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# On the Complete Information First–Price Auction and its Intuitive Solution\*

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## **Abstract**

Despite the popularity of auction theoretical thinking, it appears that no one has presented an elementary equilibrium analysis of the complete information first-price sealed-bid auction mechanism when the bidding space has a finite grid. This paper aims to remedy that omission. We show that there always exists a “high price equilibrium” which can be considered “the intuitive solution” (an agent with the highest valuation wins the auction bidding at the second-highest valuation). Although there might be other “low price equilibria”, we also show that when there are two bidders “the intuitive solution” is the unique limiting equilibrium when the grid size goes to zero and ties are randomly broken.

**Keywords:** First-price auctions, undominated Nash equilibria.

**JEL Classification Numbers:**

C72 (Noncooperative Games),

D44 (Auctions).

## 1. Introduction

It is well known among auction theorists that the first-price sealed-bid auction mechanism under complete information does not possess a pure strategy Nash equilibrium.<sup>1</sup> For instance, Moldovanu and Sela (2003, footnote 12) write that “asymmetric Bertrand games (and first-price auctions) have no equilibria in pure strategies here, but introducing a smallest money unit immediately yields the intuitive solution.” Conventional wisdom holds that in this intuitive solution an agent with the highest valuation wins the auction bidding at the second-highest valuation, which is, thus, efficient. However, despite the popularity of auction theoretical thinking, it appears that no one has presented an elementary equilibrium analysis of the first-price sealed-bid auction mechanism under these conditions. This note aims to remedy that omission and to investigate under which conditions the focus on the intuitive solution as an equilibrium to the first-price auction under complete information can be motivated by the introduction of a smallest money unit.

There are several reasons why an analysis under complete information merits a closer look. First, this setting is often considered a useful starting point of the analysis before moving to incomplete information models or to be a useful benchmark case.<sup>2</sup> Second,

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<sup>1</sup>To be fully precise, it can be shown that there is no undominated pure strategy Nash equilibrium for the first-price auction and the only case in which a pure strategy Nash equilibrium exists is when two bidder with the highest valuation have the same valuation, see our more extensive working paper version Alcalde and Dahm (2008).

<sup>2</sup>For the former see e.g. Baye et al. (1993) and (1996), Benoit and Krishna (2001), Bernheim and Whinston (1986) or Krishna and Tranaes (2002). For the latter see e.g. Anton and Yao (1989) or

auction-theoretic ways of thinking have been successfully applied to the analysis of broader economic questions (see Klemperer (2003)) and for some applications complete information has been argued to be more appropriate than an incomplete information setting.<sup>3</sup>

Although, there are alternative ways to restore existence of equilibrium—like looking for mixed-strategy equilibria—, our approach of a bidding space with a finite grid is important. First, as the above quote shows it is a very natural procedure. Second, in experimental settings there is also a smallest monetary unit. Third, this model is often viewed as a better description of reality.<sup>4</sup>

The present note offers an elementary analysis of pure strategy undominated Nash equilibria assuming fairly general tie-breaking rules and (possibly irregular) finite grids on bidding spaces. We show that there always exists the intuitive “high price equilibrium” which contrary to conventional wisdom might be inefficient. There might also be inefficient “low price equilibria” which, when the bidding space is very restrictive, might

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Moldovanu and Sela (2003)).

<sup>3</sup>For instance, Moldovanu and Sela (2003) use the first-price auction mechanism to model patent licensing. They report that, in the steel industry, competitors “know each other well, and engineers often visit competitors’ plants”.

<sup>4</sup>Simon and Zame (1990, p. 863) state this view as follows. “Games with infinitely many strategies are sometimes viewed as proxies for games with a large finite number of strategies. From this point of view it is the equilibria ... of the finite games which are of real interest; equilibria of the infinite games are merely convenient approximations.” Rapoport and Amaldoss (2004, p. 587) write “the assumption of a discrete strategy space is appropriate as firms typically consider their expenditures in discrete (e.g., thousands or millions of dollars) rather than continuous units. Indeed, continuous strategy spaces are mostly introduced to achieve tractability, not to provide a more adequate description of reality”. Other auction models using the assumption of a finite grid are O’Neill (1986), Chwe (1989) or Rapoport and Amaldoss (2004).

generate very low revenues. We also show that the focus on the intuitive solution as an equilibrium to the first-price auction under complete information can be motivated in the following way. When there are two bidders the intuitive solution is the unique limiting equilibrium when the grid size goes to zero and ties are randomly broken.

## 2. The Model and Notation

Consider the seller of an (indivisible) object (indexed by 0) and a set of potential buyers  $\mathcal{B} = \{B_1, \dots, B_i, \dots, B_n\}$ . Each agent has a valuation  $v_i$  for the object. There are at least two buyers, i.e.  $n \geq 2$ , and agents' valuations are increasingly ordered, i.e.  $v_i \leq v_j$  for all  $0 \leq i \leq j$ . All agents' valuations  $v = (v_0, v_1, \dots, v_i, \dots, v_n)$  are commonly known by all the buyers, and this is public information. As in Bernheim and Whinston (1986) or Anton and Yao (1989) the seller only has information about her own valuation of the object.

The finite bidding grid is formalized as follows. Let the (fixed) set of prices that buyers can propose be given by  $A = \{a_0, a_1, \dots, a_k, \dots, a_K\}$ , where  $a_k \in \mathbb{R}_+$  and  $a_{k+1} \geq a_k$  for all  $k = 0, 1, \dots, K - 1$ . For each such  $k$ , define  $\delta_{a_k} = a_{k+1} - a_k$ . We say that the bidding space has a finite grid if there exists  $\delta > 0$  such that  $\delta_{a_k} \geq \delta$  for all  $\delta_{a_k}$ . If all  $\delta_{a_k}$  are equal, we say that the bidding space has a constant grid of (at least) size  $\delta$ .

We formalize now the first-price auction mechanism analyzed in the present paper. Loosely speaking, the object is assigned to the buyer with highest bid, and she pays

her bid. However, when two or more buyers propose the same bid there is a function  $\pi$  establishing a probabilistic allocation rule in order to assign the object. This fixed monotonic (probabilistic) measure function  $\pi : 2^{\mathcal{B}} \rightarrow \mathbb{R}^n$  satisfies:

- (a) for all  $S \subseteq \mathcal{B}$ ,  $\sum_{i=1}^n \pi_i(S) = 1$ ,
- (b) for all  $S \subseteq \mathcal{B}$  and  $i \in \mathcal{B} \setminus S$ ,  $\pi_i(S) = 0$ ,
- (c) for all  $S \subseteq \mathcal{B}$  and  $i \in S$ ,  $\pi_i(S) > 0$ ; and
- (d) for all  $S \subseteq S' \subseteq \mathcal{B}$ , and  $i \in S$ ,  $\pi_i(S) \geq \pi_i(S')$ .

For  $A$  and  $\pi$  given, the first-price auction mechanism proceeds as follows. Each buyer simultaneously sets the price  $p_i \in A$  ( $i = 1, \dots, n$ ) that she is willing to pay for the object if it is assigned to her. This defines a vector  $p = (p_1, \dots, p_n)$ .

- (1) If  $p_i < v_0$  for all  $i = 1, \dots, n$ , the object is unassigned, i.e. the seller keeps it.
- (2) Otherwise, denote by  $S(p) = \{B_i \in \mathcal{B} : p_i \geq p_j \text{ for all } B_j \in \mathcal{B}\}$  the set of buyers proposing the highest bid. Then the object is assigned with probability  $\pi_i(S(p))$  to buyer  $B_i$  who pays  $p_i$  with this probability.

We analyze undominated Nash equilibria (in pure strategies) in the bidding game. Note that for each buyer  $B_i$ , a strategy  $\hat{p}_i$  is undominated if, and only if,  $0 \leq \hat{p}_i < v_i$ . We denote by  $w_i \in A$  agent  $B_i$ 's largest (undominated) bid strictly smaller than  $v_i$ . For simplicity we also denote  $\delta_{w_{n-1}} = \Gamma$ .



### 3. Analysis of the First-Price Auction Mechanism

It turns out that if a strategy profile is an equilibrium, then it belongs to the following class of strategy profiles.

**Definition 3.1.** Given  $a \in A$ ,  $a \leq \min\{w_{n-1} + \Gamma, w_n\}$ , we denote by  $\mathcal{P}(a)$  the set of strategy profile  $\hat{p}$  such that:

1. Buyer  $B_n$  chooses  $\hat{p}_n = a$ .
2. There exists  $B_j \in \mathcal{B} \setminus \{B_n\}$  such that  $w_j = w_{n-1}$  bidding  $\hat{p}_j = \min\{a, w_{n-1}\}$ .
3. All other bidders  $B_i \in \mathcal{B} \setminus \{B_j, B_n\}$  choose  $\hat{p}_i \leq \min\{w_i, a\}$ .

Notice that  $a \in A$  just indicates the winning bid.<sup>5</sup> Given a strategy profile  $\hat{p} \in \mathcal{P}(a)$ , we indicate the buyers bidding at least  $b \in A$  by  $W(b) = \{B_i \in \mathcal{B} \text{ s.t. } \hat{p}_i \geq b\}$ . To simplify notation we will omit  $a$  and  $b$  using  $\delta$  and  $W$  instead, whenever this notation is clear from the context.

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<sup>5</sup>We implicitly assume in what follows that  $v_0 \leq a$ .

Given a winning bid  $a$  and a strategy profile  $\hat{p} \in \mathcal{P}(a)$ , it might pay to raise or lower an individual bid. Define the two threshold values

$$\begin{aligned} \alpha &= [1 - \pi_n(W(w_{n-1}))][v_n - w_{n-1}] \quad \text{and} \\ \beta &= \left. \begin{aligned} &\max [1 - \pi_j(W(a))][v_j - a] \\ &s.t. \quad B_j \in W(a). \end{aligned} \right\} \end{aligned}$$

Notice that the definition of  $\beta$  might not be determined by  $B_n$  when the probabilistic measure function  $\pi$  is strongly biased in favor of this buyer.<sup>6</sup>

We are now in a position to characterize undominated Nash equilibria when the bidding space has a finite grid. There are three cases to be distinguished. Case (1) and case (2.2) formalize the conventional wisdom that the strongest bidder just outbids the others or ties with an equally strong bidder at their common valuation. However, case (2.1) shows that even when valuations are different it might not pay to outbid others because the required increase of the bid may be too large. Case (3) establishes that this intuition might even apply to much lower bids.

**Theorem 3.2.** *A profile of strategies  $p^*$  is an undominated Nash equilibrium for the first-price auction if, and only if,  $p^* \in \mathcal{P}(a)$  for some  $a \in A$ ; and one of the following is true:*

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<sup>6</sup>The exact threshold for  $B_n$  not to determine  $\beta$  is that there exists  $B_i \in W \setminus B_n$  such that  $\pi_n > 1 - (1 - \pi_i)(v_i - a)/(v_n - a)$ .

(1) (High price equilibrium, unique winner)  $a = w_{n-1} + \Gamma$ ,  $w_{n-1} < w_n$  and  $\Gamma \leq \alpha$ .

(2) (High price equilibrium, tie)  $a = w_{n-1}$  and either

(2.1)  $w_{n-1} < w_n$  and  $\Gamma \geq \alpha$  or

(2.2)  $w_{n-1} = w_n$ .

(3) (Low price equilibrium, tie)  $a < w_{n-1}$  and  $\delta_a \geq \beta$ .

*Proof.* (I) We show first that  $p^*$  is an undominated Nash equilibrium for the first-price auction whenever (1), (2) or (3) are true. Note that, since  $p_i^* \leq w_i$  for all  $B_i \in \mathcal{B}$ , no buyer employs a dominated strategy. We show now that  $p^*$  is a Nash equilibrium. Let us observe that the expected utility of buyers in  $\mathcal{B} \setminus W$  is zero. Moreover, given agents' bids, no buyer in  $\mathcal{B} \setminus W$  can obtain a positive (expected) utility.

Suppose (1) holds. The fact that  $p^* \in \mathcal{P}(a)$  implies that  $B_n$  wins, so

$$U_n(p^*) = v_n - a \geq 0, \text{ with } a = w_{n-1} + \Gamma.$$

Assume  $B_n$  changes  $p_n^*$  to  $\tilde{p}_n$ . Given that she cannot gain from raising her bid, suppose  $\tilde{p}_n \leq w_{n-1}$ . Notice that there exists  $B_j \neq B_n$  bidding  $\hat{p}_j(a) = w_{n-1}$ . We have

$$\begin{aligned} U_n(\tilde{p}_n, p_{-n}^*) &\leq \pi_n(W(w_{n-1})) [v_n - w_{n-1}] = v_n - w_{n-1} - \alpha \leq \\ &\leq v_n - w_{n-1} - \Gamma = v_n - a = U_n(p^*). \end{aligned}$$

And, thus,  $p_n^*$  is the best decision for agent  $B_n$ , given the others' bids.

Suppose (2) or (3) holds. For buyer  $B_j \in W$ , we have that her expected utility is

$$U_j(p^*) = \pi_j(W) [v_j - p_j^*] = \pi_j(W) [v_j - a] > 0.$$

Assume  $B_j$  changes her strategy, by setting  $\tilde{p}_j$ . If she lowers her bid, her expected utility will be zero, since  $W$  is not a singleton. Thus, suppose  $\tilde{p}_j > a$ , and note that  $B_j$  will get the object with probability one. Notice that in case (2.2)  $U_j(\tilde{p}_j, p_{-j}^*) < 0$ . Consider case (3). Observe that

$$\begin{aligned} U_j(\tilde{p}_j, p_{-j}^*) &= v_j - \tilde{p}_j \leq v_j - a - \delta_a \leq v_j - a - \beta \leq \\ &\leq v_j - a - [1 - \pi_j(W)] [v_j - a] = U_j(p^*). \end{aligned}$$

Again,  $p_j^*$  is the best decision for agent  $B_j$ , given the others' bids. The argument for case (2.1) is similar replacing  $B_j$ ,  $\delta_a$  and  $\beta$  by  $B_n$ ,  $\Gamma$  and  $\alpha$  respectively.

(II) We show now the converse. Suppose there is a Nash equilibrium  $p^*$  in which all agents employ undominated strategies. Let  $a'$  denote the highest bid.

Notice that if  $a' > \min\{w_{n-1} + \Gamma, w_n\}$ , then  $a'$  is either dominated or  $p_n^* = a'$ . In the latter case  $B_n$  can improve by lowering her bid. Hence, suppose  $a' \leq \min\{w_{n-1} + \Gamma, w_n\}$ . Notice that  $p_n^* = a'$  must hold because otherwise  $B_n$  can improve by making this bid. Suppose  $a' = w_{n-1} + \Gamma$  and that there does not exist  $B_j \in \mathcal{B} \setminus \{B_n\}$  such that

$w_j = w_{n-1}$  bidding  $p_j^* = w_{n-1}$ . Given that for bidders with lower valuations  $p_i^* = w_{n-1}$  is dominated,  $B_n$  could improve by lowering her bid. Assume  $a' \leq w_{n-1}$  and that there exists  $B_j \in \mathcal{B} \setminus \{B_n\}$  such that  $w_j \geq a'$  bidding  $p_j^* < a'$ . In this case  $B_j$  can improve by changing her bid to  $\tilde{p}_j = a'$  because

$$U_j(p^*) = 0 < \pi_j(W(a') \cup B_j) [v_j - a'] = U_j(\tilde{p}_j, p_{-j}^*)$$

This proves that  $p^* \in \mathcal{P}(a')$ . From part (I) it is clear  $p^*$  cannot be a undominated Nash equilibrium when the conditions in case (1), (2) or (3) are not fulfilled. ■

Observe that case (1) and case (2) of Theorem 3.2 imply the following.

**Corollary 3.3.** *There exists a ‘high price’ undominated Nash equilibrium in the first-price auction. In this equilibrium the winning bid  $a$  fulfills  $a \in \{w_{n-1}, w_{n-1} + \Gamma\}$ .*

However, in addition to this equilibrium there might be further low price equilibria as specified in case (3) of Theorem 3.2. We give now an example in which increasing the restrictiveness of the bidding space creates further equilibria. Notice that in applications the relevant bidding space might be very irregular and the required increments of bids might be large.

**Example 3.4.** *There are two bidders with valuations  $v_1 = 90$  and  $v_2 = 100$ . The reservation price of the seller is zero. In the case that both bidders submit the same*

bid, they obtain the object with equal probability. Suppose first that the bidding space coincide with the set of uneven integers. In this case Theorem 3.2(1) and Corollary 3.5(1) (stated below) imply that  $p^* = (89, 91)$  is the unique undominated Nash equilibrium. However, if bidding space is  $A = \{1, 51, 76, 89, 91, 99, 106, \dots\}$ , then apart from  $p^*$  there are three additional equilibria, namely,  $p^{*'} = (1, 1)$ ,  $p^{*''} = (51, 51)$  and  $p^{*'''} = (76, 76)$ . Notice that, although  $A$  is restrictive, it still leaves the bidders a fairly rich set of options.

The intuition for the existence of the low price equilibria is the following. A bidder can prevent a tie by outbidding the competitors by the minimal increase. However, when the grid is restrictive the required increase is large and does not pay.<sup>7</sup> So a natural question to ask is, How small must a smallest monetary unit be in order to make sure that low price equilibria do not exist? Note that, for some values of  $\delta$  it is possible to have situations where  $w_{n-1} = w_n$  but  $v_{n-1} < v_n$ . Assume that the tie breaking rule assigns the object with equal probability and that there are two bidders.<sup>8</sup>

**Corollary 3.5.** *Assume that there are two bidders who get the object with equal probability in case of a tie and that the bidding space has a constant grid of size  $\delta$ . For any  $\delta > 0$  the following is true:*

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<sup>7</sup>Low price equilibria may generate considerably lower revenues than high price equilibria. In this sense there is ‘collusion’. But, given that bidding strategies constitute an equilibrium, they are also ‘self-enforcing’. This contrasts with the conventional wisdom that “unlike in a second-price auction, the cartel agreement in a first-price auction is *not* self-enforcing and, hence, is somewhat fragile” (Krishna (2002), pg. 160).

<sup>8</sup>Notice that because of the monotonicity of the tie breaking rule further bidders increase the incentives to deviate from a low price equilibrium. For completeness we mention that case (1) of the next Corollary assumes that  $\Gamma \neq \alpha$ .

- (1) If  $w_{n-1} < w_n$ , the high price equilibrium is unique.
- (2) If  $w_{n-1} = w_n$ , in addition to the high price equilibrium, there exists a low price equilibrium with strategy profile<sup>9</sup>  $\hat{p}(a)$  where  $a = w_{n-1} - \delta$ .

*Proof.* Suppose  $w_{n-1} < w_n$ . The fact that  $v_n > w_n \geq w_{n-1} + \delta$  implies that  $\delta < v_n - w_{n-1}$ . We show first that the profile  $\hat{p}(a)$  with  $a = w_{n-1} - \delta$  is not an equilibrium. For this  $\delta < \beta$  must hold. Since  $\beta = \frac{1}{2}(v_n - w_{n-1} + \delta)$ ,  $\delta < \frac{1}{2}(v_n - w_{n-1} + \delta)$  must hold. Simplifying yields  $\delta < v_n - w_{n-1}$ , which we already have shown to be true. Notice that no profile  $\hat{p}(a')$  with  $a' < w_{n-1} - \delta$  can be an equilibrium because  $\beta(a) = \frac{1}{2}(v_n - a) < \frac{1}{2}(v_n - a') = \beta(a')$ .

Suppose  $w_{n-1} = w_n$ . Notice that  $v_n - w_n \leq \delta$ . This implies that  $\beta = \frac{1}{2}(v_n - w_{n-1} + \delta) \leq \delta$ . ■

We are now in a position to come back to our initial question under which conditions the focus on the intuitive solution as an equilibrium to the asymmetric first-price auction under complete information can be motivated by the introduction of a smallest monetary unit. Notice first that whenever the grid is fine enough  $w_{n-1} < w_n$  holds, as in the asymmetric game  $v_{n-1} < v_n$ . Also, decreasing the grid guarantees that Theorem 3.2(1) and Corollary 3.5(1) apply, implying that in the unique undominated Nash equilibrium the bidder with the higher valuation bids a little bit more than the valuation of the

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<sup>9</sup>Note that, for the two-bidder case, for any  $a \in A$  such that  $a \leq \min\{w_{n-1} + \Gamma, w_n\}$ ,  $\mathcal{P}(a)$  is a singleton. Therefore we can denote by  $\hat{p}(a)$  such an element.

other bidder and both bids differ only by the smallest monetary unit. The limit of this unique undominated Nash equilibrium of the discrete asymmetric first-price auction, when the grid size goes to zero, is the intuitive solution.

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