A new stopping time and American option model: a solution to the free-boundary problem

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A NEW STOPPING TIME AND AMERICAN OPTION MODEL: A SOLUTION TO THE FREE-BOUNDARY PROBLEM

ABSTRACT: We present a new model of stopping times and American options. In so doing, we solve the free-boundary problem.

1 Introduction

The existing theoretical and empirical literature on American options and stopping times is usually based on conventional partial differential equations with inequalities. Examples include Musiela and Zariphopolou (2007), Bensossan (1984), Focardi and Fabozzi (2004) and Cvitanic and Zapatero (2004), among many others. Consequently, the conventional models of American options result in a cumbersome free-boundary problem, which is impossible to solve without numerical methods. Even with numerical methods, it is generally very difficult to obtain a solution. Therefore, the theoretical and empirical pricing of American options is an unresolved problem and constitutes a major gap in the literature. Thus a new approach to stopping times and American options is needed.

In this paper, we introduce a new model of stopping times and American options, which will not yield a free-boundary problem. Moreover, we show how the optimal stopping time can be determined. In addition, we present two versions of option pricing: risk neutral pricing and non-risk-neutral pricing. This new model simplifies the pricing of American options.

2 Risk neutral pricing

As usual the stock price dynamics can be expressed as

$$dS_u = S_u \left(r_u du + \sigma \left(S_u \right) dW_{1u} \right), \tag{1}$$

where S_u is the price of stock, r_u is the risk-free rate of return, σ is the stock price volatility, W_{1u} is a standard Brownian motion defined on the usual probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$. Since the stopping time depends on the asset price, the dynamics of the random stopping time τ are given by

$$d\tau = adu + \rho dW_{1u} + \sqrt{1-\rho^2} dW_{2u}, \tau_t = y, \tag{2}$$

where W_{2u} is a standard Brownian motion and ρ is the correlation factor between the Brownian motion driving the price and the Brownian motion driving the stopping time. Thus τ can be expressed as

$$\tau = \bar{\tau} + \int_t^T \delta_u dW_{2u},$$

where $\bar{\tau}$ is the expected (deterministic) stopping time and δ is its volatility.

The option writer maximizes the expected (discounted) payoff g with respect to expected stopping time. That is, the writer chooses $\bar{\tau}^*$ (which maximizes $E_t\left[g\left(\tau\right)\right]$) and sets the option price A_t equal to $E_t\left[g\left(\bar{\tau}^*\right)\right]$

$$A\left(t,s,y\right)=\underset{\bar{\tau}}{Sup}E\left[e^{-\int\limits_t^T r_u du}g\left(S_u,\tau\right)\mid\mathcal{F}_t\right], \tag{3}$$

where $g\left(.\right)$ is bounded and differentiable, $0\leq t\leq T$, $\bar{\tau}\in\left[t,T\right], s=S_t$.

The option price satisfies the HJB PDE

$$A_t+sr_tA_s+\frac{1}{2}A_{yy}+\underset{\bar{\tau}}{Sup}\left(\bar{\tau}A_y+s\rho\sigma\left(\bar{\tau},y\right)A_{sy}+\frac{1}{2}s^2\sigma^2\left(\bar{\tau},y\right)A_{ss}\right)=0,$$

$$A\left(T,s\right)=g\left(s\right), \tag{4}$$

since $\bar{\tau}^*$ maximizes $E_t\left[g\left(.\right)\right]$. To solve for the optimal stopping time $\bar{\tau}^*$, we simply differentiate the HJB with respect to $\bar{\tau}$ and obtain

$$A_y+\left\{s\rho A_{sy}+s^2\sigma\left(\bar{\tau}^*,y\right)A_{ss}\right\}\sigma_{\bar{\tau}}\left(\bar{\tau}^*,y\right)=0. \tag{5}$$

Solving this PDE is as easy as solving the classical Black-Scholes equation.

3 Pricing without risk neutrality

Under risk aversion or risk loving, the writer's objective function is

$$V\left(t,s,y\right)=\underset{\bar{\tau}}{Sup}E\left[U\left(e^{-\int_t^T r_u du}g\left(S_u,\tau\right)\right)\mid\mathcal{F}_t\right],\tag{6}$$

where U is a bounded and differentiable utility function. The value function satisfies the HJB PDE

$$V_t+sr_tV_s+\frac{1}{2}V_{yy}+\underset{\bar{\tau}}{Sup}\left(\bar{\tau}V_y+s\rho\sigma\left(\bar{\tau},y\right)V_{sy}+\frac{1}{2}s^2\sigma^2\left(\bar{\tau},y\right)V_{ss}\right)=0,$$

$$V\left(T,s\right)=g\left(s\right).\tag{7}$$

As before the solution yields

$$V_y+\left\{s\rho V_{sy}+s^2\sigma\left(\bar{\tau}^*,y\right)V_{ss}\right\}\sigma_{\bar{\tau}}\left(\bar{\tau}^*,y\right)=0.\tag{8}$$

Once $\bar{\tau}^*$ is determined, $E_t\left[g\left(\bar{\tau}_t^*\right)\right]$ will be known and thus A_t will be known, since by construction $A_t=E_t\left[g\left(\bar{\tau}^*\right)\right]$. In contrast to the utility based indifference pricing, the solution does not yield a free-boundary problem.

References

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