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Abstract

We study discretionary equilibria in dynamic linear-quadratic rational expectations models. In contrast to the assumptions that pervade this literature we show that these models *do* have multiple equilibria in some situations. We demonstrate the existence of multiple discretionary equilibria by example. We investigate general properties of discretionary equilibria and discuss implications for numerical algorithms.

Key Words: Discretion, Multiple Equilibria, LQ RE models

JEL References: E31, E52, E58, E61, C61

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1 Introduction

In this paper we study discretionary policy in the class of infinite-horizon discrete-time linear dynamic models that is typically used to study aggregate fluctuations in macroeconomics. In such models the optimizing behaviour of the private sector is characterized by appropriate forward-looking implementability constraints, with the policy maker acting to maximize a quadratic intra-temporal objective. We discuss and illustrate the potential for the existence of multiple rational expectations equilibria in this class of models, usually termed Linear Quadratic Rational Expectations (LQ RE) models. The class of LQ RE models is of considerable applied interest because it can – under appropriate conditions – be interpreted as a quadratic approximation to the underlying non-linear optimal policy problem. If a multiplicity of equilibria arises then this can generate rich dynamics, with alternating periods of high and low volatility of aggregate macroeconomic variables.

The existence of multiple equilibria in non-linear models is now well established. Albanesi et al. (2003) and King and Wolman (2004) demonstrate how discretionary policy generates a multiplicity of equilibria, where the non-linearity gives rise to dynamic complementarities in the pricing decisions of different firms in the private sector. In standard LQ RE models the private sector is aggregated and these types of equilibria do not exist. Simply by virtue of their linearity, such models are often treated as immune to multiplicity.[1] This class of models is routinely used in policy analysis and forms much of the basis for our understanding of policy equilibria; the standard New Keynesian model is frequently used to demonstrate results that are presumed valid for the whole class of LQ RE models under discretion.[2]

We show that any such inference is fundamentally incorrect. In contrast to the assumption that pervades the literature, we show that LQ RE models *do* have multiple discretionary equilibria. Further, we describe different types of discretionary equilibria that can exist in non-degenerate LQ RE models and show how knowledge of the source of multiplicity can help us locate these different policy equilibria.

Our definition of discretionary policy is conventional and is widely used in the monetary policy literature, see e.g. Backus and Driffill (1986), Oudiz and Sachs (1985), Clarida et al. (1999), and Woodford (2003a). At the beginning of each time period the policy maker observes the state of the economy and makes an optimal decision; all agents also know that the policy maker will apply the same procedure in every subsequent period. When the private sector expects the policy maker to pursue discretionary policy in each future period, the best a policy maker can do is to pursue the discretionary policy: there is no incentive for the policy maker to deviate from it and the private sector's expectations are rational. Discretionary policy is credible by construction.

The very property of credibility creates a potential for multiplicity of discretionary equilibria and makes it impossible for the policymaker to just select the welfare-dominant one. When the state of the economy is observed, and if the policy maker acts first, then the private sector observes this policy action and reacts optimally given the state of the economy, the action of the policy

[1] In a seminal paper on discretionary policy Oudiz and Sachs (1985, p. 288) suggest an algorithm for finding an equilibrium and remark: "Although we cannot prove that the resulting function is the unique memoryless, time-consistent equilibrium, we suspect that it is in fact unique, in view of the linear-quadratic structure of the underlying problem."

[2] See Woodford (2003b), Vestin (2006), Walsh (2003), Evans and Honkapohja (2003) among many others.

maker, and its expectations of the future state of the economy. The optimizing policy maker expects the private sector to react rationally and policy is chosen accordingly. Crucially, the rational reaction of the private sector is determined by its expectations about the future state of the economy which, in turn, is affected by the future policy. So the policy maker acts taking into account the effect of the future policy decisions *as expected by the private sector*. There are two further elements to complete the story. First, if there are complementarities between the decisions of the private sector and the policy maker – complementarities that mean an optimal decision taken by a policy maker reinforces the decisions of the private sector – then multiple equilibria can arise. For this to happen the presence of state variables that are affected by agents' decisions is necessary, as this ensures the importance of future policy decisions for current actions. Second, when choosing an optimal policy the current policy maker reacts (indirectly) to the *past* actions of the private sector by observing the state of the economy. If there is a multiplicity of equilibria and (for some reason) the private sector in the past period changed its perception of current and future policy *after* the policy maker acted in that past period, then the current policy maker will find it optimal to validate these expectations and has no power to choose a different equilibrium.

We illustrate our argument first by example and then more generally. The paper is organized as follows. In the next section we present three examples of increasing complexity. They demonstrate uniqueness, the existence of multiplicity and the importance of endogenous predetermined states, and the form of multiplicity in a more general model, respectively. In Section 3 we formally define the discretionary optimization problem and derive the corresponding first-order conditions. In Section 4 we discuss the properties of discretionary equilibria in the general class of LQ RE models. In Section 5 we discuss numerical algorithms to find discretionary equilibria. Section 6 concludes.

2 Multiple Equilibria by Example

We develop our arguments thematically using three models with different features. The first of these is the standard New Keynesian model that is often used as a benchmark model for results on discretionary policy. We show the necessary *uniqueness* of the discretionary equilibrium as a consequence of a particular disconnect between decisions in current and subsequent periods. This disconnect, and the absence of important interactions between current and future policy and the private sector decisions, make this model a very special LQ RE model of discretionary policy. The uniqueness of equilibria under discretionary policy is then necessarily a special case and many policy implication derived from the standard New Keynesian model under discretion may be difficult to generalize.

In the second example we demonstrate *the existence* of multiple discretionary equilibria. To do this we must introduce an *endogenous state variable*, in this case government debt, into the New Keynesian model. This model is known to generate a multiplicity of determinate regimes under rules-based policy (Leeper (1991)) which makes it appealing to investigate the existence of different equilibria under discretion. We demonstrate how dynamic complementarities between the decisions of the private sector and the policy maker lead to a multiplicity of equilibria. We also argue that these equilibria are consistent with the empirical evidence documented in a number of studies.

Using a third example, a New Keynesian model with capital accumulation, we demonstrate that discretionary equilibria may also arise because of dynamic complementarities between the different decisions of the aggregate private sector alone. The nature of this type of multiple equilibria is different in a number of important ways, in particular implying that there can be several time-invariant responses of the aggregate private sector to the same policy action; King and Wolman (2004) call similar equilibria 'point-in-time' equilibria. These point-in-time equilibria generate multiple discretionary equilibria. As our model is linear, complementarities of this sort can only arise if, for example, the different decisions of the aggregate private sector for consumption and investment have separate effects on the marginal costs faced by firms. It should be apparent that this type of complementarity is far from uncommon.

2.1 Standard New Keynesian Model

We start with the standard New Keynesian model as in Clarida et al. (1999), Sec. 3. This model has become the workhorse for policy analysis, including the analysis of policy under discretion.[3] As we show, this model *cannot* have multiple discretionary equilibria. We begin with it to introduce definitions and the important ideas that follow, and in particular to showcase the model features that preclude multiple equilibria. It will also become apparent that these features are unlikely to exist in the general class of LQ RE models under discretionary policy.

We consider a deterministic perfect-foresight model. The law of motion of the aggregate economy can be written as

$$\pi_t = \beta\pi_{t+1} + \lambda c_t + \nu b_t, \qquad (1)$$

$$b_{t+1} = \rho b_t, \qquad (2)$$

and the initial state $\bar{b}$ is known to all agents. The only predetermined state variable in this economy is the *exogenous* autoregressive process b_t. Equation (1) is the New Keynesian Phillips curve that relates the inflation rate π_t positively to the output c_t. Parameter β is the private sector's discount factor and λ is the slope of the Phillips curve.[4] Parameter ν scales the effect of the exogenous state on inflation. The aggregate agent sets π_t in response to the evolution of c_t and b_t.

Following Clarida et al. (1999), we assume that the policy maker chooses output c_t and then, conditional on subsequent optimal evolution of c_t and π_t, decides on the value of interest rate that achieves the desired c_t and π_t.[5] The inter-temporal policy maker's welfare criterion is defined by the quadratic loss function

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \alpha c_s^2\right). \qquad (3)$$

[3] See e.g. Woodford (2003b), Vestin (2006), Walsh (2003).

[4] We assume that the Phillips curve evolves from staggered nominal price setting as in Calvo (1983). The individual firm price-setting decision, which provides the basis for the aggregate relation, is derived from an explicit optimization problem. Firms are monopolistically-competitive: When given the opportunity, each firm chooses its nominal price to maximize profits subject to constraints on the frequency of future price adjustments. Parameter λ is a function of the frequency of price adjustments.

[5] Using the interest rate as an instrument implies that consumption and price-setting decisions are made simultaneously, while in this model they are consecutive decisions taken by the relevant agent. Here and in the next example this makes no difference for the results on multiplicity.

We assume that the policy maker knows the law of motion (1)-(2) of the aggregate economy when it formulates policy. The policy maker's decision problem is to find the best policy for every period, knowing that future policy makers have the freedom to change policy, and knowing that future policy makers face the same problem.

We assume that the policy maker acts in a *discretionary* way in the following sense. At every point t in time the private sector *observes* the policy that reacts only to the current state, so can be written in the form[6]

$$c_t = c_b b_t. \tag{4}$$

The private sector *expects* that the future policy makers will apply the same decision process and will react to the contemporary state only, i.e. will implement policy (4). We assume that the aggregate decision of the private sector, taken after the policy maker has acted, can be written as the linear feedback function

$$\pi_t = \pi_b b_t. \tag{5}$$

At any time t, the policy maker reacts to the current state (4), knows that the private sector observes its action, and knows that the private sector expects all future policy makers will apply the same decision process and implement policy (4). Henceforth we shall refer to parameters that define the behaviour of the policy maker and the private sector, c_b and π_b, as 'decisions'.

Denote the response of the next-period private sector to the next-period state b_{t+1} as $\tilde{\pi}_b b_{t+1}$ where we qualify the expected private sector decision using a 'tilde'.[7] Now, from

$$\pi_{t+1} \overset{eq.(5)}{=} \tilde{\pi}_b b_{t+1} \overset{eq.(2)}{=} \tilde{\pi}_b \rho b_t \overset{eq.(1)}{=} \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}c_t - \frac{\nu}{\beta}b_t$$

it follows that the private sector's decision can also be written as

$$\pi_t = \left(\beta\rho\tilde{\pi}_b + \nu\right) b_t + \lambda c_t. \tag{6}$$

If there is a disturbance such that $b_t > 0$ and firms raise inflation to $\pi_b b_t$ then this rise is the result of the reaction to state, $\left(\beta\rho\tilde{\pi}_b + \nu\right) b_t$, and to the policy, λc_t. Because the policy maker moves first within each period and because the private sector observes the policy, the private sector takes into account the 'instantaneous' influence of the policy choice, measured by λ. The first term in (6) shows that the response to the state is also affected by the *next-period* response of the private sector to the state, $\tilde{\pi}_b$.

We now complete the definition of discretion. Policy determined by (4) is *discretionary* if the policy maker *finds it optimal* to continue to follow it in every period $s > t$, given the private sector (i) knows that in every period $s > t$ future policy makers re-optimize and use the same decision process, (ii) observes the current policy, (iii) anticipates policy (4) to be implemented in all future periods.[8]

[6] We restrict ourselves to the 'memoryless' or Markov equilibria, where agents' decisions are functions of current state only. We also assume a linear contemporaneous relationship.

[7] We shall use this notation in all our examples to denote next-period decisions.

[8] In the language of game theory we restrict our attention to time-consistent feedback equilibria with intra-period leadership, see e.g. de Zeeuw and van der Ploeg (1991), Oudiz and Sachs (1985), Cohen and Michel (1988). Here and below we simply call such equilibria as 'discretionary'.

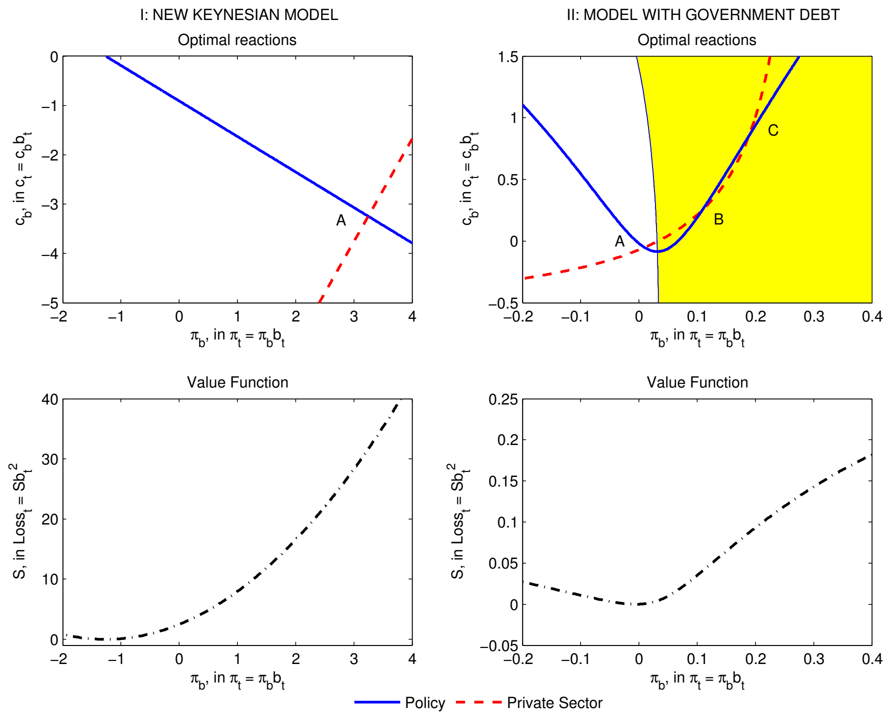

Figure 1: Discretionary Equilibria

We can write the criterion for optimality, the Bellman equation, as

$$L_t\left(b_t\right) = \min_{c_t}\left(\frac{1}{2}\left(\left(\left(\beta\rho\tilde{\pi}_b + \nu\right) b_t + \lambda c_t\right)^2 + \alpha c_t^2\right) + \beta L_{t+1}\left(\rho b_t\right)\right),$$

where we take the intra-period leadership of the policy maker into account by substituting in constraint (6). Because the per-period loss function in (3) is quadratic and because both policy maker and private sector decisions are linear in state, the discounted loss will necessarily be quadratic in b_t or

$$L_t\left(b_t\right) = \frac{1}{2} S b_t^2$$

for some value of S to be determined.

The Bellman equation, written in terms of current and future anticipated losses S and $\tilde{S}$, becomes

$$S b_t^2 = \min_{c_t}\left(\left(\left(\left(\beta\rho\tilde{\pi}_b + \nu\right) b_t + \lambda c_t\right)^2 + \alpha c_t^2\right) + \beta\tilde{S}\left(\rho b_t\right)^2\right). \tag{7}$$

Differentiation of (7) with respect to c_t yields the optimal policy response

$$c_t = -\frac{\left(\beta\rho\tilde{\pi}_b + \nu\right)\lambda}{\left(\lambda^2 + \alpha\right)} b_t = c_b b_t. \tag{8}$$

The coefficient c_b in (8) determines the optimal policy feedback on predetermined state, b_t. Notice that this coefficient is independent of $\tilde{S}$. When choosing the best policy in period t, the policy maker knows that future policy makers will re-optimize. For this model this constraint is not binding. This implies that the optimal policy feedback coefficient c_b is a *linear* function of $\tilde{\pi}_b$, the future private sector reaction. It is immediately apparent that the disconnect between *policy decisions* in times t and s for any $t \neq s$ only happens if the predetermined state variable is exogenous, so policy neither affects the state directly nor via an effect on the private sector's decision. We plot (8) as the solid line in the top chart in Panel I in Figure 1.[9]

To complete characterization of discretionary equilibrium it remains to present the optimal decision of the private sector in the feedback form (5). Substitute (4) into (6) to obtain

$$\pi_t = (\beta\rho\tilde{\pi}_b + \nu + \lambda c_b)\, b_t = \pi_b b_t. \tag{9}$$

The next-period response to any disturbance positively affects the current-period response to the same disturbance, but the effect is linear in the decisions of *both* agents. The effect is linear because the future state is determined by parameter ρ alone, depending on neither policy nor private sector decisions.

If the policy is expected to be the same in the next period, the time-invariant aggregate private sector response is the same, $\tilde{\pi}_b = \pi_b$, and there is a unique solution $\pi_b = \pi_b(c_b)$ which is the linear relation

$$\pi_b = \frac{\nu + \lambda c_b}{1 - \beta\rho} \tag{10}$$

We plot (10) with the dashed line in the top chart in Panel I in Figure 1.

To summarize, we have two linear response functions, $c_b = c_b(\pi_b)$ and $\pi_b = \pi_b(c_b)$ given by (8) and (10). There is a unique solution, given by

$$c_b = -\frac{\lambda\nu}{\lambda^2 + \alpha(1 - \beta\rho)}, \qquad \pi_b = \frac{\alpha\nu}{\lambda^2 + \alpha(1 - \beta\rho)}.$$

This was obtained by Clarida et al. (1999), equations (3.4) and (3.5), using a different approach.[10]

In order to find the value function S we substitute the optimal solution (8) into Bellman equation (7) and, using that in the discretionary equilibrium the next-period policy is expected to be the same and so $S = \tilde{S}$, obtain

$$S = \frac{\alpha(\beta\rho\pi_b + \nu)^2}{(\alpha + \lambda^2)(1 - \beta\rho^2)},$$

which is a quadratic function of the private sector's response, π_b. The stronger the response of firms to any disturbance, π_b, the more costly it is to stabilize the economy, as the optimal policy has to offset the effect on prices with bigger reduction in output, c_t, see (8). Therefore, the overall

[9] We use parameter values: $\beta = 0.99$, $\lambda = 0.1$, $\alpha = 0.1$, and $\rho = 0.8$.

[10] Clarida et al. (1999), Sec. 3 do not discuss the details of the interactions between the private sector and the policy maker, primarily because they are absent in this model. Because of this, the authors can use a simple way to solve the model that is not valid in a more general case. See their footnote 27.

loss, as measured by (3), rises as π_b^2. We plot it in the bottom chart in Panel I in Figure 1. We further note that the actual speed of stabilization of the economy is determined solely by the persistence parameter ρ and is exogenous.

The coefficients π_b, c_b and S describe the solution to the discretionary optimization problem outlined above. They uniquely define the trajectories $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ for any given $b_0 = \bar{b}$. Conversely, if the sequence $\{b_t, \pi_t, c_t\}_{t=0}^{\infty}$ solves the discretionary policy outlined above then there is a unique triplet $\{\pi_b, c_b, S\}$ that satisfies (4), (5) and (7). We call the triplet of coefficients $\{\pi_b, c_b, S\}$ a *discretionary equilibrium.* It follows from the linearity of all agents' decisions in each other's decision variables that the discretionary equilibrium always exists and is unique.

Finally, for this model it is instructive to compute the welfare (equal to minus loss (3)) for *arbitrary* decisions c_b and π_b, but still exploiting the intra-period order of moves. Write this as

$$\begin{aligned} W_t &= -\frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2+\alpha c_s^2\right) = -\frac{1}{2}\left(((\beta\rho\pi_b+\nu)+\lambda c_b)^2+\alpha c_b^2\right)\sum_{s=t}^{\infty}\beta^{s-t}b_s^2 \\ &= -\frac{1}{2}\frac{\left((\beta\rho\pi_b+\lambda c_b+\nu)^2+\alpha c_b^2\right)}{1-\beta\rho^2}b_t^2. \end{aligned}$$

The second derivative of this is

$$\frac{\partial^2 W_t}{\partial \pi_b \partial c_b} = -\frac{\beta\rho\lambda}{1-\beta\rho^2}b_t^2,$$

which is negative for any π_b, c_b. If it is expected that firms increase inflation in response to a disturbance to b_t then this reduces the marginal return to the policy that increases output in response to the same disturbance. Cooper and John (1988) define decisions of the private sector and the policy maker as *dynamic substitutes* if the optimal decision of the policy maker is decreasing in the subsequent decision of the private sector, i.e. it is optimal to reduce output if inflation is expected to rise. Condition $\frac{\partial^2 W_t}{\partial \pi_b \partial c_b} < 0$ implies dynamic substitutability. In our model uniqueness is ensured by the linearity of responses, but global dynamic substitutability and the existence of an equilibrium would also ensure uniqueness.

2.2 New Keynesian Model with Government Debt

2.2.1 Quick Overview of the Example

In our previous example the discretionary equilibrium was unique. For the New Keynesian model with no endogenous state variable there is a complete separation between the current and future decisions of agents. This implies the linearity of agents' decisions and, as we have seen, necessarily implies the uniqueness of the solution.

In this section we present an example that by contrast has multiple discretionary equilibria. As before, we consider a deterministic New Keynesian model, but instead include an endogenous state variable. We assume that the agents observe *and can affect* the accumulation of the real government debt. As before, we study the interactions of a single monetary policy maker and an aggregate private sector.

The accumulation of government debt must depend on the fiscal stance. So in the model there is a *non-optimizing* fiscal authority that faces a stream of exogenous public consumption. These expenditures are financed by levying income taxes and by issuing one-period risk-free nominal bonds. We assume that the fiscal authority imposes a simple proportional rule for the tax rate: if the real debt is higher (lower) than in the steady state then the tax rate rises (falls). We shall refer to the tax rate as the 'taxes' and to the parameter of the proportional rule as the 'fiscal feedback'. The size of the fiscal feedback measures the strength of fiscal stabilization of the debt and, as we shall show, plays an important role in the model. The presence of the non-optimizing fiscal authority in the economy can be captured by this single parameter.

If this parameter is relatively large then an increase in public debt is practically eliminated by fiscal policy within few periods. The equilibrium behavior of the discretionary monetary policy maker and of the private sector is, therefore, similar to the one in the standard New Keynesian model. In order to reduce the effect of temporarily high taxes on inflation the policy maker lowers consumption, so that demand falls. This is a low-inflation-volatility equilibrium as the firms set relatively low inflation anticipating low consumption in the future.

If the fiscal feedback on debt is zero, then an initial increase in the public debt results in a debt spiral unless one or both agents intervene. If the policy maker intervenes debt is stabilized only by higher consumption-fuelled demand engineered to raise income and so tax revenues. This implies a high-inflation-volatility equilibrium as firms set inflation relatively high, reacting to anticipated high demand in the future.

We shall demonstrate that if the fiscal feedback on debt is moderate – and so debt is only slowly stabilized by fiscal policy alone – then both equilibria are possible. If the private sector *believes* that the future demand will be low and sets low inflation, the next-period policy maker *finds it optimal to validate those beliefs* and will reduce demand. Conversely, if the private sector *believes* that the future demand will be high and inflates, the next-period policy maker *finds it optimal to validate those beliefs* and will increase demand. Following any initial debt displacement the economy can follow one of several paths, each of which satisfies the conditions imposed on discretionary policy, i.e. time-invariance and optimality, given that the future policy makers are assumed to re-optimize.

As we discuss later in Section 2.2.4 the described economic behavior is familiar from the literature on the fiscal theory of the price level, and is consistent with both the theoretical and empirical findings.[11] A crucial difference is that we study discretionary policy, not just simple monetary rules. However, because of the similarity with rules-based policy models, we anticipate our results and expect to find a switch from one equilibrium to another when we vary the strength of the fiscal feedback. If we do find this then the empirical evidence makes such an example rather an appealing way to demonstrate the potential existence of multiple equilibria.

2.2.2 Discretionary Equilibria

Aggregate behavior of private agents and the policy maker's actions. We adopt the model of Woodford (2001), Benigno and Woodford (2004). We delegate all technical details to the Online Appendix[12] and only present the model log-linearized about the steady state. The

[11] See e.g. Leeper (1991), Woodford (2001), Davig and Leeper (2006b), Favero and Monacelli (2005).

[12] The Online Appendix and all necessary MATLAB programs are available from www.people.ex.ac.uk/tkirsano or upon request from the authors.

model consists of equations that describe the aggregate behaviour of the private sector and the evolution of debt. As before, we assume that the policy maker chooses consumption c_t.

We assume that *nominal* debt is observed at the beginning of period t, and define real debt b_t as the nominal beginning-of-period debt deflated by the end-of-previous-period price. The begining-of-period real debt b_t is then the aggregate predetermined state variable in period t. The economy evolves according to

$$\pi_t = \beta\pi_{t+1} + \lambda c_t + \nu b_t, \tag{11}$$

$$b_{t+1} = \rho b_t - \eta c_t, \tag{12}$$

and the initial state $\bar{b}$ is known to all agents. As before, the aggregate agents' decision variable is inflation, π_t.

Equation (11) is the appropriate New Keynesian Phillips curve.[13] Equation (12) describes the evolution of real debt.[14] In contrast to the standard New Keynesian model the state will now be affected by policy.

The inter-temporal welfare criterion of the policy maker is defined by the quadratic loss function[15]

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + \alpha c_s^2\right). \tag{13}$$

The policy maker knows the laws of motion (11)-(12) of the aggregate economy and takes them into account when formulating policy. The policy maker finds the best action every period, knows that future policy makers have freedom to change policy, and knows that future policy makers will apply the same decision process.

Under discretion, at every point in time t decision rules of each agent are linear functions of the current state

$$c_t = c_b b_t, \tag{14}$$

$$\pi_t = \pi_b b_t. \tag{15}$$

Now we find that

$$\pi_{t+1} \overset{eq.(15)}{=} \tilde{\pi}_b b_{t+1} \overset{eq.(12)}{=} \tilde{\pi}_b\left(\rho b_t - \eta c_t\right) \overset{eq.(11)}{=} \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}c_t - \frac{\nu}{\beta}b_t,$$

[13] In contrast to the previous example, now marginal cost is a function of consumption and taxes, but taxes are determined by debt so we have the additional term νb_t in (11). If the underlying fiscal feedback rule is $\tau_t = \mu b_t$ where τ_t is log-linearized tax rate with steady state level τ_o, then $\nu = \mu\kappa\tau_o/(1-\tau_o)$ and $\lambda = \kappa(1/\sigma + \theta/\psi)$ where σ and ψ are parameters of private sector utility function, β is the private sector's discount factor, θ is the steady-state consumption to output ratio and λ is the slope of the Phillips curve. Public spending is a part of marginal cost too, but it is assumed to be constant and so does not enter the log-linearized version of the model.

[14] It states that real debt at the beginning of the next period $t+1$ is equal to real debt at the beginning of period t less taxes collected in period t, and this accumulates at the rate $1/\beta$, which is the real interest rate in the steady state. Collected taxes change because of either a change in the tax rate, or because of a change in the tax base. Parameter $\rho = (1-\mu\tau_o)/\beta$ is a function of the tax rate, and with stronger feedback μ the debt is stabilized faster. Parameter $\eta = \theta\tau_o/\beta$ describes the sensitivity of debt to the tax base. We assume that the steady state level of debt is zero, which eliminates the first-order effect of the interest rate and inflation on debt in (12).

[15] The criterion is derived under the assumption of steady state labour subsidy. Here parameter α is a function of model parameters, $\alpha = \theta\lambda/\epsilon$, and ϵ is the elasticity of substitution between any pair of monopolistically produced goods.

implying the private sector reaction function can also be written as

$$\pi_t = (\beta\rho\tilde{\pi}_b + \nu)\, b_t + (\lambda - \beta\eta\tilde{\pi}_b)\, c_t. \tag{16}$$

So, because the policy maker moves first within each period and because the private sector observes the policy, the private sector takes into account the 'instantaneous' influence of the policy choice, here measured by $(\lambda - \beta\eta\tilde{\pi}_b)$. The current-period responses of the private sector to the state and to the policy action are determined in part by the *next-period* response of the private sector to the state, $\tilde{\pi}_b$.

Following the same procedure as before, the Bellman equation characterizing discretionary policy becomes

$$Sb_t^2 = \min_{c_t} \left(\left(((\beta\rho\tilde{\pi}_b + \nu)\, b_t + (\lambda - \beta\eta\tilde{\pi}_b)\, c_t)^2 + \alpha c_t^2 \right) + \beta\tilde{S}\,(\rho b_t - \eta c_t)^2 \right). \tag{17}$$

Discretionary policy. Differentiation of (17) with respect to c_t yields the optimal policy response

$$c_t = -\frac{\left((\beta\rho\tilde{\pi}_b + \nu)\,(\lambda - \tilde{\pi}_b\eta\beta) - \beta\rho\eta\tilde{S} \right)}{\left((\lambda - \beta\eta\tilde{\pi}_b)^2 + \alpha + \beta\eta^2\tilde{S} \right)} b_t = c_b b_t. \tag{18}$$

The coefficient c_b in (18) determines the optimal policy feedback on the predetermined state, b_t, but now the feedback coefficient is a function of $\tilde{S}$. As $\tilde{S}$ determines the next-period loss and depends on the whole future path of the state, it is a function of future policy decisions. When the current-period policy maker chooses the optimal policy, it knows that the next-period policy maker will re-optimize. In this model this constraint is binding because $\eta \neq 0$. This implies the non-linearity of $c_b = c_b(\tilde{\pi}_b)$.

As before, in order to find the equilibrium value function S we substitute the optimal solution (18) into the Bellman equation (17) and, under the equilibrium condition $S = \tilde{S}$, obtain the following equation for the value function S

$$S = (\beta\rho\tilde{\pi}_b + \nu)^2 + \beta\rho^2 S - \frac{((\beta\rho\tilde{\pi}_b + \nu)\,(\lambda - \beta\eta\tilde{\pi}_b) - \beta\eta\rho S)^2}{((\lambda - \beta\eta\tilde{\pi}_b)^2 + \alpha + \beta\eta^2 S)}.$$

After some straightforward algebra we obtain a quadratic equation for S with a positive leading coefficient and a negative constant term. This equation has only one nonnegative solution

$$\begin{aligned} S \;=\; & \frac{1}{2\beta\eta^2}\left(-\left((\lambda - \beta\eta\tilde{\pi}_b)^2 - \beta\,(\rho\lambda + \nu\eta)^2 - \alpha\left(\beta\rho^2 - 1\right) \right) \right. \\ & \left. + \sqrt{\left((\lambda - \beta\eta\tilde{\pi}_b)^2 - \beta\,(\rho\lambda + \nu\eta)^2 - \alpha\,(\beta\rho^2 - 1) \right)^2 + 4\alpha\beta\eta^2(\beta\rho\tilde{\pi}_b + \nu)^2} \right) \geq 0. \end{aligned}$$

We plot S as a function of the equilibrium private sector decision $\tilde{\pi}_b = \pi_b$ in the lower chart of Panel II in Figure 1.[16] We comment on the shape later.

[16] The parameter values are $\beta = 0.99$, $\lambda = 0.0582$, $\nu = 0.0025$, $\xi = 0.925$, $\eta = 0.1875$ and $\alpha = 0.0087$. We report calibration of deep structural parameters in Online Appendix.

In order to obtain the optimal *policy* we substitute the equilibrium value of S into the feedback coefficient in (18) so

$$\begin{aligned} c_b &= -\frac{(\rho\lambda+\nu\eta)(\beta\rho\tilde{\pi}_b+\nu)}{(\rho\alpha+(\lambda-\beta\eta\tilde{\pi}_b)(\rho\lambda+\nu\eta))}+\Big(-\beta\left((\lambda-\beta\eta\tilde{\pi}_b)^2-\beta(\rho\lambda+\nu\eta)^2-\alpha\left(\beta\rho^2-1\right)\right) \\ &\quad +\sqrt{\left((\lambda-\tilde{\pi}_b\eta\beta)^2-\beta(\rho\lambda+\nu\eta)^2-\alpha\left(\beta\rho^2-1\right)\right)^2+4\beta\alpha\eta^2(\beta\rho\tilde{\pi}_b+\nu)^2}\Big) \\ &\quad /(2\eta\beta(\rho\alpha+(\lambda-\beta\eta\tilde{\pi}_b)(\rho\lambda+\nu\eta))) . \end{aligned} \tag{19}$$

The optimal policy feedback coefficient as a function of the equilibrium decision of the private sector, $\tilde{\pi}_b = \pi_b$, is plotted using a solid line in the top chart in Panel II in Figure 1. In contrast to the standard New Keynesian model, the decision function is now U-shaped in $\tilde{\pi}_b$, as we now explain.

Suppose that debt b_t is positive and the policy maker knows that firms believe that in the next period firms will react with higher inflation, $\tilde{\pi}_b \gg 0$. The policy maker also knows that current-period firms take into account the 'instantaneous' influence of the policy choice, as is measured by $(\lambda - \beta\eta\tilde{\pi}_b)$. The profit maximization problem for the firms implies that if any future response of inflation to debt is strong then higher demand (consumption) implies lower inflation, so $(\lambda - \beta\eta\tilde{\pi}_b) < 0$ if $\tilde{\pi}_b \gg 0$. Hence, in order to reduce the cost of inflation, optimal policy should increase consumption in response to a higher debt, $c_b > 0$. This also contributes to the debt stabilization. But a large movement in consumption is also costly, and this imposes a finite optimal value of c_b. In equilibrium, if π_b is large then c_b also rises. The social loss S is high, as the response to the positive debt by both consumption and inflation is consequently large.

Now suppose the policy maker knows that firms believe that in the next period firms will react to positive debt with lower inflation, $\tilde{\pi}_b \ll 0$. The profit maximization problem for the firms now implies that if future inflation is low then an increase of demand (consumption) will raise inflation, so $(\lambda - \beta\eta\tilde{\pi}_b) > 0$ if $\tilde{\pi}_b \ll 0$. Hence, the optimal policy is to increase consumption and $c_b > 0$. This will stabilize debt and moderate the fall in inflation. This regime is also characterized by high social loss S because of large response of consumption to debt.

Now suppose there is an 'intermediate' scenario and in the next period firms are expected to react only weakly to debt, so $\tilde{\pi}_b \simeq 0$. Now current-period firms will keep inflation low if the effect of higher taxation on marginal cost (via ν) is approximately offset by the negative effect of demand. This can only happen if consumption falls, i.e. if $c_b < 0$. A fall in demand slows down the debt stabilization, but still ensures debt stability. As the effect of taxation is going to reduce in the future, the next-period profit-optimizing firms are expected to keep inflation low. Hence a small $c_b < 0$ for small $\pi_b \simeq 0$ can characterize the optimal policy. The loss is small as in a response to the higher b_t the implied responses of π_t and c_t are small.

Private sector response. In order to obtain the time-invariant optimal decision of the private sector we substitute (14) into (16) and obtain

$$\pi_t = (\beta\rho\tilde{\pi}_b + \nu + (\lambda - \beta\eta\tilde{\pi}_b) c_b) b_t = \pi_b b_t. \tag{20}$$

Future decisions of the private sector $\tilde{\pi}_b$ affect the current decisions of the private sector π_b. In a discretionary equilibrium $\tilde{\pi}_b = \pi_b$ which yields

$$\pi_b = \frac{\nu + \lambda c_b}{(1 - \beta(\rho - \eta c_b))}. \tag{21}$$

The response of inflation to b_t is determined by policy c_b. Higher demand increases inflation, but the current price-setting decisions of firms depend on their future firms' decisions, which depend on the future policy and on future debt. With stronger reaction of consumption to debt, c_b, the parameter $\beta(\rho - \eta c_b) \in (0, 1)$ is smaller and debt is stabilized faster, from (12). This implies that the discounted effect of the future debt on current inflation is smaller. Hence, the total effect of debt on inflation $\pi_b = \pi_b(c_b)$ is *non-linear* and concave. We plot the optimal response of the private sector using a dashed line in the top chart of Panel II in Figure 1.

Dynamic complementarities and multiple discretionary equilibria. As in the previous example, we can search for dynamic complementarities, which enable us to better understand the nature of the discretionary equilibria. Following Cooper and John (1988), we say that we have a *dynamic complementarity* if the optimal decision of one agent is increasing in the decision of the other. In order to demonstrate complementarities it is convenient to look at the properties of the welfare function.

As before, we can derive welfare given each agent's response along the debt-stabilizing solution that starts at given b_t as

$$\begin{aligned} W_t &= -\frac{1}{2}\sum_{s=s}^{\infty}\beta^{s-t}\left(\pi_s^2 + \alpha c_s^2\right) \overset{eqs.(14,16)}{=} -\frac{1}{2}\left(\left(\beta\rho\pi_b + \nu + \lambda c_b - \beta\eta\pi_b c_b\right)^2 + \alpha c_b^2\right)\sum_{s=s}^{\infty}\beta^{s-t}b_s^2 \\ &\overset{eqs.(12,14,15)}{=} -\frac{1}{2}\left(\left(\beta\rho\pi_b + \nu + \lambda c_b - \beta\eta\pi_b c_b\right)^2 + \alpha c_b^2\right) b_t^2 \sum_{s=t}^{\infty}\left(\beta\left(\rho - \eta c_b\right)^2\right)^{s-t} \\ &= -\frac{1}{2}\frac{\left(\left(\beta\rho\pi_b + \lambda c_b + \nu - \beta\eta\pi_b c_b\right)^2 + \alpha c_b^2\right)}{\left(1 - \beta\left(\rho - \eta c_b\right)^2\right)} b_t^2. \end{aligned}$$

We also assume that welfare is finite so $(\rho - \eta c_b)^2 < 1/\beta$. The second derivative is

$$\frac{\partial^2 W_t}{\partial \pi_b \partial c_b} = \left(-\frac{\beta\rho\lambda + \beta\nu\eta}{1 - \beta\left(\rho - \eta c_b\right)^2} + 2\frac{\beta^2\eta\left(\left(\nu + \beta\rho\pi_b\right) + \left(\lambda - \beta\eta\pi_b\right)c_b\right)}{\left(1 - \beta\left(\rho - \eta c_b\right)^2\right)^2}\right) b_t^2. \tag{22}$$

If $\frac{\partial^2 W_t}{\partial \pi_b \partial c_b} > 0$ then we have dynamic complementarities between the private sector and the policy decisions: Higher inflation, set by the firms in response to a higher debt level, increases the marginal return to a policy decision that increases consumption in response to the higher debt. The shaded area in the top chart of Panel II in Figure 1 shows where the decisions of the private sector and of the policy maker are complementary whilst in the unshaded area the decisions

Table 1: New Keynesian Model with Debt Accumulation: Characteristics of Discretionary Equilibria

Discretionary Equilibria		'Active' A	Weakly 'passive' B	'Passive' C
Private	π_b	0.0095	0.1088	0.1885
sector	$\frac{\partial \pi_t}{\partial b_t} = (\beta\rho\pi_b + \nu)$	0.0124	0.0993	0.1690
response	$\frac{\partial \pi_t}{\partial c_t} = (\lambda - \beta\eta\pi_b)$	0.0565	0.0378	0.0229
Policy	c_b	-0.0513	0.2514	0.8505
Speed of adjustment	b_b	0.8935	0.8362	0.7228
Loss	S	0.0005	0.0403	0.0867
Complementarities	$\text{sign}\big(\partial^2 W_t/\partial\pi_b\partial c_b\big)$	$-$	$+$	$+$
Numerical methods		OS, BD, PP	PP	PP, BD*

Notes:
* Obtained with initialization $S = 1$, $\pi_b = 0$, Söderlind (1999)

are substitutable. Because of complementarities between the decisions of agents now multiple equilibria can arise, see Cooper and John (1988), Vives (2005).

As the first chart in Panel II in Figure 1 demonstrates, for our parameterization we have three discretionary equilibria. Two of them, labelled B and C, are in the area where there is dynamic complementarity of the decisions of the aggregate private sector and of the policy maker. The remaining equilibrium A is in the area where those decisions are substitutable. In Table 1 we report the numerical characteristics of these equilibria.

If the policy maker knows that any next-period firm will raise inflation *sufficiently high* in response to positive debt values, then it is optimal for the policy maker to increase consumption. The higher future inflation is going to be, the higher consumption needs to be set; higher consumption increases demand and thus further increases inflation, but higher consumption also slows down the rise of inflation as the firms' response to lower *future* taxes will reduce the response to higher current demand. Dynamic complementarities are at work and two discretionary equilibria, B and C, arise.

If the policy maker knows that the next-period firms will raise inflation only weakly or even deflate in a response to a positive debt, then with stronger *deflationary* decisions by firms, the policy maker will optimally raise consumption and demand and stabilize inflation. Consumption and inflation are dynamic substitutes and only one equilibrium A arises.

Equilibrium trajectories. An initial deviation of debt from its steady state value, $\bar{b} \neq 0$, is the only reason for the future dynamic adjustment of the economy in our model. Panel I in Figure 2 plots responses of the economy to an initial unit-deviation of debt from its steady state value. The three solutions discussed above correspond to the three different paths towards the steady state.

In equilibrium A the policy maker expects that firms anticipate that in the next period firms will set low inflation and optimally respond with low consumption. The slow convergence of debt is consistent with depressed demand and low inflation along the path.

In equilibria B and C the policy maker expects that firms anticipate that the next-period firms will set high inflation and so consumption optimally increases. Higher consumption contributes to faster debt stabilization. All responses are stronger in equilibrium C than in equilibrium B, as we discussed above.

Numerical solution. Because of the simplicity of this model we are able to substitute (21) into (19) and obtain the resulting expression in the form of a univariate polynomial with given coefficients. We can then employ a standard numerical technique to find the roots of this polynomial.[17] We could instead specify parameters of the dynamic system (11)-(12) and of the policy objective (13) and employ some numerical dynamic programming routines to obtain discretionary equilibria directly. Two almost always used solution methods by Oudiz and Sachs (1985) and Backus and Driffill (1986) employ essentially the same *iterative algorithm* that requires initial values for π_b and S and then updates them *simultaneously*.[18] Oudiz and Sachs (1985) suggest using a particular initialization to begin iterations, while Backus and Driffill (1986) suggest that any initialization should work. If the equilibrium is unique, as Oudiz and Sachs (1985) conjecture, and the algorithm converges, then the choice of initialization should not matter indeed; Söderlind (1999), who presents a popular implementation of this algorithm, also suggests using any positive S and zero π_b as a first guess.

With multiple equilibria different initializations of the same algorithm can lead to different solutions. However, random initializations of a particular algorithm may not find all of possible equilibria, and different algorithms with different asymptotic stability properties need to be constructed. The last line in Table 1 reports whether the equilibrium can be obtained by numerical routines. In addition to the Oudiz and Sachs (1985) and Backus and Driffill (1986) algorithm with different initializations, herewith labelled 'OS' and 'BD' correspondingly, we checked which equilibria we obtain if for an arbitrary policy c_b^0 we find π_b^0 using (21), then find c_b^1 using (19), and use it to compute π_b^1 and so on until convergence.[19] We label the latter algorithm 'PP' for policy–private sector iterations and it is based on *consecutive* update of agents' decisions and is very different in stability properties from the OS/BD algorithm.[20] Again, different initializations of c_b^0 can generate different equilibria, not necessarily those that are obtained by the OS/BD algorithm.

All equilibria can be obtained by at least one of these iterative procedures, given some appropriate initializations and other parameters.

2.2.3 Switching equilibria

In this model with government debt the optimizing policy maker at time t chooses policy c_b knowing that the private sector's decision π_b explicitly depends on the private sector's decision $\tilde{\pi}_b$ in period $t+1$. All agents know that at any time $s > t$ the optimizing policy maker will choose the best policy, based on the current decision rule of the private sector that itself depends on its own next-period decision rule. The period-$(s+1)$ decision of the private sector is a function

[17] We used available MATLAB routines.

[18] Currie and Levine (1993) suggest a similar algorithm to solve continuous-time problems.

[19] Here superscrit is an iteration count.

[20] The stability properties of the PP algorithm crucially depend on the choice of damping between iterations.

of period-$(s+1)$ policy and hence of period-$(s+2)$ private sector decisions. Thus, any policy maker chooses current policy c_b conditioned on the aggregated private sector's beliefs about future policy $\tilde{c}_b$. The existing dynamic complementarities between the decisions of the private sector and the policy maker create the possibility of different beliefs about the future course of policy; with different beliefs, multiple discretionary equilibria arise.

In contrast to our first example, the presence of government debt eliminates the disconnect between current and future periods, and different beliefs about future policy result in different current actions by both sets of agents. Therefore, three different sets of beliefs correspond to three different adjustment paths of the economy, with each of these paths differing in speed of convergence back to equilibrium. Remember that in linear models all variables adjust at the same speed (this is uniquely determined by ρ in the standard New Keynesian model). In this example the presence of government debt is not only sufficient for multiplicity, but also necessary, as different paths are only possible if the adjustment of the predetermined state (debt) is affected by agents' decisions.

The dynamic structure of interactions between the private sector and the policy maker implies it is easy to move the economy from one equilibrium to another. Suppose that at time t agents observe some positive b_t. The monetary policy maker, which (as leader) is required to decide first, reduces demand because (i) it knows that current-period firms are expecting tight monetary policy in all (current and future) periods, and (ii) it anticipates that current-period firms, when they make their current-period decisions, will rationally lower inflation given their expectations of tight future policy. The policy maker rationally reduces demand and next-period debt remains high.

Now suppose that after the current-period policy maker has acted, and *if multiple equilibria exist*, current-period firms change their beliefs about future policy. Instead they now increase inflation relative to the level that would be consistent with their former beliefs. They still react to realized policy and to the state, but the strength of their reaction depends on their beliefs about future policy. The next-period policy maker will observe high debt and high inflation and will know that firms expect lax policy in all subsequent periods, and it will anticipate that all future firms will increase inflation after the policy maker acts. We have shown that it will be optimal for the next-period discretionary policy maker to validate the beliefs held by the private sector at the time the next-period policy decision is made and stimulate demand.

Firms choose inflation based on beliefs, but they are free to change beliefs when making decisions in the face of an exogenous event. Although the private sector observes current-period policy and rationally responds to it, no precommitment of future policy makers to *the-same-as-observed* policy coefficient c_b is expected. The private sector only expects that the policy maker will re-optimize, apply the same decision process, react to the current state and deliver *the same policy as expected at the time when the private sector makes the decision*. Although the policy maker moves first within each period, it cannot manipulate the beliefs of the private sector about *future* policy. Albanesi et al. (2003) and King and Wolman (2004) characterize similar situations as 'expectations traps'.

This framework does not suggest which equilibria are more realistic or those that should be preferred. Nothing in this model indicates how the private sector sets its beliefs about which equilibrium will prevail if an exogenous event occurs. The framework implies that if there are several possible equilibria and the private sector does change its beliefs about which one will prevail, then it is optimal for the policy maker to change policy. If some sufficiently 'bad' event occurs and the private sector changes its beliefs, then the best the policy maker can do is to validate the new beliefs of the private sector, and the welfare-dominant equilibrium becomes unattainable.

2.2.4 Robustness of Results and their Empirical Relevance

The strength of fiscal control of debt and 'active'/'passive' policies. Parameters ν and ρ of the model in (11)-(12) are functions of parameter μ, the strength of fiscal feedback in the underlying tax rule $\tau_t = \mu b_t$. We vary μ between zero and some relatively large number and plot π_b, c_b and welfare against μ in Panel II in Figure 2.[21] Different discretionary equilibria as functions of μ correspond to different branches. Panel II in Figure 2 demonstrates that only equilibrium C survives for small μ, in particular for $\mu = 0$; as μ increases only equilibrium A survives. If μ is sufficiently large then monetary policy reacts to debt only weakly and stabilizes inflation in a conventional way, so can be classified as 'active'. Fiscal policy can be classified as 'passive' as it is devoted entirely to the control of domestic debt. If μ is zero or relatively small then monetary policy controls the debt tightly in order to ensure that it will converge back to the steady state, so it can be classified as 'passive'. Fiscal policy can be classified as 'active' as it pursues some other targets but not the control of debt. In this regime inflation is accommodated as this helps to reduce the real debt. Our classification resembles the one in Leeper (1991), but it differs in policy design: we consider discretionary monetary policy, not a policy formulated in terms of simple rules. We discover three equilibria for a weak fiscal feedback $0 < \mu \ll \infty$. Equilibria B and C are qualitatively similar, and only differ by the strength of reactions. It may be difficult to distinguish them empirically. One of them may be implausible and further research in equilibrium selection mechanisms may help to eliminate one of them. Because the issues of equilibrium selection are beyond the scope of this paper, in the rest of this section we simply distinguish between two types of interactions, and use equilibrium C to illustrate 'passive' monetary policy, as equilibrium B does not survive for very small values of μ.

Empirical relevance. Do we observe such multiple equilibria in practice? Some authors have argued that we observe multiple *regimes*.[22] In particular, Davig and Leeper (2006b) (herewith DL) document fluctuating policies in the United States in terms of policy rules. DL identify four different regimes depending on the interactions of monetary and fiscal policy. They assume that economic policy is formulated in terms of simple rules and each policy maker can implement either an 'active' or a 'passive' rule. As a result, there are four possible combinations of active/passive monetary policy (we label them AM and PM correspondingly) and active/passive fiscal policy (labelled AF and PF). The authors estimate these rules and document the sequence of movements from one such regime to another in the post-war US history. We plot the sequence of regimes in Panel III in Figure 2. This picture is adapted from Figure 5 in DL. For every year the relative width of every color corresponds to the probability that each regime prevails. For example, in

[21] We plot welfare as minus loss, hence negative numbers. The loss is measured in percentage of steady state consumption; the loss is small as we work with the deterministic model, assuming that the only sourse of displacement is the higher debt level (1%) in the initial moment.

[22] See, for example, Engel and Hamilton (1990), Clarida et al. (2003), Davig and Leeper (2006a,b).

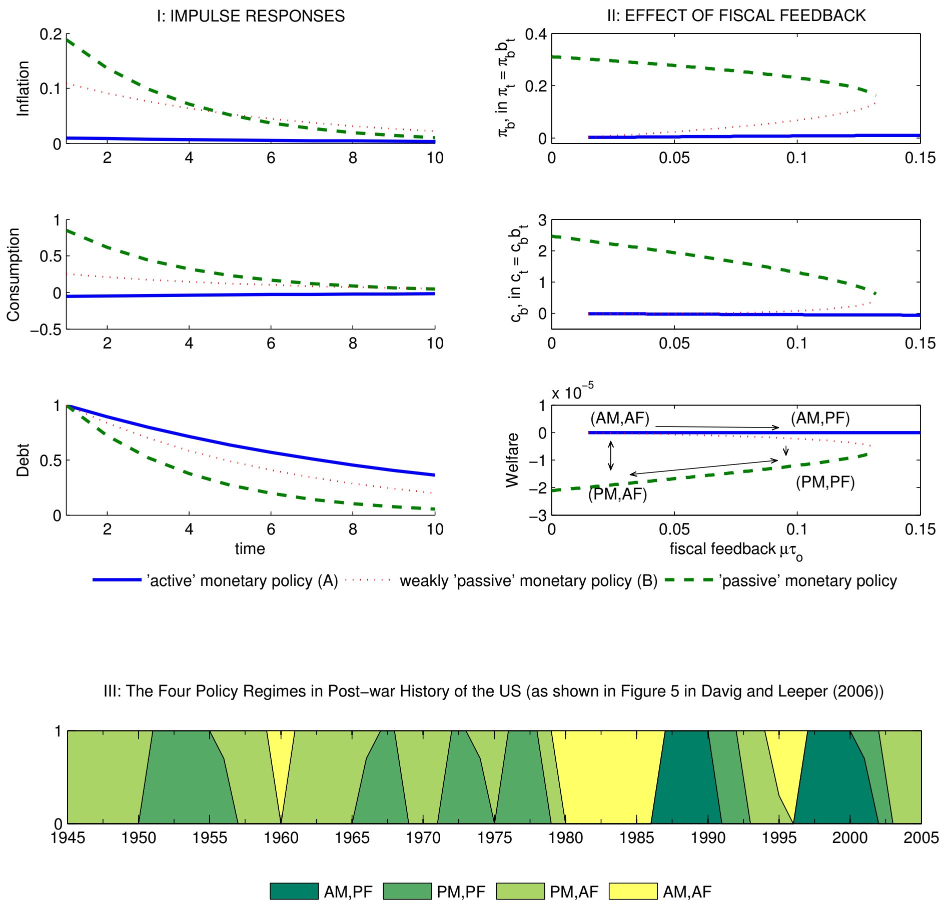

Figure 2: Discretionary Equilibria and Policy Regimes

1956 we have approximately a 70% probability of (PM, PF) regime and a 30% probability of (PM, AF) while in 1985 there is an estimated 100% probability of (AM, AF). In 1945 the US economy starts at (PM, AF) and it arrives back into the same state in 2005.

Davig and Leeper (2006a) discuss that it is generally difficult to justify *optimal* switching policies, and so *ad hoc* simple-rules-based regime switches dominate the analysis. In this paper we demonstrate that each of switching policies can be optimal under discretion. We argue that the Markov-switching model in DL implies the same regimes that we identified as equilibria A and C. Using our model we can interpret these movements as switches between the two discretionary equilibria. Our fiscal feedback parameter μ is similar in effect to the fiscal feedback on output in the estimated fiscal rules in DL. We use the bottom chart in Panel II in Figure 2 and label where the four policy regimes identified by DL might lie. Arrows show all realized movements between the states that happened in the post-war period in the US. Some of the changes in regimes can be seen as switches between the two equilibria. Our model is silent the nature of the sunspot that causes the 'regime shifts', but we note instead a rather particular sequence of 'jumps' and 'falls' between the two equilibria. The welfare-dominant equilibrium is attained in 'jumps' from the state with lowest welfare only. There are three such 'jumps' from (PM, AF) to (AM, AF), in 1959, 1980 and in 1994, one of which resulted in an immediate 'fall' back. It is also apparent that the route from the best to the worst equilibrium is via (AM, PF) to (PM, PF), and this route is characterized by the minimal loss in welfare, as the distance between two branches in the welfare chart in Panel II of Figure 2 is the smallest. These moves happened in 1991 and 2001, both post-recession years in which the Federal Reserve lowered interest rates. The observed regularity invites new research to explain the pattern, but further exploration is beyond the scope of this paper.

2.3 New Keynesian Model with Capital Accumulation

The previous model demonstrates the *existence* of multiple equilibria in LQ RE models under discretion and the role of endogenous predetermined states. It was deliberately chosen to as simple as possible to enable us to do this in an effective way. More realistic models will have several decision variables of agents and potentially several states. In most cases we invariably resort to numerical methods of finding discretionary equilibria.

We now turn to a model with capital accumulation. This example demonstrates a further feature of policy under discretion, and shows that LQ RE models can have *two different types* of multiple equilibria. We have already shown that multiple equilibria can arise because of the dynamic complementarity of the decisions of the private sector and the policy maker. Now we show that the dynamic complementarity between the *different decisions of the private sector* can also lead to multiple equilibria. King and Wolman (2004) obtain equilibria of this second type in a non-linear model; they originate from what they call multiple 'point-in-time' equilibria.

The point-in-time equilibria are likely to generate multiple discretionary equilibria in any moderately complex model, but they are difficult to find with conventional numerical algorithms. In order to develop a successful algorithm we need to know the properties of equilibria we should search for.

The primary objective of this example is to demonstrate the existence of multiple point-in-time equilibria and show how their existence implies the existence of multiple discretionary equilibria. We shall generalize our results in Section 4. These results allow us to develop a numerical algorithm that finds *all* point-in-time equilibria associated with any given policy and reduces the number of discretionary equilibria that would otherwise remain undiscovered.

In order to demonstrate this type of multiplicity we need a model that has several private sector decision variables. We use a New Keynesian model with capital accumulation, adopted from Woodford (2003a), Woodford (2005) and Sveen and Weinke (2005), that has just this feature. As before, we delegate all technical details to the Online Appendix.

2.3.1 Discretionary Policy

Log-linearized about the steady state, the equations that describe the aggregate decisions of the private sector and the evolution of the state can be written as

$$c_t = c_{t+1} - \sigma(i_t - \pi_{t+1}), \tag{23}$$

$$\Delta k_{t+1} = \beta \Delta k_{t+2} + \frac{1}{\varepsilon_\psi} \left(\left(1 - \beta\left(1-\delta\right)\right) ms_{t+1} - \left(i_t - \pi_{t+1}\right)\right), \tag{24}$$

$$\pi_t = \beta \pi_{t+1} + \lambda_c c_t + \lambda_o k_{t+1} - \lambda_k k_t. \tag{25}$$

The aggregate state variable in this economy is the real capital stock, k_t and the initial state $\bar{k}$ is known to all agents. Equation (23) is the standard Euler equation for aggregate consumption expenditure c_t, i_t is the nominal interest rate and π_t is the inflation rate. Equation (24) describes capital accumulation with depreciation rate δ, adjustment cost parameter ε_ψ, and real marginal savings $ms_t = \frac{\phi+1}{(1-\alpha)} \left(\zeta c_t + \frac{1-\zeta}{\delta} \left(k_{t+1} - (1-\delta) k_t\right)\right) + \frac{1}{\sigma} c_t - \frac{\alpha\phi+1}{(1-\alpha)} k_t$.[23] Equation (25) is the appropriate New Keynesian Phillips curve. The private sector chooses consumption, c_t, inflation, π_t, and next-period capital, k_{t+1}.

The policy maker's control variable is nominal interest rate i_t. The inter-temporal policy maker's welfare criterion is defined by the quadratic loss function

$$L_t = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \left(\pi_s^2 + \omega y_s^2\right) = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \left(\pi_s^2 + \omega \left(\zeta c_s + \frac{1-\zeta}{\delta} \left(k_{s+1} - \left(1-\delta\right) k_s\right)\right)^2 \right),$$

where we substituted out output y_t using the national income identity. Parameter ω is treated as given.[24]

The policy maker knows the laws of motion (23)-(25) of the aggregate economy and takes them into account when formulating its policy. The policy maker finds the best action every period, knows that future policy makers have the freedom to change policy, and knows that future policy makers will apply the same decision process.

[23] Parameters ζ, ϕ, σ and α are the steady state consumption to output ratio, elasticity of labour supply, inverse elasticity of inter-temporal substitution of consumption, and capital share in production function, correspondingly. Parameter β is the private sector's discount factor. The system is log-linearized about the zero-inflation steady state. We assume capital can be rented. All derived parameters are given in Appendix A.

[24] We use 'traditional' welfare metric for this case. This choice is unimportant for our results and is a reasonable metric that an actual policymaker might choose for a complex model.

Under discretion, at every point in time t the decision rule of each agent is a linear function of the current state

$$
\begin{aligned}
i_t &= \iota_k k_t, & (26)\\
k_{t+1} &= k_k k_t & (27)\\
c_t &= c_k k_t & (28)\\
\pi_t &= \pi_k k_t. & (29)
\end{aligned}
$$

We lead (27)-(29) one period and use (23)-(25) to write private sector decisions as a response to the state and response to policy

$$
\begin{aligned}
k_{t+1} &= \tilde{k}_S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) k_t + \tilde{k}_P\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) i_t, & (30)\\
\pi_t &= \tilde{\pi}_S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) k_t + \tilde{\pi}_P\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) i_t, & (31)\\
c_t &= \tilde{c}_S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) k_t + \tilde{c}_P\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) i_t, & (32)
\end{aligned}
$$

where we use the subscripts S for state and P for policy. We also put tildes over the coefficients to emphasize that they depend on the *next-period* decisions of the private sector, $\tilde{k}_k$, $\tilde{c}_k$ and $\tilde{\pi}_k$. Written this way, (30)-(32) isolate the 'instantaneous' influence of policy on private sector decisions. We report the exact form of the coefficients in Appendix A.

The Bellman equation characterizing discretionary policy becomes

$$
\begin{aligned}
Sk_t^2 = \min_{i_t} \Bigg(& \left(\tilde{\pi}_S k_t + \tilde{\pi}_P i_t\right)^2 + \omega \left(\left(\zeta \tilde{c}_S + \frac{1-\zeta}{\delta} \left(\tilde{k}_S - (1-\delta) \right) \right) k_t \right. & (33)\\
& \left. + \left(\zeta \tilde{c}_P + \frac{1-\zeta}{\delta} \tilde{k}_P \right) i_t \right)^2 + \beta \tilde{S} \left(\tilde{k}_S k_t + \tilde{k}_P i_t \right)^2 \Bigg).
\end{aligned}
$$

Differentiation of (33) with respect to i_t yields the optimal policy response

$$
i_t = -\frac{\left(\tilde{\pi}_P \tilde{\pi}_S + \left(\zeta \tilde{c}_P + \frac{1-\zeta}{\delta} \tilde{k}_P \right) \omega \left(\zeta \tilde{c}_S + \frac{1-\zeta}{\delta} \left(\tilde{k}_S - (1-\delta) \right) \right) + \beta \tilde{S} \tilde{k}_P \tilde{k}_S \right)}{\left(\tilde{\pi}_P^2 + \omega \left(\zeta \tilde{c}_P + \frac{1-\zeta}{\delta} \tilde{k}_P \right)^2 + \beta \tilde{S} \tilde{k}_P^2 \right)} k_t = \iota_k k_t. \quad (34)
$$

The coefficient ι_k in (34) determines the optimal policy feedback on the predetermined state, k_t, with the feedback coefficient a function of $\tilde{S}$. $\tilde{S}$ determines the next-period loss, which depends on the whole future path of the state, which in turn is a function of future policy decisions. When the current-period policy maker chooses the best policy, it knows that the next-period policy maker will re-optimize. In this model this constraint is binding. It ensures the non-linearity of $\iota_k = \iota_k \left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k \right)$.

In order to find the equilibrium value function S we substitute the optimal solution (34) into the Bellman equation (33) and, under the equilibrium condition $S = \tilde{S}$, obtain the following quadratic equation with positive leading coefficient and a negative constant term

$$
\beta S^2 + \mu S - \omega \left(\frac{(1-\zeta)}{\delta} \left(\tilde{\pi}_S - \left(\tilde{k}_S - (1-\delta) \right) \frac{\tilde{\pi}_P}{\tilde{k}_P} \right) + \zeta \frac{\left(\tilde{\pi}_S \tilde{c}_P - \tilde{\pi}_P \tilde{c}_S \right)}{\tilde{k}_P} \right)^2 = 0,
$$

where coefficient $\mu = \mu\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right)$ is given in Appendix A. This equation has only one nonnegative solution

$$\begin{aligned} S &= -\frac{1}{2\beta}\left(\mu + \sqrt{\mu^2 + 4\beta\omega\left(\frac{(1-\zeta)}{\delta}\left(\tilde{\pi}_S - \left(\tilde{k}_S - (1-\delta)\right)\frac{\tilde{\pi}_P}{\tilde{k}_P}\right) + \zeta\frac{(\tilde{\pi}_S\tilde{c}_P - \tilde{\pi}_P\tilde{c}_S)}{\tilde{k}_P}\right)^2}\right) \\ &= S\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) = S\left(k_k, c_k, \pi_k\right). \end{aligned}$$

The last equality holds for the time-invariant private sector response.

In order to obtain the optimal policy we substitute S into (34) to give

$$\iota_k = \iota_k\left(\tilde{k}_k, \tilde{c}_k, \tilde{\pi}_k\right) = \iota_k\left(k_k, c_k, \pi_k\right). \tag{35}$$

By construction, for every triplet $\{k_k, c_k, \pi_k\}$ that describes a time-invariant private sector response we obtain a *unique* ι_k that describes the policy decision.

2.3.2 Private Sector Response: Point-in-Time Equilibria

We substitute equation (26) into (30)-(32) and, after some manipulations, obtain the following system that describes time-invariant optimal response of the private sector

$$k_k = \frac{1}{\beta\nu_k + \lambda_k\nu_r}\left(\left(\beta\left(1 - \nu_o\tilde{k}_k\right) + \lambda_o\nu_r\right)k_k - \nu_r\pi_k + \left(\lambda_c\nu_r - \beta\nu_c\right)c_k + \beta\nu_r\iota_k\right), \tag{36}$$

$$\pi_k = \left(\beta\tilde{\pi}_k + \lambda_o\right)k_k + \lambda_c c_k - \lambda_k, \tag{37}$$

$$c_k = \frac{1}{\beta + \sigma\lambda_c}\left(\left(\beta\tilde{c}_k - \sigma\lambda_o\right)k_k + \sigma\pi_k + \sigma\left(\lambda_k - \beta\iota_k\right)\right). \tag{38}$$

where all coefficients ν are given in Appendix A. From the first two equations it is immediately apparent that for a given policy ι_k, in a response to a positive state k_t higher consumption raises inflation but it also makes profit optimizing firms increase next-period capital stock in order to meet anticipated increased demand; higher next-period capital raises inflation too. The decisions to raise consumption and to increase the next-period capital stock are dynamic complements.

In the previous example the decision of the private sector was unique given policy. Here we have complementarity between the effects of different decisions of the private sector and the multiplicity of *point-in-time equilibria* is likely to arise: there can be several private sector responses to *the same* policy decision. The nature of the multiplicity is now different.

We now simplify system (36)-(38) by assuming the equilibrium conditions $k_k = \tilde{k}_k$, $\pi_k = \tilde{\pi}_k$ and $c_k = \tilde{c}_k$. We solve the last equation with respect to π_k

$$\pi_k = \frac{1}{\sigma}\left(\sigma\lambda_o k_k + \left(\beta + \sigma\lambda_c - \beta k_k\right)c_k + \sigma\left(\beta\iota_k - \lambda_k\right)\right), \tag{39}$$

and substitute into the first two equations. We obtain the following system

$$c_k = \sigma\frac{\left(\nu_k - k_k + \nu_o k_k^2\right)}{\left(\nu_r k_k - \left(\nu_r + \sigma\nu_c\right)\right)}, \tag{40}$$

$$c_k = \sigma\frac{\left(\lambda_o k_k^2 - \left(\lambda_k - \beta\iota_k\right)k_k - \iota_k\right)}{\left(\beta k_k^2 - \left(1 + \beta + \sigma\lambda_c\right)k_k + 1\right)}. \tag{41}$$

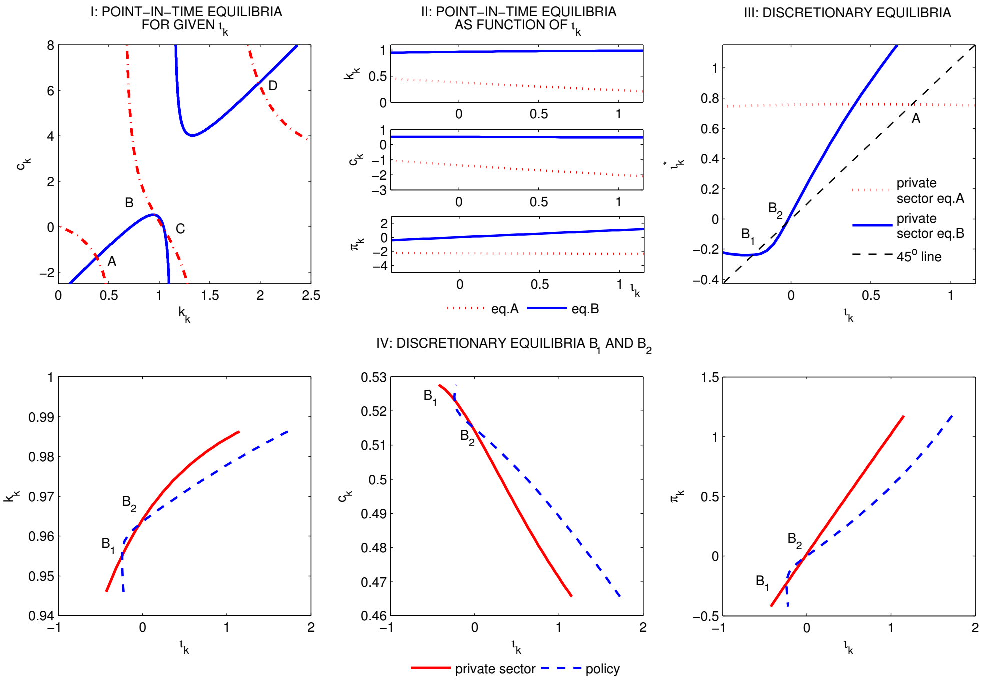

Figure 3: Discretionary Equilibria in the New Keynesian Model with Capital Accumulation

Solutions to this system for a given value of ι_k are plotted in Panel I in Figure 3. We plot the response $c_k(k_k)$, given by equation (40), with a solid line. We plot the response $k_k(c_k)$ given by equation (41) with a dash-dotted line.[25] Now there are four pairs $\{c_k, k_k\}$ that solve (40)-(41) and are labelled as point-in-time equilibria A, B, C and D. For every pair $\{c_k, k_k\}$ we can calculate the unique $\pi_k(c_k, k_k)$ using (39). For all values of ι_k in equilibria A and B the economy is stable as $|k_k| < 1$, while it is unstable in equilibria C and D with $k_k > 1$. We do not consider equilibria C and D further and plot how $\{c_k, k_k, \pi_k\}$ change with ι_k in equilibria A and B in Panel II in Figure 3. We use dotted and solid lines for equilibria A and B correspondingly.

It is apparent that in response to a *given* policy ι_k the aggregate private sector coordinates on one of the two point-in-time equilibria. Suppose the capital stock is higher than in the steady state. In equilibrium A consumption and investment fall so that excessive capital stock is quickly reduced. Inflation falls relatively fast because demand is low and the high level of productive capital also has a deflationary effect. In equilibrium B consumption rises by only a little and the next-period capital is chosen to ensure a slow reduction of the excessive capital stock. A combination of the high level of productive capacity with high demand ensures only slight fall in inflation. All variables converge back to the steady state much more slowly.

For a given ι_k we compute the two triplets $\{c_k(\iota_k), k_k(\iota_k), \pi_k(\iota_k)\}$ that describe point-in-time equilibria A and B. For every point-in-time equilibrium the discretionary policy maker will *optimally* choose the *unique* $\iota_k^* = \iota_k(c_k, k_k, \pi_k)$, see equation (35). For each triplet $\{c_k(\iota_k), k_k(\iota_k), \pi_k(\iota_k)\}$

[25] We calibrate the model as $\beta = 0.99$, $\sigma = 1$, $\phi = 1$, $\varepsilon = 11$, $\varepsilon_\psi = 3$, $\alpha = 0.36$ and $\delta = 0.025$.

Table 2: New Keynesian Model with Capital Accumulation: Characteristics of Discretionary Equilibria

Discretionary Equilibria		'Fast' A	'Moderately Slow' B_1	'Slow' B_2
Private sector response	k_k	0.2625	0.9552	0.9630
	c_k	-2.3367	-0.2257	-0.0145
	π_k	-1.8590	0.5232	0.5155
Policy	ι_k	0.7577	-0.2391	-0.0331
Speed of adjustment	k_k	0.2625	0.9552	0.9630
Loss	S	6.4064	0.5302	0.0102
Complementarities	$\text{sign}\left(\frac{\partial^2 W_t}{\partial k_k \partial \iota_k}, \frac{\partial^2 W_t}{\partial \pi_k \partial \iota_k}, \frac{\partial^2 W_t}{\partial c_k \partial \iota_k}\right)$	$(+,-,+)$	$(+,+,-)$	$(+,+,+)$
Numerical methods		BD*, PP	PP	OS, BD, PP

Notes:
* Obtained with initialization $S = 1$, $\pi_k = c_k = k_k = 0$, Söderlind (1999).

we plot $\iota_k^* = \iota_k^* (c_k (\iota_k), k_k (\iota_k), \pi_k (\iota_k)) \equiv \iota_k^* (\iota_k)$ with dotted and solid lines for point-in-time equilibria A and B correspondingly in Panel III in Figure 3. Points of intersection of lines $\iota_k^* = \iota_k^* (\iota_k)$ with the 45-degree line are points of *discretionary policy equilibria.*

The dotted line (point-in-time equilibrium A) intersects the 45-degree line once. If the policy maker expects that the private sector coordinates on equilibrium A, it will choose *positive* ι_k. Acting consistently with current and future positive response of interest rate to the higher-than-steady-state-level of capital, the private sector coordinates on the equilibrium with a large contraction in demand and investment and low inflation. Capital is stabilized quickly.

The solid line (point-in-time equilibrium B) intersects the 45-degree line twice, points labelled B_1 and B_2 in Panel III in Figure 3. If the policy maker expects that the point-in-time equilibrium B will prevail, it chooses *negative* ι_k. Acting consistently with current and future negative response of the interest rate to higher-than-steady-state-level of capital, the private sector coordinates on an equilibrium with a small expansion in demand, a small contraction in future capital stock and a small fall in inflation. However, in this case there are dynamic complementarities between the decisions of the private sector and the policy maker, so *two* discretionary equilibria arise, similar to the previous example with government debt. The mechanism is also similar: lower marginal cost results in lower inflation which implies a lower interest rate and so results in lower marginal cost.

Specifically, suppose the real capital stock is above the steady state, $k_t > 0$. Suppose that the policy maker knows that the private sector believes that the future policy maker will lower interest rate in response to any positive k_t. Suppose the policy maker expects that in a response to the current-period fall in interest rate the aggregated private sector coordinates on equilibrium B with higher consumption, slightly lower next-period capital and small fall in inflation. It will be optimal for the policy maker to reduce the interest rate and validate the beliefs of the private sector. However, assume a further fall in interest rate. It results in even higher consumption, which crowds out investment even more and so the next-period capital stock is even lower. The deflationary effect of even lower next-period capital stock on marginal cost outweighs the effect of consumption, so inflation falls by more, that makes the assumed further fall in the interest rate both rational and optimal. Multiple discretionary policy equilibria B_1 and B_2 arise, further illustrated in Panel IV in Figure 3.[26] All characteristics of the three equilibria A, B_1 and B_2 are reported in Table 2.

As in our previous examples we can formally check whether the private sector and policy maker decisions are dynamic complements. Straightforward substitutions yield the following welfare function

$$
\begin{aligned}
W_t &= -\frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2+\omega\left(\zeta c_s+\frac{1-\zeta}{\delta}\left(k_{s+1}-(1-\delta)\,k_s\right)\right)^2\right)\\
&= -\frac{1}{2}\left(\left(\pi_S+\pi_P\iota_k\right)^2+\omega\left(\zeta\left(c_S+c_P\iota_k\right)+\frac{1-\zeta}{\delta}\left(k_S+k_P\iota_k-(1-\delta)\right)\right)^2\right)\frac{k_t^2}{1-\beta k_k^2}.
\end{aligned}
$$

We compute the sign of each of the second order derivatives $\frac{\partial^2 W_t}{\partial k_k \partial \iota_k}$, $\frac{\partial^2 W_t}{\partial \pi_k \partial \iota_k}$ and $\frac{\partial^2 W_t}{\partial c_k \partial \iota_k}$ for each of the discretionary equilibria A, B_1 and B_2 and report them in Table 2. It is apparent that in each equilibrium at least one derivative is positive. Hence, the private sector and the policy maker decisions are dynamic complements and each point-in-time equilibrium can generate more than one discretionary equilibrium, as we have demonstrated.

2.3.3 Numerical Solution

The simplicity of this particular model and the prior knowledge of the number and location of the equilibria allows us to use polynomial root-finding algorithms to find them. However, as in our previous example, we also checked which equilibria can be obtained with numerical algorithms OD and BD. We present the results in the last line of Table 2. This algorithm can deliver both equilibria A and B_2.

We can also compute *all* point-in-time equilibria for some policy decision ι_k^0. Then, for *each* point-in-time equilibrium that stabilizes the economy we can use (34) and compute optimal policy ι_k^1. We then use ι_k^1 to compute *the same* point-in-time equilibrium and so on until convergence.[27] If this iterative procedure converges, then we label the resulting equilibrium with 'PP' in Table 2. We are able to get equilibria A, B_1 and B_2 in this way.

To motivate subsequent generalizations in Sections 3-5, note that the PP iterative algorithm is different from the OS and BD algorithm in its asymptotic stability properties and identical initializations of the PP and of the OS/BD algorithm can lead to different equilibria. It is also important that we know *how many* point-in-time equilibria exist before searching for discretionary equilibria. If we use a particular *iterative* algorithm to find all responses of the private sector, the multiplicity of point-in-time equilibria implies that we may not be able to find all of them, regardless of initialization. The knowledge of the nature of the solution we seek helps to design different algorithms, with different stability properties.

[26] Because of multi-variate private sector's response, Panel IV plots optimal policy given the private sector's *response to a policy.* In the previous example in Figure 1 we plotted the optimal policy for an *arbitrary* decision of the private sector.

[27] Here superscrit is an iteration count.

3 The General Framework

In the previous section we demonstrated the existence of multiple equilibria by examining particular models and were able to use the useful expository device of dynamic complementarities to describe the mechanisms at work. Now we extend some of our results for the entire class of non-singular LQ RE models. First, we prove that in an economy without endogenous predetermined state variables the discretionary equilibrium is unique, as in our first example. Second, we show how many policy-induced 'point-in-time' private sector equilibria exist and discuss how to find all of them. This result eliminates the need to search for dynamic complementarities between the private sector's actions in order to assess the possibility of multiple point-in-time equilibria. Third, we prove that for a given private sector response, the optimal policy decision is unique. Forth, we demonstrate that (for most economic applications) there is a finite number of locally isolated, or determinate, equilibria. These results have important implications for the design of the numerical algorithms needed to locate and isolate many of the equilibria that cannot be found using more conventional methods. We also present an algorithm that was able to locate all equilibria in our examples.

3.1 Discretionary Policy

We assume a non-singular linear deterministic rational expectations model, augmented by a vector of control instruments. Specifically, the evolution of the economy is explained by the linear system

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [u_t], \tag{42}$$

where y_t is an n_1-vector of predetermined variables with initial conditions y_0 given, x_t is n_2-vector of non-predetermined (or jump) variables with $\lim_{t\to\infty} x_t = 0$, and u_t is a $k-$vector of policy instruments of the policy maker. For notational convenience we define the n-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$. We assume A_{22} is non-singular.

Typically, the second block of equations in this system represents an aggregation of the first order conditions to the optimization problems of the private sector, which has decision variables x_t. Additionally, there is a first block of equations which explains the evolution of the predetermined state variables y_t. (Such predetermined states include observed shocks, lagged decision variables, including inflation and interest rates, as well as stock variables.) These two blocks describe the 'evolution of the economy' as observed by the policy maker.

The inter-temporal policy maker's welfare criterion is defined by the quadratic loss function

$$L_t = \frac{1}{2}\sum_{s=t}^{\infty} \beta^{s-t} g_s' \mathcal{Q} g_s = \frac{1}{2}\sum_{s=t}^{\infty} \beta^{s-t} \left(z_s' Q z_s + 2 z_s' P u_s + u_s' R u_s \right). \tag{43}$$

The elements of vector g_s are the goal variables of the policy maker, $g_s = \mathcal{C}(z_s', u_s')'$. Matrix $\mathcal{Q}$ is assumed to be symmetric and positive semi-definite.[28]

[28] It is standard to assume that R is symmetric positive definite (see Anderson et al. (1996), for example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $\mathcal{Q}$ being positive definite can be weakened to $\mathcal{Q}$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer (2003)). The analysis in this paper is valid for $R \equiv 0$.

Assumption 1 *Suppose that at any time t the private sector and the policy maker respond only to the current state*

$$u_t = \mathcal{F}(y_t) = -Fy_t, \quad (44)$$
$$x_t = \mathcal{N}(y_t) = -Ny_t. \quad (45)$$

This assumption rules out non-stationarity of policy and private sector decisions, i.e. any time-dependence from the more general formulation $u_t = \mathcal{F}(t; y_t, y_{t-1}, .., y_{t-k}, ..)$, $x_t = \mathcal{N}(t; y_t, y_{t-1}, .., y_{t-k}, ..)$, and restricts policy decisions to memoryless feedback rules. We also assume that rules are *linear* in the state.

We define discretionary policy as satisfying several constraints. We want to assume that the policy maker can implement (or truthfully announce) at each point of time its policy decision *before* the private sector selects its own action x_t.

Assumption 2 *At each time t the private sector observes the current decision u_t and expects that future policy makers at any time $s > t$ will re-optimize, will apply the same decision process and implement policy (44).*

Proposition 1 *Given Assumption (2) the current aggregate decision of the private sector can be written as a linear feedback function*

$$x_t = -Jy_t - Ku_t, \quad (46)$$

where

$$J(N) = (A_{22} + NA_{12})^{-1}(A_{21} + NA_{11}), \quad (47)$$
$$K(N) = (A_{22} + NA_{12})^{-1}(B_2 + NB_1). \quad (48)$$

Proof. Relationship (45) can be taken with one lead forward and y_{t+1} is substituted from the first equation (42). We obtain:

$$\begin{aligned} x_{t+1} &= -\tilde{N}y_{t+1} = -\tilde{N}(A_{11}y_t + A_{12}x_t + B_1u_t) \\ &= A_{21}y_t + A_{22}x_t + B_2u_t, \end{aligned} \quad (49)$$

from where it follows:

$$x_t = -(A_{22} + \tilde{N}A_{12})^{-1}[(A_{21} + \tilde{N}A_{11})y_t + (B_2 + \tilde{N}B_1)u_t] = -J\left(\tilde{N}\right)y_t - K\left(\tilde{N}\right)u_t. \quad (50)$$

where $J\left(\tilde{N}\right)$ and $K\left(\tilde{N}\right)$ are defined as in (47)-(48). Invertibility of A_{22} ensures invertibility of $A_{22} + \tilde{N}A_{12}$ almost surely. ■

Proposition 1 implies that the policy maker, which moves before the private sector, takes into account its 'instantaneous' influence on the choice of x_t, which is measured by $-K\left(\tilde{N}\right)$ and which also depends on the future response of the private sector to the state, $\tilde{N}$.

Assumption 3 *At each point in time t the policy maker knows Assumptions 1 and 2.*

Definition 1 *Policy determined by (44) is discretionary if the policy maker finds it optimal to continue to follow it in every period $s > t$ given Assumptions (1) – (3).*

Hence, we look for a policy u_t that satisfies the following Bellman equation

$$L_t(y_t) = \min_{u_t} \left(\frac{1}{2} \left(y_s' Q^* y_t + 2y_t' P^* u_t + u_t' R^* u_s \right) + \beta L_{t+1} \left(A^* y_t + B^* u_t \right) \right),$$

with

$$Q^* = Q_{11} - Q_{12}J - J'Q_{21} + J'Q_{22}J, \qquad P^* = J'Q_{22}K - Q_{12}K + P_1 - J'P_2, \tag{51}$$

$$R^* = K'Q_{22}K + R - K'P_2 - P_2'K, \qquad A^* = A_{11} - A_{12}J, \qquad B^* = B_1 - A_{12}K. \tag{52}$$

Because of the quadratic nature of the per-period objective in (43) and because policy and private sector decisions are both linear in the state, the discounted loss will necessarily have quadratic form in the state

$$L_t(y_t) = \frac{1}{2} y_t' S y_t. \tag{53}$$

The Bellman equation characterizing discretionary policy, therefore, becomes

$$y_t' S y_t = \min_{u_t} \left(y_s' \left(Q^* + \beta A^{*\prime} S A^* \right) y_t + 2y_t' \left(P^* + \beta A^{*\prime} S B^* \right) u_t + u_t' \left(R^* + \beta B^{*\prime} S B^* \right) u_t \right). \tag{54}$$

We have outlined a deterministic setup, both here and in the examples in Section 2. However, none of the results depend on this, as we can always add an appropriate vector of shocks and appeal to the certainty equivalence property of LQ models. This, however, would complicate the analysis unnecessarily in order to demonstrate the main point.[29]

For a policy F and the private sector response N, the evolution of the state variable satisfies the following equation

$$y_{t+1} = My_t, \tag{55}$$

where $M = A_{11} - A_{12}N - B_1F$.

3.2 Discretionary equilibrium as a 'triplet' of matrices

Given y_0 and system matrices A and B, matrices N and F define the trajectories $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ in a unique way and vise versa: if we know that $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ solve discretionary optimization problem stated above then, by construction, there are unique time-invariant linear relationships between them which we label by N and F. Matrix S defines the cost-to-go along a trajectory. Given the one-to-one mapping between equilibrium trajectories and $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ and the triplet of matrices $\mathcal{T} = \{N, S, F\}$, it is convenient to continue with definition of policy equilibrium in terms of $\mathcal{T}$, not trajectories.

The following Proposition derives the first order conditions for a discretionary optimization problem.

[29] Shocks can be included into vector y_t, see e.g. Anderson et al. (1996).

Proposition 2 (First order conditions) *The first-order conditions to the discretionary optimization problem (42) – (43) can be written in the following form*

$$N = (A_{22} + NA_{12})^{-1}((A_{21} - B_2F) + N(A_{11} - B_1F)), \tag{56}$$
$$F = (R^* + \beta B^{*\prime}SB^*)^{-1}\left(P^{*\prime} + \beta B^{*\prime}SA^*\right), \tag{57}$$
$$S = Q^* + F'R^*F - F'P^{*\prime} - P^*F + \beta\left(A^* - B^*F\right)' S\left(A^* - B^*F\right), \tag{58}$$

where matrices Q^*, P^*, R^*, A^*, and B^* are defined in (47)-(48) and (51)-(52)

Proof. From relationships (45) and (46) it immediately follows that

$$N = J - KF. \tag{59}$$

A straightforward substitution of (47)-(46) into (59) leads to (56)

The discretionary policy can be determined from (54) by differentiating with respect to u_t

$$u_t = -(R^* + \beta B^{*\prime}SB^*)^{-1}(P^{*\prime} + \beta B^{*\prime}SA^*)y_t = -Fy_t,$$

from where the policy maker's reaction function is defined by (57). Now, we substitute policy $u_t = -Fy_t$ into (54) and obtain equation (58) for S. ■

Definition 2 *The triplet $\mathcal{T} = \{N, S, F\}$ is a discretionary equilibrium if it satisfies the system of first order conditions (56)-(58).*

Definition 2 implicitly assumes that the first order conditions are necessary and sufficient conditions of optimality. We proceed with this assumption and demonstrate later in Proposition 5 that, under the assumption of symmetric positive semi-definite $\mathcal{Q}$, the second order conditions for the minimum are always satisfied.

4 Properties of discretionary equilibria

This section describes some properties of discretionary equilibria that can help to locate them.

We start the section on *multiple* equilibria with demonstration of a particular case where the discretionary equilibrium is *unique.* [30]

Proposition 3 (A Special Case) *Suppose A_{22} is non-singular, $A_{12} = 0$ and $B_1 = 0$. Then if the discretionary equilibrium exists it is unique.*

Proof. Formulae (46) suggests $K = A_{22}^{-1}B_2$ so it does not depend on N. It follows that R^* does not depend on N and, generally speaking, is non-singular.

Equation (59) can be written as

$$A_{22}N - NA_{11} = A_{21} - B_2F,$$

[30] Proposition 3 proves what appears to be a well known fact, but we were unable to find a published proof. Typically, when dealing with a particular problem in this class of models, researchers easily find the particular solution, and it is clear that it is unique by construction, see e.g. Clarida et al. (1999).

so there is no quadratic term in N. Similarly, equation (57) collapses to

$$F = (R^*)^{-1} \left(\left(B_2' A_{22}^{-1\prime} \mathcal{Q}_{22}' - \mathcal{P}_2' \right) \left(N + A_{22}^{-1} B_2 F \right) - \left(\mathcal{Q}_{12} A_{22}^{-1} B_2 - \mathcal{P}_1 \right)' \right),$$

and hence S does not affect F. Both equations together constitute a linear in coefficients of F and N system of $(k+n_2)n_1$ equations (after applying the vec-operator). If the system is non-singular, the solution is unique. Having determined F we can find corresponding S from equation (58), which always has a unique symmetric solution, as we demonstrate in later in this section. ■

In this example, $B_1 = 0$ suggests that predetermined state variables cannot be affected by policy and $A_{12} = 0$ suggests that they cannot be affected by private sector's decisions. A typical example of models in this class is a system where the only predetermined variables are potentially (auto-)correlated shocks, which are *exogenous* predetermined state variables. This is by no means uncommon in models that omit potentially important *endogenous* predetermined state variables such as capital or debt for the sake of simplicity.

Formally, our assumptions result in two of three non-linear first order conditions, (56)-(58), becoming linear and disconnected from the third equation in this special case. It is clear that this is unlikely to happen under more general conditions. In what follows, we shall study the first-order conditions in their most general form. Proposition 3 suggests that the model has to have predetermined endogenous state variables in order to be able to generate multiple equilibria under discretion.

The absence of endogenous predetermined state variables ensures complete disconnect between time periods. Discretionary policy maker knows that all future policy makers will re-optimize but without endogenous states this has no implications for the current policy choice.

Example in Section 2.3 demonstrates existence of multiple point-in-time equilibria. We explain their existence by dynamic complementarities of decisions of the private sector, given policy. In general case, it is more difficult to find complementarities among private sector's decisions than to find *all* point-in-time equilibria directly.

For a given policy response written in the form of linear rule $u_t = -Fy_t$ the coefficients of (56) depend only on the structural system matrices A and B. The next Proposition describes all solutions $N = N(F)$.

Proposition 4 *Under the following conditions:*

i) matrix $C = \begin{bmatrix} A_{11} - B_1 F & A_{12} \\ A_{21} - B_2 F & A_{22} \end{bmatrix}$ *has all distinct eigenvalues, and* V_{22} *in diagonalization* $C = V^{-1} \Lambda V$ *is invertible for any permutation of eigenvalues in* Λ*;*

ii) for some $\rho \in \mathbb{R}$, $\rho > 0$, m *is the number of eigenvalues of* C *that are strictly less than* ρ *in modulus;*

one of the three following situations is almost always possible.

1. *If* $m = n_2$ *then there is a unique solution* N *to (56) such that all eigenvalues of transition matrix* $M = A_{11} - A_{12}N - B_1 F$ *in*

$$y_{t+1} = (A_{11} - B_1 F)\, y_t + A_{12} x_t = M y_t$$

are strictly less than ρ *in modulus.*

2. *If $m > n_2$ then there are no solutions to (56) such that all eigenvalues of matrix $M(N)$ are strictly less than ρ in modulus.*

3. *If $m < n_2$ then there are at most $\binom{n-m}{n_2-m}$ of different solutions N to (56) such that all eigenvalues of matrix $M(N)$ are strictly less than ρ in modulus. Only one solution is locally asymptotically stable. All solutions can be found using standard eigenvalue decomposition methods.*

This proposition generalizes the well known Blanchard and Kahn (1980) condition for rational expectations equilibrium to the class of time-invariant solutions. We prove it in Appendix B.

Condition i) rules out a continuum of solutions to (56), as shown by Freiling (2002). Condition ii) leaves the choice of parameter ρ to a researcher. Parameter ρ defines the asymptotic growth rate of y_t. Because in infinite-horizon optimization problems transversality conditions are *necessary* conditions, i.e. they follow from optimization, setting $\rho = 1/\sqrt{\beta}$ is the most nonrestrictive.[31]

In all examples in Section 2 the policy maker had only one instrument, either demand or interest rate. For these examples we demonstrated the uniqueness of optimal policy response, given time-invariant private sector response. The next Proposition states that the result holds even for multi-variate policy instrument. The proof is delegated to Appendix C.

Proposition 5 *Suppose N is given. The following two results hold:*

1. *There is a unique symmetric positive semi-definite solution S to (58) if the matrix pair (A^*, B^*) is controllable, i.e. if the controllability matrix $[B^*, A^*B^*, A^{*2}B^*, ..., A^{*n_1-1}B^*]$ has full row rank;*[32]

2. *The policy function F, which is uniquely determined from (57) for given S, is such that all eigenvalues of transition matrix M (that defines the evolution of the system under control)*

$$\begin{aligned} y_{t+1} &= A_{11}y_t + A_{12}x_t + B_1u_t = (A_{11} - A_{12}J)\,y_t + (B_1 - A_{12}K)\,u_t \\ &= (A^* - B^*F)\,y_t = M\,(F(S))\,y_t, \end{aligned} \tag{60}$$

   are strictly less than $1/\sqrt{\beta}$ in modulus.

It follows that $\lim_{t\to\infty} \beta^{-t/2} y_t = 0$. Thus, the policy reaction function ensures finite loss. It also follows that necessary conditions for optimality (58)-(57) are sufficient, because with symmetric and positive semi-definite matrix S the second-order conditions for minimum are always satisfied.

Propositions 4 and 5 demonstrate that the private sector and the policy maker are 'non-symmetric' agents: while multiple private sector equilibria are possible if there are several decision instruments, the optimal response of the policy maker is always unique regardless the number of policy instruments.

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[31] It is sometimes suggested to set $\rho = 1$ (which is the case in Blanchard and Kahn (1980)). The over-restrictiveness of $\rho = 1$ is discussed, for example, in Sims (2001) for a similar class of LQ RE problems.

[32] The requirement of controllability of (A^*, B^*) is standard for the linear-quadratic optimal control. We use this condition as a sufficient condition. We do not discuss whether this is necessary condition.

Corollary 1 *Propositions 4 and 5 suggest that distinct discretionary equilibria have different matrices N and vice versa. Therefore, we can label a discretionary equilibrium not by a triplet of matrices, $T = \{N, S, F\}$ but by a single matrix 'identifier' N.*

Finally, we claim that all discretionary equilibria are determinate in the following sense.

Definition 3 *Discretionary equilibrium is determinate if, for given initial conditions, $\{z_t\}_{t=0}^{\infty}$ is a path under optimal discretionary policy then there exist no other path $\{\tilde{z}_t\}_{t=0}^{\infty}$ such that $\|z_t - \tilde{z}_t\| < \varepsilon$ in each period, where $\varepsilon > 0$ is any arbitrary small real number.*

Thus determinacy is viewed as a property of trajectories and not of their limit points.

Proposition 6 *Suppose we can find a discretionary equilibrium and compute the Jacobian of the system of first order conditions (56)-(58), $\mathcal{J}$. If $\det(\mathcal{J}) \neq 0$ then:*

1. *There can be at most a finite number of other discretionary equilibria.*
2. *All discretionary equilibria are determinate.*

Proof. There is a one-to-one correspondence between a trajectory and a triplet $T = \{N, S, F\}$. $q = (n + k) \times n_1$ coefficients of T solve polynomial system (56)-(58) of q equations. If the determinant of the Jacobian of the *polynomial* system of first order conditions is not equal to zero *identically*, then the system can *only* have a finite number of locally isolated solutions T. The local isolation is equivalent to determinacy. ■

We can say nothing if $\det(\mathcal{J}) = 0$ as it can either be equal to zero *identically* and so a continuum of solutions is possible, or it might be that $\det(\mathcal{J})$ has an isolated zero in this point and, again, we have the finite number of locally isolated discretionary equilibria. Condition $\det(\mathcal{J}) \neq 0$ is likely to be satisfied in most economic applications.[33]

5 Finding Equilibria

Examples 2 and 3 in Section 2 demonstrate that the number of discretionary equilibria can be greater than the number of point-in-time equilibria if there are dynamic complementarities between the policy decision and private sector decisions. Although a search for complementarities can help to understand *why* multiplicity arises, it is difficult to use this information in order to *find* the equilibria, or to prove that there is no more than $\binom{n-m}{n_2-m}$ equilibria.

Our examples show that even simple models lead to very complex systems of polynomial equations to solve. Practically, we have to rely on numerical solutions. The current literature on discretionary policy, which uses a numerical approach to find equilibria, almost always uses some variations of the OS and BD iterative algorithms based on a *simultaneous update* of S and N. The examples in Section 2 show that *different initializations* of the same algorithm can converge to different solutions.

However, as we have seen, *different algorithms*, that are based on different updating schemes, may converge to different solutions even if we start with the same initialization. Propositions

[33] The Online Appendix gives the expression for the Jacobian.

4 and 5 have practical implications for the design of different numerical routines. Proposition 4 states how to find *all* point-in-time equilibria for an arbitrary policy F. The proposition also implies that *iterative* algorithms to find N are unhelpful – they only find the unique locally asymptotically stable solution – but eigenvalue decomposition methods find all solutions. Proposition 5 implies that for any private sector response, N, we can use *any* algorithm to search for for the optimal policy, F, as the solution is unique, no local extrema exists and so the initialization does not matter. One can iterate between F and the *corresponding* N as we did in examples (the PP algorithm), and find discretionary equilibria that are impossible to locate with the OS/BD algorithm.[34] Propositions 4 and 5 provide a guide to locating and separating discretionary solutions.

Finally, when solving a stochastic model we can substantially reduce the size of the problem. In stochastic LQ RE models we write shocks as part of the vector of predetermined state variables.[35] The resulting number of predetermined states may be substantial. However, because exogenous state variables are not the source of multiplicity in LQ RE models, we prove in the Online Appendix the following Proposition.

Proposition 7 *A stochastic LQ RE model can be solved in two steps. First, solve the deterministic model where all shock variables are excluded. Second, compute the coefficients on the stochastic components that make up matrices N and F and the necessary components of matrix S in a unique way.*

Hence, the search for multiple solutions in models of discretionary policy can only involve their endogenous deterministic components; this often keeps the dimension of the problem relatively low.

6 Conclusion

We have described discretionary equilibria in the general class of LQ RE models. We illustrated the potential for existence of multiple rational expectations equilibria. Because decisions of the policymaker depend on expectations of the private sector that are based on future policy, dynamic complementarities between decisions of agents can create multiplicity, and different beliefs about future policy correspond to different discretionary equilibria. The policy maker cannot control the expectations of the private sector about future policy, and current policy decisions have to accommodate the expectations set by the past-period private sector. A sunspot, that changes private sector beliefs about future policy, determines which equilibrium will prevail.

These interactions and sunspot-driven changes between equilibria can generate rich dynamics, similar to those often observed in aggregate data.

We generalized several results to the entire class of non-singular LQ RE models. We described all types of equilibria that can arise in these models. Our analysis can be used to develop numerical methods that find most discretionary equilibria.

[34] We discuss numerical algorithms and search strategies in more details in the Online Appendix.

[35] See e.g. Anderson et al. (1996).

A Parameters of Models in Examples

$$\begin{aligned}
\lambda_c &= \kappa\left(\frac{(\phi+\alpha)\zeta}{1-\alpha}+\frac{1}{\sigma}\right), \lambda_o = \kappa\frac{(\phi+\alpha)(1-\zeta)}{(1-\alpha)\delta}, \lambda_k = \kappa\left(\frac{(\phi+\alpha)(1-\zeta)(1-\delta)}{\delta(1-\alpha)}+\frac{\alpha(1+\phi)}{(1-\alpha)}\right)\\
\zeta &= 1-\frac{\delta\alpha(\varepsilon-1)}{\varepsilon(\delta-\ln\beta)}, \bar{\nu} = \left(\varepsilon_\psi(1+\beta)+\frac{(1-\beta(1-\delta))}{1-\alpha}\left(\frac{(\phi+1)(1-\zeta)(1-\delta)}{\delta}+\alpha\phi+1\right)\right)^{-1},\\
\nu_o &= \left(\varepsilon_\psi\beta+\frac{(1-\beta(1-\delta))(\phi+1)(1-\zeta)}{(1-\alpha)\delta}\right)\bar{\nu}, \nu_c = (1-\beta(1-\delta))\left(\frac{(\phi+1)\zeta}{1-\alpha}+\frac{1}{\sigma}\right)\bar{\nu},\\
\nu_r &= \left(1-(1-\beta(1-\delta))\left(\frac{(\phi+1)\zeta\sigma}{(1-\alpha)}+1\right)\right)\bar{\nu}, \nu_k = \varepsilon_\psi\bar{\nu}, \xi = 1-(\nu_r+\sigma\nu_c)\pi_k-\nu_c c_k-\nu_o k_k,\\
k_S &= \frac{\nu_k}{\xi}, k_P = -\frac{(\nu_r+\sigma\nu_c)}{\xi}, c_S = \frac{\nu_k c_k+\nu_k\sigma\pi_k}{\xi}, c_P = \frac{-\sigma-\nu_r c_k+\sigma\nu_o k_k}{\xi},\\
\pi_S &= \frac{1}{\xi}\left(-\lambda_k+\nu_k\lambda_o+((\beta+\sigma\lambda_c)\nu_k+(\nu_r+\sigma\nu_c)\lambda_k)\pi_k+(\lambda_c\nu_k+\nu_c\lambda_k)c_k+\lambda_k\nu_o k_k\right),\\
\pi_P &= -\frac{1}{\xi}\left(\sigma\lambda_c+\lambda_o\nu_r+\sigma\nu_c\lambda_o+\beta(\nu_r+\sigma\nu_c)\pi_k+\lambda_c\nu_r c_k-\sigma\lambda_c\nu_o k_k\right)\\
\mu &= \left(\frac{\pi_P^2}{k_P^2}+\omega\left(\eta\frac{c_P}{k_P}+\gamma\right)^2-\beta\left(\left(\pi_P\frac{k_S}{k_P}-\pi_S\right)^2+\omega\left(\eta\left(c_S-c_P\frac{k_S}{k_P}\right)-\gamma(1-\delta)\right)^2\right)\right)
\end{aligned}$$

B Proof of Proposition (4)

First, all solutions of equation (56) also solve the Non-Symmetric Continuous Algebraic Riccati Equation

$$NC_{12}N + C_{22}N - NC_{11} - C_{21} = 0. \tag{61}$$

Indeed, we multiply both sides of (56) by $(A_{22}+NA_{12})$ and, at most, we also acquire all solutions of $A_{22}+NA_{12}=0$. Matrix C was defined in Proposition (4).

By assumption, matrix C can be diagonalized as $C = V^{-1}\Lambda V$.[36] Matrix V is the matrix of left eigenvectors which correspond to eigenvalues Λ. Arrange the eigenvalues so that Λ_u is a diagonal matrix of size n_2 and Λ_s a diagonal matrix of size $n_1 = n - n_2$. Rearrange similarly V and partition it to give

$$\Lambda = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix} \text{ and } V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Now, construct $N = V_{22}^{-1}V_{21}$. Matrix V_{22} is invertible by assumption, but this assumption is unlikely to be restrictive.

It is known from the control literature (see e.g. Medanic (1982) Th. 1, Freiling (2002) Th. 3.3) that *any* solution of (61) can be represented in the form of $V_{22}^{-1}V_{21}$ for some adequate Jordan

[36]In what follows we always assume that matrix C is *simple*, i.e. all its eigenvalues are of geometric multiplicity one, and the column rank of M is equal to n. This case is of practical interest; but Freiling (2002) discusses implications of higher geometric multiplicity.

basis of C. If all eigenvalues of C are simple then there are at most $\binom{n}{n_2}$ of different solutions N. Note that we did not make any assumptions about matrices Λ_s and Λ_u apart from assuming that they are of particular size.

We fix some value $\rho > 0$ and rearrange rows of V such that Λ_u collects all eigenvalues that are greater than ρ in modulus. Suppose there are $m \leq n_2$ of them, so Λ_u might also have $n_2 - m$ eigenvalues that are not greater than ρ in modulus. For any solution N in the form of $V_{22}^{-1}V_{21}$ matrix $M = C_{11} - C_{12}V_{22}^{-1}V_{21}$. Freiling (2002), Blake (2004) prove that the eigenvalues of $C_{11} - C_{12}V_{22}^{-1}V_{21}$ are Λ_s and the eigenvalues of $C_{11} + C_{22}V_{22}^{-1}V_{21}$ are Λ_u.

It follows that if $m = n_2$ then Λ_u is uniquely determined and $V_{22}^{-1}V_{21}$ is a unique solution (as in Blanchard and Kahn (1980)). If $m < n_2$ we can construct Λ_u in at most $\binom{n-m}{n_2-m}$ ways, collecting different combinations of smaller than ρ in modulus eigenvalues into Λ_u, and correspondingly rearranging rows of matrix V.

Medanic (1982) demonstrates that only one of these $\binom{n-m}{n_2-m}$ solutions, the dichotomic solution, is locally asymptotically stable.

C Proof of Proposition (5)

First, equation (58) is equivalent to the following Discrete Algebraic Riccati Equation

$$S = Q^* + \beta A^{*\prime} S A^* - \left(P^{*\prime} + \beta B^{*\prime} S A^*\right)' \left(R^* + \beta B^{*\prime} S B^*\right)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^*\right), \tag{62}$$

provided we can use (57). Indeed, start with (62) and add and subtract additional terms:

$$\begin{aligned} S &= Q^* + \beta A^{*\prime} S A^* - (\beta B^{*\prime} S A^* + P^{*\prime})' F = Q^* + F' R^* F - F' P^{*\prime} - P^* F \\ &\quad + \beta (A^* - B^* F)' S (A^* - B^* F) + F' P^{*\prime} - F' R^* F + \beta F' B^{*\prime} S (A^* - B^* F) \\ &= Q^* + F' R^* F - F' P^{*\prime} - P^* F + \beta (A^* - B^* F)' S (A^* - B^* F) \\ &\quad + F' \left[\beta B^{*\prime} S (A^* - B^* F) + P^{*\prime} - R^* F\right]. \end{aligned}$$

The term in square brackets is zero because of (57) and we obtain (58).

Second, properties of solutions to equation (62) are known from the control literature. If the pair matrices $(\beta^{\frac{1}{2}} A^*, \beta^{\frac{1}{2}} B^*)$ in (62) is controllable (i.e. if the $k \times n_1$ controllability matrix $[\beta^{\frac{1}{2}} B^*, \beta^{\frac{2}{2}} A^* B^*, \beta^{\frac{3}{2}} A^{*2} B^*, ..., \beta^{\frac{n_1}{2}} A^{*n_1 - 1} B^*]$ has rank n_1 or full row rank) then the solution pair $\{S, F\}$ to (62) and (57) exists and unique, and all eigenvalues of matrix $\beta^{\frac{1}{2}} M = \beta^{\frac{1}{2}} (A^* - B^* F)$ are strictly inside the unit circle, see e.g. (Kwakernaak and Sivan (1972, Ch. 6)). The controllability of $(\beta^{\frac{1}{2}} A^*, \beta^{\frac{1}{2}} B^*)$ is equivalent to the controllability of (A^*, B^*).

Third, it is also a textbook result that matrix S is symmetric and positive semi-definite if $\hat{Q}^* = \begin{bmatrix} Q^* & P^* \\ P^{*\prime} & R^* \end{bmatrix}$ is symmetric and positive semi-definite. One can easily demonstrate that $\hat{Q}^* = (\mathcal{C}\Psi)' \mathcal{Q} (\mathcal{C}\Psi)$, where $\Psi = \begin{bmatrix} I & 0 \\ -J & -K \\ 0 & I \end{bmatrix}$ and $\mathcal{C}$ is defined in Section 3.1. Because $\mathcal{Q}$ is symmetric and positive semi-definite by assumption then $\hat{Q}^*$ has the same properties. Hence S is symmetric and positive semi-definite.

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