Existence of optimal strategies in linear multisector models with several consumption goods

Freni, Giuseppe and Gozzi, Fausto and Salvadori, Neri

University Partenope, Naples (Italy), University LUISS, Rome (Italy), University of Pisa, Pisa (Italy)

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Giuseppe Freni · Fausto Gozzi · Neri Salvadori

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Abstract. In this paper we give a sufficient and almost necessary condition for the existence of optimal strategies in linear multisector models when time is continuous and the the instantaneous utility function of the representative agent has two properties: (a) the intertemporal elasticity of substitution is constant over time and (b) preferences are concave and homothetic.

Key words: Endogenous growth, optimal control with mixed constraints, von Neumann growth model.

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Giuseppe Freni
Dipartimento di Studi Economici, Facoltà di Economia, Università Parthenope,
via Medina 40, 80133 Napoli, Italy.
E-mail: giuseppe.freni@uninav.it

Fausto Gozzi
Dipartimento di Scienze Economiche ed Aziendali, Facoltà di Economia, Università LUISS - Guido Carli,
vial Pola 12, 00198 Roma, Italy.
E-mail: fgozzi@luiss.it

Neri Salvadori
Dipartimento di Scienze Economiche, Facoltà di Economia, Università di Pisa,
via Ridolfi 10, 56124 Pisa, Italy.
E-mail: nerisal@ec.unipi.it

1 Introduction

In 1981 Magill [12] explored existence of optimal strategies in linear multisector models when time is continuous and the preferences of the representative agent are characterized by two parameters: the rate of time discount ρ and the constant elasticity of substitution $\sigma > 0$, within a more general formulation in which technology is not necessarily linear. He proved ([12], Theorem 9.15, p. 703) that in a von Neumann technology with constant returns if

$$\Gamma_0 > \frac{\Gamma_0 - \rho}{\sigma},$$

then an optimal strategy exists, where Γ_0 is the maximum rate of growth (Magill [12] did not provide any non-existence results; but see [13]). However Magill used an assumption on "regularity" ([12], Assumption T.2, p. 703) justified on the basis of the Gale [9] indecomposability assumption implying that the upper bound of the uniform over time rates of reproduction of any commodity equals the maximum rate of growth.

In a more recent paper, Freni et alii [7] analyzed in greater depth the existence of optimal strategies in linear multisector models when time is continuous and proved, on the assumption that only one commodity is consumed, that if

$$\Gamma_1 > \frac{\Gamma_1 - \rho}{\sigma},$$

then an optimal strategy exists, whereas if

$$\Gamma_1 < \frac{\Gamma_1 - \rho}{\sigma},$$

then no optimal strategy exists, where Γ_1 is the upper bound of the uniform over time rate of reproduction of commodity 1, which is the only commodity which is consumed. Freni et alii [7] also considered the case in which

$$\Gamma_1 = \frac{\Gamma_1 - \rho}{\sigma},$$

and provided further results of the existence or non existence according to the size of σ. Therefore what matters is not the maximum rate of growth, but the upper bound of the uniform over time rates of reproduction of the consumption good *if only one commodity is consumed.* In this paper we aim to generalize this result to the case in which several consumption goods exist. More precisely we will prove that if

$$\Gamma_\nu > \frac{\Gamma_\nu - \rho}{\sigma}, \tag{1}$$

then an optimal strategy exists, whereas if

$$\Gamma_\nu < \frac{\Gamma_\nu - \rho}{\sigma}, \tag{2}$$

then no optimal strategy exists, where Γ_ν is an average of the upper bounds of the uniform over time rates of reproduction of consumption commodities. Finding the right definition of the average Γ_ν and studying its properties was indeed the main technical difficulty of the paper. The average Γ_ν is defined by the instantaneous utility function only and is totally independent of technology whereas the upper bound of the uniform over time rate of reproduction of a commodity depends only on technology and is independent of consumer preferences.[1]

Our result depends on the fact that the instantaneous utility function has two properties: (a) the intertemporal elasticity of substitution is constant over time and (b) preferences are concave and homothetic. Hence the instantaneous utility function u_σ depends on a parameter $\sigma > 0$ (the elasticity of substitution) and is given by

$$\begin{array}{ll} u_\sigma(\nu) = \frac{\nu^{1-\sigma}-1}{1-\sigma} & for \quad \sigma > 0,\ \sigma \neq 1 \\ u_1(\nu) = \log \nu & for \quad \sigma = 1 \end{array} \tag{3}$$

(with the agreement that $u_\sigma(0) = -\infty$ for $\sigma \geq 1$), where $\nu : \mathbb{R}^k_+ \to \mathbb{R}$ is continuous, increasing on every component, concave, and homogeneous. With no loss of generality we assume that ν is homogeneous of degree 1. The assumption of homothetic preferences is justified by the fact that otherwise income distribution among consumers matters and, as a consequence, it would be appropriate to avoid an analysis in terms of a representative agent. However, if nonhomothetic preferences are modelled through Stone-Geary utility functions, as in Kongsamut et al. [11], then results similar to those presented here can be obtained.

The analysis presented here is most relevant when the upper bound of the uniform over-time rates of reproduction of different consumption goods are not equal (because of decomposability of technology). This analysis introduces the possibility of investigating structural change in relation to the different growth rates Γ_i and their average Γ_ν found here. Structural change, mainly in exogenous growth models, is an increasingly widely studied issue in the literature: see [17] and [1]. The fact that the average defined here Γ_ν depends on the instantaneous utility function alone and is wholly independent of technology suggests that such an average should emerge even in different analytic contexts in which technology is not linear and inequalities among growth rates are not related to the decomposability of technology.

From the technical point of view the results obtained in this paper can be seen as generalizations of those obtained elsewhere [7]. Indeed, while some preliminary Lemmas (see e.g. Lemma 4.1) are straightforward extensions of results contained in [7], the technical core of the present paper needs new ideas and techniques to be dealt with. The main reason is the fact that as we want to find "sharp" results (i.e. existence when (1) holds and nonexistence when (2) holds), we have to understand precisely the role of the function ν in the existence of optimal trajectories. It would be very easy to extend the result of the paper [7] and find only rough sufficient conditions for the existence (e.g. substituting Γ_ν with the maximum of the Γ_i's) and for the non-existence (e.g. substituting Γ_ν with the minimum of the Γ_i's). But to get our sharp results we need to understand what is

[1]We consider also the case in which $\Gamma_\nu = \frac{\Gamma_\nu - \rho}{\sigma}$, and provide further results of existence or non existence.

the right average, and then carefully prove all the various estimates needed for existence and nonexistence. For this reason these estimates, as happens for all sharp estimates, are structurally different from those needed when there is only one consumption good.

The paper is structured as follows: first we describe the model and assess the main assumptions in Section 2. In Section 3 we enunciate the main results and provide examples to show the complexity of the limiting cases. Section 4 is devoted to proving the main results. Finally the Appendix aims to prove some useful results concerning the average Γ_ν.

2 The Model

There are $n \geq 1$ commodities, and k of them are consumed, say commodities $1, \dots, k$. Preferences with respect to consumption over time are such that they can be described by a single intertemporal utility function U_σ, which is the usual C.E.S. (Constant Elasticity of Substitution) function: for a given consumption path $\mathbf{c} : [0, +\infty) \to \mathbb{R}^k$, ($\mathbf{c}_t \geq \mathbf{0}$ a.e.), we set

$$U_\sigma (\mathbf{c}(\cdot)) = \int_0^{+\infty} e^{-\rho t} u_\sigma (\nu(\mathbf{c}(t)))\, dt \tag{4}$$

where $\rho \in \mathbb{R}$ is the rate of time discount of the representative agent, and the instantaneous utility function $u_\sigma : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ has the form (3) and $\nu : \mathbb{R}^k_+ \to \mathbb{R}$ is continuous, increasing on every component, concave, and homogeneous of degree 1. Possible examples of function ν are the following:

$$\nu(\mathbf{c}) = c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k}, \qquad \alpha_i \in (0,1),\ i = 1, \dots k, \quad \sum_{i=1}^k \alpha_i = 1 \tag{5}$$

$$\nu(\mathbf{c}) = \min \{\alpha_1 c_1, \alpha_2 c_2, \dots, \alpha_k c_k\}, \qquad \alpha_i \in (0,1),\ i = 1, \dots k, \quad \sum_{i=1}^k \alpha_i = 1 \tag{6}$$

$$\nu(\mathbf{c}) = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k, \qquad \alpha_i \in (0,1),\ i = 1, \dots k, \quad \sum_{i=1}^k \alpha_i = 1 \tag{7}$$

$$\nu(\mathbf{c}) = \{\alpha_1 c_1^q + \alpha_2 c_2^q + \cdots + \alpha_k c_k^q\}^{1/q}, \qquad \alpha_i \in (0,1),\ i = 1, \dots k, \quad \sum_{i=1}^k \alpha_i = 1, q < 1. \tag{8}$$

In the first case preferences are Cobb-Douglas, in the second consumed commodities are perfect complements, in the third consumed commodities are perfect substitutes, in the fourth preferences are CES.

For the sake of simplicity we will drop the additive constant $-(1-\sigma)^{-1}$ in the following since this will not affect the optimal paths.

Technology is fully described by a pair of nonnegative matrices (the $m \times n$ material input matrix $\mathbf{A}$ and the $m \times n$ material output matrix $\mathbf{B}$, $m \geq 0$) and by a uniform rate of depreciation $\delta_\mathbf{x}$ of capital goods used for production. The rate of depreciation for goods not employed in production is $\delta_{\mathbf{z}}$. If $m = 0$, we say that matrices $\mathbf{A}$ and $\mathbf{B}$ are void. In this degenerate case production does not hold and all capital goods decay at rate $\delta_{\mathbf{z}}$: the model reduces to the standard one-dimensional AK model with $A = -\delta_{\mathbf{z}} \leq 0$.

The amounts of commodities available as capital at time t are defined by the vector $\mathbf{s}_t$. They may be either used for production (if $m > 0$) or disposed of. That is

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \mathbf{z}_t^T,$$

where $\mathbf{x} \geq \mathbf{0}$ denotes the vector of the intensities of operation and $\mathbf{z} \geq \mathbf{0}$ the vector of the amounts of goods which are disposed of. Production consists in combining the productive services from the stocks to generate flows that add to the existing stocks. Decay and consumption, on the other hand, drain away the stocks:

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T \left[\mathbf{B} - \delta_{\mathbf{x}} \mathbf{A}\right] - \delta_{\mathbf{z}} \mathbf{z}_t^T - \hat{\mathbf{c}}_t^T; \qquad \hat{\mathbf{c}}_t \geq 0 \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

where $\hat{\mathbf{c}}_t$ is the $n \times 1$ vector obtained from the $k \times 1$ consumption vector $\mathbf{c}_t$ and adding a zero component for each pure capital good at places $k+1, \dots, n$. By eliminating the variable $\mathbf{z}$ and setting $\delta = -\delta_{\mathbf{z}} + \delta_{\mathbf{x}}$, we obtain

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T \left[\mathbf{B} - \delta \mathbf{A}\right] - \delta_{\mathbf{z}} \mathbf{s}_t^T - \hat{\mathbf{c}}_t^T; \tag{9}$$

with the initial condition

$$\mathbf{s}_0 = \bar{\mathbf{s}} \geq \mathbf{0} \tag{10}$$

and the constraints

$$\mathbf{x}_t \geq \mathbf{0}, \qquad \mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}, \qquad c_t \geq 0. \tag{11}$$

If we also add the constraint

$$\mathbf{x}_t^T \mathbf{B} \geq \hat{\mathbf{c}}_t^T \tag{12}$$

the proof of existence here provided would be simplified since the constraint (12) would imply that the set of admissible control strategies is relatively compact in the space of integrable functions with a suitable weight. As a consequence a simpler procedure to prove existence could be used (see Remark 4.4 after the proof of Lemma 4.2). The economic interpretation of the constraint (12) is the following: commodities which in principle can be used both as consumption and as capital (the first k commodities in our case) cannot be converted to consumption once they are installed as capital. One of the aims of this paper is to show that a constraint of this type is not needed.[2]

Our problem is then to maximize the intertemporal utility (4) over all production-consumption strategies $(\mathbf{x}, \mathbf{c})$ that satisfy the constraints (9), (10) and (11). This is an

[2]On the contrary, constraints of this type are used by Magill [12], Becker *et alii* [4], and Balder [2]. In [12], Definition 4.1 and Assumption 1, p. 686 (then in Section 9, Definition 9.5 and subsequent results) allow us to obtain the existence of what Magill calls an expansion function (Definition 5.1 and Assumption 3, p.687, [12]) which is a key assumption for proving the existence theorem. In [4], Section 4.3, the same setting as [12], Section 9, is used. By this means, it can be proved that the Technology Conditions (i) and (ii), p. 81 occur and again this is a key point to prove the existence theorem. In [2] we find the Growth Condition 2.4 (p. 424) to be essential for the proof of existence (together with the compactness of $A(0)$).

optimal control problem where $\mathbf{s}$ is the state variable and $\mathbf{x}$ and $\mathbf{c}$ are the control variables. We now describe this problem more formally.

A production-consumption strategy $(\mathbf{x}, \mathbf{c})$ is defined as a measurable and locally integrable function of $t : \mathbb{R}^+ \rightarrow \mathbb{R}^m \times \mathbb{R}^k$ (we will denote by $L^1_{\mathrm{loc}}\left(0, +\infty; \mathbb{R}^{m+k}\right)$ the set of such functions). Then the differential equation (9) has a unique solution : $\mathbb{R}^+ \mapsto \mathbb{R}^n$ which is absolutely continuous (we will denote by $W^{1,1}_{\mathrm{loc}}\left(0, +\infty; \mathbb{R}^n\right)$ the set of such functions). As this solution clearly depends on the initial datum $\overline{\mathbf{s}}$ and on the production-consumption strategy $(\mathbf{x}, \mathbf{c})$, it will be denoted by the symbol $\mathbf{s}_{t;\overline{\mathbf{s}},(\mathbf{x},\mathbf{c})}$, omitting the subscript $\overline{\mathbf{s}}, (\mathbf{x}, \mathbf{c})$ when it is clear from the context.

Given an initial endowment $\overline{\mathbf{s}}$ we will say that a strategy $(\mathbf{x}, \mathbf{c})$ is admissible from $\overline{\mathbf{s}}$ if the triple $\left(\mathbf{x}, \mathbf{c}, \mathbf{s}_{t;\overline{\mathbf{s}},(\mathbf{x},\mathbf{c})}\right)$ satisfies the constraints (11) and $U_1(\mathbf{c})$ is well defined[3]. The set of admissible control strategies starting at $\overline{\mathbf{s}}$ will be denoted by $\mathcal{A}(\overline{\mathbf{s}})$. We adopt the following definition of optimal strategies.

Definition 2.1 *A strategy* $(\mathbf{x}^*, \mathbf{c}^*) \in \mathcal{A}(\overline{\mathbf{s}})$ *will be called optimal if we have* $U_\sigma(\mathbf{c}^*) > -\infty$ *and*

$$+\infty > U_\sigma(\mathbf{c}^*) \geq U_\sigma(\mathbf{c})$$

for every admissible control pair $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\overline{\mathbf{s}})$*.*

We now comment on a set of assumptions that will be used throughout the paper.

Assumption 2.2 *Each row of matrix* $\mathbf{A}$ *is semipositive.*

This assumption means that no commodity can be produced without using some commodity as an input.

Assumption 2.3 *Each row of matrix* $\mathbf{B}$ *is semipositive.*

This assumption means that each process produces something: i.e. that pure destruction processes are not dealt with as production processes.

Assumption 2.4 *The initial datum* $\overline{\mathbf{s}} \geq \mathbf{0}$ *and the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are such that there is an admissible strategy* $(\mathbf{x}^*, \mathbf{c}^*) \in \mathcal{A}(\overline{\mathbf{s}})$ *and a time* $t^* > 0$ *such that* $\nu\left(\tilde{\mathbf{s}}_{t^*;\overline{\mathbf{s}},(\mathbf{x}^*,\mathbf{c}^*)}\right) > 0$*, where* $\tilde{\mathbf{s}}$ *is the subvector of vector* $\mathbf{s}$ *consisting of the first* k *elements.*

If this assumption does not hold, then every admissible strategy starting from $\overline{\mathbf{s}}$ must have $\nu(\mathbf{c}_t) = 0$ a.e. This case holds little interest for investigation purposes.

[3] The condition on $U_1(\mathbf{c})$ is relevant only when $\sigma = 1$. Note that for $\sigma \in (0,1)$ the function $t \rightarrow e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))$ is always nonnegative so it is always semiintegrable (with the integral eventually $+\infty$). On the other hand for $\sigma > 1$ the function $t \rightarrow e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))$ is always negative (and may be $-\infty$ when $\nu(\mathbf{c})_t = 0$) and again it is always semiintegrable (with the integral eventually $-\infty$). This means that the intertemporal utility U_σ is always well defined for $\sigma \neq 1$. For $\sigma = 1$ the function $t \rightarrow e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))$ may change sign so it may be not semiintegrable on $[0, +\infty)$. This is the reason why we need to require that $U_1(\mathbf{c})$ is well defined in order to define the admissibility of $\mathbf{c}$.

Assumption 2.5 *The initial datum* $\bar{\mathbf{s}} \geq \mathbf{0}$ *and the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are such that there is an admissible strategy* $(\mathbf{x}^*, \mathbf{c}^*) \in \mathcal{A}(\bar{\mathbf{s}})$ *and a time* t^* *such that* $\mathbf{s}_{t^*;\bar{\mathbf{s}},(\mathbf{x}^*,\mathbf{c}^*)}$ *is positive.*

Assumption 2.5 implies that all commodities are available at any time $t > 0$ and, in particular, that Assumption 2.4 is satisfied. Moreover Assumptions 2.4 and 2.5 could be stated in terms of the zero components of the initial datum $\bar{\mathbf{s}}$ and of the structure of matrices $\mathbf{A}$ and $\mathbf{B}$. On this point, see Appendix D of [8]. Further, it can be shown that Assumption 2.5 is not really restrictive, provided that Assumption 2.4 holds, in the sense that when it does not hold, matrices $\mathbf{A}$ and $\mathbf{B}$, vector $\bar{\mathbf{s}}$, and consumption goods can be redefined in order to obtain an equivalent model in which Assumption 2.5 holds. Assume, in fact, that Assumption 2.5 does not hold. Then there is a commodity j which is not available at any time $t \geq 0$ ($\mathbf{s}_t^T \mathbf{e}_j = 0$ for every $t \geq 0$). In this case any production process i in which commodity j is employed ($a_{ij} > 0$) cannot be used. The model is then equivalent to one in which matrices $\mathbf{B}$ and $\mathbf{A}$ and vector $\mathbf{s}$, in the state equation (9), are substituted with matrices $\mathbf{D}$ and $\mathbf{C}$ and vector $\mathbf{s}'$, respectively, where matrix $\mathbf{C}$ is obtained from $\mathbf{A}$ by deleting the j-th column and all rows which on the j-th column have a positive element, matrix $\mathbf{D}$ is obtained from matrix $\mathbf{B}$ by deleting the corresponding rows and the j-th column, and vector $\mathbf{s}'$ is obtained from vector $\mathbf{s}$ by deleting the j-th element. (If commodity j is a consumption good, it is also deleted from the list of consumption goods.) Note that if in the new equivalent model the Assumption 2.5 does not hold and matrices $\mathbf{C}$ and $\mathbf{D}$ are not void, the argument can be iterated. If matrices $\mathbf{C}$ and $\mathbf{D}$ are void, then an equivalent model satisfying Assumption 2.5 is obtained by deleting the nought elements of vector $\mathbf{s}'$. In any case the algorithm is able to determine an equivalent model in which Assumption 2.5 does hold. We will refer to the equivalent model found in this way as the *truncated model* and to the corresponding technology as the *truncated technology*, which then depends on $\bar{\mathbf{s}}$. It can easily be proved that if Assumptions 2.2, 2.3, 2.4 hold in the original technology, then they hold in the truncated technology too (see Appendix D of [8]). If not otherwise mentioned, all the following assumptions refer to the truncated technology.

Let us define

$$\mathcal{G}_0 := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{0}\right\}, \qquad \Gamma_0 = \max \mathcal{G}_0$$

and, for $i = 1, \ldots, k$,

$$\mathcal{G}_i := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{e}_i^T\right\}, \qquad \Gamma_i = \sup \mathcal{G}_i$$

Γ_0 is clearly the maximum among the uniform over time rates of growth feasible for this economy and corresponds to what von Neumann [16] found both as growth rate and as rate of profit. Γ_i is the upper bound of the uniform over time rates of reproduction of the $i-$th consumption good. Obviously $\Gamma_i \leq \Gamma_0$ for every $i = 1, \ldots, k$.

Magill [12], Assumption T.2, p. 703, assumed that if $\mathbf{x} \in \mathcal{X}$, then $\mathbf{x}^T \mathbf{B} > \mathbf{0}^T$, where

$$\mathcal{X} = \left\{\mathbf{x} | \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\Gamma_0 + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{0}^T\right\}$$

It is easily checked that, under this assumption, if $\mathbf{x} \in \mathcal{X}$ and $\mathbf{x}^T \mathbf{A} \mathbf{e}_i = 0$, then there is $\alpha > 0$ such that

$$\mathbf{x}^T \left[\mathbf{B} - (\Gamma_0 + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \alpha \mathbf{e}_i^T$$

whereas if $\mathbf{x} \in \mathcal{X}$ and $\mathbf{x}^T \mathbf{A}\mathbf{e}_i > 0$, then for any $\varepsilon > 0$ there is $\alpha > 0$ such that

$$\mathbf{x}^T \left[\mathbf{B} - (\Gamma_0 - \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \alpha \mathbf{e}_i^T$$

In any case

$$\sup \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{e}_i^T\right\} = \Gamma_0$$

that is, the upper bound of the uniform over time rates of reproduction of any commodity equals the maximum rate of growth. In this paper we will not make any assumption on indecomposability.

It is easily proved that the Γ_i's relative to the truncated technology are not greater than the corresponding Γ_i's relative to the original one. If either $\mathbf{B}\mathbf{e}_i = \mathbf{A}\mathbf{e}_i = \mathbf{0}$ or matrices $\mathbf{A}$ and $\mathbf{B}$ are void, then $\Gamma_i = -\infty$. Moreover if $\mathbf{B}\mathbf{e}_i = \mathbf{0}$ and $\mathbf{A}\mathbf{e}_i \neq \mathbf{0}$ then $\Gamma_i = -\delta_{\mathbf{x}}$. Finally, if $\mathbf{B}\mathbf{e}_i \neq \mathbf{0}$ and if commodity j is available at time 0 ($\bar{\mathbf{s}}^T \mathbf{e}_j > 0$) and is essential to the reproduction of the $i-$th consumption good, then $\Gamma_i = -\delta_{\mathbf{x}}$.[4] (see [8] Proposition 4.4).

Assumption 2.6 $\mathbf{B}\mathbf{e}_i \neq \mathbf{0}$ *for each consumption good* i *and* $\delta_{\mathbf{z}} < \delta_{\mathbf{x}}$.

Assumption 2.6 is not necessary, but it helps in simplifying the exposition since it implies that $\Gamma_i > -\delta_{\mathbf{z}}$ for each consumption good i. The first part means that all consumption goods are technologically producible. The second part implies that using commodities in production dominates their storing; i.e. it is not convenient to produce something just in order to dispose something else.

We call

$$\Gamma_{max} := \max_{i=1,\dots,k} \Gamma_i, \qquad \Gamma_{min} := \min_{i=1,\dots,k} \Gamma_i > 0$$

Moreover we introduce the following number

$$\Gamma_\nu = \inf \left\{\eta \in \mathbb{R} : \lim_{t \to +\infty} e^{-\eta t} \nu \left(e^{\Gamma_1 t}, e^{\Gamma_2 t}, \dots, e^{\Gamma_k t}\right) = 0\right\}$$

$$= \sup \left\{\eta \in \mathbb{R} : \lim_{t \to +\infty} e^{-\eta t} \nu \left(e^{\Gamma_1 t}, e^{\Gamma_2 t}, \dots, e^{\Gamma_k t}\right) = +\infty\right\}$$

Since, from the homogeneity of degree 1 and the monotonicity of ν, we have

$$e^{\Gamma_{min} t} \nu(1, 1, \dots, 1) \leq \nu \left(e^{\Gamma_1 t}, e^{\Gamma_2 t}, \dots, e^{\Gamma_k t}\right) \leq e^{\Gamma_{max} t} \nu(1, 1, \dots, 1)$$

then it is easy to see that $\Gamma_{min} \leq \Gamma_\nu \leq \Gamma_{max}$.

It is also easy to check that, for the four examples given in (5), (6), (7), and (8),we have:

[4] We say that commodity j is essential to the reproduction of the $i-$th consumption good when

$$\left(\mathbf{x} \geq 0, \varepsilon > 0, \mathbf{x}^T \left[\mathbf{B} - \varepsilon \mathbf{A}\right] \geq \mathbf{e}_i\right) \Rightarrow \mathbf{x}^T \mathbf{A}\mathbf{e}_j \neq 0.$$

- in the example (5), $\Gamma_\nu = \sum_{i=1}^{k} \alpha_i \Gamma_i$,
- in the example (6), $\Gamma_\nu = \Gamma_{min}$,
- in the example (7), $\Gamma_\nu = \Gamma_{max}$,
- in the example (8), $\Gamma_\nu = \Gamma_{max}$ if $0 < q < 1$ whereas $\Gamma_\nu = \Gamma_{min}$ if $q < 0$.

As mentioned in the introduction, this paper is mainly devoted to showing the role that the following assumption plays in the existence of optimal strategies of the problem under analysis.

Assumption 2.7

$$\Gamma_\nu > \frac{\Gamma_\nu - \rho}{\sigma}$$

The reader will have noticed that we used the convoluted expression "the upper bound of the uniform over time rates of reproduction of the $i-$th consumption good" instead of the more straightforward "the upper bound of the rates of reproduction of the $i-$th consumption good". This is because, for particular forms of matrices, growth rates of consumption might be found which are higher, but not uniform over time. In [7] we provided the following example, with details, to clarify this point.

Example 2.8 $k = 1$, $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} \in (0,1)$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is immediately recognized that $\Gamma_1 = 1 - \delta_{\mathbf{x}} > 0$. *Nevertheless, consumption can grow at the rate*

$$\frac{\dot{c}}{c} = \Gamma_1 + \frac{\beta}{\alpha + \beta t} > \Gamma_1$$

where α and β are positive constants.

We finally observe that, from the definition of Γ_ν the value of the limit

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu \left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right),$$

may be any $L \in [0, +\infty]$. However, as shown in the Appendix when the law of the function ν is made only by a finite number of sums, products, powers, and/or max/min over a finite number of terms - which is e.g. the case of the examples (5), (6), (7), (8) above - then $L \in (0, +\infty)$. In the same appendix we provide examples in which either $L = 0$ or $L = +\infty$. The relevance of these two cases is explored in Theorem 3.3.

3 The main results

The main goal of this paper is to show that in the general context outlined by Assumptions 2.2, 2.3, 2.5, 2.6, we have substantially an if and only if condition for the existence of optimal strategies. In this paper we will prove the following results:

Theorem 3.1 *If Assumptions 2.2, 2.3, 2.5, 2.6 and 2.7 hold, then there is an optimal strategy* $(\mathbf{x}, \mathbf{c})$ *for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*. Moreover this strategy is unique in the sense that if* $(\tilde{\mathbf{x}}, \tilde{\mathbf{c}})$ *is another optimal strategy, then* $\nu(\tilde{\mathbf{c}}) \equiv \nu(\mathbf{c})$*. If* ν *is strictly concave, then we also have* $\tilde{\mathbf{c}} \equiv \mathbf{c}$*.*

Theorem 3.2 *Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. If*

$$\Gamma_\nu < \frac{\Gamma_\nu - \rho}{\sigma}$$

then no optimal strategy exists for problem (P_σ) *starting at* $\bar{\mathbf{s}}$*.*

Theorem 3.3 *Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Let*

$$\Gamma_\nu = \frac{\Gamma_\nu - \rho}{\sigma}.$$

Then we have the following:

1. *Let* $\sigma = 1$*.*

   - *If* $\Gamma_\nu < 0$*, then all strategies have value* $-\infty$ *and so no optimal strategy exists for problem* (P_1) *starting at* $\bar{\mathbf{s}}$*.*
   - *If* $\Gamma_\nu > 0$ *there exists an admissible strategy with value* $+\infty$ *and so no optimal strategy exists for problem* (P_1) *starting at* $\bar{\mathbf{s}}$*.*
   - *If* $\Gamma_\nu = 0$*, each* Γ_i *is not a maximum and*

     $$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right) < +\infty, \tag{13}$$

     then all strategies have value $-\infty$ *and so no optimal strategy exists for problem* (P_1) *starting at* $\bar{\mathbf{s}}$*.*
   - *If* $\Gamma_\nu = 0$*, each* Γ_i *is a maximum,*

     $$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right) > 0, \tag{14}$$

     and $\bar{\mathbf{s}}$ *satisfy*[5]

     $$\bar{\mathbf{s}}^T \mathbf{e}_j > \sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A} \mathbf{e}_j, \qquad j = 1, \ldots n.$$

     (where $\mathbf{x}_i$ *is the maximum point of* $\mathcal{G}_i$*), then there exists an admissible strategy with value* $+\infty$ *and so no optimal strategy exists for problem* (P_1) *starting at* $\bar{\mathbf{s}}$*.*

[5]Note that this is always true if all components of $\bar{\mathbf{s}}$ are very large.

2. *Let $\sigma \in (0,1)$. If each Γ_i is a maximum and (14) holds then there exists an admissible strategy with value $+\infty$ and so no optimal strategy exists for problem (P_σ) starting at $\bar{\mathbf{s}}$.*

3. *Let $\sigma > 1$. If each Γ_i is not a maximum and (13) holds, then all strategies have value $-\infty$ and so no optimal strategy exists for problem (P_σ) starting at $\bar{\mathbf{s}}$.*

The limit cases where $\Gamma_\nu = \frac{\Gamma_\nu - \rho}{\sigma}$ and

1. $\sigma = 1$, $\Gamma_\nu = 0$, and at least one Γ_i is a maximum or (13) does not hold;
2. $\sigma = 1$, $\Gamma_\nu = 0$, and at least one Γ_i is not a maximum or (14) does not hold or $\bar{\mathbf{s}}$ is not big enough;
3. $\sigma \in (0,1)$ and at least one Γ_i is not a maximum or (14) does not hold;
4. $\sigma > 1$ and at least one Γ_i is a maximum or (13) does not hold;

are intrinsically more complex than the others. Indeed in such cases we can have existence or nonexistence depending on the value of σ. In our paper [7] (see also [8] for details) we provided two examples of matrices A and B and scalars ρ, $\delta_{\mathbf{x}}$, $\delta_{\mathbf{z}}$ showing counterexamples when $k = 1$ and $\Gamma_\nu = \Gamma_1 > 0$. Here we provide an example of the case when $\sigma = 1$, $k = 1$, $\Gamma_\nu = \Gamma_1 = 0$ is a maximum, and (14) holds, showing what happens for different values of $\bar{\mathbf{s}}$.

Example 3.4 *$k = 1$, $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} = 1$ and*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is immediately recognized that $\Gamma_\nu = \Gamma_1 = 0$ and that it is a maximum. The maximum point in the definition of Γ_1 is $(x_1, 1)$ for every $x_1 \geq 0$. The constant strategy $\mathbf{x}_t = \alpha(0,1)$, $c_t = \alpha$, for any $\alpha > 0$, is admissible if and only if $\bar{s}_2 \geq \alpha$. So, if $\bar{s}_1 \geq 0$ and $\bar{s}_2 \geq 1$, then such an admissible strategy has value $+\infty$. On the other hand, the state control constraints imply that $s_1' \leq s_2 - c$ and $s_2' \leq 0$ so. Hence, by integrating, we get, for $t \geq 0$,

$$\int_0^t c_s ds \leq \bar{s}_1 + t\bar{s}_2$$

which implies

$$\int_0^t \ln c_s ds \leq t \ln \left[\frac{\bar{s}_1}{t} + \bar{s}_2 \right].$$

The above implies that, when $\bar{s}_2 < 1$ then $U(c) = -\infty$ for every admissible strategy. Moreover, when $\bar{s}_2 = 1$, then the strategy above has value zero and $U(c) \leq \bar{s}_1$ for every admissible strategy. In this last case there exists an optimal strategy.

Remark 3.5 *If Γ_ν is invariant to a small change in Γ_i, consumption good i is said to be* marginally inessential *to the definition of Γ_ν; otherwise it is* marginally essential. *In the above Theorem 3.3, each time all Γ_i's are required to be maximum or not, this requirement can be restricted to the goods that are marginally essential to the definition of Γ_ν.*

4 Proofs of the main results

In this section, we provide the proofs of the main results stated in section 3. The proofs require a set of preliminary results which we discuss in Subsection 4.1. In Subsection 4.2 we prove the existence and nonexistence results stated above as Theorems 3.1, 3.2 and 3.3.

Throughout this section we will assume that Assumptions 2.2, 2.3, 2.5, 2.6 hold without explicitly mentioning them. We observe that some results hold in the more general framework when Assumption 2.6 does not hold; on this point see also Appendix A and B of [8].

4.1 Preliminary lemmata

The following Lemma provides the basis for estimates of the state and control trajectories.

Lemma 4.1 *Let $i \in \{1, \ldots, k\}$. If Γ_i is not a maximum*

$$\gamma \geq \Gamma_i \Longleftrightarrow \exists \mathbf{v}_F^i \geq \mathbf{0} : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_F^i \leq \mathbf{0}, \quad \mathbf{e}_i^T \mathbf{v}_F^i = 1. \tag{15}$$

If Γ_i is a maximum

$$\gamma > \Gamma_i \Longleftrightarrow \exists \mathbf{v}_F^i \geq \mathbf{0} : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_F^i \leq \mathbf{0}, \quad \mathbf{e}_i^T \mathbf{v}_F^i = 1, \tag{16}$$

Moreover

$$\exists \mathbf{v}_S^i \geq \mathbf{0} : \quad (\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_S^i \leq \mathbf{0}, \mathbf{v}_S^i \neq \mathbf{0}, \tag{17}$$

and

$$\mathbf{e}_i^T \mathbf{v}_S^i = \mathbf{y}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \mathbf{v}_S^i = 0, \tag{18}$$

where

$$\mathbf{y} \in \left\{\mathbf{x} | \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{e}_i^T\right\}.$$

Proof. Statements (15) and (16) are obvious applications of the Farkas Lemma (see for instance Gale's theorem for linear inequalities; [10] or [14], pp. 33-34). Assume now that statement (17) does not hold and obtain, once again from the Farkas Lemma (see for instance Motzkin's theorem of the alternative; [15] or [14], pp. 28-29), that

$$\exists \mathbf{w} \geq \mathbf{0} : \mathbf{w}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] > \mathbf{0}^T.$$

Hence there is $\phi > 0$ so large and $\eta > 0$ so small that

$$\phi \mathbf{w}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{e}_i^T + \eta \phi \mathbf{w}^T \mathbf{A}$$

Hence a contradiction since $\Gamma_i = \sup \mathcal{G}_i$. By remarking that

$$0 \geq \mathbf{y}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \mathbf{v}_S^i \geq \mathbf{e}_i^T \mathbf{v}_S^i \geq 0$$

the proof is completed. ∎

The next lemma and the subsequent corollary give various estimates for the state and control variables that will be the basis for the proof of existence and nonexistence. Note that for the case $\sigma \in (0,1)$ we are interested in an estimate from above of the integral $\int_0^t e^{-\rho s}\nu(\mathbf{c}_s)^{1-\sigma}ds$ giving finiteness of the value function for $\rho - \Gamma_\nu(1-\sigma) > 0$ (so we need terms that remain bounded when $t \to +\infty$), while for the case $\sigma \in (1,+\infty)$ we are interested in an estimate from below of the same integral to show that the value function is equal to $-\infty$ when $\rho - \Gamma_\nu(1-\sigma) < 0$, (so we need terms that explode when $t \to +\infty$). These different targets require the use of different estimates with different methods of proof. Of course, both methods can be applied to both cases, albeit yielding estimates that are not useful for our target. In order to simplify notation we will set, for $\varepsilon \geq 0$,

$$\overline{\Gamma}_i := \max\{-\delta_{\mathbf{z}}, \Gamma_i\} \tag{19}$$

$$\overline{\Gamma}_{i,\varepsilon} := \max\{-\delta_{\mathbf{z}}, \Gamma_i + \varepsilon\} \tag{20}$$

$$a_{i,\varepsilon} = \rho - \overline{\Gamma}_{i,\varepsilon}(1-\sigma) \tag{21}$$

$$a_{\nu,\varepsilon} = \rho - (\Gamma_\nu + \varepsilon)(1-\sigma). \tag{22}$$

Obviously, if Assumption 2.6 holds, $\overline{\Gamma}_i = \Gamma_i$ and $\overline{\Gamma}_{i,\varepsilon} := \Gamma_i + \varepsilon$.

Lemma 4.2 *Let $i \in \{1,\ldots,k\}$ and $\sigma > 0$. Fix $\varepsilon > 0$ when Γ_i is a maximum and $\varepsilon = 0$ when Γ_i is not a maximum; call $\mathbf{v}^i_{F,\varepsilon}$ the vector given by Lemma 4.1. For every $0 \leq t < +\infty$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \geq 0$ we have, for every admissible control strategy $(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$,*

$$\mathbf{s}_t^T\mathbf{v}^i_{F,\varepsilon} \leq e^{\overline{\Gamma}_{i,\varepsilon}t}\bar{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}, \tag{23}$$

and, for $\eta \in \mathbb{R}$

$$\begin{aligned} &\textstyle\int_0^t e^{-\eta s}\mathbf{s}_s^T\mathbf{v}^i_{F,\varepsilon}ds \leq \bar{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}\frac{e^{\left(\overline{\Gamma}_{i,\varepsilon}-\eta\right)t}-1}{\overline{\Gamma}_{i,\varepsilon}-\eta}; \qquad \eta \neq \overline{\Gamma}_{i,\varepsilon};\\ &\textstyle\int_0^t e^{-\eta s}\mathbf{s}_s^T\mathbf{v}^i_{F,\varepsilon}ds \leq \bar{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}t; \qquad \eta = \overline{\Gamma}_{i,\varepsilon}; \end{aligned} \tag{24}$$

and, setting $I_{i,\varepsilon}(t) := \int_0^t e^{-\overline{\Gamma}_{i,\varepsilon}s}\hat{\mathbf{c}}_s^T\mathbf{v}^i_{F,\varepsilon}ds$,

$$I_{i,\varepsilon}(t) + e^{-\overline{\Gamma}_{i,\varepsilon}t}\mathbf{x}_t^T\mathbf{A}\mathbf{v}^i_{F,\varepsilon} \leq \bar{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}. \tag{25}$$

Moreover, for $0 \leq \tau \leq t < +\infty$, and $\eta \in \mathbb{R}$,

$$\mathbf{x}_t^T\mathbf{A}\mathbf{v}^i_{F,\varepsilon}e^{-\eta t} + \int_\tau^t e^{-\eta s}\hat{\mathbf{c}}_s^T\mathbf{v}^i_{F,\varepsilon}ds \leq e^{-\eta\tau}\bar{\mathbf{s}}_\tau^T\mathbf{v}^i_{F,\varepsilon}e^{\left(\overline{\Gamma}_{i,\varepsilon}-\eta\right)^+(t-\tau)}. \tag{26}$$

Finally there exists a constant $\lambda > 0$ (depending only on the matrices $\mathbf{A}$ and $\mathbf{B}$) such that, for every $t \geq 0$

$$\left|\left|\mathbf{x}_t^T\mathbf{A}\right|\right| \leq ||\mathbf{s}_t|| \leq e^{\lambda t}\,||\bar{\mathbf{s}}||\,, \qquad ||\mathbf{x}_t|| \leq Ce^{\lambda t}\,||\bar{\mathbf{s}}|| \tag{27}$$

for suitable real number $C > 0$ (depending only on the matrix $\mathbf{A}$).

Proof. We prove the five inequalities (23)–(27) in order of presentation. We give only a sketch. To avoid heavy notation we will write $\mathbf{v}_F^i$ for $\mathbf{v}_{F,\varepsilon}^i$ along this proof.

(1) Let $i \in \{1, \ldots, k\}$. First we observe that, by multiplying the state equation (9) by $\mathbf{v}_F^i$ we obtain

$$\begin{aligned} \dot{\mathbf{s}}_t^T \mathbf{v}_F^i &= -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i + \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}] \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \quad t \in (0, +\infty), \\ \mathbf{s}_0^T \mathbf{v}_F^i &= \bar{\mathbf{s}}^T \mathbf{v}_F^i \geq 0. \end{aligned}$$

Now for every $\mathbf{x}$ and ε,

$$\mathbf{x}^T [\mathbf{B} - \delta \mathbf{A}] = \mathbf{x}^T [\mathbf{B} - (\Gamma_i + \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}] + (\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}^T \mathbf{A}$$

Moreover for $\mathbf{x} \geq 0$ we have by (15) and (16) $\mathbf{x}^T [\mathbf{B} - (\Gamma_i + \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}] \mathbf{v}_F^i \leq 0$ with the agreement that $\varepsilon = 0$ when Γ_i is not a maximum. Then

$$\begin{aligned} \dot{\mathbf{s}}_t^T \mathbf{v}_F^i &= -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i + \mathbf{x}_t^T [\mathbf{B} - (\Gamma_i + \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}] \mathbf{v}_F^i + (\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \\ &\leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i + (\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}^T \mathbf{A} \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \end{aligned}$$

If $\Gamma_i + \varepsilon \geq -\delta_{\mathbf{z}}$, from the constraint $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ and from the non-negativity of $\hat{\mathbf{c}}_t$, we get

$$\dot{\mathbf{s}}_t^T \mathbf{v}_F^i \leq (\Gamma_i + \varepsilon) \mathbf{s}_t^T \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \leq (\Gamma_i + \varepsilon) \mathbf{s}_t^T \mathbf{v}_F^i \qquad t \in (0, +\infty), \tag{28}$$

and so, by integrating on $[0, t]$ and using the Gronwall lemma (see e.g. [3, p. 218]) we get the first claim (23).

Take now $\Gamma_i + \varepsilon < -\delta_{\mathbf{z}}$ in this case we have $(\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}^T \mathbf{A} \mathbf{v}_F^i \leq 0$ which gives

$$\dot{\mathbf{s}}_t^T \mathbf{v}_F^i \leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i, \qquad t \in (0, +\infty), \tag{29}$$

and so the claim (in this case we clearly can take $\varepsilon = 0$).

(2) Inequalities (24) are proved by multiplying the inequality (23) by $e^{-\eta s}$ and integrating on $[0, t]$.

(3) From (28) (taking $\varepsilon = 0$ when allowed)

$$\dot{\mathbf{s}}_s^T \mathbf{v}_F \leq \overline{\Gamma}_{i,\varepsilon} \mathbf{s}_s^T \mathbf{v}_F^i - \hat{\mathbf{c}}_s^T \mathbf{v}_F^i \qquad \forall s \in [0, t] \tag{30}$$

so that, by the comparison theorem for ODE's

$$\mathbf{s}_t^T \mathbf{v}_F^i \leq \bar{\mathbf{s}}^T \mathbf{v}_F^i e^{\overline{\Gamma}_{i,\varepsilon} t} - \int_0^t e^{\overline{\Gamma}_{i,\varepsilon}(t-s)} \hat{\mathbf{c}}_s^T \mathbf{v}_F^i ds$$

From the inequality $\mathbf{x}_t^T \mathbf{A} \mathbf{v}_F^i \leq \mathbf{s}_t^T \mathbf{v}_F^i$ we get inequality (25) by rearranging the terms.

(4) For simplicity we take the case $\tau = 0$. Inequality (26) easily follows by multiplying both sides of (30) by $e^{-\eta s}$ and then integrating. Indeed we have

$$0 \le e^{-\eta s} \hat{\mathbf{c}}_s^T \mathbf{v}_F^i \le e^{-\eta s} \left[\overline{\Gamma}_{i,\varepsilon} \mathbf{s}_s^T \mathbf{v}_F^i - \dot{\mathbf{s}}_s^T \mathbf{v}_F^i \right] \qquad \forall s \in [0,t]$$

Now we integrate the above expression, then we integrate by parts and use that $\mathbf{x}_t^T \mathbf{A} \mathbf{v}_F^i \le \mathbf{s}_t^T \mathbf{v}_F^i$:

$$\begin{aligned} \int_0^t e^{-\eta s} \hat{\mathbf{c}}_s^T \mathbf{v}_F^i ds &\le \int_0^t e^{-\eta s} \left[\overline{\Gamma}_{i,\varepsilon} \mathbf{s}_s^T \mathbf{v}_F^i - \dot{\mathbf{s}}_s^T \mathbf{v}_F^i \right] ds \\ &= \int_0^t e^{-\eta s} \overline{\Gamma}_{i,\varepsilon} \mathbf{s}_s^T \mathbf{v}_F^i ds - e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_F^i + \bar{\mathbf{s}}^T \mathbf{v}_F^i - \eta \int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F^i ds \\ &\le \bar{\mathbf{s}}^T \mathbf{v}_F^i - e^{-\eta t} \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F^i + \left(\overline{\Gamma}_{i,\varepsilon} - \eta \right) \int_0^t e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_F^i ds \end{aligned}$$

Now, if $\eta \ge \overline{\Gamma}_{i,\varepsilon}$ the above inequality gives the claim immediately. If $\eta < \overline{\Gamma}_{i,\varepsilon}$ we get the claim by using (24).

(5) The inequality (27) comes as follows. By Assumption 2.2 for every i there exists j such that $a_{ij} > 0$ so that

$$\mathbf{x}^T \mathbf{e}_i \le a_{ij}^{-1} \mathbf{s}^T \mathbf{e}_j$$

and we can find a nonnegative matrix $n \times m$ $\mathbf{C}$ with exactly one positive element for every column such that $\mathbf{x}^T \le \mathbf{s}^T \mathbf{C}$. Consequently we have, for $\mathbf{x}^T \mathbf{A} \le \mathbf{s}^T$,

$$\mathbf{x}^T \mathbf{B} \le \mathbf{s}^T \mathbf{C} \mathbf{B}.$$

Now the matrix $\mathbf{D} = \mathbf{C}\mathbf{B}$ is $n \times n$ and has only positive elements. From the state equation (9) it follows that for every admissible strategy we have

$$\dot{\mathbf{s}}_t^T \le \mathbf{s}_t^T \mathbf{D} - \delta_{\mathbf{z}} \mathbf{s}_t^T - \hat{\mathbf{c}}_t^T.$$

Since the control $\hat{\mathbf{c}}$ is positive and $\mathbf{D}$ has only positive elements one gets

$$\mathbf{s}_t^T \le \bar{\mathbf{s}}^T e^{t[\mathbf{D} - \delta_{\mathbf{z}} \mathbf{I}]}$$

so the claim easily follows taking any $\lambda > \max \{ Re\mu, \mu \text{ eigenvalue of } \mathbf{D} \} - \delta_{\mathbf{z}}$ ∎

For $\eta \in \mathbb{R}$ define, for every $0 \le s < +\infty$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \ge 0$ and $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$, the quantity

$$I_\eta(s) := \int_0^s e^{-\eta r} \nu \left(c_{1r}, c_{2r}, \ldots, c_{kr} \right) dr. \tag{31}$$

The following estimates hold.

Lemma 4.3 *Let $t \geq 0$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \geq 0$ and $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$. We have, for $\sigma \in (0,1)$,*

$$\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \leq t^\sigma \left[I_{\frac{\rho}{1-\sigma}}(t)\right]^{1-\sigma} \tag{32}$$

while, for $\sigma \in (1, +\infty)$,

$$\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \geq t^\sigma \left[I_{\frac{\rho}{1-\sigma}}(t)\right]^{1-\sigma}. \tag{33}$$

Moreover let $\eta \in \mathbb{R}$. Then, for $\sigma = 1$ we have

$$\begin{aligned} \int_0^t e^{-\rho s} \log\left[\nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)\right] ds \ \leq \ & t e^{-\rho t}\left[\frac{\eta}{2} t + \log\left(\frac{I_\eta(t)}{t}\right)\right] \\ & + [\rho]^+ \int_0^t s e^{-\rho s}\left[\frac{\eta}{2} s + \log\left(\frac{I_\eta(s)}{s}\right)\right] ds \end{aligned} \tag{34}$$

while, for $\sigma \in (0,1)$,

$$\begin{aligned} & \int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \\ \leq \ & I_\eta(t)^{1-\sigma} t^\sigma e^{-(\rho - \eta(1-\sigma))t} + [\rho - \eta(1-\sigma)]^+ \int_0^t I_\eta(s)^{1-\sigma} s^\sigma e^{-(\rho-\eta(1-\sigma))s} ds \end{aligned} \tag{35}$$

and, for $\sigma > 1$, and $\rho > \eta(1-\sigma)$,

$$\begin{aligned} & \int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \\ \geq \ & I_\eta(t)^{1-\sigma} t^\sigma e^{-(\rho - \eta(1-\sigma))t} + (\rho - \eta(1-\sigma)) \int_0^t I_\eta(s)^{1-\sigma} s^\sigma e^{-(\rho-\eta(1-\sigma))s} ds \end{aligned} \tag{36}$$

Proof.

(1) Concerning inequality (35) we take $\eta \in \mathbb{R}$. Setting

$$h_\eta(s) := \int_0^s \left[e^{-\eta r} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)\right]^{1-\sigma} dr$$

we have, by Jensen's inequality, for $\sigma \in (0,1)$

$$h_\eta(s) \leq s\left[\frac{1}{s}\int_0^s e^{-\eta r}\nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right) dr\right]^{1-\sigma} = s^\sigma I_\eta(s)^{1-\sigma} \tag{37}$$

Now integrating by parts we obtain

$$\begin{aligned} \int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \ &= \ \int_0^t e^{-(\rho-\eta(1-\sigma))s}\left[e^{-\eta s}\nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)\right]^{1-\sigma} ds \\ = e^{-(\rho-\eta(1-\sigma))t} h_\eta(t) \ &+ \ \int_0^t \left(\rho - \eta\left(1-\sigma\right)\right) e^{-(\rho-\eta(1-\sigma))s} h_\eta(s) ds \qquad (38) \\ \leq e^{-(\rho-\eta(1-\sigma))t} s^\sigma I_\eta(t)^{1-\sigma} \ &+ \ \left[\rho - \eta\left(1-\sigma\right)\right]^+ \int_0^t e^{-(\rho-\eta(1-\sigma))s} s^\sigma I_\eta(s)^{1-\sigma} ds. \end{aligned}$$

which gives the claim. Inequality (32) follows observing that

$$\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds = h_{\frac{\rho}{1-\sigma}}(t) \tag{39}$$

and using inequality (37).

(2) To prove inequality (33) dealing with the case when $\sigma \in (1, +\infty)$ we still observe that (39) holds and then apply the Jensen inequality. Since the power function $x \to x^{1-\sigma}$ is convex we get the inequality (37) with $\geq$ and so the claim.

Similarly inequality (36) follows integrating by part exactly as for proving (35) and then applying the reversed Jensen inequality.

(3) Inequality (34) follows by similar arguments. In fact, calling, for $\eta \in \mathbb{R}$

$$h(s) := \int_0^s \log \nu(\mathbf{c}_r) dr = \int_0^s \eta r dr + \int_0^s \log\left(e^{-\eta r} \nu(\mathbf{c}_r)\right) dr$$

we have, because of Jensen's inequality

$$h(s) \leq \eta \frac{s^2}{2} + s \log\left[\frac{I_\eta(s)}{s}\right]. \tag{40}$$

Now, integrating by parts as in (38), we obtain

$$\int_0^t e^{-\rho s} \log \nu(\mathbf{c}_s) ds = e^{-\rho t} h(t) + \int_0^t \rho e^{-\rho s} h(s) ds. \tag{41}$$

which, together with (40) and (25), gives the claim.

■

Remark 4.4 *We observe that, if the constraint (12) is assumed to hold then the proof of the above lemma would be simpler. Indeed all estimates on the integrals containing the consumption strategy (25)–(34) would be immediately true since, thanks to (23) and (27) we would have an estimate of the type* $c_t \leq Ce^{\lambda t} \|\bar{\mathbf{s}}\|$.

Moreover an estimate of this kind would allow us to prove the existence result more simply, using the technique of proof of the existence Theorem 2.8 of [2] (see also [4, 12]), based on the compactness of the derivatives of the stock (Theorem 4.2) in the space of absolutely continuous functions (which, in our model, would be equivalent to the compactness of the set of admissible strategies in a suitable weighted space of integrable functions).

Since we do not have this property we employ a different technique that exploits the structure of our problem. In the case when $\sigma \in (0, 1)$*, we change variables to get compactness in the new variable and then we go back to the old variable; in the other cases we use a result that strongly exploits the structure of the problem, in particular the monotonicity of the functions involved.* ■

4.2 Proof of existence and nonexistence theorems

We now prove the above Theorem 3.1 about existence and Theorems 3.2, 3.3 about nonexistence of optimal strategies. The proof of nonexistence consists in providing suitable estimates for the value of admissible strategies; the proof of existence requires a "dual" version of such estimates and then uses compactness arguments. Due to the complexity of the problem (that combines the difficulties of solving inequalities for positive matrices with the dynamic optimization problem), to our knowledge the results given in the literature cannot apply to this case (see [5] and [19] for similar results). For this reason we give a complete proof.

The structure of the proof is a little complex since various cases need to be analyzed. To be precise, for existence we need to prove that:

1 the admissible strategies always have the value $< +\infty$ (this is obvious for $\sigma > 1$ as $u_\sigma \geq 0$, but nontrivial for $\sigma \leq 1$)

2 at least one admissible strategy has the value $> -\infty$ (this is obvious for $\sigma < 1$ as $u_\sigma > 0$, but nontrivial for $\sigma \geq 1$);

3 suitable compactness arguments can be applied.

For the nonexistence proof we need to prove that

1′ in the case when $\sigma < 1$ (or $\sigma = 1$ and $\Gamma_\nu > 0$) either at least one admissible strategy has the value $= +\infty$ or there exists a sequence of admissible strategies with values converging to $+\infty$;

2′ in the case $\sigma > 1$ (or $\sigma = 1$ and $\Gamma_\nu \leq 0$), all admissible strategies have the value $= -\infty$.

The techniques needed to prove points **1** and **2′** are very similar. Moreover, the techniques needed to prove points **2** and **1′** are also very similar. So we give first Proposition 4.5 where points **1** and **2′** are dealt with. Then in Proposition 4.7 points **2** and **1′** are treated. These two Propositions prove the nonexistence Theorems 3.2, 3.3 and provide elements for the proof of Theorem 3.1. In order to complete the proof of the existence Theorem 3.1 we still have to tackle point **3** which is the aim of Proposition 4.9. In the statement below we denote by Γ_E the Euler Gamma function.

Proposition 4.5 *Given any $\bar{\mathbf{s}} \geq \mathbf{0}$, satisfying Assumption 2.5 the following hold.*

1. *Let $\sigma \in (0,1)$. Fix $\varepsilon > 0$ ($\varepsilon = 0$ when Γ_i is not a maximum for every $i = 1, \ldots, k$) such that $\frac{\rho}{1-\sigma} > \Gamma_\nu + \varepsilon$. Then for any $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ and η such that $\frac{\rho}{1-\sigma} > \eta > \Gamma_\nu + \varepsilon$ we have*

$$0 \leq U_\sigma(\mathbf{c}) \leq \frac{\rho - \eta(1-\sigma)}{1-\sigma} \cdot \frac{\Gamma_E(1+\sigma)}{(\rho - \eta(1-\sigma))^{1+\sigma}} \left[C_\varepsilon \sum_{i=1}^{k} \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon} \right]^{1-\sigma} < +\infty \qquad (42)$$

for a suitable C_ε independent of the initial datum and of the control strategy.

2. *Let $\sigma = 1$ (in this case for every ε we have $a_{\nu,\varepsilon} = \rho$). If $\rho > 0$ and $\eta > \Gamma_\nu + \varepsilon$ then for every $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ we have*

$$U_\sigma(\mathbf{c}) \leq \rho \int_0^{+\infty} e^{-\rho s} s \left[\frac{\eta}{2} s + \log \left(\frac{C_\varepsilon^1 \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}_{F,\varepsilon}^i}{s} \right) \right] ds < +\infty. \tag{43}$$

for a suitable C_ε^1 independent of the initial datum and the control strategy. If $\rho \leq 0$ and $\Gamma_\nu < 0$ then $U_\sigma(\mathbf{c}) = -\infty$ for every $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$. The same if $\rho \leq 0$, $\Gamma_\nu = 0$,

$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu \left(e^{\bar{\Gamma}_1 t}, \ldots, e^{\bar{\Gamma}_k t} \right) < +\infty \tag{44}$$

and Γ_i is not a maximum for every $i = 1, \ldots k$.

3. *If $\sigma > 1$, then*

$$U_\sigma(\mathbf{c}) \leq 0.$$

Moreover if $a_{\nu,0} < 0$ then $U_\sigma(c) = -\infty$ for every $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$. The same holds if $a_{\nu,0} = 0$, (44) holds and Γ_i is not a maximum for every $i = 1, \ldots k$.

Proof.

(0) We first prove a key estimate for $I_\eta(t)$. Setting, for $i = 1, \ldots, k$, $\varepsilon > 0$ ($\varepsilon = 0$ if each Γ_i is not a maximum), $s \geq 0$,

$$\omega_{i,\varepsilon,s} := e^{-\bar{\Gamma}_{i,\varepsilon} s} c_{is}, \qquad \omega_{max,\varepsilon,s} := \max\{\omega_{i,\varepsilon,s},\ i = 1, \ldots, k\},$$

we have, using (25) and the fact that $\mathbf{e}_i^T \mathbf{v}_{F,\varepsilon}^i = 1$,

$$\int_0^t \omega_{max,\varepsilon,s} ds \leq \sum_{i=1}^k \int_0^t \omega_{i,\varepsilon,s} ds \leq \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}_{F,\varepsilon}^i, \qquad \forall t \geq 0. \tag{45}$$

Now, for $\eta \in \mathbb{R}$, we have

$$\begin{aligned}
I_\eta(t) &= \int_0^t e^{-\eta s} \nu \left(c_{1s}, c_{2s}, \ldots, c_{ks} \right) ds \\
&= \int_0^t e^{-(\eta - \Gamma_\nu)s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_{1,\varepsilon} s} \omega_{1,\varepsilon,s}, e^{\bar{\Gamma}_{2,\varepsilon} s} \omega_{2,\varepsilon,s}, \ldots, e^{\bar{\Gamma}_{k,\varepsilon} s} \omega_{k,\varepsilon,s} \right) ds \\
&\leq \int_0^t e^{-(\eta - (\Gamma_\nu + \varepsilon))s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_1 s} \omega_{1,\varepsilon,s}, e^{\bar{\Gamma}_2 s} \omega_{2,\varepsilon,s}, \ldots, e^{\bar{\Gamma}_k s} \omega_{k,\varepsilon,s} \right) ds \\
&\leq \int_0^t e^{-(\eta - (\Gamma_\nu + \varepsilon))s} \omega_{max,\varepsilon,s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_1 s}, \ldots, e^{\bar{\Gamma}_k s} \right) ds
\end{aligned} \tag{46}$$

If $\eta > \Gamma_\nu + \varepsilon$ then, by the definition of Γ_ν, we have that the function

$$t \to e^{-(\eta - (\Gamma_\nu + \varepsilon))s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_1 s}, \ldots, e^{\bar{\Gamma}_k s} \right)$$

is bounded on $[0, +\infty)$ (say by a constant C_ε independent of the initial datum and on the control strategy) and so, by putting (45) into (46) we get

$$I_\eta(t) \le C_\varepsilon \sum_{i=1}^{k} \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}, \qquad \forall t \ge 0. \tag{47}$$

Similarly, if

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu\left(e^{\bar{\Gamma}_1 t}, \ldots, e^{\bar{\Gamma}_k t}\right) < +\infty$$

and Γ_i is not a maximum for every $i = 1, \ldots k$, then we can choose $\varepsilon = 0$ and $\eta = \Gamma_\nu$ in (46) and still get (47) with $\varepsilon = 0$, $\eta = \Gamma_\nu$.

(1) Now we prove estimate (42) using (35). Take $\eta < \frac{\rho}{1-\sigma}$, put the estimate (47) into (35) and let $t \to +\infty$. We get

$$U_\sigma(\mathbf{c}) \le \frac{\rho - \eta(1-\sigma)}{1-\sigma} \left[C_\varepsilon \sum_{i=1}^{k} \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}\right]^{1-\sigma} \int_0^{+\infty} s^\sigma e^{-(\rho-\eta(1-\sigma))s} ds$$

$$\le \frac{\rho - \eta(1-\sigma)}{1-\sigma} \cdot \frac{\Gamma_E(1+\sigma)}{(\rho - \eta(1-\sigma))^{1+\sigma}} \left[C_\varepsilon \sum_{i=1}^{k} \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}\right]^{1-\sigma}$$

so the claim (42) follows.

(2) If $\sigma = 1$ and $\rho > 0$ we get (43) taking $\eta > \Gamma_\nu + \varepsilon$, putting (47) into (34) and letting $t \to +\infty$.

If $\rho = 0$, then from (34) and (47) we get, for $\eta > \Gamma_\nu + \varepsilon$

$$\int_0^t \log \nu(\mathbf{c}_s) ds \le t \left[t\frac{\eta}{2} + \log\left(\frac{C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}}{t}\right)\right]$$

so if $\Gamma_\nu < 0$ we take $\varepsilon > 0$, $\eta < 0$ such that $\eta > \Gamma_\nu + \varepsilon$. Then we get, in the limit for $t \to +\infty$, that $U_1(\mathbf{c}) = -\infty$.

Let finally $\Gamma_\nu = 0$,

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu\left(e^{\bar{\Gamma}_1 t}, \ldots, e^{\bar{\Gamma}_k t}\right) < +\infty$$

and Γ_i is not a maximum for every $i = 1, \ldots k$. Then by part (0) of this proof we can take $\eta = \Gamma_\nu = 0$ and $\varepsilon = 0$ in (47) so the estimate (34) becomes

$$\int_0^t \log \nu(\mathbf{c}_s) ds \le t \log\left(\frac{C_0 \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,0}}{t}\right)$$

so, in the limit for $t \to +\infty$, we still get that $U_1(\mathbf{c}) = -\infty$. If $\rho < 0$ and $\Gamma_\nu < 0$ (or $\Gamma_\nu = 0$ when possible) then

$$\int_0^t e^{-\rho s} \log \nu(\mathbf{c}_s) ds = \int_0^t e^{-\rho s} \left[\log \nu(\mathbf{c}_s)\right]^+ ds + \int_0^t e^{-\rho s} \left[\log \nu(\mathbf{c}_s)\right]^- ds.$$

Now, thanks to the nonpositivity of the negative part and to the fact that $e^{-\rho s} \geq 1$,

$$\int_0^t e^{-\rho s} \left[\log \nu(\mathbf{c}_s)\right]^- ds \leq \int_0^t \left[\log \nu(\mathbf{c}_s)\right]^- ds.$$

Since the right hand side goes to $-\infty$ as $t \to +\infty$ (thanks to the case $\rho = 0$) we have $\int_0^{+\infty} e^{-\rho s} \left[\log \nu(\mathbf{c}_s)\right]^- ds = -\infty$. By admissibility this implies that the integral of the positive part is finite and so $U_1(\mathbf{c}) = -\infty$.

(3) When $\sigma > 1$ it is obvious that $U_\sigma(\mathbf{c}) \leq 0$ by construction. Moreover using (33) and (47) with $\eta = \frac{\rho}{1-\sigma} > \Gamma_\nu + \varepsilon$ we get

$$\frac{1}{1-\sigma} \int_0^t e^{-\rho s} \nu(c_{1s}, c_{2s}, \ldots, c_{ks})^{1-\sigma} ds \leq \frac{t^\sigma}{1-\sigma} \left(C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon} \right)^{1-\sigma}$$

and letting $t \to +\infty$ we get $U_\sigma(c) = -\infty$ for every admissible strategy. Finally if $a_{\nu,0} = 0$, (44) holds and Γ_i is not a maximum for every $i = 1, \ldots k$, we know from part (0) of this proof that (47) still holds so we can still let $t \to +\infty$ and get $U_\sigma(c) = -\infty$ for every admissible strategy. ∎

Remark 4.6 *The above result shows in particular that, when $a_{\nu,0} > 0$ and $\sigma \in (0,1)$, the intertemporal utility functional $U_\sigma(\mathbf{c})$ is finite and uniformly bounded for every admissible production-consumption strategy (while for $\sigma \geq 1$ it is only bounded from above). In the cases when*

1. *$\sigma = 1$, $\rho \leq 0$, $\Gamma_\nu < 0$;*
2. *$\sigma = 1$, $\rho \leq 0$ and $\Gamma_\nu = 0$, each Γ_i is not a maximum and (44) holds;*
3. *$\sigma > 1$, $a_{\nu,0} < 0$;*
4. *$\sigma > 1$, $a_{\nu,0} = 0$, each Γ_i is not a maximum and (44) holds;*

Proposition 4.5 shows that there are no optimal strategies in the sense of Definition 2.1 since all strategies have utility $-\infty$. ∎

Proposition 4.7 *Let $\bar{\mathbf{s}} \geq \mathbf{0}$.*

1. *Let $\sigma \in (0,1)$ and, either $a_{\nu,0} < 0$, or $a_{\nu,0} = 0$, each Γ_i is a maximum for $i = 1, \ldots, k$ and*
$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right) > 0. \tag{48}$$
Then there exists an admissible strategy $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ such that $U_\sigma(\mathbf{c}) = +\infty$.

2. Let $\sigma = 1$, $a_{\nu,0} = \rho \leq 0$, $\Gamma_\nu > 0$. Then there exists an admissible strategy $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ such that $U_\sigma(\mathbf{c}) = +\infty$.

Moreover, if $a_{\nu,0} = \rho = 0$ and $\Gamma_\nu = 0$, each Γ_i is a maximum, (14) holds and $\bar{\mathbf{s}}$ satisfy

$$\bar{\mathbf{s}}^T \mathbf{e}_j > \sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A} \mathbf{e}_j, \qquad j = 1, \ldots n.$$

(where $\mathbf{x}_i$ is the maximum point of $\mathcal{G}_i$), then there exists an admissible strategy with value $+\infty$. If the limit in (14) is $+\infty$ then the above holds for every $\bar{\mathbf{s}}$.

3. Let $\sigma \geq 1$ and $a_{\nu,0} > 0$, then there exists an admissible strategy $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ with $U_\sigma(\mathbf{c}) > -\infty$.

Proof. We prove the three points separately.

Proof of 1. Consider first the case when $\sigma \in (0,1)$ and $a_{\nu,0} < 0$. First let the system evolve to reach a state $\mathbf{s}_0 > \mathbf{0}$ (this is possible since Assumption 2.5 holds). This means that we can take from the beginning $\bar{\mathbf{s}} > \mathbf{0}$. At this point we observe that for any $i = 1, \ldots, k$ and $\varepsilon > 0$ ($\varepsilon = 0$ if Γ_i is a maximum for every i) we can find $\mathbf{x}_{i,\varepsilon} \geq \mathbf{0}$ such that

$$\mathbf{x}_{i,\varepsilon}^T \left(\mathbf{B} - \left(\Gamma_i - \varepsilon + \delta_{\mathbf{x}}\right) \mathbf{A}\right) \geq \mathbf{e}_i^T \qquad \Rightarrow \qquad \mathbf{x}_{i,\varepsilon}^T \left(\mathbf{B} - \delta \mathbf{A}\right) \geq \mathbf{e}_i^T + \left(\Gamma_i - \varepsilon + \delta_{\mathbf{z}}\right) \mathbf{x}_{i,\varepsilon}^T \mathbf{A}.$$

Take now $\beta_0 > 0$ and $\beta_1, \ldots \beta_k$ such that $\beta_i \geq 0$. Set

$$\mathbf{x}_{\varepsilon,s} := \beta_0 \sum_{i=1}^{k} \beta_i \mathbf{x}_{i,\varepsilon} e^{(\Gamma_i - \varepsilon)s}, \qquad s \geq 0.$$

We clearly have that $\mathbf{x}_{\varepsilon,s} \geq \mathbf{0}$, $\mathbf{x}_{\varepsilon,s} \neq \mathbf{0}$ for every $s \geq 0$ and

$$\mathbf{x}_{\varepsilon,s}^T \left(\mathbf{B} - \delta \mathbf{A}\right) \mathbf{e}_j = \beta_0 \sum_{i=1}^{k} \beta_i e^{(\Gamma_i - \varepsilon)s} \mathbf{x}_{i,\varepsilon}^T \left(\mathbf{B} - \delta \mathbf{A}\right) \mathbf{e}_j \geq \beta_0 \sum_{i=1}^{k} \beta_i e^{(\Gamma_i - \varepsilon)s} \left[\mathbf{e}_i^T \mathbf{e}_j + \left(\Gamma_i - \varepsilon + \delta_{\mathbf{z}}\right) \mathbf{x}_{i,\varepsilon}^T \mathbf{A} \mathbf{e}_j\right]$$

Consider now the control strategy $\mathbf{x}_t = \mathbf{x}_{\varepsilon,t}$, $\mathbf{c}_t = \beta_0 (\beta_1 e^{(\Gamma_1 - \varepsilon)t}, \ldots, \beta_k e^{(\Gamma_k - \varepsilon)t})$ for each $t \geq 0$. Since, for $t \geq 0$, we have

$$\mathbf{x}_{\varepsilon,t}^T \left(\mathbf{B} - \delta \mathbf{A}\right) \mathbf{e}_j - \hat{\mathbf{c}}_t^T \mathbf{e}_j \geq \beta_0 \sum_{i=1}^{k} \beta_i e^{(\Gamma_i - \varepsilon)t} \left(\Gamma_i - \varepsilon + \delta_{\mathbf{z}}\right) \mathbf{x}_{i,\varepsilon}^T \mathbf{A} \mathbf{e}_j$$

and the associated solution of the state equation (9) is given by:

$$\mathbf{s}_t^T = e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_{\varepsilon,s}^T \left[\mathbf{B} - \delta \mathbf{A}\right] ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \hat{\mathbf{c}}_s ds$$

$$= e^{-\delta_{\mathbf{z}} t} \left[\bar{\mathbf{s}}^T + \int_0^t \left(\mathbf{x}_{\varepsilon,s}^T \left(\mathbf{B} - \delta \mathbf{A}\right) - \hat{\mathbf{c}}_s^T\right) e^{\delta_{\mathbf{z}} s} ds\right],$$

then

$$\mathbf{s}_t^T\mathbf{e}_j \geq e^{-\delta_\mathbf{z} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j + \int_0^t \beta_0 \sum_{i=1}^k \beta_i e^{(\delta_\mathbf{z}+\Gamma_i-\varepsilon)s}\left(\Gamma_i-\varepsilon+\delta_\mathbf{z}\right)\mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j\, ds\right]$$

$$= e^{-\delta_\mathbf{z} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j + \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j \int_0^t e^{(\delta_\mathbf{z}+\Gamma_i-\varepsilon)s}\left(\Gamma_i-\varepsilon+\delta_\mathbf{z}\right) ds\right]$$

$$= e^{-\delta_\mathbf{z} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j + \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j \left(e^{(\delta_\mathbf{z}+\Gamma_i-\varepsilon)t}-1\right)\right]$$

$$= e^{-\delta_\mathbf{z} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j - \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j\right] + \beta_0 \sum_{i=1}^k \beta_i e^{(\Gamma_i-\varepsilon)t}\mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j$$

$$= e^{-\delta_\mathbf{z} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j - \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j\right] + \mathbf{x}_{\varepsilon,t}^T\mathbf{A}\mathbf{e}_j$$

In this case it is clear that the constraints $\mathbf{s}_t^T \geq \mathbf{x}_t^T\mathbf{A}$ are satisfied if, for every $j = 1, \ldots, n$,

$$\bar{\mathbf{s}}^T\mathbf{e}_j - \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T\mathbf{A}\mathbf{e}_j \geq 0. \tag{49}$$

Since $\bar{\mathbf{s}} > 0$, the above is true if we set β_0 sufficiently small. So our control strategy is admissible. Moreover setting $\beta_1 = \ldots = \beta_k = 1$ we have

$$U(\mathbf{c}) = \frac{\beta_0^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{-\rho t}\nu\left(e^{(\Gamma_1-\varepsilon)t}, \ldots, e^{(\Gamma_k-\varepsilon)t}\right)^{1-\sigma} dt$$

$$= \frac{\beta_0^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{[-\rho+(\Gamma_\nu-\varepsilon)(1-\sigma)]t}\left[e^{-\Gamma_\nu t}\nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right)\right]^{1-\sigma} dt$$

Using the definition of Γ_ν and the fact that $a_{\nu,0} < 0$ we get that, for ε sufficiently small, the above integral is $+\infty$ and so the claim.

The case $a_{\nu,0} = 0$ follows simply observing that, in the above equation, since Γ_i is a maximum for $i = 1, \ldots, k$, we can take $\varepsilon = 0$ and, thanks to (48), we can take $\rho = \Gamma_\nu(1-\sigma)$.

Proof of 2. Take now the case when $\sigma = 1$ and $a_{\nu,0} \leq 0$, $\Gamma_\nu > 0$. Since $\Gamma_\nu > 0$ let ε such that $\Gamma_\nu > 2\varepsilon$. Then we take the above control strategy so that

$$U_1(\mathbf{c}) = \int_0^{+\infty} e^{-\rho t}\log\nu\left(e^{(\Gamma_1-\varepsilon)t}, \ldots, e^{(\Gamma_k-\varepsilon)t}\right) ds =$$

$$= \int_0^{+\infty} e^{-\rho t}\left[\log\beta_0 + (\Gamma_\nu - 2\varepsilon)t + \log\left(e^{-(\Gamma_\nu-\varepsilon)t}\nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right)\right)\right] dt.$$

Clearly, for $a_{\nu,0} = \rho \geq 0$ the last integrand is locally bounded, definitely positive, and goes to $+\infty$ for $t \to +\infty$. Then for this strategy we have $U_1(\mathbf{c}) = +\infty$.

In the case when $\sigma = 1$, $a_{\nu,0} = \rho = 0$, $\Gamma_\nu = 0$, each Γ_i is a maximum, and (14) holds, the strategy defined in the proof of point 1 above is still admissible. However it has value $+\infty$ only if β_0 is big enough (i.e. $\beta_0 L > 1$, where L is defined at the end of Section 2). This means that also $\bar{\mathbf{s}}$ must be big enough so (49) is verified. In particular, if $L = +\infty$, again for every $\bar{\mathbf{s}}$ the claim holds.

Proof of 3. Let $a_{\nu,0} > 0$ and $\sigma \in [1, +\infty)$. We observe that (since admissibility does not depend on the value of σ) the strategy found in point 1 above is admissible. We then have, for $\sigma \in (1, +\infty)$

$$U(\mathbf{c}) = \frac{\beta_0^{1-\sigma}}{1-\sigma} \int_0^{+\infty} e^{[-\rho+(\Gamma_\nu-\varepsilon)(1-\sigma)]t} \left[e^{-\Gamma_\nu t} \nu \left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)\right]^{1-\sigma} dt$$

so, if $\varepsilon > 0$ is such that $-\rho + (\Gamma_\nu - \varepsilon)(1 - \sigma) < 0$ we get the claim.

For $\sigma = 1$ we have

$$U_1(\mathbf{c}) = \int_0^{+\infty} e^{-\rho t} \log \nu \left(e^{(\Gamma_1-\varepsilon)t}, \dots, e^{(\Gamma_k-\varepsilon)t}\right) ds =$$

$$= \int_0^{+\infty} e^{-\rho t} \left[\log \beta_0 + \Gamma_\nu t + \log \left(e^{-(\Gamma_\nu+\varepsilon)t} \nu \left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)\right)\right] dt.$$

Since the last integrand is less than polynomially growing and $\rho = a_{\nu,0} > 0$ then the integral above is finite, so $U_1(\mathbf{c}) > -\infty$. ∎

Remark 4.8 *The above result shows in particular that, when $a_{\nu,0} > 0$ and $\sigma \in [1, +\infty)$, the intertemporal utility functional $U_\sigma(\mathbf{c})$ is not always $-\infty$ so it is bounded from below (recall that from Proposition 4.5 we already know that in these case $U_\sigma(\mathbf{c})$ is bounded from above). Moreover in the cases when*

1. *$\sigma \in (0, 1)$ and $a_{\nu,0} < 0$;*
2. *$\sigma \in (0, 1)$, $a_{\nu,0} = 0$, each Γ_i is a maximum and (48) holds;*
3. *$\sigma = 1$, $a_{\nu,0} \le 0$ and $\Gamma_\nu > 0$;*
4. *$\sigma = 1$, $a_{\nu,0} = 0$, $\Gamma_\nu = 0$, each Γ_i is a maximum (48) holds and $\bar{\mathbf{s}}$ is big enough;*

Proposition 4.7 shows that there are no optimal strategies in the sense of Definition 2.1 since the supremum of the utility is $+\infty$. ∎

Summing up the informations taken from Propositions 4.5 and 4.7 we can say the following.

- In the cases when $a_{\nu,0} > 0$ we know that the functional is uniformly bounded (case $\sigma \in (0, 1)$) or bounded from from above and not identically $-\infty$ (case $\sigma \ge 1$);

- In the cases when $a_{\nu,0} \leq 0$ we have nonexistence when

  1. $\sigma \in (0,1)$ and $a_{\nu,0} < 0$;
  2. $\sigma \in (0,1)$, $a_{\nu,0} = 0$, each Γ_i is a maximum and (48) holds;
  3. $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu \neq 0$
  4. $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu = 0$, each Γ_i is not a maximum and (44) holds;
  5. $\sigma = 1$, $a_{\nu,0} = 0$, $\Gamma_\nu = 0$, each Γ_i is a maximum (48) holds and $\bar{\mathbf{s}}$ is big enough;
  6. $\sigma > 1$ and $a_{\nu,0} < 0$
  7. $\sigma > 1$, $a_{\nu,0} = 0$, each Γ_i is not a maximum and (44) holds.

We observe that, to end the treatment of nonexistence result one should deal with a complete treatment of the following limiting cases:

- $\sigma \in (0,1)$ and $a_{\nu,0} = 0$;
- $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu = 0$;
- $\sigma > 1$ and $a_{\nu,0} = 0$

Proof of Theorem 3.2. It follows directly from Proposition 4.5, Proposition 4.7 and the remarks above. ∎

Now we come to prove existence when $a_{\nu,0} > 0$ using compactness arguments. To do this we need first to prove suitable properties of the set $\mathcal{A}(\bar{\mathbf{s}})$ which are given in the next proposition. First we recall a simple definition: given a measurable function $g_1 : \mathbb{R}^+ \to \mathbb{R}$ we denote by $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m)$ the set of measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^m$ such that the product $f \cdot g_1$ is bounded on $\mathbb{R}^+$. Moreover given a measurable function $g_2 : \mathbb{R}^+ \to \mathbb{R}^k$ we denote by $L^1_{g_2}\left(0,+\infty;\mathbb{R}^k\right)$ the set of measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^k$ such that the product $f_i \cdot g_{2,i}$ is integrable on $\mathbb{R}^+$ for each $i = 1,\ldots,k$. We set $g_1(t) = e^{\lambda t}$ (λ is given by (27) of Lemma 4.2) and $g_2(t) = (e^{(\Gamma_1+\varepsilon)t},\ldots,e^{(\Gamma_k+\varepsilon)t})$ for $\varepsilon > 0$ such that $a_{\nu,\varepsilon} > 0$.

Proposition 4.9 *Let Assumptions 2.2 and 2.3 be verified. Let also $\sigma \in (0,1) \cup (1,+\infty)$. Given any $\bar{\mathbf{s}} \geq \mathbf{0}$ the set $\mathcal{A}(\bar{\mathbf{s}})$ of admissible control strategies starting at $\bar{\mathbf{s}}$ is a closed, bounded, convex subset of the space $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m) \times L^1_{g_2}\left(0,+\infty;\mathbb{R}^k\right)$. Moreover*

$$(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}}), \quad \lambda \in [0,1] \Rightarrow (\lambda\mathbf{x},\lambda\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}}) \tag{50}$$

Finally, if ν is strictly concave the functional U_σ is strictly concave with respect to the argument $\mathbf{c}$. The same holds when $\sigma = 1$ and $\rho > 0$.

Proof. Convexity. Let $i=1,2$ and let $(\mathbf{x}_i,\mathbf{c}_i)\in\mathcal{A}(\overline{\mathbf{s}})$, and $\lambda\in[0,1]$. Calling

$$(\mathbf{x}_\lambda,\mathbf{c}_\lambda)=\lambda(\mathbf{x}_1,\mathbf{c}_1)+(1-\lambda)(\mathbf{x}_2,\mathbf{c}_2)$$

then due to the linearity of the state equation (9)

$$\mathbf{s}_{t,\overline{\mathbf{s}},(\mathbf{x}_\lambda,\mathbf{c}_\lambda)}=\lambda\mathbf{s}_{t,\overline{\mathbf{s}},(\mathbf{x}_1,\mathbf{c}_1)}+(1-\lambda)\mathbf{s}_{t,\overline{\mathbf{s}},(\mathbf{x}_2,\mathbf{c}_2)}.$$

Since all constraints on $(\mathbf{s},(\mathbf{x},\mathbf{c}))$ (i.e. $\mathbf{x}\geq\mathbf{0}$, $\mathbf{c}\geq 0$, $\mathbf{x}^T\mathbf{A}\leq\mathbf{s}^T$) are linear it follows that, since $(\mathbf{x}_i,\mathbf{c}_i)$ $(i=1,2)$ satisfy them, then so does $(\mathbf{x}_\lambda,\mathbf{c}_\lambda)$. This yields $(\mathbf{x}_\lambda,\mathbf{c}_\lambda)\in\mathcal{A}(\overline{\mathbf{s}})$ when $\sigma\in(0,1)\cup(1,+\infty)$. If $\sigma=1$ we also have to prove that $(\mathbf{x}_\lambda,\mathbf{c}_\lambda)$ is semiintegrable. This follows from point (2) of Proposition 4.5. Indeed if $\rho>0$, thanks to estimate (43) we know all admissible strategies are upper semiintegrable, so also their convex combinations are upper semiintegrable. Boundedness follows from the estimates of Lemma 4.2.

Closedness follows from the fact that all constraints are linear so all of them preserve in the limit in the topology of $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m)\times L^1_{g_2}\left(0,+\infty;\mathbb{R}^k\right)$. For $\sigma=1$ we need to know that the limit of semiintegrable sequences is again semiintegrable. For $\rho>0$ this follows from the estimate (43).

Homogeneity (50) follows from convexity and from the fact that the strategy $(\mathbf{0},\mathbf{0})$ is always admissible.

Strict concavity of the functional U_σ is a standard result (see e.g. [6]) and we omit the proof. ∎

Now we move on to the proof of Theorem 3.1.

Proof of Theorem 3.1. The uniqueness property follows from the strict concavity of U_σ proved in Proposition 4.9. The existence result follows applying a suitable modification of Theorems 21 and 22 in [19, p. 406] (see also [18]). We divide it into three cases depending on the value of σ.

Case $\sigma>1$.

Here we can apply directly Theorem 22 and note 26 of [19, p. 406] plus [19, note 20, p.137]. In fact this theorem asks the following:

1. the set U where the controls take values is closed (in our case U is $\mathbb{R}^m_+\times\mathbb{R}^k_+$ i.e. the positive orthant of $\mathbb{R}^{m+k}$);
2. the functions defining the running utility $((\mathbf{s},(\mathbf{x},\mathbf{c}),t)\to e^{-\rho t}u_\sigma(\nu(\mathbf{c})))$, the dynamics of the state equation $((\mathbf{s},(\mathbf{x},\mathbf{c}),t)\to-\delta_{\mathbf{z}}\mathbf{s}^T+\mathbf{x}^T(\mathbf{B}-\delta\mathbf{A})-\hat{\mathbf{c}})$ and the constraints $((\mathbf{s},(\mathbf{x},\mathbf{c}),t)\to\mathbf{s}^T-\mathbf{x}^T\mathbf{A})$ are defined on the set

$$S=\left\{(\mathbf{s},(\mathbf{x},\mathbf{c}),t)\in\mathbb{R}^n_+\times\left[\mathbb{R}^m_+\times\mathbb{R}^k_+\right]\times\mathbb{R}_+:\mathbf{s}^T-\mathbf{x}^T\mathbf{A}\geq\mathbf{0}\right\},$$

are linear (or sum of linear and nondecreasing) in the variable $\mathbf{s}$ and continuous on the set

$$S'=\left\{(\mathbf{s},(\mathbf{x},\mathbf{c}),t)\in\mathbb{R}^n_+\times\left[\mathbb{R}^m_+\times\mathbb{R}^k_+\right]\times\mathbb{R}_+:\mathbf{s}^T-\mathbf{x}^T\mathbf{A}\geq\mathbf{0}\right\};$$

3. for each $t \geq 0$ the set

$$S'(t) = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \mathbb{R}_+^n \times \left[\mathbb{R}_+^m \times \mathbb{R}_+^k \right] : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$

is contained in the closure $\overline{S^0(t)}$ of the set

$$S^0(t) = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \mathbb{R}_+^n \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} > \mathbf{0} \right\};$$

4. for each $n \in \mathbb{N}$ and $t \geq 0$ the set

$$\Gamma_t^n = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x},\mathbf{c}) \in U, \right.$$

$$\left. \left|\left| \left(e^{-\rho t} u_\sigma (\nu(\mathbf{c})), -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}} \right) \right|\right| \leq n, \right\}$$

is closed and is contained in $S'(t)$. The same for the set

$$\Gamma^n = \{ \mathbf{s} : (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \Gamma_t^n \};$$

5. there exists an admissible strategy with finite value;

6. the set

$$N(\mathbf{s}, U, t) = \left\{ \left(e^{-\rho t} u_\sigma (\nu(\mathbf{c})) + \gamma, -\delta_{\mathbf{z}} \mathbf{s} + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}} + \gamma \right) : \right.$$

$$\left. (\gamma, \gamma) \leq 0, \mathbf{s} - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x},\mathbf{c}) \in U \right\}$$

is convex for all $(\mathbf{s},t) \in \mathbb{R}^n \times [0, +\infty)$;

7. the set $N(\mathbf{s}, U, t)$ has closed graph for each t as a function of $\mathbf{s} \in \Gamma^n$. Closed graph means that

$$\mathbf{s}_n \in \Gamma^n, \mathbf{v}_n \in N(\mathbf{s}, U, t), \mathbf{s}_n \to \mathbf{s}, \mathbf{v}_n \to \mathbf{v} \Rightarrow \mathbf{s} \in \Gamma^n.$$

8. there exists $\mathbf{q}' \in \mathbb{R}^{n+1}$, $\mathbf{q}' \geq 0$ such that for every $\mathbf{q} \geq \mathbf{q}'$ ($\mathbf{q} = (q_0, q_1, ..., q_n) = (q_0, \mathbf{q}_1)$) there exists locally integrable functions $\phi_{\mathbf{q}}$ and $\psi_{\mathbf{q}}$ defined for $t \in [0, +\infty)$ such that

$$e^{-\rho t} u_\sigma (\nu(\mathbf{c})) q_0 + \left(-\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}} \right) \mathbf{q}_1 \leq \phi_{\mathbf{q}}(t) + \psi_{\mathbf{q}}(t) \cdot \max \left[0, \mathbf{s}^T \mathbf{e}_1, ..., \mathbf{s}^T \mathbf{e}_n \right]$$

for every $(\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \in S$.

9. for every $i = 1, ..., n$ and every admissible state trajectory $\mathbf{s}_t$, we have $\mathbf{s}_t^T \mathbf{e}_i \geq 0$ for every $t \geq 0$. Moreover for every $q_0 \in \mathbb{R}$, $q_0 \geq 0$, there exists an integrable function ν_{q_0} defined for $t \in [0, +\infty)$ such that, ,

$$e^{-\rho t} u_\sigma (\nu(\mathbf{c}_t)) (1 + q_0) \leq \nu_{q_0}(t),$$

for every admissible strategy $(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\overline{\mathbf{s}})$.

All points 1-4 and 6-7 are easily checked in our case thanks to the linearity of the state equation and of the constraints. We omit the verification of them for brevity. Point 5 is known from previous results (Proposition 4.5 and Proposition 4.7). Point 9 comes simply recalling that for $\sigma > 1$ the utility is negative and so one can choose $\nu_{q_0}(t) = 0$ for every $t \geq 0$. Point 8 is more delicate. Setting

$$g(\mathbf{c}) = e^{-\rho t} u_\sigma(\nu(\mathbf{c}))\, q_0 - \hat{\mathbf{c}}^T \mathbf{q}_1$$

we have

$$g(\mathbf{c}) \leq 0$$

Moreover

$$-\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) = -\delta_{\mathbf{z}} \left(\mathbf{s}^T - \mathbf{x}^T \mathbf{A} \right) + \mathbf{x}^T (\mathbf{B} - \delta_{\mathbf{x}} \mathbf{A}) \leq \mathbf{x}^T \mathbf{B}.$$

Now, recalling the proof of (27) we have that

$$\mathbf{x}^T \mathbf{B} \leq \mathbf{s}^T \mathbf{D} \leq M \max\left[0, \mathbf{s}^T \mathbf{e}_1, ..., \mathbf{s}^T \mathbf{e}_n\right]$$

where M depends only on the coefficient of $\mathbf{D}$. Setting, for $t \geq 0$, $\phi_{\mathbf{q}}(t) = 0$ and $\psi_{\mathbf{q}}(t) = M$ we see that $\phi_{\mathbf{q}}$ and $\psi_{\mathbf{q}}$ are locally integrable functions and satisfy point 8.

Case $\sigma = 1$ and $\sigma \in (0, 1)$.

Also this case goes applying Theorem 22 and note 26 of [19, p. 406] (see also [18] or [19, Exercise 6.8.3, p.410]). In fact this theorem asks the following:

1. the set U where the controls take values is closed (in our case U is $\mathbb{R}_+^m \times \mathbb{R}_+^k$ i.e. the positive orthant of $\mathbb{R}^{m+k}$);

2. the functions defining the running utility $((\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \to e^{-\rho t} u_\sigma(c))$, the dynamics of the state equation $((\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \to -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}})$ and the constraints $((\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \to \mathbf{s}^T - \mathbf{x}^T \mathbf{A})$ are defined on the set

   $$S = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \in \mathbb{R}_+^n \times U \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$

   are linear (or sum of linear and nondecreasing) in the variable $\mathbf{s}$ and continuous on the set

   $$S' = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \in \mathbb{R}_+^n \times U \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\};$$

3. for each $t \geq 0$ the set

   $$S'(t) = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \mathbb{R}_+^n \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$

   is contained in the closure $\overline{S^0(t)}$ of the set

   $$S^0(t) = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \mathbb{R}_+^n \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} > \mathbf{0} \right\};$$

4. for each $n \in \mathbb{N}$ and $t \geq 0$ the set

$$\Gamma_t^n = \big\{ (\mathbf{s}, (\mathbf{x}, \mathbf{c})) : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x}, \mathbf{c}) \in U,$$

$$\big|\big| \left(e^{-\rho t} u_\sigma (\nu(\mathbf{c})), -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}} \right) \big|\big| \leq n, \big\}$$

is closed and is contained in $S'(t)$. The same for the set

$$\Gamma^n = \{ \mathbf{s} : (\mathbf{s}, (\mathbf{x}, \mathbf{c})) \in \Gamma_t^n \};$$

5. there exists an admissible strategy with finite value;

6. the set

$$N(\mathbf{s}, U, t) = \big\{ \left(e^{-\rho t} u_\sigma (\nu(\mathbf{c})) + \gamma, -\delta_{\mathbf{z}} \mathbf{s} + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}}^T + \gamma \right) :$$

$$(\gamma, \gamma) \leq 0, \mathbf{s} - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x}, \mathbf{c}) \in U \big\}$$

is convex for all $(\mathbf{s}, t) \in \mathbb{R}^n \times [0, +\infty)$;

7. the set $N(\mathbf{s}, U, t)$ has closed graph for each t as a function of $\mathbf{s} \in \Gamma^n$. Closed graph means that

$$\mathbf{s}_n \in \Gamma^n, \mathbf{v}_n \in N(\mathbf{s}, U, t), \mathbf{s}_n \to \mathbf{s}, \mathbf{v}_n \to \mathbf{v} \Rightarrow \mathbf{s} \in \Gamma^n.$$

8. there exists $\mathbf{q}' \in \mathbb{R}^{n+1}$, $\mathbf{q}' \geq 0$ such that for every $\mathbf{q} \geq \mathbf{q}'$ ($\mathbf{q} = (q_0, q_1, ..., q_n) = (q_0, \mathbf{q}_1)$) there exists locally integrable functions $\phi_{\mathbf{q}}$ and $\psi_{\mathbf{q}}$ defined for $t \in [0, +\infty)$ such that

$$e^{-\rho t} u_\sigma (\nu(\mathbf{c})) q_0 + \left(-\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}}^T \right) \mathbf{q}_1 \leq \phi_{\mathbf{q}}(t) + \psi_{\mathbf{q}}(t) \cdot \max \left[0, \mathbf{s}^T \mathbf{e}_1, ..., \mathbf{s}^T \mathbf{e}_n \right]$$

for every $(\mathbf{s}, (\mathbf{x}, \mathbf{c}), t) \in S$.

9. for every $i = 1, ..., n$ and every admissible state trajectory $\mathbf{s}_t$, we have $\mathbf{s}_t^T \mathbf{e}_i \geq 0$ for every $t \geq 0$. Moreover for every $q_0 \in \mathbb{R}$, $q_0 \geq 0$, there exists an integrable function ν_{q_0}, continuous functions $\chi_{q_0}^i$ and $\theta_{q_0}^i$ ($i = 1, ..., n$) defined for $t \in [0, +\infty)$ such that, for every admissible strategy $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$,

$$e^{-\rho t} u_\sigma (\nu(\mathbf{c}_t)) (1 + q_0) + \sum_{i=1}^{n} \chi_{q_0}^i (t) \left(-\delta_{\mathbf{z}} \mathbf{s}_t^T + \mathbf{x}_t^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}}_t^T \right) \mathbf{e}_i \leq \nu_{q_0}(t), \quad (51)$$

and

$$-\int_s^{+\infty} \chi_{q_0}^i (t) \left(-\delta_{\mathbf{z}} \mathbf{s}_t^T + \mathbf{x}_t^T (\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}}_t^T \right) \mathbf{e}_i dt \leq \theta_{q_0}^i (s) \quad (52)$$

where $\lim_{s \to +\infty} \theta_{q_0}^i (s) = 0$.

All points 1-4 and 6-7 are easily checked in our case thanks to the linearity of the state equation and of the constraints. We omit the verification of them for brevity. Point 5 is known from previous results (Proposition 4.5 and Proposition 4.7). Point 8 follows arguing exactly as in the case $\sigma > 1$ except for the estimate of the function $g(\mathbf{c})$ which is done as follows. First observe that $g(\mathbf{c})$ goes to $-\infty$ as any $c_i \to +\infty$, then observe that g is positive on a compact set depending on $\mathbf{q}$ and t which is bounded uniformly when $\mathbf{q}$ and t belong to a bounded set. This is enough to guarantee that g has a maximum point and that the value of the maximum is uniformly bounded for $\mathbf{q}$ and t on bounded sets.

Point 9 is more delicate. Take first the case $\sigma \in (0,1)$. Set, for $i = 1, \dots, n$, $\chi^i_{q_0}(t) = \mathbf{e}_i^T \left(\sum_{j=1}^k \mathbf{v}_F^j e^{-d_j t} \right)$ where $d_j = \Gamma_j + \varepsilon_0$ for suitable ε_0 to choose later. First consider, in (51), the terms containing $\mathbf{c}_t$. They are

$$e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))(1+q_0) - \sum_{i=1}^n \hat{\mathbf{c}}_t^T \mathbf{e}_i \chi^i_{q_0}(t).$$

Since we have

$$\sum_{i=1}^n \hat{\mathbf{c}}_t^T \mathbf{e}_i \chi^i_{q_0}(t) = \sum_{j=1}^k \mathbf{c}_t^T \mathbf{v}_F^j e^{-d_j t} \geq \sum_{j=1}^k \mathbf{c}_t^T \mathbf{e}_j e^{-d_j t}$$

(using Lemma 4.1 to get the last inequality) then, setting

$$g(t, \mathbf{c}) := e^{-\rho t} u_\sigma(\nu(\mathbf{c}))(1+q_0) - \sum_{i=1}^n \hat{\mathbf{c}}^T \mathbf{e}_i \chi^i_{q_0}(t).$$

we have

$$g(t, \mathbf{c}) \leq e^{-\rho t} u_\sigma(\nu(\mathbf{c}))(1+q_0) - \sum_{j=1}^k e^{-d_j t} \mathbf{c}^T \mathbf{e}_j.$$

Setting $\omega_j = e^{-d_j t} \mathbf{c}^T \mathbf{e}_j$ we have that

$$e^{-\rho t} u_\sigma(\nu(\mathbf{c}))(1+q_0) - \sum_{j=1}^k e^{-d_j t} \mathbf{c}^T \mathbf{e}_j = e^{-\rho t} u_\sigma(\nu(\omega_1 e^{d_1 t}, \dots, \omega_k e^{d_k t}))(1+q_0) - \sum_{j=1}^k \omega_j.$$

So, calling

$$g_1(t, \omega) = e^{-\rho t} u_\sigma(\nu(\omega_1 e^{d_1 t}, \dots, \omega_k e^{d_k t}))(1+q_0) - \sum_{j=1}^k \omega_j,$$

we have

$$\sup_{\mathbf{c} \geq \mathbf{0}} g(t, \mathbf{c}) \leq \sup_{\omega \geq \mathbf{0}} g_1(t, \omega)$$

Now, setting $\omega_{\max} = \max\{\omega_1, \dots, \omega_k\}$

$$\nu(\omega_1 e^{d_1 t}, \dots, \omega_k e^{d_k t}) \leq \omega_{\max} \nu(e^{d_1 t}, \dots, e^{d_k t}) = \omega_{\max} e^{\varepsilon_0 t} \nu(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}) = \omega_{\max} e^{\varepsilon_0 t} \nu(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t})$$

So, calling, for $\varepsilon > 0$

$$C_\varepsilon = \sup_{t\geq 0}\{e^{-(\Gamma_\nu+\varepsilon)t}\nu(e^{\Gamma_1 t},\dots,e^{\Gamma_k t})\} < +\infty$$

we have

$$\nu(\omega_1 e^{d_1 t},\dots,\omega_k e^{d_k t}) \leq C_\varepsilon \omega_{\max} e^{(\Gamma_\nu+\varepsilon+\varepsilon_0)t}.$$

It follows

$$g_1(t,\omega) \leq C_\varepsilon^{1-\sigma}\omega_{\max}^{1-\sigma}\frac{1+q_0}{1-\sigma}e^{-[\rho-(\Gamma_\nu+\varepsilon+\varepsilon_0)(1-\sigma)]t} - \sum_{j=1}^k \omega_j.$$

Since

$$\omega_{\max}^{1-\sigma} \leq \sum_{j=1}^k \omega_j^{1-\sigma}$$

we have

$$g_1(t,\omega) \leq \sum_{j=1}^k C_\varepsilon^{1-\sigma}\frac{1+q_0}{1-\sigma}e^{-[\rho-(\Gamma_\nu+\varepsilon+\varepsilon_0)(1-\sigma)]t}\omega_j^{1-\sigma} - \omega_j.$$

The maximum value of the left hand side can be easily calculated and is

$$\frac{\sigma}{1-\sigma}C_\varepsilon^{\frac{1}{\sigma}-1}(1+q_0)^{\frac{1}{\sigma}}e^{-[\rho-(\Gamma_\nu+\varepsilon+\varepsilon_0)(1-\sigma)]t}$$

So if we take ε and ε_0 such that $\rho > (\Gamma_\nu+\varepsilon+\varepsilon_0)(1-\sigma)$ we have integrability of $\sup_{\mathbf{c}\geq\mathbf{0}} g(t,\mathbf{c})$ and so also of $g(t,\mathbf{c}_t)$ for every admissible consumption strategy $\mathbf{c}_t$.

Now consider the terms without $\mathbf{c}_t$ in (51)

$$-\delta_{\mathbf{z}}\mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A}) = -\delta_{\mathbf{z}}\left(\mathbf{s}_t^T - \mathbf{x}_t^T\mathbf{A}\right) + \mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A}) \leq \mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A}).$$

Then

$$\sum_{i=1}^n \chi_{q_0}^i(t)\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A})\right)\mathbf{e}_i \leq \sum_{i=1}^n \chi_{q_0}^i(t)\left(\mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A})\right)\mathbf{e}_i = \sum_{j=1}^k\left(\mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A})\right)\mathbf{v}_F^j e^{-d_j t}$$

From Lemma 4.1 we have, for every $\varepsilon > 0$

$$\left(\mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A})\right)\mathbf{v}_F^j e^{-d_j t} \leq \mathbf{x}_t^T\mathbf{A}\mathbf{v}_F^j(\Gamma_j+\varepsilon)e^{-d_j t} \leq e^{(-d_j+\Gamma_j+\varepsilon)t}\overline{\mathbf{s}}^T\mathbf{v}_F^j(\Gamma_j+\varepsilon).$$

So, taking $\varepsilon < \varepsilon_0$ the first part of point 9 is true.

Now we go to (52) observing that

$$\begin{aligned}&-\int_s^{+\infty}\chi_{q_0}^i(t)\cdot\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A}) - \hat{\mathbf{c}}_t^T\right)\mathbf{e}_i dt\\ \leq &\int_s^{+\infty}\chi_{q_0}^i(t)\cdot\left(\delta_{\mathbf{z}}\mathbf{s}_t^T + \delta_{\mathbf{x}}\mathbf{x}_t^T\mathbf{A} + \hat{\mathbf{c}}_t^T\right)\mathbf{e}_i dt.\end{aligned}$$

From the estimates (23) and (25) we get, for every $j = 1, \ldots, k$,

$$\mathbf{e}_i^T \mathbf{v}_F^j \cdot \left(\delta_{\mathbf{z}} \mathbf{s}_t^T + \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A}\right) \mathbf{e}_i \leq \left(\delta_{\mathbf{z}} + \delta_{\mathbf{x}}\right) \mathbf{s}_t^T \mathbf{v}_F^j \leq \left(\delta_{\mathbf{z}} + \delta_{\mathbf{x}}\right) \overline{\mathbf{s}}^T \mathbf{v}_F^j e^{(\Gamma_j + \varepsilon)t}$$

so that

$$\int_s^{+\infty} \chi_{q_0}^i (t) \cdot \left(\delta_{\mathbf{z}} \mathbf{s}_t^T + \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A}\right) \mathbf{e}_i dt \leq \frac{e^{-(\varepsilon_0 - \varepsilon)s}}{\varepsilon_0 - \varepsilon} \left(\delta_{\mathbf{z}} + \delta_{\mathbf{x}}\right) \sum_{j=1}^k \overline{\mathbf{s}}^T \mathbf{v}_F^j,$$

which goes to zero as $t \to +\infty$ if $\varepsilon < \varepsilon_0$. Finally, arguing as above, thanks to (23)-(26) we get, for $\varepsilon < \varepsilon_0$

$$\int_s^{+\infty} \chi_{q_0}^i (t) \cdot \hat{\mathbf{c}}_t^T \mathbf{e}_i dt \leq e^{-(\varepsilon_0 - \varepsilon)s} \sum_{j=1}^k \overline{\mathbf{s}}^T \mathbf{v}_F^j$$

and this completes the proof.

Case $\sigma = 1$. The proof of (52) and of the estimate for the part not containing $\mathbf{c}$ of (52) is the same as in the case $\sigma \in (0, 1)$. Concerning the remaining estimate we set

$$g(t, \mathbf{c}) := e^{-\rho t} \log(\nu(\mathbf{c})) (1 + q_0) - \sum_{i=1}^n \hat{\mathbf{c}}^T \mathbf{e}_i \chi_{q_0}^i (t).$$

Again we observe that

$$\sup_{\mathbf{c} \geq \mathbf{0}} g(t, \mathbf{c}) \leq \sup_{\omega \geq \mathbf{0}} g_1(t, \omega)$$

where

$$g_1(t, \omega) = e^{-\rho t} \log(\nu(\omega_1 e^{d_1 t}, \ldots, \omega_k e^{d_k t})) (1 + q_0) - \sum_{j=1}^k \omega_j,$$

and, arguing as above we get

$$g_1(t, \omega) \leq e^{-\rho t} (1 + q_0) \log(C_\varepsilon \omega_{\max} e^{(\Gamma_\nu + \varepsilon + \varepsilon_0)t}) - \sum_{j=1}^k \omega_j,$$

and from now on we estimate $g_1(t, \omega)$ as above getting the claim. ∎

5 Appendix

In this Appendix we study the limit

$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu \left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right) =: L_\nu, \tag{53}$$

First of all we prove the following result.

Proposition 5.1 *Assume that the law of ν contains a finite number of sums, products, powers and min operation. Then $L \in (0, +\infty)$.*

Proof. We prove the result by induction on the number of elementary operations (sums, products, power, min). If there is only one elementary operation the claim is obvious. Assume now that the the claim is true when ν is done by n elementary operations. Consider then ν with the law done by $n+1$ elementary operations. This means, in particular, that we can write, either

$$\nu(\mathbf{c}) = f(\nu_1(\mathbf{c}), \nu_2(\mathbf{c})) \qquad \forall \mathbf{c} \in \mathbb{R}_+^k,$$

where $f(a,b) = a+b, ab, a \wedge b$ and ν_1, ν_2 are done by n elementary operations; or

$$\nu(\mathbf{c}) = [\nu_0(\mathbf{c})]^\alpha \qquad \forall \mathbf{c} \in \mathbb{R}_+^k,$$

for some $\alpha \in \mathbb{R} - \{0\}$ and ν_0 done by n elementary operations.

First case: $f(a,b) = a+b$. In this case we have

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right) = \lim_{t\to+\infty} e^{-\Gamma_\nu t}\left[\nu_1\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right) + \nu_2\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)\right]$$

By the positivity of ν_1, ν_2 we have

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu_1\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right) + \lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu_2\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)$$

and the same when we substitute Γ_ν with $\Gamma_\nu \pm \varepsilon$. So we have

$$\Gamma_\nu = \Gamma_{\nu_1} \vee \Gamma_{\nu_2}.$$

So we must have either $L_\nu = L_{\nu_1}$, or $L_\nu = L_{\nu_2}$, or $L_\nu = L_{\nu_1} + L_{\nu_2}$.

Second case: $f(a,b) = ab$. In this case we have that both ν_1, ν_2 must be homogeneous of constant degree η and $1-\eta$, respectively. Then

$$\nu_1 \cdot \nu_2 = \left[\nu_1^{1/\eta}\right]^\eta \cdot \left[\nu_2^{1/(1-\eta)}\right]^{1-\eta} =: \bar{\nu}_1^\eta \cdot \bar{\nu}_2^{1-\eta}$$

Moreover arguing as above we observe that it must be

$$\Gamma_\nu = \eta \Gamma_{\bar{\nu}_1} + (1-\eta)\Gamma_{\bar{\nu}_2}$$

so

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right) = \lim_{t\to+\infty} \left[e^{-\Gamma_{\bar{\nu}_1} t} \bar{\nu}_1\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)\right]^\eta \cdot \left[e^{-\Gamma_{\bar{\nu}_2} t} \nu_2\left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)\right]^{1-\eta}$$

and the claim follows as in the first case.

Third case: $f(a,b) = a \wedge b$. One argues as in the first case using that here

$$\Gamma_\nu = \Gamma_{\nu_1} \wedge \Gamma_{\nu_2}.$$

So we must have $L_\nu = L_{\nu_1} \wedge L_{\nu_2}$.

Fourth case: the power. One argues as in the second case. ∎

Now we provide two examples where $L = 0$ and $L = +\infty$.

Example 5.2 *Let $k = 2$ and let $\{\alpha_i\}_{i\in\mathbb{N}}$ be a sequence such that $\alpha_i \in (0,1)$ for every $i \in \mathbb{N}$ and $\lim_{i\to+\infty} \alpha_i = 0$. Assume that $0 < \Gamma_1 < \Gamma_2$ and let*

$$\nu(c_1, c_2) = \sum_{i=1}^{+\infty} c_1^{\alpha_i} c_2^{1-\alpha_i}$$

It is easy to check that $\Gamma_\nu = \Gamma_2$ and that $L = 0$.

Example 5.3 *Let $k = 2$ and let $\alpha_i = 1 - \frac{1}{i+1}$ for $i \in \mathbb{N}$. Assume that $0 < \Gamma_1 < \Gamma_2$ and let*

$$\nu(c_1, c_2) = \inf_{i\in\mathbb{N}} \left\{ i c_1^{\alpha_i} c_2^{1-\alpha_i} \right\}$$

It is easy to check that $\Gamma_\nu = \Gamma_1$ and that $L = +\infty$.

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