., 2002. Human Capital and the Switch from Agriculture to Industry. Journal of Economic Dynamics and Control, 27(2), pp. 207-242.

[23] Vandenbroucke, G., 2005. The Trends in Hours: The U.S. from 1900 to 1950. Journal of Economic Dynamics and Control, forthcoming in Journal of Economic Dynamics and Control.