General closed-form solutions to the dynamic optimization problem in incomplete markets

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GENERAL CLOSED-FORM SOLUTIONS TO THE DYNAMIC OPTIMIZATION PROBLEM IN INCOMPLETE MARKETS

ABSTRACT. In this paper, we provide general closed-form solutions to the incomplete-market random-coefficient dynamic optimization problem without the restrictive assumption of exponential or HARA utility function. Moreover, we explicitly express the optimal portfolio as a function of the optimal consumption and show the impact of optimal consumption on the optimal portfolio.

Key words: stochastic, incomplete markets

1 Introduction

Dynamic optimization has been used extensively in the economic and financial literature. Examples include incomplete markets, stochastic volatility and random coefficients models. The contemporary literature usually adopts random coefficient models (the parameters of the model are dependent on a random external economic factor) or non-tradable assets models. Examples include Musiela and Zariphopoulou (2007), Cvitanic and Zapatero (2004), Focardi and Fabozzi (2004), Fleming (2004) and Pham (2002).

In order to derive explicit solution to the optimization problem, the previous studies relied exclusively on exponential or HARA utility functions. This assumption is restrictive and sometimes unrealistic, since other common and more appropriate functional forms exist.

In this paper, we relax the exponential or HARA utility assumption. In doing so, we derive general closed form solutions to the random-coefficient incomplete-market dynamic optimization problem without imposing restrictions on the functional form of utility. Furthermore, we explicitly derive a functional relationship between the optimal portfolio and optimal consumption and show the impact of consumption on the optimal portfolio.

2 The model

We consider a standard investment-consumption model, which includes a risky asset, a risk-free asset and a random external economic factor (see, for example, Fleming (2004)). This implies a two-dimensional standard Brownian motion $\{(W_s^1, W_s^2), \mathcal{F}_s\}_{t \leq s \leq T}$ based on the probability space $(\Omega, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t \leq s \leq T}$ is the augmentation of filtration. The risk-free asset price process is $S_0 = e^{\int_t^T r(Y_s) ds}$, where $r(Y_s) \in C_b^2(R)$ is the rate of return and Y_s is the economic factor.

The risky asset price process is given by

$$dS_s = S_t \left\{ \mu(Y_s) \, dt + \sigma(Y_s) \, dW_s^1 \right\}, \tag{1}$$

where $\mu(Y_t)$ and $\sigma(Y_t)$ are the rate of return and the volatility, respectively. The economic factor process is given by

$$dY_s = b(Y_s) \, dt + \rho dW_s^1 + \sqrt{1-\rho^2} dW_s^{(2)}, Y_t = y, \tag{2}$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions and $b(Y_s) \in C^1(R)$ with bounded derivative.

The wealth process is given by

$$X_T^{\pi,c} = x + \int_t^T \{r(Y_s) X_s^{\pi,c} + (\mu(Y_s) - r(Y_s)\pi_s) - c_s\} ds + \int_t^T \pi_s \sigma(Y_s) dW_s^1, \tag{3}$$

where x is the initial wealth, $\{\pi_t, \mathcal{F}_s\}_{t \leq s \leq T}$ is the portfolio process and $\{c_t, \mathcal{F}_s\}_{t \leq s \leq T}$ is the consumption process, with $\int_t^T \pi_s^2 ds < \infty$, $\int_t^T c_s ds < \infty$ and $c \geq 0$. The trading strategy $(\pi_s, c_s) \in \mathcal{A}(x, y)$ is admissible (that is, $X_s^{\pi,c} \geq 0$).

The investor's objective is to maximize the expected utility of wealth and consumption

$$V(t, x, y) = \underset{\pi_t, c_t}{SupE} \left[u_1 (X_T^{\pi,c}) + \int_t^T u_2(c_s) ds \mid \mathcal{F}_t \right], \tag{4}$$

where $V(.)$ is the value function, $u(.)$ is a continuous, bounded and strictly concave utility function.

The value function satisfies the Hamiltonian-Jacobi-Bellman PDE

$$V_t + r(y) x V_x + g(y) V_y + \frac{1}{2} V_{yy} +$$

$$\underset{\pi_t, c}{Sup}\left\{\frac{1}{2}\pi_t^2\sigma^2(y)V_{xx} + [\pi_t\sigma(y)\theta(y) - c_t]V_x + \rho\sigma(y)\pi_t V_{xy} + u_2(c_t)\right\} = 0,$$

$$V(T, x, y) = u(x), \tag{5}$$

where $\theta(Y_t) \equiv \sigma^{-1}(Y_t)(\mu(Y_t) - r(Y_t))$. Hence, the optimal solutions are

$$\pi_t^* = -\sigma^{-1}(y)\frac{\theta(y)V_x + \rho V_{xy}}{V_{xx}}, \tag{6}$$

$$u_2'(c_t^*) = V_x. \tag{7}$$

Lemma. We can obtain an *exact* fixed-coeffiecient Taylor expansion of $u_2(c_t^*)$.

Proof. Consider the following Taylor expansion around a

$$u_2(c_t) = u_2(a) + u_2'(a)(c_t - a) + \frac{1}{2}u_2''(a)(c_t - a)^2 + R(c_t), \tag{8}$$

where $R(c_t)$ is the remainder. Our objective is to minimize $R(c_t)$

$$\min_{c_t}\left\{R(x) = \left(u_2(c_t) - \left[u_2(a) + u_2'(a)(c_t - a) + \frac{1}{2}u_2''(a)(c_t - a)^2\right]\right)\right\}.$$

The solution yields

$$R'(\hat{c}_t) = u_2'(\hat{c}_t) - u_2'(a) - u_2''(a)(\hat{c}_t - a) = 0, \tag{9}$$

and thus

$$u_2'(\hat{c}_t) = u_2'(a) + u_2''(a)(\hat{c}_t - a). \tag{10}$$

Now since $\hat{c}_t$ depends on the value of a, choose a specific value of $a = \bar{a}$ such that $\hat{c}_t = c_t^*$; hence

$$u_2'(c_t^*) = u_2'(c_t^*) + u_2'(\bar{a}) + u_2''(\bar{a})(c_t^* - \bar{a}).\square \tag{11}$$

The above equation can be rewritten as

$$u_2'(c_t^*) = b_1 + b_2 c_t^*, \tag{12}$$

where b_i is a constant. Using the same procedure we obtain the following exact expansion of $V_x(.)$

$$V_x(.) = V_x(\alpha) + V_{xx}(\alpha)(x - \alpha_1) + V_{xy}(\alpha)(y - \alpha_2) \equiv b_3 + b_4 x + b_5 y. \tag{13}$$

Since $u_2'(c_t^*) = V_x(.)$, we obtain

$$c_t^* = (b_6 + b_4 x + b_5 y)/b2. \tag{14}$$

Substituting $(12) - (13)$ into (6) yields

$$\pi_t^* = -\sigma^{-1}(y) \frac{\theta(y)(b_1 + b_2 c_t^*) + b_5 \rho}{b_4} \tag{15}$$

This is a general explicit formula that holds for any utility function. Moreover, this formula allows us to determine the impact of consumption on the portfolio. To show this

$$\frac{\partial \pi_t^*}{\partial c_t^*} = -\sigma^{-1}(y) \frac{b_2}{b_4} \equiv -\sigma^{-1}(y) \frac{u_2''(\bar{a})}{V_{xx}(\alpha)} < 0$$

by the concavity of u. Hence, there is a trade-off between consumption and investment.

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