Hidden Markov models with t components. Increased persistence and other aspects

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Abstract

Hidden Markov models have been applied in many different fields during the last decades, including econometrics and finance. However, the lion's share of the investigated models is Markovian mixtures of Gaussian distributions. We present an extension to conditional t-distributions, including models with unequal distribution types in different states. It is shown that the extended models, on the one hand, reproduce various stylized facts of daily returns better than the common Gaussian model. On the other hand, robustness to outliers and persistence of the visited states increases significantly.

Keywords: Hidden Markov model, Markov-switching model, state persistence, t-distribution, daily returns.

JEL classification codes: C22, C51, C52, E44.

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1 Introduction

The hidden Markov model (HMM) was introduced in the late sixties (Baum & Petrie 1966, Baum et al. 1970) and since then applied in many fields, such as biology (Koski 2001, Durbin et al. 1998), environmental time series (MacDonald & Zucchini 1997), and speech recognition (Rabiner 1989). Applications related to financial econometrics followed mainly after the seminal works of Hamilton (1989, 1990) on Markov-switching models (a synonym for the HMM).
Amongst the early articles in finance is also Turner et al. (1989), who first considered a Markov mixture of normal distributions to model return series. The presumably best-known article on daily return series and the HMM is authored by Rydén et al. (1998), who showed that a Markovian mixture of normal variables reproduces most of the stylized facts for daily return series introduced by Granger & Ding (1995*a*,*b*). Other works followed (e.g., Linne 2002, Bialkowski 2003). However, as in many other applications the hidden Markov models (HMMs) considered mainly focus on mixtures of Gaussian distributions.

In this paper, we present an extension of the HMM by replacing the conditional Gaussian distribution stepwise by conditional t-distributions, which are more suitable, in particular, for states representing periods of high volatility. By means of daily returns series of the S&P 500 from 1928-2007, we show that the extended models are, on the one hand, preferred by model selection criteria and outlier location tests. Moreover, they are able to reproduce most of the stylized facts better than or comparably well as the model with Gaussian components. This includes, in particular, the slow decay of the autocorrelation function of absolute returns. On the other hand, an analysis of various international indices shows that the introduction of conditional t-distributions often increases the state persistency significantly, resulting in longer and more stable volatility periods. This has considerable effects on the estimated state sequence, which is often utilized to link certain economic patterns to particular periods. Finally, the extended models with non-zero conditional mean confirm the link between periods of high volatility and falling stock prices. In contrast to other extensions of the commonly used Gaussian HMM, e.g. duration-dependent parameters (Maheu & McCurdy 2001, Peria 2002) or semi-Markovian models (Bulla & Bulla 2006), the estimation requires only a very moderate increase in computational complexity.

The remainder of this article is organized as follows. Section 2 introduces HMMs and presents the extended models. In Section 3 we give a short description of the data. In Section 4 the results are analyzed while Section

5 concludes. Appendix A presents the full estimation results and Appendix B contains mathematical details on the estimation procedures.

2 Hidden Markov models

We provide a brief introduction to HMMs and their estimation in Section 2.1. Section 2.2 is dedicated to the specific models investigated in our analysis.

2.1 Model setup and estimation

Hidden Markov Models are a class of models for time series $\{X_0, \ldots, X_T\}$ where the probability distribution of X_t is determined by the unobserved states of a homogeneous and irreducible finite-state Markov chain S_t with $m \geq 2$ states. In many cases, the implicit assumption of models switching between different regimes is that the data result from a process that undergoes abrupt changes. These may be induced, e.g., by political or environmental events. The switching behavior is governed by a $m \times m$ *transition probability matrix* (TPM). Under the assumption of a model with two states, the TPM is of the form

$$\mathbf{\Pi} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},$$

where $p_{ij},\ i, j \in \{1, 2\}$ denote the probability of being in state j at time $t+1$ given a sojourn in state i at time t. The distribution of the observation at time t is specified by the *conditional* or *component distributions* $P(X_t = x_t \,|\, S_t = s_t)$. That is, the distribution of X_t depends on S_t only. Assuming, for instance, a two-state model with Gaussian component distributions yields

$$X_t = \mu_{s_t} + \epsilon_{s_t},\ \ \epsilon_{s_t} \sim N(0, \sigma^2_{s_t}),$$

where

$$\mu_{s_t} = \begin{cases} \mu_1 & \text{if} \quad s_t = 1 \\ \mu_2 & \text{if} \quad s_t = 2 \end{cases} \qquad \text{and} \qquad \sigma^2_{s_t} = \begin{cases} \sigma^2_1 & \text{if} \quad s_t = 1 \\ \sigma^2_2 & \text{if} \quad s_t = 2 \end{cases}.$$

HMMs and related models, often also referred to as regime switching models, have a couple of appealing properties. These models segment the data into blocks corresponding to consecutive time intervals (or regimes), whose evolution over time is modelled by an unobserved Markov chain, in addition to the evolution within regimes. That is, the different time scales in the data are separately modelled within a compact framework with a rather simple structure. The good interpretability of the results also permits the use of HMMs as exploratory tool to help guide appropriate specification of other model-based methods.

The parameters of a HMM are generally estimated using the method of maximum-likelihood. The likelihood function is available in a convenient form:

$$L(\theta) = \boldsymbol{\pi}\boldsymbol{P}(x_1)\boldsymbol{\Pi}\boldsymbol{P}(x_2)\boldsymbol{\Pi}\ldots\boldsymbol{P}(x_{T-1})\boldsymbol{\Pi}\boldsymbol{P}(x_T)\mathbf{1'}\,, \tag{1}$$

where $\boldsymbol{P}(x_t)$ represents a diagonal matrix with the state-dependent conditional densities as entries. The initial distribution of the Markov chain is denoted by $\boldsymbol{\pi}$ and the model parameters by θ. In the following we deal with stationary models, i.e., $\boldsymbol{\pi}$ is the stationary distribution associated with $\boldsymbol{\Pi}$.

The two most popular approaches to maximize the log-likelihood are direct numerical maximization using, e.g., Newton-type methods (see MacDonald & Zucchini 2009) and the Baum-Welch algorithm, a special case of what subsequently became known as the Expectation Maximization (EM) algorithm (Baum et al. 1970, Dempster et al. 1977, Rabiner 1989). The EM algorithm consists of two steps, *E-* and *M-step*. The *E-step* requires the computation of the so-called Q-function, which calculates the conditional expectation of the complete-data log-likelihood given the observations and $\theta^{(k)}$, the current estimate of the parameter vector θ.

$$Q(\theta,\theta^{(k)}) = \mathbf{E}\left[\log P\left(X_1^T = x_1^T, S_1^T = s_1^T \,|\, \theta\right) \,|\, X_1^T = x_1^T, \theta^{(k)}\right],$$

where $X_1^T := \{X_1, \cdots, X_T\}$ and S_1^T is defined analogously. The *M-step* maximizes $Q(\theta,\theta^{(k)})$ w.r.t. θ to determine the next set of parameters $\theta^{(k+1)}$:

$$\theta^{(k+1)} = \arg\max_{\theta}\, Q(\theta,\theta^{(k)}).$$

After assigning initial values to the parameters, these steps are successively iterated until convergence is achieved. For further details on the EM algorithm, in particular the M-step for the stationary HMMs used in the following, we refer to Appendix B.

The estimation procedures we used base on a hybrid algorithm. This approach combines the EM algorithm with a rapid algorithm with strong local convergence as follows: the estimation procedure starts with the EM algorithm and switches to a Newton-type algorithm when a certain stopping criterion is fulfilled (Redner & Walker 1984, Bulla & Berzel 2008). The resulting algorithm exhibits a large circle of convergence from the EM algorithm along with superlinear convergence of the Newton-type algorithm in the neighbourhood of the maximum.

It may be noted that the hybrid algorithm shows a high robustness towards poor initial values. We explored the effect of different initial values by grid searches and discovered stable convergence to the global maximum for most models and data series, although the stability got weaker for models with four and more states. To reduce the computational effort, we allowed only values lower than 40 for the degrees of freedom of the t-distribution. On the one hand, this restriction prevents the algorithms from diverging to infinity, and thus carrying out large numbers of needless iterations. On the other hand, the value 40 is high enough to conclude that the t-distribution entails no significant advantage w.r.t. a Gaussian component.

2.2 Non-Gaussian conditional distributions

The class of Markov-switching models was introduced to financial econometrics by Hamilton (1989, 1990). Since that time, many applications followed. One field treated by several authors is the modelling of return series in Markov-switching frameworks. Turner et al. (1989) were the first considering Markov-switching mixtures of Gaussian distributions, and other studies followed, e.g., Cecchetti et al. (1990), Rydén et al. (1998), Linne (2002), Bialkowski (2003). However, the initially proposed model basically remained unchanged, and almost all researchers focus on Gaussian conditional distributions.

We propose an alternative approach, which extends the Gaussian model and can be implemented with moderate effort. In view of the application to return series, which are often heavy-tailed and leptokurtic (see, e.g., Gettinby et al. 2004, Harris & Küçüközmen 2001), a possible candidate for an extension of the Gaussian is the t-distribution. For this distribution, the M-step requires some attention, because a closed form solution is not available for all parameters. However, the estimation procedure is still well-feasible compared to other parametric alternatives. The data investigated are daily returns from the S&P 500, which already formed the basis for the analysis of Rydén et al. (1998) (subsequently abbreviated by RY) and are presented in detail in Section 3. The model of RY, a Markovian mixture of Gaussian variables with zero means, is denoted by M_{RY} in the following. We investigate two extensions: On the one hand, the conditional means may take any value, allowing for skewed marginal distributions. The model with Gaussian distributions and variable means is denoted by M_N. On the other hand, we introduce conditional t-distributions. The model denoted by M_{Nt} is characterized by a $m-1$ Gaussian distributions and one t-distribution in m^{th} state, i.e.

$$X_t = \mu_{s_t} + \epsilon_{s_t},\ \epsilon_{s_t} \sim \begin{cases} N(0, \sigma_i^2) & \text{for} \quad S_t \in \{1, \ldots, m-1\} \\ t(0, \sigma_m^2, \nu) & \text{for} \quad S_t = m \end{cases} .$$

The choice of only one t-distribution is motivated by the application to daily returns: the m^{th} state is supposed to represent that regime characterized by highest volatility and extreme observations. The last model is M_t and has m conditional t-distributions. In view of Robert & Titterington (1998) we require $\sigma_i < \sigma_{i+1} \, \forall \, i = 1, \ldots, m-1$ for all models considered to ensure their identifiability. Without this condition, changing of state labels would yield equivalent models and thus violate the well-definedeness of all models. Moreover, note that the class of finite mixtures of Gaussian/t-distributions, varying in mean and variance, is identifiable. For further details we refer to the HMM-books of Cappé et al. (2007), Section 12.4.3 and Titterington et al. (1985), Section 3.1, as well as the work of Peel & McLachlan (2000) on mixtures of multivariate t-distributions.

3 The data

The main data analyzed in this paper are the daily returns calculated for the S&P500 index, covering the period from January 3$^{\text{rd}}$, 1928 to August 13$^{\text{th}}$, 2007. We segmented this long time series into periods of the length of eight calendar years, starting with 1928-1935 and ending with 2000-2007, which allows analyzing the performance of different models in many different time periods. The segmentation yields ten periods, each of which contains roughly 2000 daily returns (with the exception of the slightly shorter last period). The chosen length is not too different to the settings of Rydén et al. (1998), which serves as reference for this work (these authors utilized sub-series of length 1700).

The returns are calculated by $R_t = \ln(P_t) - \ln(P_{t-1})$, where P_t represents the index closing price on day t and ln is the natural logarithm. The periods 1928-1935 and 1984-1991 contain three very extreme observations, 'Black Tuesday' on October 29$^{\text{th}}$ 1928, the first trading day after Roosevelt started the 'New Deal' legislation on March 15$^{\text{th}}$, 1933, and the 'Black Monday' on October 19$^{\text{th}}$ 1987. On these days the S&P500 changed by -17.5%, 15.4%, and -22.8%, respectively. To prevent these unique events from imposing a significant bias to our analyses, we replaced the value by plus/minus six times the standard deviation of the respective period.

Table 1 provides descriptive statistics for the data.

Table 1: Descriptive statistics of daily returns

This table summarizes the daily returns data of the S&P500 index, covering the period from 3 January 1928 to 13 August 2007.

No.	Period	N	Mean$\cdot 10^4$	S.D.$\cdot 10^2$	Skew.	Kurt.	JB
1	1928-1935	1992	-2.57	2.1	-0.1	7.71	1852
2	1936-1943	2006	-0.7	1.33	-0.167	10.6	4811
3	1944-1951	1996	3.56	0.886	-0.782	11.3	5987
4	1952-1959	2013	4.59	0.697	-0.519	10.2	4505
5	1960-1967	2013	2.37	0.649	-0.567	14.2	10577
6	1968-1975	1992	-0.338	0.908	0.315	5.5	553
7	1976-1983	2022	2.99	0.849	0.183	4.6	228
8	1984-1991	2022	5.38	1.01	-0.536	12.7	8025
9	1992-1999	2022	6.23	0.872	-0.434	9.63	3781
10	2000-2007	1759	-0.201	1.13	0.104	5.61	507

All indices are leptokurtic. The Jarque-Bera statistic confirms the departure from normality for all return series at the 1% level of significance. In the following, we refer to the different periods by the numbers 1-10, indicated in the first column.

4 Results

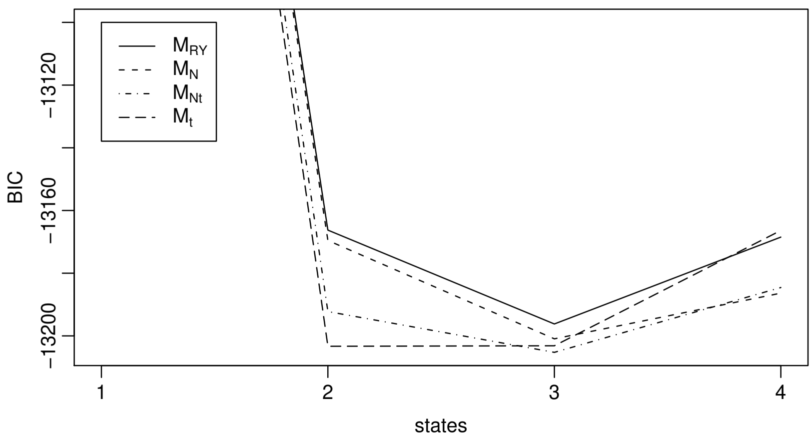
The results are presented in five parts. The first part, Section 4.1 addresses model selection aspects, i.e., choice of the number of states and conditional distributions of a model. Section 4.2 summarizes some basic estimation results, and in Section 4.3 we present an analysis of the stylized facts established by Granger & Ding (1995*a*). The content of the last part, Section 4.4, is mainly dealing with an analysis of the state-persistence of the different models.

4.1 Model selection

For comparing HMMs with an identical number of states and nested conditional distributions or testing parameter constraints, the likelihood ratio statistic is a useful tool. However, this statistic cannot be applied anymore for models with different numbers of states, as these are not hierarchically nested anymore (Visser et al. 2002). Alternatively, model selection criteria such as AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), and modifications of both may be used (MacDonald & Zucchini 1997). Simulation studies showed that AIC tends to select models more complex than the true model (Visser et al. 2002), which is why we chose the BIC as main model selection tool. The following Figure 1 provides a first impression as to which models are generally preferred. It shows the mean BIC over all sub-series for the four models with two to four states (lines going up on the left side result from a single Gaussian distribution). On average, M_{Nt} with three and M_t with two or three states show the lowest BIC values. The BIC of M_{RY} is permanently high, however, the values for M_N are relatively low, in particular for the models with more than 2 states.

Figure 1: BIC for 2- to 4-state-models

This figure shows the average BIC calculated from the 10 sub-series for the 4 models considered.



To compare the models in compact form, we denote the number of states by a superscript in what follows. Table 2 displays those three models which attain the lowest BIC together with the value of the criterion. The purely Gaussian models are selected only three times, M_{RY} for series 7 and 10, and M_N for series 6. In the remaining cases, a model with t component(s) performs best. Moreover, 20 of the 30 models displayed are not purely Gaussian, which also indicates that M_{Nt} and M_t should be considered for further analysis. For the complete BIC values for all models and states, we refer to Table 12 in the appendix.

Table 2: BIC of the best fitted models

The three models with the lowest BIC, evaluated for each of the ten sub-series. The superscript indicates the number of states.

No.	1		2		3		4		5	
1	M^3_{Nt}	-10563	M^2_t	-12313	M^2_t	-13686	M^2_{Nt}	-14637	M^4_{Nt}	-15287
2	M^3_N	-10556	M^3_t	-12303	M^3_t	-13674	M^2_t	-14632	M^4_N	-15283
3	M^3_t	-10550	M^3_{RY}	-12296	M^3_N	-13672	M^3_{Nt}	-14631	M^3_{Nt}	-15282
No.	6		7		8		9		10	
1	M^4_N	-13557	M^2_{RY}	-13692	M^2_t	-13312	M^2_t	-13957	M^3_{RY}	-11248
2	M^4_{Nt}	-13550	M^3_{RY}	-13691	M^3_t	-13284	M^3_t	-13936	M^3_N	-11232
3	M^3_{RY}	-13548	M^2_{Nt}	-13684	M^3_{Nt}	-13279	M^3_{Nt}	-13929	M^3_{Nt}	-11227

For the remainder of the paper, M_{RY} serves as reference. To select the correct number of states, we carry out two steps. At first, we chose those models which perform best according to the BIC criterion. That is, for every series we determine the preferred M^i_{RY}, $i \in \{2, 3, 4\}$, by the lowest criterion value and similarly conduct the selection of one of the models M^i_{Nt} and M^i_t. Secondly, we check the stability of the estimated parameters by means of their standard error. As the Hessian does not provide numerically stable results for long time series, the standard errors are computed by a parametric bootstrap approach (Visser et al. 2000). We can partially confirm the observation of RY that the three-state models 'are less similar to each other' and that 'the estimation results seem heavily dependent on outlying observations'. In our setting, the preferred 3-state models for the series 3, 8, and 9 showed a high parameter instability in the TPM (low persistence of at least one diagonal element and standard errors of up to 0.10-0.24) and thus in these cases a 2-state model is selected. For the models with t-components, the degrees of freedom often display very high standard errors. In particular, for all models with three and more states and for some M^2_t, the 95%-confidence band of the parameter ν ranged up to or close to the value 40, and thus the more stable model M^2_{Nt} is preferred. Therefore, all models selected with t-components possess two states.

Table 3: Parameter estimates for 10 sub-series of the S&P500

Parameter estimates for the preferred models with and without t components, selected according to the BIC and parameter stability. Note that the standard deviation in the states with t-distributions requires an adjustment by the factor $\sqrt{\nu/(\nu-2)}$ for direct comparison with Gaussian states.

	$M_{RY}^{2/3}$		$M_{Nt/t}^{2}$			
no.	P	$\sigma \cdot 10^3$	P	$\sigma \cdot 10^3$	$\mu \cdot 10^4$	ν
1	0.973 0.027 0.000	8.89	0.993 0.007	10.10	1.11	8.21
	(0.010) (0.010) (0.000)	(0.340)	(0.003) (0.003)	(0.352)	(3.32)	(3.14)
	0.025 0.958 0.016	16.32	0.011 0.989	24.19	-2.44	5.55
	(0.010) (0.012) (0.006)	(0.625)	(0.005) (0.005)	(1.152)	(10.5)	(2.87)
	0.000 0.026 0.974	34.94				
	(0.000) (0.010) (0.010)	(1.240)				
2	0.985 0.012 0.003	7.46	0.990 0.010	6.97	4.69	5.59
	(0.006) (0.006) (0.003)	(0.210)	(0.003) (0.003)	(0.234)	(2.22)	(1.18)
	0.026 0.952 0.022	12.83	0.026 0.974	17.03	-14.84	5.84
	(0.011) (0.015) (0.010)	(0.636)	(0.009) (0.009)	(1.061)	(8.65)	(2.84)
	0.000 0.074 0.926	27.59				
	(0.000) (0.010) (0.010)	(1.240)				
3	0.966 0.033	6.56	0.947 0.053	5.18	12.26	-
	(0.007) (0.007)	(0.137)	(0.014) (0.014)	(0.203)	(2.34)	
	0.288 0.712	19.68	0.046 0.954	7.64	1.87	3.70
	(0.053) (0.053)	(1.183)	(0.018) (0.018)	(0.453)	(3.55)	(0.49)
4	0.960 0.040	5.20	0.956 0.044	4.58	14.03	-
	(0.010) (0.010)	(0.124)	(0.010) (0.010)	(0.128)	(1.66)	
	0.202 0.798	12.57	0.074 0.926	7.44	-12.22	5.27
	(0.044) (0.044)	(0.721)	(0.019) (0.019)	(0.457)	(4.01)	(1.40)
5	0.970 0.030 0.000	3.32	0.968 0.032	3.65	9.46	-
	(0.009) (0.009) (0.000)	(0.114)	(0.007) (0.007)	(0.094)	(1.13)	
	0.027 0.957 0.016	6.11	0.055 0.945	7.28	-10.62	5.17
	(0.009) (0.011) (0.006)	(0.208)	(0.014) (0.014)	(0.406)	(3.56)	(1.56)
	0.000 0.093 0.907	15.28				
	(0.000) (0.043) (0.043)	(1.197)				
6	0.988 0.012 0.000	5.17	0.991 0.009	5.92	1.33	-
	(0.006) (0.006) (0.001)	(0.165)	(0.003) (0.003)	(0.129)	(1.74)	
	0.012 0.980 0.007	8.94	0.016 0.984	11.97	-5.49	14.8
	(0.006) (0.008) (0.005)	(0.307)	(0.007) (0.007)	(0.547)	(5.02)	(10.29)
	0.003 0.022 0.976	16.37				
	(0.008) (0.052) (0.052)	(1.112)				
7	0.992 0.008	6.79	0.994 0.006	6.47	0.93	-
	(0.004) (0.004)	(0.165)	(0.004) (0.004)	(0.198)	(2.31)	
	0.017 0.983	11.37	0.007 0.993	9.30	4.67	9.93
	(0.009) (0.009)	(0.408)	(0.006) (0.006)	(0.430)	(3.77)	(5.50)
8	0.988 0.012	7.85	0.985 0.015	7.55	7.94	-
	(0.003) (0.003)	(0.146)	(0.005) (0.005)	(0.166)	(2.02)	
	0.111 0.889	21.90	0.068 0.932	12.58	0.66	3.86
	(0.032) (0.032)	(1.386)	(0.026) (0.026)	(1.354)	(10.02)	(2.92)
9	0.993 0.007	5.70	0.997 0.003	4.71	5.71	5.73
	(0.003) (0.003)	(0.127)	(0.002) (0.002)	(0.177)	(1.68)	(1.35)
	0.012 0.988	12.23	0.006 0.994	9.65	9.14	6.07
	(0.127) (0.364)	(0.364)	(0.005) (0.005)	(0.487)	(4.34)	(3.13)
10	0.993 0.007 0.000	6.12	0.998 0.002	5.98	6.17	-
	(0.005) (0.005) (0.000)	(0.212)	(0.003) (0.003)	(0.187)	(2.33)	
	0.009 0.979 0.011	9.77	0.003 0.997	9.70	-1.49	8.46
	(0.007) (0.010) (0.007)	(0.438)	(0.002) (0.002)	(0.536)	(5.06)	(4.61)
	0.000 0.013 0.987	17.45				
	(0.000) (0.010) (0.010)	(1.240)				

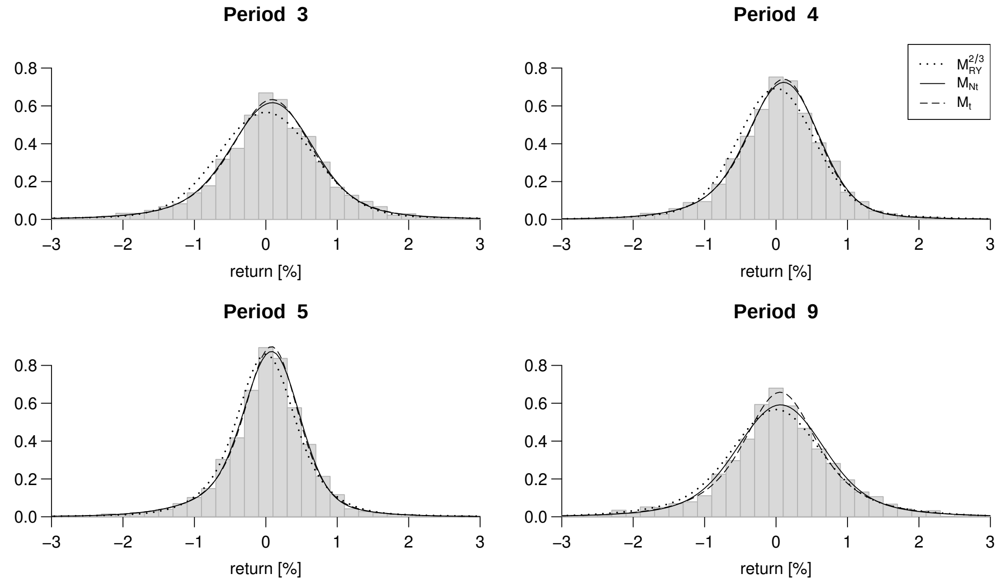
Table 3 shows the parameter estimates for the selected models and the standard errors of the parameters. We observe the following: First, comparing the TPM of models with and without t-component, the matrix of the latter models has more persistent states and lower standard errors with only very few exceptions (note in particular Series 4). Second, the more volatile state of M_{nt}/M_t has a lower conditional mean for eight of the ten series, and all mean estimates are subject to high standard errors. We further investigate this aspect in 4.4. Third, the degrees of freedom estimates take a high value and additionally are subject to high standard deviation in Periods 6 and 7. This may indicate that in these periods an extension towards t-components may not entail in a significantly better fitted model.
Additionally to the BIC and stability checks, we apply a likelihood ratio test (LRT) to further investigate the differences between hierarchically nested models. As only the 2-state models provide stable results for most of the series, we carry out sequential tests (always at 1%-level) for these models, starting with M^2_{RY} and M^2_N. For this comparison, M_N turns out to be the a significantly better model for six of the ten periods. Proceeding with the comparison of M^2_N and M^2_{Nt}, the latter is preferred in nine periods. As the parameter estimates already indicated, the simpler model is preferred in Period 6. The last test between M^2_{Nt} and M^2_t shows that the more complex model is selected in five cases.

Summarizing this section, the extension from the common HMM utilizing Gaussian components with identical means to models with varying means and at least one conditional t-distribution seems to be reasonable. Often these models are more parsimonious than a Gaussian 3-state alternative, provide more stable parameter estimates, and are preferred by both the BIC and the LRT.

4.2 Basic statistics

To continue the analysis, we present some basic statistics on the returns and model selection criteria. For most series, the degrees of freedom ν of the t-distributions take low values, indicating a departure from normality in the components. As to the fit of the models to the empirical distribution of the returns, Table 4 summarizes empirical and model skewness, and kurtosis. All models experience minor problems in reproducing a positive skewness, while negatively skewed series are mostly reproduced well. Moreover, it should be noted that M_{RY} is, by construction of the model, not able to reproduce any skewness, as the marginal distribution is symmetric. We omit detailed results on the mean and the standard deviation, because the empirical mean and

Figure 2: Empirical distribution and model densities for selected periods
Empirical distributions with model densities, Periods 3, 4, 5, and 9. In Period 5, M_{RY} is a 3-state model. We do not display M_N, because there is only a very small visual difference to M_{RY}.



the mean of the fitted models both lie very close to zero for all series, and the standard deviation almost coincides with the empirical value for each model. Table 4 displays skewness and kurtosis for the models considered. Application of Friedman's rank sum test to test for the equality of mean, standard deviation, skewness and kurtosis location of data and models rejects the hypothesis for skewness and kurtosis. However, further investigation of the skewness of the sample data and each model by paired two sample Wilcoxon tests does not reveal any significant differences. As to the kurtosis, the Wilcoxon test rejects the equality hypothesis for the data and M_{RY} as well as M_N at 1% and 5% level, respectively. This confirms the first impression from the Figure 2 that the Gaussian models do not seem to reproduce the kurtosis.

Additionally, regarding Series 1, 2, 5, 6, and 10, the Gaussian models with three states do not seem to reproduce the Kurtosis better than M_{RY}^2/M_N^2 (the sample size is too small to perform formal tests).

Table 4: Skewness and kurtosis for the data and fitted models

Skewness and kurtosis of the returns and the four fitted models M_{RY} (skewness omitted, equals zero), M_{NN}, M_{Nt}, and M_{tt} (by Monte Carlo approximation).

	Skewness				Kurtosis				
No.	Data	M_N^2	M_{Nt}^2	M_t^2	Data	$M_{RY}^{2/3}$	M_{NN}^2	M_{Nt}^2	M_{tt}^2
1	0.10	-0.16	-0.20	-0.22	7.7	6.4	5.6	11.3	11.5
2	-0.17	-0.21	-0.18	-0.18	10.6	7.8	6.2	13.0	11.5
3	-0.78	-0.71	-0.10	-0.43	11.3	8.3	8.0	36.2	15.3
4	-0.52	-0.41	-0.38	-0.37	10.2	6.0	5.4	10.2	10.4
5	-0.57	-0.45	-0.39	-0.40	14.2	8.8	6.4	13.0	13.4
6	0.31	-0.05	-0.08	-0.10	5.5	5.7	4.9	5.6	5.7
7	0.18	-0.02	0.03	0.03	4.6	3.9	3.9	4.7	4.7
8	-0.54	-0.25	-0.10	-0.12	12.7	7.4	7.5	30.2	21.1
9	-0.43	-0.01	0.02	0.04	9.6	4.7	4.8	8.3	8.7
10	0.10	-0.08	-0.09	-0.09	5.6	5.2	4.5	6.2	6.2

The paper of Breunig et al. (2003) presents tests on mean, variance and peak location. The basic idea of these encompassing tests is to check the hypothesis that a parameter $\hat{\gamma}$ that has been estimated from the data is reproduced by the model. The null hypothesis is that the model is correct, and the test statistic is

$$R = (\hat{\gamma} - \gamma_M(\hat{\theta}))^t [\text{var}(\hat{\gamma})]^{-1} (\hat{\gamma} - \gamma_M(\hat{\theta})),$$

following a $\chi^2_{dim(\hat{\gamma})}$ distribution. The quantity $\hat{\theta}$ is the MLE estimate of the model parameters, and $\gamma_M(\hat{\theta})$ the model quantity corresponding to $\hat{\gamma}$. According to the proposals of Breunig et al. (2003), $\gamma_M(\hat{\theta})$ and $\text{var}(\hat{\gamma})$ are estimated by simulation techniques. An analysis of mean and variance does not reveal any differences between the models, all perform comparably well. However, the authors also propose a test statistic

$$\hat{\phi} = T^{-1} \sum_{t=1}^{T} \mathbf{1}_{(-k,k)}(x_t),$$

which measures the proportion of observations lying between -k and and k. In their paper, they chose $k = 2\%$ to cover roughly 50% of the observations. For our data, values of 2%, 1% and 0.5% did not show any difference between the models. However, we modified the statistic $\hat{\phi}$ to measure the fraction of extreme values lying outside the interval $(-k, k)$. For k we selected the value of 4%, because some of the series do not contain any observations for bigger integer values.

Table 5: Measures for outlier fraction in data and models

Results for quantity $\hat{\phi}^* = T^{-1} \sum_{t=1}^{T} \mathbf{1}_{[4,\infty)}(|x_t|)$ (in %) and corresponding values of R. The critical value for R at 5% and 1% level is 3.84 and 6.63, respectively.

	Data	$M_{RY}^{2/3}$	R	M_N^2	R	M_{Nt}^2	R	M_t^2	R
1	6.07	6.66	5.43	7.03	13.91	6.02	0.06	6.15	0.11
2	1.69	1.80	0.61	1.80	0.57	1.67	0.05	1.71	0.01
3	0.40	0.43	0.24	0.39	0.02	0.42	0.10	0.46	0.89
4	0.10	0.02	23.46	0.01	126.55	0.10	0.00	0.10	0.00
5	0.10	0.07	0.99	0.01	131.48	0.09	0.07	0.10	0.00
6	0.20	0.19	0.08	0.08	20.44	0.16	1.11	0.16	0.97
7	0.10	0.01	53.82	0.02	41.89	0.07	1.01	0.07	1.07
8	0.54	0.67	2.42	0.66	2.19	0.62	0.92	0.59	0.33
9	0.25	0.04	117.10	0.04	104.97	0.23	0.20	0.23	0.15
10	0.57	0.61	0.26	0.39	7.79	0.66	1.39	0.67	1.43

The models M_{RY} and M_N are rejected 4 and 7 times, respectively, at 5% level. The models with t-distributions are not rejected at all, indicating that they are more capable to reproduce daily return series with extreme observations.

Summarizing, M_{Nt} and M_t allow for skewed distributions, and reproduce the kurtosis as well as extreme observations better than their competitors with Gaussian components. In this connection, note that three-state Gaussian models are also affected by the weaker performance.

4.3 Stylized facts

In their article on stylized facts of daily return series and the HMM, Rydén et al. (1998) analyze their model's ability to reproduce four temporal and three distributional properties of daily returns. Their main result is that M_{RY} reproduces most of the properties quite well, with exception of the very slow decay of the autocorrelation function of absolute or squared returns.
In this section, we check these properties for M_N, M_{Nt}, and M_t. The stylized facts, established by Granger & Ding (1995a,b) and further analyzed by Granger et al. (2000) are

TP1: Returns r_t are not autocorrelated (except for, possibly, at lag one)
TP2: $|r_t|$ and r_t^2 are 'long-memory', i.e., their autocorrelation functions decay slowly starting from the first autocorrelation, and $\text{corr}(|r_t|, |r_{t-k}|) > \text{corr}(r_t^2, r_{t-k}^2)$. The autocorrelations remain positive for many lags and the decay is much slower than the exponential rate of a typical stationary ARMA model.
TP3: The Taylor effect $\text{corr}(|r_t|, |r_{t-k}|) > \text{corr}(|r_t|^\theta, |r_{t-k}|^\theta)$, $\theta \neq 1$ (Taylor 1986). Autocorrelations of powers of absolute returns are highest at power one.
TP4: The autocorrelations of $\text{sign}(r_t)$ are negligibly small.

The three distributional properties are:
DP1: $|r_t|$ and $\text{sign}(r_t)$ are independent.
DP2: Mean $|r_t|$ = standard deviation $|r_t|$.
DP3: The marginal distribution of $|r_t|$ is exponential (after outlier correction).

Note that an exponentially distributed variable (DP3) x_t has the following properties.
PED1: $E(x_t) = Var(x_t)$ (same as DP2).
PED2: $E\left(x_t - E(x_t)\right)^3 = 2$.
PED3: $E\left(x_t - E(x_t)\right)^4 = 9$.

In their analysis, RY showed that M_{RY} satisfies TP1, and that TP4 is not violated in practice. Moreover, DP1 holds by construction of the model. Although M_N, M_{Nt}, and M_t have means unequal to zero, all conditional means take values very close to zero. As expected, a preliminary analysis showed that none of the estimated models violates TP1, TP4 or DP1.
We firstly analyze PED1-PED3: Table 6 presents the mean-standard deviation ratio, skewness, and kurtosis of the absolute returns and the fitted models (we omit M_N, as the results are almost similar to M_{RY}). The ratio of mean and standard deviation (PED1/DP2) is close to one for all series and all fitted models, however, sometimes slightly overestimated by the models with two Gaussian components. This is in line with the analysis of RY, who noted that PED1 'has to be relaxed somewhat (the mean has to be allowed to be slightly larger than the standard deviation) if we at the same time want PED2 and PED3 to be satisfied'. For the original data, M_{RY} and M_N underestimate skewness and kurtosis in all periods. The M_{Nt} and M_t reproduce

these stylized facts quite well with a slightly better performance of the latter one. Skewness and kurtosis are reproduced considerably well by all models. For some series, M_{RY} and M_N slightly underestimate these moments, while M_{Nt} and M_t sometimes overestimate them.
To summarize the above findings, M_{Nt} and M_t reproduce PED1-PED3 as well as or better than M_{RY} and M_N for the original data.

Table 6: Statistics of the absolute returns and the estimated models
Mean-standard deviation ratio, skewness and kurtosis of the absolute returns estimated from the ten data series and from the fitted models M_{RY}, M_{Nt}, and M_t (by Monte Carlo approximation)

	Mean/standard deviation				Skewness				Kurtosis			
No.	Data	$M_{RY}^{2/3}$	M_{Nt}^2	M_t^2	Data	$M_{RY}^{2/3}$	M_{Nt}^2	M_t^2	Data	$M_{RY}^{2/3}$	M_{Nt}^2	M_t^2
1	0.95	0.97	0.95	0.92	2.50	2.16	3.05	3.02	12.1	9.2	22.3	22.1
2	0.92	0.98	0.95	0.93	2.99	2.57	3.30	3.05	19.4	13.1	27.6	22.0
3	0.95	1.02	0.94	0.94	3.21	2.80	4.78	3.66	21.9	15.7	82.5	33.3
4	1.03	1.08	1.05	1.04	2.94	2.17	2.87	2.88	23.2	10.4	23.5	23.9
5	0.94	0.97	0.96	0.95	3.62	2.81	3.25	3.30	30.2	15.7	27.7	28.5
6	1.06	1.07	1.08	1.06	1.94	2.04	2.01	2.01	8.7	9.4	9.2	9.1
7	1.17	1.22	1.18	1.17	1.69	1.42	1.72	1.72	7.9	5.8	8.2	8.1
8	0.96	1.06	0.99	0.97	3.46	2.59	4.83	3.96	26.6	14.3	79.0	50.5
9	0.99	1.11	1.04	1.00	2.81	1.72	2.60	2.60	19.2	6.9	17.1	16.9
10	1.05	1.07	1.06	1.05	1.97	1.86	2.14	2.13	8.6	7.6	10.9	10.8

The two remaining stylized facts are TP3 and TP2. For TP3, the Taylor effect, we estimate the coefficient θ for every period by maximizing the first-order autocorrelation of $|r_t|^\theta$ utilizing numerical optimization routines. Following the approach of RY, the value of θ maximizing the first-order autocorrelation for the models was estimated over the range $\{0.1, 0.2, \ldots, 2.0\}$ by Monte-Carlo approximation. Table 7 summarizes the results, and again the results for M_N are not displayed as they are similar to those of M_{RY}. On the one hand, maximizing values of θ for the data series are significantly different to one, which is also the case for the Gaussian models (t-test, $\alpha = 0.05$). On the other hand, the values for models with conditional t-distributions do not significantly differ from one.

Table 7: Taylor coefficient of the returns and the estimated models
Values of θ maximizing the first-order autocorrelation of $|r_t|^\theta$ estimated from the ten return series and the fitted models.

No.	Data	M_{RY}	M_{Nt}	M_t
1	1.46	0.9	0.8	0.7
2	0.77	1.1	0.8	0.8
3	1.14	1.4	0.7	1.2
4	1.14	1.4	1.0	0.9
5	1.72	1.2	0.8	0.8
6	1.27	1.2	1.0	1.0
7	1.70	1.5	1.2	1.2
8	1.84	1.5	1.0	1.0
9	2.10	1.1	0.9	0.8
10	1.41	1.0	0.8	0.8

According to RY, the slow decay of the ACF for series of absolute daily returns, which is stylized fact TP2, cannot be reproduced by the HMM because the decay of the autocorrelations is (much) faster than that observed in reality. They considered this stylized fact to be 'the most difficult [...] to reproduce with a HMM'. Figures 3 and 4 show the empirical ACF and the ACF of the the fitted models (we do not display M_N, because it is visually indistinguishable from M_{RY}). The left and right panels display models with 2 and 3 states, respectively. The solid line represents the ACF of M_{RY}, the dashed corresponds to M_{Nt} and the dotted lines to M_t.
In most cases, the ACF of M_{RY} shows a much stronger decay of the autocorrelations than the decay of the empirical ACF, which confirms the results of RY. The models with t-components reproduce this stylized fact much better, although their fit show slight deficiencies for lower lags in most of the periods. However, it also seems that models with three states provide a better fit than models with two states. To verify these visual impressions, we measure the fit of the ACF by the mean squared error (MSE). Table 8 displays the results. The average MSE over all periods is denoted by $\overline{MSE}$; $\overline{MSE_l}$ and $\overline{MSE_h}$ represent the MSE for the 'lower' lags 1-20 and 'higher' lags 21-100, respectively. The MSE confirm the visual impression that models with three and at least one conditional t-distribution provide the best fit, especially for the lags of higher order. If a model with two states is preferred, M_t would be the first choice.

Figure 3: Empirical and model ACF of absolute returns for Series 1-5, lag 1-100

The panels show the empirical ACF of absolute returns (grey bars) and the model ACF (straight line for M_{RY}, dashed line for M_{Nt}, and dotted line for M_t). Models with two and three states are displayed in the left and right panels, respectively. We omit M_N, because there is almost no visual difference to M_{RY}.

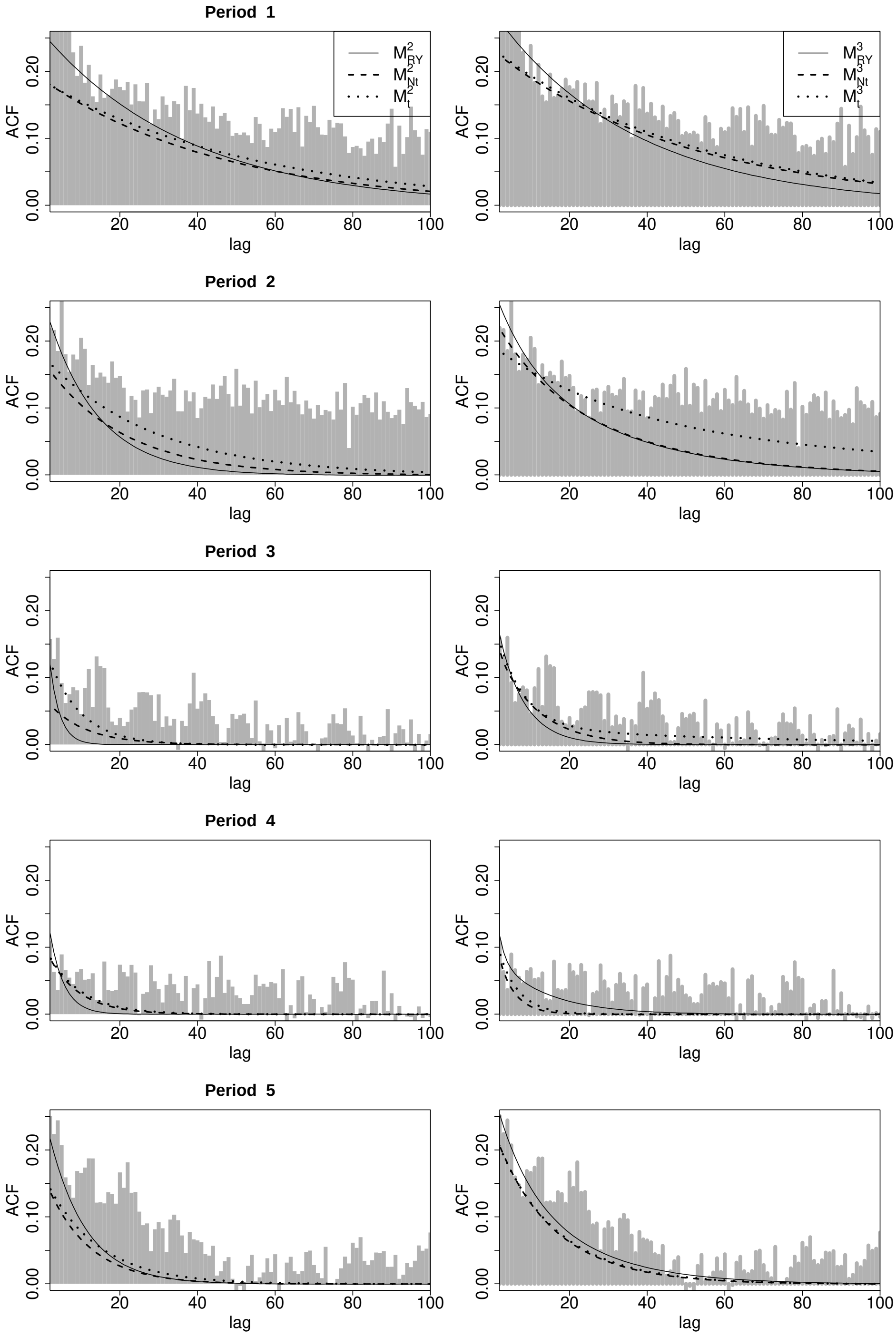


Figure 4: Empirical and model ACF of absolute returns for Series 6-10, lag 1-100

The panels show the empirical ACF of absolute returns (grey bars) and the model ACF (straight line for M_{RY}, dashed line for M_{Nt}, and dotted line for M_t). Models with two and three states are displayed in the left and right panels, respectively. We omit M_N, because there is almost no visual difference to M_{RY}.

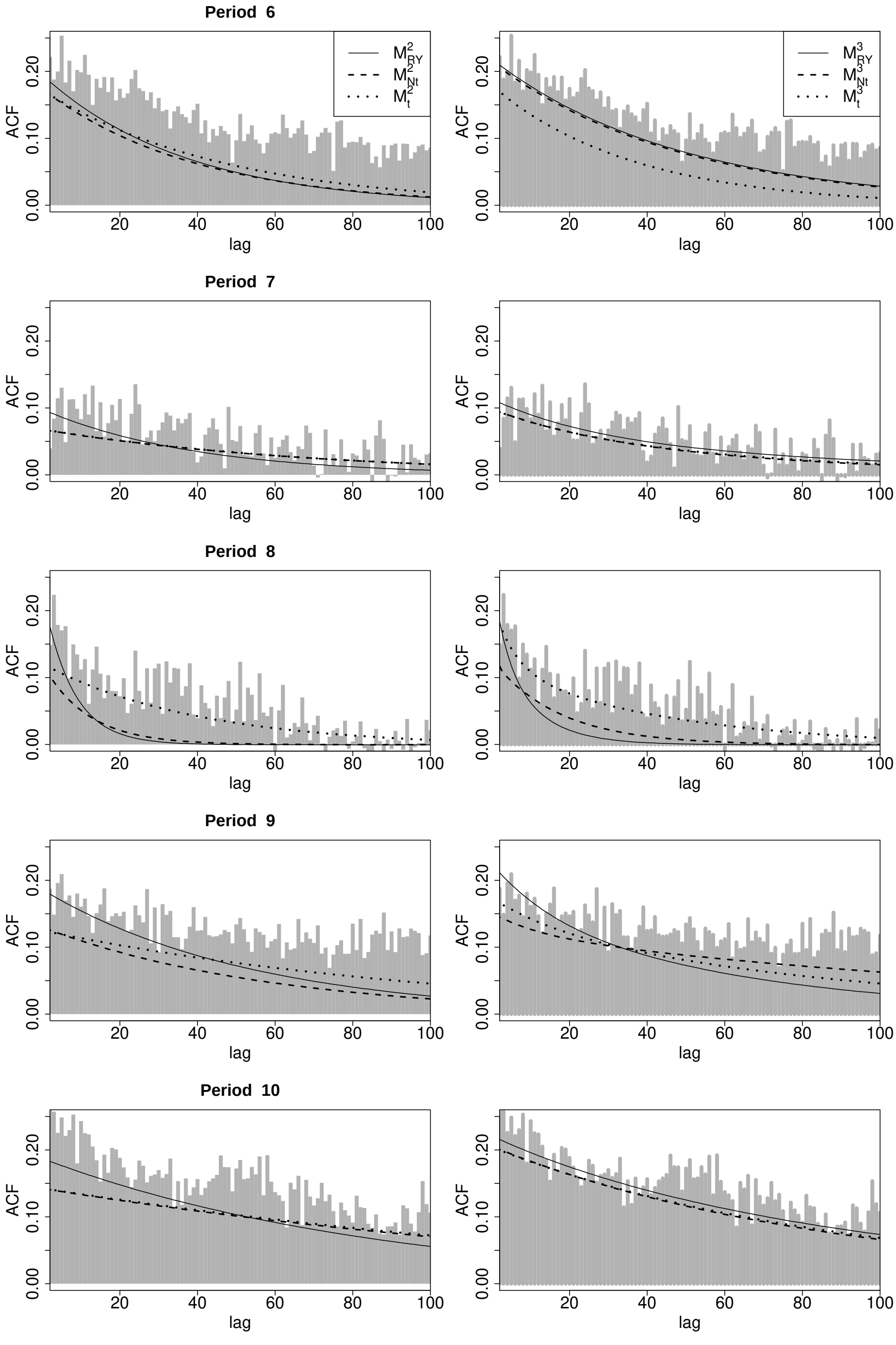


Table 8: Average mean squared error of the ACF for absolute returns
Mean squared error of the empirical and the model ACF, averaged over the ten periods of the S&P500. The error for the lags 1-100 is denoted by MSE, while MSE_l and MSE_h represent the mean squared error for the lags 1-20 and 21-100, respectively. All errors are scaled by 10^4.

	2 states			3 states		
Criterion	$\overline{MSE}$	$\overline{MSE_l}$	$\overline{MSE_h}$	$\overline{MSE}$	$\overline{MSE_l}$	$\overline{MSE_h}$
M_{RY}	3.48	2.93	3.62	2.3	1.34	2.54
M_{Nt}	3.96	5.21	3.64	2.19	1.95	2.25
M_t	2.97	4.03	2.70	1.95	1.80	1.99

Summarizing, M_{Nt} and M_t reproduce most of the temporal and distributional properties as well as or better than M_{RY} and M_N. In particular, the models with conditional t-distributions are able to reproduce the slow decay of the ACF of absolute returns much better than the models with two Gaussian distributions.

A final remark on the impact of outliers: According to Chan (1995) extreme outliers could jeopardize the specification power of the ACF. To analyze outlier effects, we followed the approach of Granger & Ding (1995a) and generated a second data set by setting values outside the interval $[\bar{r}_t - 4\hat{\sigma}, \bar{r}_t + 4\hat{\sigma}]$ equal to the value of the closest interval boundary. Here, $\hat{\sigma}$ and $\bar{r}_t$ denote the estimated standard deviation and mean, respectively. However, the results from this second data set have not produced much additional insight: Concerning the ACF, the outlier-corrected data show similar results. As to the Taylor effect, we can confirm the observation of RY that outlier-correction weakens the Taylor effect (the median of θ increases from 1.43 to 1.57). With respect to the statistics on distributional properties of absolute returns, the outlier-correction causes a reduction of the differences between the models.

4.4 Persistence of stock market volatility

As shown, e.g., by Schwert (1989), the volatility of stock markets tends to be persistent, and mainly two effects can be observed. On the one hand, periods of high/low volatility often last very long, even periods of several months frequently occur. On the other hand, periods of high volatility tend to coincide with periods of falling stock prices. The author explains these facts by the liaison of the stock market with economic variables, which themselves are, in most cases, highly persistent (e.g. inflation). The market's volatility itself can also be used to predict changes in the economic variables, such as GDP growth (Campbell et al. 2001). In the following, we investigate the ability of our models to reproduce these two findings by means of the ten sub-series of the S&P 500, and an additional analysis of the S&P500, the German DAX30, French CAC40, Swiss SMI, and Japanese Nikkei225 for a 15-year period from 1993 to 2007.

Similar to Section 4.1, the models are selected by the BIC and parameter stability in terms of low standard error. To keep the results from different models comparable for each index, we fit models with two states to the S&P500, SMI, and Nikkei, and 3-state models to DAX and CAC. Table 12 and 13 in the Appendix show the parameter estimates and BIC values, respectively (we omit the results for M_N as they are very close to M_{RY}). Note that in case of the DAX, the third state of M_{RY} is rather non-persistent - however, for the models with t-component all three states are persistent. According to the BIC criterion, either M_t or M_{Nt} or both are preferred for the S&P500, DAX, CAC, whereas for SMI and Nikkei the Gaussian models seem sufficient.
The ability of M_N, M_{Nt}, and M_t to link periods of high volatility to periods of falling stock prices can be deduced directly from the estimated parameters. As shown in Tables 3 and 12, the conditional mean of the state with low/medium standard deviation is higher than the conditional mean of the state with high standard deviation (in the following, we refer to these states by 'low-risk state', 'medium-risk state' and 'high-risk state'). Moreover, the conditional mean of the high-risk state is negative and the mean for the low/medium-risk state positive for the large majority of indices and periods considered. An exception seems to be Periods 7 and 9, where the models incur difficulties to establish the link between high volatility and low return. As the conditional means of M_{RY} are both zero, it is not possible to establish a direct relation between high volatility and low returns for this model.

In what follows, we focus on the so-called 'smoothing probabilities', which are given by

$$P(S_t = i \,|\, X_1^T)$$

for $i \in \{1, ..., m\}$ and $t \in \{1, \cdots, T\}$. These probabilities are a by-product of the EM algorithm, for their derivation we refer to Appendix B and the references mentioned therein. The evolution of the hidden state sequence is often a key analysis tool, as the states are linked to an economic interpretation (see, e.g. Guidolin & Timmermann 2005, Linne 2002, Maheu & McCurdy 2001). The following Tables 9 and 10 illustrate the effect of including conditional t-distributions on the estimated sojourn times, i.e., the duration of a state visit. The state visited is determined by $\arg\max_j P(S_t = j \mid X_1^T)$. Again, M_N is omitted as the difference to M_{RY} is very low. The states are denoted by 'lr' and 'hr' for high- and low-risk, for 3-state models additionally 'mr' (medium-risk) exists. A notable effect is that the persistence of all states increases when M_t is used instead of M_{RY}. For M_{Nt}, the results differ slightly: Compared to M_{RY}, the persistence of the high-risk state consistently increases. However, this is not always the case for the low-/medium-risk states. Note that for some sub-series the effects are even stronger.

Table 9: Average estimated sojourn times, 2-state models

Estimated sojourn times per state in trading days for the ten periods (averaged) and S&P500, SMI, and Nikkei for the period 1993-2007. The states denoted 'lr' (low risk) and 'hr' (high risk) correspond lower respectively higher conditional standard deviations.

	sub-series		S&P500		SMI		Nikkei	
Model	lr	hr	lr	hr	lr	hr	lr	hr
M_{RY}	111	54	115	72	218	36	128	59
M_{Nt}	132	102	126	126	138	49	127	83
M_t	201	111	350	280	254	66	148	92

Table 10: Average estimated sojourn times, 3-state models

Estimated sojourn times per state in trading days for DAX and CAC for the period 1993-2007. The states denoted 'lr' (low risk), 'mr' (medium risk), and 'hr' (high risk) correspond lower, medium, and higher conditional standard deviations, respectively.

	DAX			CAC		
Model	lr	mr	hr	lr	mr	hr
M_{RY}	230	27	5	94	60	24
M_{Nt}	171	21	25	91	59	54
M_t	305	46	14	174	81	58

The reason for the increased persistence of the models with conditional t-distribution(s) may most likely result from the excess kurtosis of the t-distributed component. Regarding the high-volatile state, the augmented probability mass around zero increases the persistence of this state in short periods of low volatility, while heavier tails still allow for catching extreme outliers. The argumentation for the low-risk state is similar: compared to the Gaussian distribution, heavier tails increase the state's persistence, because

they allow for a higher robustness towards short periods of observations with comparably high volatility.
Figure 5 visualized the effect of adding conditional t-distributions by plotting the smoothing probabilities and resulting state classifications. The top eight panels display the returns and smoothing probabilities for the S&P500 on the left and for the Nikkei on the right. For better identification, the background of the periods with $P(S_t = 2) > P(S_t = 1)$ is shaded light gray. These two 2-state models visualize how large respectively small the effect of conditional t-distributions can be. The state classification of the S&P500 changes completely as the number of transitions (or state switches) reduces from 41 (M_{RY}) to 23 (M_{Nt}) and finally 5 (M_t). In case of the Nikkei, however, the (optical) difference between the models is much smaller. The lower eight panels show corresponding quantities resulting from the two 3-state models for CAC and DAX. The solid and dotted lines represent $P(S_t = 2)$ and $P(S_t = 3)$ respectively, and the background of the high-risk state is shaded dark grey. For both indices, the evolution of the estimated state sequence changes considerably.

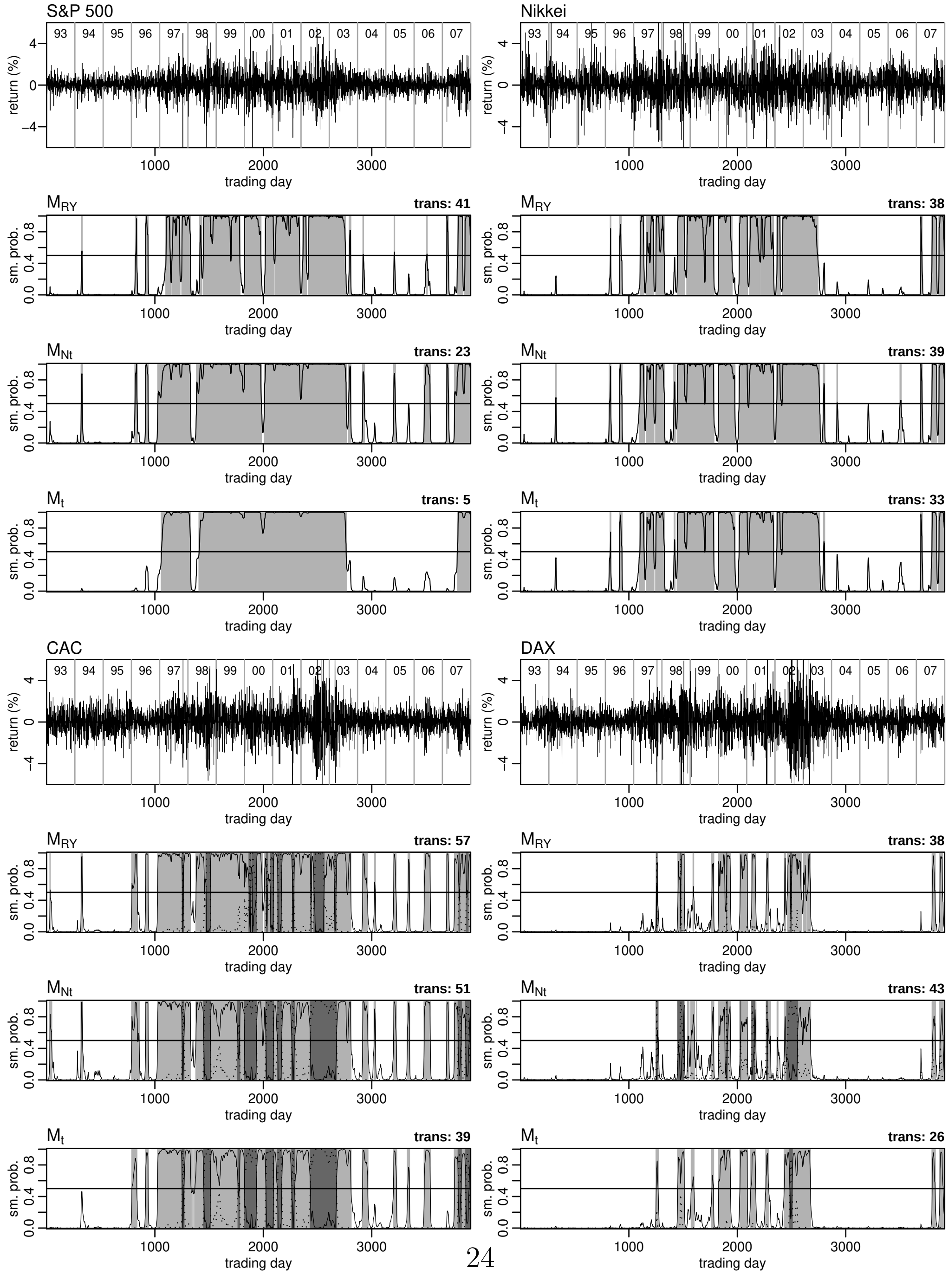
Recapitulating, the parameter estimates for M_N, M_{Nt}, and M_t confirm the link between periods of high volatility and falling stock prices. Moreover, the persistence of most regimes increases significantly when substituting the commonly used Gaussian conditional distributions by at least one t-distributed component. This gives rise to longer estimated periods with high, respectively low and medium stock volatility. Finally, the state evolution of the models with conditional t-distribution, which are often preferred by the BIC, changes considerably compared to M_{RY}.

5 Conclusion

In this paper we present an amelioration of the common HMM with conditional Gaussian distributions by introducing conditional t-distributions. This includes models with varying distributions in different states, which have not been analyzed yet in the context of daily return series.
By means of an application to the S&P 500 index, we show that, in particular, HMMs with varying conditional mean and at least one conditional t-distribution represents a useful extension. These models are preferred by the model selection criterion BIC, and reproduce most of the temporal and distributional properties as well as or better than the commonly utilized model of RY, including the slow decay of the ACF of absolute returns. The main reason for this improved performance may lie in the augmented proba-

Figure 5: International indices with smoothing probabilities, 1993-2007

The figure shows percentage returns of the S&P 500, Nikkei, CAC, and DAX from 1993 to 2007. Below each return series, three panels display the corresponding the smoothing probabilities $P(S_t = i \,|\, X_1^T)$ for M_{RY}, M_{Nt}, and M_t, respectively. The background of periods with $\hat{s}_t = 2$ is shaded light gray. For the two 3-state models (DAX and CAC), the background of periods with $\hat{s}_t = 3$ is shaded dark gray. The smoothing lines themselves are solid and dotted for state 2 and 3, respectively. We omit M_N, because there is almost no visual difference to M_{RY}.



bility mass around zero and heavy tails of the t-distribution. It allows for a higher robustness towards short periods lower or higher volatility when the actual regime actually is a high- respectively low-volatile regime residing in a (very) short period of contrary pattern.

An analysis of various international indices equally shows that the inclusion of conditional t-distributions often lowers the BIC significantly. More importantly, the evolution of the estimated state sequence by smoothing probabilities changes considerably. As the estimated state sequence is often utilized to link certain economic patterns to particular periods, conditional t-distributions may have relevant impact on these interpretations. In particular, stock volatility may be more persistent than Gaussian models suggest, or state classification patterns may change significantly. Finally, HMMs with varying conditional means establish a link between periods of high volatility and falling stock prices in daily return series.

APPENDIX

A Estimation Results

Table 13: BIC values for return series 1993-2007, 3-state models

BIC values for the four models M_{RY}, M_N, M_{Nt}, and M_t. For DAX and CAC results are from 3-state models, for the other series from 2-state models.

Model	S&P500	DAX	CAC	SMI	Nikkei
$M_{RY}^{2/3}$	-26103	-22275	-26005	-24625	-25908
$M_N^{2/3}$	-26107	-22293	-26007	-24638	-25906
$M_{Nt}^{2/3}$	-26104	-22297	-26011	-24630	-25896
$M_t^{2/3}$	-26134	-22304	-26048	-24620	-25885

Table 11: BIC values for S%P500 sub-series, 2- to 4-state models

BIC values for the four models M_{RY}, M_N, M_{Nt}, and M_t. The upper part of the table displays 2-state-models, the middle and lower part show 3- and 4-state-models, respectively.

States	Model	1	2	3	4	5	6	7	8	9	10
2	M_{RY}	-10473	-12265	-13630	-14563	-15175	-13517	-13692	-13255	-13874	-11218
	M_N	-10464	-12257	-13652	-14599	-15203	-13503	-13680	-13256	-13877	-11206
	M_{Nt}	-10489	-12286	-13662	-14637	-15253	-13500	-13684	-13273	-13924	-11214
	M_t	-10501	-12313	-13686	-14632	-15251	-13497	-13676	-13312	-13957	-11208
3	M_{RY}	-10535	-12296	-13656	-14558	-15248	-13548	-13691	-13263	-13920	-11248
	M_N	-10556	-12283	-13672	-14612	-15279	-13529	-13673	-13257	-13915	-11232
	M_{Nt}	-10563	-12284	-13667	-14631	-15282	-13521	-13669	-13279	-13929	-11227
	M_t	-10550	-12303	-13674	-14619	-15267	-13534	-13653	-13284	-13936	-11211
4	M_{RY}	-10528	-12262	-13634	-14515	-15225	-13513	-13643	-13235	-13917	-11214
	M_N	-10545	-12243	-13661	-14605	-15283	-13557	-13629	-13249	-13899	-11191
	M_{Nt}	-10536	-12248	-13665	-14611	-15287	-13550	-13624	-13242	-13898	-11183
	M_t	-10515	-12251	-13636	-14594	-15268	-13525	-13600	-13229	-13890	-11160

Table 12: Parameter estimates for international indices, 1993-2007

Parameter estimates for the preferred models M_{RY}, M_{nt}, and M_t. Note that the standard deviation in the states with t-distributions requires an adjustment by the factor $\sqrt{\nu/(\nu-2)}$ for direct comparison with Gaussian states.

	$M_{RY}^{2/3}$		$M_{Nt}^{2/3}$				$M_t^{2/3}$			
Index	P	$\sigma \cdot 10^3$	P	$\sigma \cdot 10^3$	$\mu \cdot 10^4$	ν	P	$\sigma \cdot 10^3$	$\mu \cdot 10^4$	ν
S&P500	0.990 0.010	6.36	0.992 0.008	5.93	7.19		0.998 0.002	5.34	6.90	6.52
	(0.004) (0.004)	(0.14)	(0.004) (0.004)	(0.15)	(2.05)		(0.002) (0.002)	(0.22)	(1.99)	(2.51)
	0.016 0.984	13.74	0.009 0.991	10.15	0.49	5.5	0.003 0.997	10.24	1.45	5.38
	(0.006) (0.006)	(0.42)	(0.004) (0.004)	(0.45)	(4.19)	(1.56)	(0.004) (0.004)	(0.49)	(4.31)	(1.66)
DAX	0.978 0.021 0.001	9.49	0.972 0.013 0.014	9.12	13.52		0.985 0.014 0.001	8.71	14.50	11.67
	(0.007) (0.007) (0.003)	(0.28)	(0.012) (0.013) (0.007)	(0.33)	(3.63)		(0.006) (0.006) (0.003)	(0.37)	(3.34)	(8.91)
	0.031 0.953 0.016	18.14	0.059 0.933 0.008	15.83	26.25		0.018 0.976 0.006	15.55	2.49	8.32
	(0.005) (0.008) (0.006)	(0.77)	(0.038) (0.047) (0.026)	(1.32)	(14.45)		(0.009) (0.026) (0.022)	(1.01)	(8.87)	(8.55)
	0.000 0.178 0.822	41.13	0.000 0.039 0.961	18.42	-32.84	4.48	0.000 0.044 0.956	24.73	-53.46	4.33
	(0.000) (0.103) (0.103)	(5.40)	(0.001) (0.047) (0.047)	(1.73)	(18.05)	(2.83)	(0.005) (0.086) (0.085)	(5.86)	(75.44)	(12.49)
CAC	0.991 0.009 0.000	5.60	0.989 0.011 0.000	5.44	7.08		0.994 0.006 0.000	4.62	6.69	5.51
	(0.004) (0.004) (0.000)	(0.15)	(0.006) (0.006) (0.001)	(0.17)	(2.13)		(0.005) (0.005) (0.000)	(0.23)	(2.12)	(2.09)
	0.010 0.980 0.010	10.48	0.013 0.978 0.009	9.79	3.42		0.007 0.983 0.009	9.33	4.78	23.43
	(0.005) (0.008) (0.006)	(0.36)	(0.006) (0.010) (0.006)	(0.37)	(3.87)		(0.005) (0.028) (0.027)	(0.43)	(4.73)	(12.03)
	0.000 0.046 0.954	20.08	0.000 0.021 0.979	14.90	-9.32	8.14	0.000 0.021 0.979	14.98	-10.18	8.31
	(0.000) (0.059) (0.059)	(1.97)	(0.002) (0.021) (0.021)	(1.37)	(11.63)	(9.93)	(0.000) (0.064) (0.064)	(1.55)	(20.46)	(11.38)
SMI	0.994 0.006	8.35	0.989 0.011	7.10	10.00		0.992 0.008	6.94	9.8	10.21
	(0.002) (0.002)	(0.17)	(0.004) (0.004)	(0.16)	(2.23)		(0.003) (0.003)	(0.23)	(2.03)	(5.17)
	0.021 0.979	22.7	0.018 0.982	15.23	-5.93	5.68	0.018 0.982	17.05	-9.0	6.70
	(0.009) (0.009)	(0.78)	(0.006) (0.006)	(0.75)	(6.36)	(1.71)	(0.007) (0.007)	(0.90)	(8.38)	(4.37)
Nikkei	0.992 0.008	6.69	0.992 0.008	6.37	6.13		0.992 0.008	6.24	6.16	40.0
	(0.003) (0.003)	(0.14)	(0.003) (0.003)	(0.14)	(1.89)		(0.003) (0.003)	(0.16)	(1.95)	(9.46)
	0.018 0.982	15.26	0.013 0.987	12.61	-3.75	8.61	0.013 0.987	12.69	-3.72	8.7
	(0.007) (0.007)	(0.50)	(0.006) (0.006)	(0.62)	(5.15)	(4.79)	(0.006) (0.006)	(0.62)	(5.64)	(5.76)

B Re-estimation formulae

The EM algorithm has been treated in detail by many authors and we therefore omit explicit derivations of the Q-function and re-estimation formulae. For a short overview, the article of Ephraim & Merhav (2002) and the sources mentioned therein provide a good introduction. A more comprehensive and detailed survey on HMMs was written by Cappé et al. (2007).

The Q-function of a HMM is given by

$$Q(\theta, \theta^{(k)}) = \underbrace{\sum_{i=1}^{m} log\, \pi_i\, \gamma_i(1)}_{A} + \underbrace{\sum_{i,j=1}^{m} \sum_{t=1}^{T-1} log\, p_{ij}\, \xi_{ij}(t)}_{B} + \underbrace{\sum_{i=1}^{m} \sum_{t=1}^{T} \gamma_i(t) \log b_i(x_t)}_{C},$$

where

$$\begin{aligned} \gamma_i(t) &:= P(S_t = i \mid X_1^T = x_1^T), \\ \xi_{ij}(t) &:= P\left(S_t = i, S_{t+1} = j \mid X_1^T = x_1^T\right), \\ b_i(x_t) &:= P\left(X_t = x_t \mid S_t = i\right) \end{aligned}$$

for $i = 1, \ldots, m$ and $t = 1, \ldots, T$.

Transition and initial probabilities

Most implementations of the EM algorithm in the context of HMMs base on the algorithms of Baum et al. (1970). These allow to fit a homogeneous, but non-stationary HMM. For the non-stationary case, the Q-function is split up into the three additive parts denoted by A (initial component), B (transition component) and C (observation component), which are maximized separately.
In order to fit a stationary Markov chain, component A and B have to be treated simultaneously, respecting a stationarity constraint. Then, the joint M-step for these two components becomes

$$\max_{p_{ij} \in \mathbf{\Pi}} \left(\sum_{i=1}^{m} log\, \pi_i\, \gamma_i(1) + \sum_{i,j=1}^{m} \sum_{t=1}^{T-1} log\, p_{ij}\, \xi_{ij}(t) \right)$$

$$\text{with } \boldsymbol{\pi}\tilde{\mathbf{\Pi}} = (0, \ldots, 0, 1).$$

The matrix $\tilde{\mathbf{\Pi}}$ is obtained by replacing the last column of $\mathbf{1} - \mathbf{\Pi}$ by the vector $(1, \ldots, 1)^t$ of length m. The calculation of a closed solution to this system of equations is more difficult to solve than it appears at first glance. However, solving it with numerical methods is straightforward. Compared to the original algorithms of Baum et al. (1970), the estimation does not slow down significantly. For further details on this part we refer to Bulla & Berzel (2008).

Conditional Gaussian distribution

The re-estimation formulae for the conditonal mean μ_i and variance σ_i^2 of normal component distributions are

$$\begin{aligned}
\mu_i^{(k+1)} &= \frac{\sum_{t=1}^T \gamma_i(t) x_t}{\sum_{t=1}^T \gamma_i(t)}, \\
\sigma_i^{2(k+1)} &= \frac{\sum_{t=1}^T \gamma_i(t) \left(x_t - \mu_j^{(k+1)}\right)^2}{\sum_{t=1}^T \gamma_i(t)}.
\end{aligned}$$

Conditional t-distribution

The maximization of the Q-function for t-distributed variables is slightly more difficult than the Gaussian case. Peel & McLachlan (2000) present some techniques for the estimation of mixtures of t-distributions, which can be adopted to HMMs with a reasonable amount of effort. For details to this step, see Bulla & Bulla (2006), where the approach is discussed for hidden semi-Markov models.

Let the density of the t-distribution with mean $\boldsymbol{\mu}$, ν degrees of freedom and positive definite inner product matrix $\boldsymbol{\Sigma}$ be given by

$$f(\boldsymbol{x_t}) = \frac{\Gamma(\frac{\nu+p}{2})|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(\pi\nu)^{\frac{1}{2}p}\Gamma(\frac{\nu}{2})\left\{1 + \delta(\boldsymbol{x}_t, \boldsymbol{\mu}, \boldsymbol{\Sigma})/\nu\right\}^{\frac{1}{2}(\nu+p)}},$$

where $\delta(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the Mahalanobis distance, defined by

$$\delta(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (\boldsymbol{x} - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}),$$

and p the dimension of the observations. The re-estimation formulae for $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ can be derived explicitly as

$$\boldsymbol{\mu}_i^{(k+1)} = \frac{\sum_{t=1}^T \gamma_i(t) u_i^{(k)}(t) \boldsymbol{x}_t}{\sum_{t=1}^T \gamma_i(t) u_i^{(k)}(t)}$$

and

$$\boldsymbol{\Sigma}_i^{(k+1)} = \frac{\sum_{t=1}^T \gamma_i(t) u_i^{(k)}(t)(\boldsymbol{x}_t - \boldsymbol{\mu}_i^{(k+1)})(\boldsymbol{x}_t - \boldsymbol{\mu}_i^{(k+1)})^T}{\sum_{t=1}^T \gamma_i(t)},$$

where $u_i^{(k)}(t)$ denotes an auxiliary variable defined

$$u_i^{(k)}(t) \quad := \quad \frac{\nu_j{}^{(k)} + p}{\nu_j{}^{(k)} + \delta(\boldsymbol{x}_t^{(k)}, \boldsymbol{\mu}^{(k)}, \boldsymbol{\Sigma}^{(k)})}.$$

The denominator in the re-estimation equation for $\boldsymbol{\Sigma}_i^{(k+1)}$ may also be replaced by $\sum_{t=1}^T \gamma_i(t) u_i^{(k)}(t)$ to increase the speed of convergence (Kent et al. 1994).
The re-estimation of $\nu_i^{(k+1)}$ is not possible explicitly. It requires determining the (unique) solution of the equation

$$\begin{aligned}
&-\psi\left(\frac{1}{2}\nu_i^{(k)}\right) + \log\left(\frac{1}{2}\nu_i^{(k)}\right) + 1 \\
&+ \frac{1}{\sum_{t=1}^T \gamma_i(t)} \left[\sum_{t=1}^T \gamma_i(t) \left(\log u_i^{(k)}(t) - u_i^{(k)}(t)\right)\right] \\
&+ \psi\left(\frac{\nu_i^{(k)} + p}{2}\right) - \log\left(\frac{\nu_i^{(k)} + p}{2}\right) = 0.
\end{aligned}$$

The solution can be determined without relevant complications, e.g., by a bisection algorithm or quasi-Newton methods, because the function on the left hand side is monotonically increasing in $\nu_i^{(k)}$.

Conditional Gaussian and t-distributions in different states

The extension to state-varying conditional distributions is straightforward. The forward-backward pass through the observations has to be carried out respecting the different conditional distributions. The calculation of the quantities $\gamma_i(t)$, $\xi_{ij}(t)$, and $b_i(t)$ follows directly from this step. The *M-step* is combined from the re-estimations for Gaussian and t components described above.

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