Externalities, income taxes and indeterminacy in OLG models

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Abstract

Using an aggregate two-periods overlapping generations model with endogenous labor, consumption in both periods of life, homothetic preferences and productive external effects [Lloyd-Braga et al., 2007. Indeterminacy in dynamic models: When Diamond meets Ramsey. Journal of Economic Theory 134, 513-536], we examine the effects of alternative government financing methods on the range of values of increasing returns leading to indeterminacy. We show that under a large enough share of first period consumption over the wage income, local indeterminacy can easily occur for mild externalities if constant government expenditure is financed through either labor or capital income taxes. More precisely, we show that, with labor income taxes and mild externalities, small government expenditures are helpful to local indeterminacy, while large government expenditures are useful to stabilize the economy. With capital income taxes and mild externalities, local indeterminacy always exists. Moreover, we explore how our previous results are changed once government expenditure is endogenously determined for fixed rates on labor and capital income under the balanced-budget rule.

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1. Introduction

A large body of literature has suggested that local indeterminacy can arise in dynamic general equilibrium models with market distortions. Since Benhabib and Farmer (1994), many authors have used a one-sector Ramsey growth model augmented to include endogenous labor supply and productive externalities to analyze the expectation-driven fluctuations. They find that local indeterminacy can easily emerge with mild externalities, provided that the elasticity of capital-labor substitution, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply are large enough.[1] Using the similar framework, Schmitt-Grohe and Uribe (1997) find that local indeterminacy can easily occur for empirically plausible values of income tax rates due to the dynamic effects of fiscal policies. And they use numerical simulations to show their main results. Guo and Harrison (2004) further show that indeterminacy can not appear once the government finances endogenous government spending and transfer with constant tax rates.

Recent works extend the Diamond (1965) model by adding endogenous labor supply, external effects and/or fiscal policy into the overlapping generations framework. Following Reichlin (1986), most of those works have focussed on a special case without first period consumption (for instance, Cazzavillan (2001) and Gokan (2009a, 2009b)). However, some other works consider a general case where consumptions in both periods of life and endogenous labor supply exist (for instance, Cazzavillan and Pintus (2004, 2006), Lloyd-Braga et al. (2007) and Chen and Zhang (2009a, 2009b)). Cazzavillan and Pintus (2004, 2006) consider an OLG model with totally separable preferences over both young and old consumptions and observe how the ratio of first period consumption over the

[1] For a complete discussion on this issue, see Pintus (2006).

wage income influences local indeterminacy. Lloyd-Braga et al. (2007) consider an OLG model with non-separable preferences over both young and old consumptions, and analyze the effects of labor and capital externalities in production on local indeterminacy. Chen and Zhang (2009a, 2009b) instead explore the dynamic effects of government fiscal policy in the very similar OLG models.

To our knowledge, Gokan (2009a) is the first paper that studied how local dynamics are affected by changes in government expenditure, depending on the degree of productive externalities, within an OLG framework. However, his study concentrates on the particular OLG model without first period consumption as studied in Cazzavillan (2001). He finds that as for consumption taxes, fixed tax rates are always recommended relative to endogenous tax rates. In contrast, as for capital income taxes, the sizes of increasing returns are important in analyzing which budget policy is more effective for mitigating the extent to which aggregate activity fluctuates. Our paper instead aims to study how alternative government financing methods influence aggregate fluctuations driven by self-fulfilling prophecies in an extended OLG economy with labor and capital externalities in production, as studied in Lloyd-Braga et al. (2007). We find that provided that the share of first period consumption over the wage income is larger than 1/2, local indeterminacy of equilibria can easily arise for mild externalities when constant government expenditure is financed through either labor or capital income taxes. In the case with labor income taxes, it is shown that for mild externalities, small government expenditures are helpful to local indeterminacy, while large government expenditures are useful to stabilize the economy. In the case with capital income taxes, local indeterminacy always exists for mild externalities. In addition, we consider another kind of fiscal policy specification. Suppose that government expenditures are endogenously determined for fixed rates on labor and capital income under a balanced-budget rule. We find that the range of values of increasing returns leading to local indeterminacy is independent of the constant tax rates on labor and capital income. In contrast, Guo and Lansing (2002) find that in a Ramsey model, constant tax rates on labor and capital income influence the minimum levels of increasing returns leading to indeterminacy.

The paper is organized as follows. In Section 2, we set up the model. In Sections 3 and 4, we study the cases with either labor or capital income taxes, and analyze how the size of government expenditures influences the local dynamics of the normalized steady state, depending on the magnitude of increasing returns. In Section 5, we provide economic interpretations behind our indeterminacy results. In Section 6, we consider another kind of fiscal policy in which tax rates on labor and capital income are constant and government expenditures are endogenous. Section 7 concludes the paper.

2. The model

This paper introduces constant government expenditure financed by either labor or capital income taxes in a competitive, non-monetary, overlapping generations model with production externalities as studied in Lloyd-Braga, Nourry and Venditti (2007). Identical agents live for two periods, consume in both periods (c when young, and $\widehat{c}$ when old). Each agent maximizes her lifetime utility

$$\max_{c_t,\, l_t,\, \widehat{c}_{t+1}} \left[u(c_t, \widehat{c}_{t+1}) - v\left(l_t/B\right)\right], \tag{1}$$

subject to the constraints

$$c_t + k_{t+1} = \left(1 - \tau_{w_t}\right) w_t l_t, \tag{2}$$

$$\widehat{c}_{t+1} = \left[1 - \delta + (1 - \tau_{r_{t+1}}) r_{t+1}\right] k_{t+1} \tag{3}$$

$$c_t \geq 0,\ \widehat{c}_{t+1} \geq 0,\ L \geq l_t \geq 0, \text{ for all } t \geq 0,$$

where l_t, c_t and k_{t+1} are labor, consumption and saving (the amount of capital), respectively, of the individual of the young generation, $\widehat{c}_{t+1}$ is the consumption of the same individual when old. $w_t > 0$ $r_{t+1} > 0$ are the real wage rate at time t and the marginal product of capital at time t+1. τ_{w_t} and

$\tau_{r_{t+1}} \in (0,1)$ are the tax rates levied on labor income and capital income respectively. $B > 0$ and $L > 0$ denote a scaling parameter and the maximum amount of labor supply, respectively.

The preferences satisfy the following conditions as in Lloyd-Braga et al. (2007).

Assumption 1. (i) $u(c_t, \widehat{c}_{t+1})$ is C^r over R^2_+ for r large enough, increasing with respect to each argument ($u_1(c_t, \widehat{c}_{t+1}) > 0$, $u_2(c_t, \widehat{c}_{t+1}) > 0$), concave and homogeneous of degree one over R^2_{++}. Moreover, for all c_t, $\widehat{c}_{t+1} > 0$, $lim_{\widehat{c}_{t+1}/c_t \to 0} u_1/u_2 = 0$ and $lim_{\widehat{c}_{t+1}/c_t \to +\infty} u_1/u_2 = +\infty$, where u_1/u_2 stands for $u_1(1, \frac{\widehat{c}_{t+1}}{c_t})/u_2(1, \frac{\widehat{c}_{t+1}}{c_t})$. (ii) $v\left(l_t/B\right)$ is C^r over $[0, L/B]$ for r large enough, increasing ($v'\left(l_t/B\right) > 0$) and convex ($v"\left(l_t/B\right) > 0$) over $(0, L/B)$. Moreover, $lim_{l_t \to 0} v'(l_t/B) = 0$ and $lim_{l_t \to l} v'(l_t/B) = +\infty$.

We introduce homogeneity in order to express the capital accumulation equation as a function of the ratio between the young agent's consumption and the after-tax wage income. The first order conditions of the agent's optimization problem are stated as follows:

$$\frac{u_1(1, \frac{\widehat{c}_{t+1}}{c_t})}{u_2(1, \frac{\widehat{c}_{t+1}}{c_t})} \equiv g(\frac{\widehat{c}_{t+1}}{c_t}) = \widetilde{R}_{t+1}, \tag{4}$$

$$u_1(1, \frac{\widehat{c}_{t+1}}{c_t})\left(1 - \tau_{w_t}\right) w_t = \frac{v'(l_t/B)}{B}, \tag{5}$$

$$c_t + \widehat{c}_{t+1}/\widetilde{R}_{t+1} = \left(1 - \tau_{w_t}\right) w_t l_t, \tag{6}$$

$$k_{t+1} = \left(1 - \tau_{w_t}\right) w_t l_t - c_t, \tag{7}$$

where $\widetilde{R}_{t+1} \equiv 1 - \delta + \left(1 - \tau_{r_{t+1}}\right) r_{t+1}$ is the after-tax real gross rate of return on capital stock.

Since $g'(\frac{\widehat{c}_{t+1}}{c_t}) > 0$, we can derive that

$$\frac{\widehat{c}_{t+1}}{c_t} = g^{-1}(\widetilde{R}_{t+1}) \equiv h(\widetilde{R}_{t+1}). \tag{8}$$

Combining (4), (6), (8) with Euler's identity $c_t u(1, \widehat{c}_{t+1}/c_t) \equiv c_t u_1(1, \widehat{c}_{t+1}/c_t) + \widehat{c}_{t+1} u_2(1, \widehat{c}_{t+1}/c_t)$, we can get:

$$c_t = \frac{u_1(1, h(\widetilde{R}_{t+1}))}{u(1, h(\widetilde{R}_{t+1}))} (1 - \tau_{w_t}) w_t l_t \equiv \eta(\widetilde{R}_{t+1}) (1 - \tau_{w_t}) w_t l_t, \tag{9}$$

where $\eta(\widetilde{R}) \in (0,1)$ is the propensity to consume of the young, or equivalently the share of first period consumption over the after-tax wage income. Then, Eq. (7) becomes

$$k_{t+1} = (1 - \eta(\widetilde{R}_{t+1})) (1 - \tau_{w_t}) w_t l_t. \tag{10}$$

We can compute the elasticity of intertemporal substitution in consumption $\gamma\left(\widetilde{R}\right)$ and the elasticity of the labor supply ε_l:

$$\gamma\left(\widetilde{R}\right) = \frac{\widetilde{R}}{g'\left(h(\widetilde{R})\right) h(\widetilde{R})} = -\left[\frac{u_{11}(1, h(\widetilde{R}))}{u_1(1, h(\widetilde{R}))} + \frac{u_{22}(1, h(\widetilde{R}))}{u_2(1, h(\widetilde{R}))} h\left(\widetilde{R}\right)\right]^{-1} > 0, \tag{11-1}$$

$$\varepsilon_l(l_t/B) = \frac{v'(l_t/B)}{v''(l_t/B)(l_t/B)} > 0. \tag{11-2}$$

It is easy to have the identity $\eta\left(\widetilde{R}\right) = 1/\left(1 + h\left(\widetilde{R}\right)/\widetilde{R}\right)$, or $\widetilde{R}/h\left(\widetilde{R}\right) = \eta\left(\widetilde{R}\right)/\left(1 - \eta\left(\widetilde{R}\right)\right)$. And the elasticity of the propensity to consume $\eta\left(\widetilde{R}\right)$ is:

$$\frac{\eta'\left(\widetilde{R}\right)\widetilde{R}}{\eta\left(\widetilde{R}\right)} = \left(1 - \eta\left(\widetilde{R}\right)\right)\left(1 - \gamma\left(\widetilde{R}\right)\right). \tag{12}$$

The saving function is then increasing with $\widetilde{R}$ iff $\gamma\left(\widetilde{R}\right) > 1$. As in Lloyd-Braga et al. (2007), we assume gross substitutability, i.e. $\gamma > 1$, in the rest of our paper.

On the production side, the perishable output y_t is produced using capital k_t and labor l_t with a Cobb-Douglas production function:

$$y_t = Ak_t^a l_t^b K_t^\zeta L_t^\lambda, \text{ where } a + b = 1,\ \zeta > 0,\ \lambda > 0,$$

where A is a scaling parameter, K_t and L_t denote the average economy wide use of capital and labor, which are taken as given by individual firms. ζ and λ are the degrees of the external effects derived from the average economy wide use of capital and labor, respectively.[2]

Focusing on the symmetric equilibrium, we have that $K_t = k_t$ and $L_t = l_t$. Therefore, the social production function is $y_t = Ak_t^\alpha l_t^\beta$, where $\alpha = a + \zeta$ and $\beta = b + \lambda$. Then the real wage rate and the marginal product of capital are given by

$$\begin{aligned} w_t &= bAk_t^\alpha l_t^{\beta-1}, && (13) \\ r_t &= aAk_t^{\alpha-1} l_t^\beta. && (14) \end{aligned}$$

In the following analysis, we rule out the production technology associated with endogenous growth.

Assumption 2. $\alpha \neq 1$.

The government finances its constant expenditures through either labor or capital income taxes,

$$\tau_{w_t} w_t l_t + \tau_{r_t} r_t k_t = g_t \equiv g, \qquad (15)$$

where $g > 0$ is the constant government expenditure.

[2] The sources of capital and labor externalities are stated in Gokan (2008, p. 1653). Lloyd-Braga et al. (2007) consider a Cobb Douglas production technology with externalities in Section 6.

Hence, we can easily derive the intertemporal competitive equilibrium paths:

$$k_{t+1} = (1-\eta(\widetilde{R}_{t+1}))\,(1-\tau_{w_t})\,w_t l_t, \tag{16-1}$$

$$v'(l_t/B)/B = u_1(1,h(\widetilde{R}_{t+1}))\,(1-\tau_{w_t})\,w_t, \tag{16-2}$$

where $\widetilde{R}_{t+1} = 1-\delta+\left(1-\tau_{r_{t+1}}\right)r_{t+1}$, $w_t = bAk_t^{\alpha}l_t^{\beta-1}$, $r_t = aAk_t^{\alpha-1}l_t^{\beta}$, and $\tau_{w_t}w_t l_t + \tau_{r_t}r_t k_t = g$.

3. Labor income tax finance

We assume total depreciation of capital in the rest of paper. In this section, we consider the case in which government expenditures are financed by labor income taxes, i.e., $\tau_{r_t} = 0$, $\widetilde{R}_{t+1} = r_{t+1} = aAk_{t+1}^{\alpha-1}l_{t+1}^{\beta}$, and $g = \tau_{w_t}w_t l_t$. Then, the intertemporal competitive equilibrium paths (k_t, l_t) can be given by

$$k_{t+1} = (1-\eta(r_{t+1}))(w_t l_t - g), \tag{17-1}$$

$$v'(l_t/B)l_t/B = u_1(1,h(r_{t+1}))(w_t l_t - g), \tag{17-2}$$

where $w_t = bAk_t^{\alpha}l_t^{\beta-1}$ and $r_t = aAk_t^{\alpha-1}l_t^{\beta}$.

3.1. Steady state existence

From (17), a steady state is a pair $\left(\overline{k},\overline{l}\right)$ such that,

$$\overline{k} = \left(1-\eta(aA\overline{k}^{\alpha-1}\overline{l}^{\beta})\right)\left(bA\overline{k}^{\alpha}\overline{l}^{\beta} - g\right), \tag{18-1}$$

$$v'(\overline{l}/B)\overline{l}/B = u_1(1,h(aA\overline{k}^{\alpha-1}\overline{l}^{\beta}))(bA\overline{k}^{\alpha}\overline{l}^{\beta} - g). \tag{18-2}$$

To ease the analysis, we consider a normalized steady state $\left(\overline{k},\overline{l}\right) = (1,1)$. Following the procedure used in Lloyd-Braga et al. (2007), we use the scaling parameters A and B to give conditions for the existence of the normalized steady state (NSS in the sequel).

Proposition 1. *Under Assumptions 1-2, let* $V(B) = v'(1/B)/B$*. Then* $(\overline{k}, \bar{l}) = (1,1)$ *is a normalized steady state of the dynamic system (17) if and only if* $lim_{A\to+\infty}(1-\eta(aA))(bA-g) > 1$*. The scaling parameters* A*,* B *are set at the levels* $A^* > 0$*,* $B^* > 0$ *that satisfy the following equations:*

$$1 = (1-\eta(aA))(bA-g), \quad \text{(19-1)}$$

$$B = V^{-1}\{u_1[1, h(aA)](bA-g)\}. \quad \text{(19-2)}$$

Proof. It is similar to the proof of Proposition 1 in Chen and Zhang (2009b). ■

There may exist multiple steady states. However, for brevity, we just analyze the local dynamics around the NSS. In the rest of this section we assume that the conditions of Proposition 1 hold in order to ensure the existence of the NSS.

Assumption 3. $lim_{A\to+\infty}(1-\eta(aA))(bA-g) > 1$, $A = A^*$ and $B = B^*$.

3.2. Local dynamics

We linearize the dynamical system (17) around the NSS and examine the local stability of the linearized dynamic system. We can have the following proposition.

Proposition 2. *The two-dimensional system (17) defines uniquely a local dynamics near the NSS* $(\overline{k}, \bar{l}) = (1,1)$*. The linearized dynamics for the deviations* $dk_t = k_t - \overline{k}$*,* $dl_t = l_t - \bar{l}$ *are determined by the determinant* D_W *and the trace* T_W *of the Jacobian matrix. And the expressions of* D_W *and*

T_W are given by

$$D_W = \frac{\alpha\Phi}{\beta}\frac{1+(1-\eta)\,g}{1-\eta}, \tag{20-1}$$

$$T_W = \frac{-\eta\gamma\,[1+(1-\eta)\,g]}{1-\eta} - \Phi\frac{\eta(1-\gamma)\,(1-\alpha)-1}{\beta\,(1-\eta)}, \tag{20-2}$$

where $\Phi \equiv \frac{1+\varepsilon_l}{\varepsilon_l} > 1$.

Proof. See Appendix 1. ■

Before we study the point $(T_W(g), D_W(g))$, we need figure out the range of the parameter g. Since $g = bA^*\tau_w^*$ and $A^* = \frac{1}{b(1-\eta)(1-\tau_w^*)}$ (when η and b are fixed), we have that $g = \frac{\tau_w^*}{(1-\eta)(1-\tau_w^*)}$. This implies that g is an increasing function of τ_w^*. As τ_w^* varies in $(0,1)$, the range of g is $(0,+\infty)$.

Depending on the degree of increasing returns to scale, local dynamics will be affected by a change in government expenditure. As in Gokan (2008), we will study how the trace and the determinant of the Jacobian matrix vary in the (T_W, D_W) plane, when g increases.[3] From (20), we can obtain the following lemma.

Lemma 1. *As the government expenditure varies in* $(0,+\infty)$*, the point* $(T_W(g), D_W(g))$ *is then defined by the following linear relationship* Δ_W*:*

$$D_W = \overline{\mathcal{S}}_W T_W + \overline{\mathcal{S}}_W \Phi\frac{\eta(1-\gamma)\,(1-\alpha)-1}{\beta\,(1-\eta)}, \tag{21}$$

[3] A simple way to analyze the local dynamics of the normalized steady state is to observe the variation of the trace T and the determinant D in the (T,D) plane as some parameters are made vary continuously. In particular, we are interested in the two roots of the characteristic polynomial $Q(\pi) = \pi^2 - T\pi + D$. There is a local eigenvalue which is equal to $+1$ when $1 - T + D = 0$. It is represented by the line (AC) in Fig. 1. Moreover, one eigenvalue is -1 when $1+T+D=0$. That is to say, in this case, (T,D) lies on the line (AB). Finally, the two roots are complex conjugate of modulus 1, whenever (T,D) belongs to the segment [BC] which is defined by $D=1$, $|T| \leq 2$. Since both roots are zero when both T and D are 0, then, by continuity, they have both a modulus less than one iff (T,D) lies in the interior of the triangle ABC, which is defined by $|T| < |1+D|$, $|D| < 1$. The steady state is then locally indeterminate given that there is a unique predeterminate variable K_t. If $|T| > |1+D|$, the stationary state is a saddle-point. Finally, in the complementary region $|T| < |1+D|, |D| > 1$, the steady state is a source.

The diagram can also be used to study local bifurcations. When the point (T,D) crosses the interior of the segment [BC], a *Hopf bifurcation* is expected to occur. If, instead, the point crosses the line (AB), one root goes through -1. In that case, a *flip bifurcation* is expected to occur. Finally, when the point crosses the line (AC), one root goes through $+1$, one expects an exchange of stability between the NSS and another steady state through a *transcritical bifurcation*.

where the slope of Δ_W *is* $\overline{\mathcal{S}}_W = \frac{-1}{\eta\gamma}\frac{\alpha\Phi}{\beta} < 0$.

As $g \in (0, \infty)$, only a part of Δ_W is relevant (see Figure 1). To find the location of Δ_W, we need figure out the starting and end points of the pair $(T_W(g), D_W(g))$:

$$\begin{aligned}
\lim_{g\to\infty} D_W &= +\infty, \ \lim_{g\to\infty} T_W = -\infty, \\
D^0 &\equiv \lim_{g\to 0} D_W = \frac{\alpha\Phi}{\beta(1-\eta)} > 0, \\
T^0 &\equiv \lim_{g\to 0} T_W = -\frac{\eta\gamma}{1-\eta} + \Phi\frac{\eta(\gamma-1)(1-\alpha)+1}{\beta(1-\eta)}.
\end{aligned}$$

In graphical terms, under Assumption 3, since $D_W(g)$ increases with g, the relevant part of Δ_W describes a half-line which starts in $\left(T^0, D^0\right)$ for $g = 0$ and points upwards to the left as g increases from 0 to $+\infty$. Since Δ_W points upward, a necessary condition for the existence of local indeterminacy, i.e., for one part of (T_W, D_W) belonging to the interior of the triangle ABC, is that the starting point $\left(T^0, D^0\right)$ lies in the interior of the triangle ABC. To ensure the necessary condition, we need the following restrictions:

$$D^0 < 1, \tag{22-1}$$

$$D^0 > T^0 - 1, \text{ and } D^0 > -T^0 - 1. \tag{22-2}$$

To ease computations and focus on the empirically plausible values of those parameters, in the rest of paper, we consider the case with small externalities and a significant share of first period consumption over the wage income (η). We assume that α, β, and η satisfy the following conditions.

Assumption 4. $\alpha < 1/2 < \eta$, and $\Phi/\beta > 1$.

The assumption $\alpha < 1/2 < \eta$ is used as we consider the presence of *small capital externalities* (see Benhabib and Farmer (1996, p. 434)) and a large enough share of first period consumption over the wage income (see Lloyd-Braga et al. (2007, p. 529)). Note that the level of labor externalities considered here is not large.[4]

Lemma 2. *Under Assumptions 1-4, we find that $D^0 < 1$, the point $\left(T^0, D^0\right)$ belongs to the interior of the triangle ABC, and $\left|\overline{\mathcal{S}}_W\right| < 1$ if*

$$1 < \gamma < \frac{1+\eta}{\eta},\ 1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}. \tag{23}$$

Proof. See Appendix 2. ■

In Lemma 2, the latter inequality implies that $(1-\eta)/\alpha > 1$, or $\eta < 1-\alpha$. Therefore, under these conditions $\alpha < 1/2 < \eta < 1-\alpha$, $1 < \gamma < \frac{1+\eta}{\eta}$, and $1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}$, the half-line Δ_W will intersect the line BC and the line AB. A critical issue is to study the intersections of Δ_W with the lines BC and AB. As shown in Proposition 3, these intersections may arise in two simple cases depending on whether Δ_W crosses the interior of the segment BC or not.

In order to get the bifurcation values of g, we calculate the intersection points of the half-line Δ_W with the line AB and the segment BC. First, as Δ_W crosses the line AB, the coordinate of the intersection point is

$$D_{AB} \equiv D_W\left(g^{flip}\right) = \frac{-\overline{\mathcal{S}}_W}{1+\overline{\mathcal{S}}_W}\left[1 + \frac{\Phi}{\beta}\frac{\eta(\gamma-1)(1-\alpha)+1}{1-\eta}\right], \tag{24}$$

$$T_{AB} \equiv T_W\left(g^{flip}\right) = -D_{AB} - 1. \tag{25}$$

As $\overline{\mathcal{S}}_W$ lies in $(-1, 0)$, we can show that D_{AB} is larger or less than 1 depending on these parameter values. When $D_{AB} > 1$, case (1.1) will occur (see Figure 1). When $D_{AB} < 1$, case (1.2) will occur (see Figure 2). When the half-line Δ_W intersects the segment BC, we have $D_W\left(g^{Hopf}\right) = 1$.

[4] For example, the level of labor externalities considered in Wen (1998, pp. 13,20)'s simulations-that is, 0.11- falls into this region.

Simple algebra gives us these two bifurcation values: $g^{Hopf} = \left[(1-\eta)\frac{\beta}{\alpha\Phi} - 1\right] / (1-\eta)$ and $g^{flip} = \left[D_{AB}(1-\eta)\frac{\beta}{\alpha\Phi} - 1\right] / (1-\eta)$. As a result, we can have the following proposition.

Proposition 3. *Under Assumptions 1-4, when* $\alpha < 1/2 < \eta < 1-\alpha$, $1 < \gamma < \frac{1+\eta}{\eta}$, *and* $1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}$, *the following holds.*

(1) Case 1.1: When $D_{AB} > 1$, *the steady state* $(1,1)$ *is a sink for* $g < g^{Hopf}$, *undergoes a Hopf bifurcation at* $g = g^{Hopf}$, *becomes a source for* $g^{Hopf} < g < g^{flip}$, *undergoes a flip bifurcation at* $g = g^{flip}$, *and becomes a saddle for* $g > g^{flip}$.

(2) Case 1.2: When $D_{AB} < 1$, *the steady state* $(1,1)$ *is a sink for* $g < g^{flip}$, *undergoes a flip bifurcation at* $g = g^{flip}$, *and becomes a saddle for* $g > g^{flip}$.

In Proposition 3, $1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}$ implies that $\max\left\{\frac{\alpha\Phi}{1-\eta}, (1-\alpha)\Phi\right\} < \beta < \Phi$ and $\Phi \geq 1$. As ε_l goes to $+\infty$, Φ converges to 1.[5] Thus, local indeterminacy may occur with mild externalities, i.e., $\alpha < 1/2$ and $\max\left\{\frac{\alpha\Phi}{1-\eta}, (1-\alpha)\Phi\right\} < \beta < \Phi$, and a large share of first period consumption over the after-tax wage income, i.e., $1/2 < \eta$. This is in contrast with the result in Chen and Zhang (2009b), in which the share of first period consumption over the after-tax wage income to generate local indeterminacy should be less than 1/2. Moreover, Proposition 3 shows that, for mild externalities, small government expenditures are helpful to local indeterminacy, while large government expenditures are useful to stabilize the economy.

4. Capital income tax finance

In this section, we consider the case in which constant government expenditures are financed by capital income taxes, i.e., $\tau_{w_t} = 0$ and $g = \tau_{r_t} r_t k_t$. The intertemporal competitive equilibrium paths

[5] Note that the supremum of β's constraint tends to be 1 and the infimum tends to be $\max\left\{\frac{\alpha}{1-\eta}, 1-\alpha\right\}$, which is less than 1.

can be written as

$$k_{t+1} = (1-\eta(\widetilde{R}_{t+1}))w_t l_t, \tag{26-1}$$

$$v'(l_t/B)l_t/B = u_1(1, h(\widetilde{R}_{t+1}))w_t l_t, \tag{26-2}$$

where $\widetilde{R}_{t+1} = r_{t+1} - g/k_{t+1}$, $w_t = bAk_t^{\alpha}l_t^{\beta-1}$, and $r_t = aAk_t^{\alpha-1}l_t^{\beta}$.

4.1. Steady state existence

A steady state is a pair $\left(\overline{k}, \overline{l}\right)$ such that,

$$\overline{k} = \left(1-\eta(aA\overline{k}^{\alpha-1}\overline{l}^{\beta} - g/\overline{k})\right) bA\overline{k}^{\alpha}\overline{l}^{\beta}, \tag{27-1}$$

$$v'(\overline{l}/B)\overline{l}/B = u_1(1, h(aA\overline{k}^{\alpha-1}\overline{l}^{\beta} - g/\overline{k}))bA\overline{k}^{\alpha}\overline{l}^{\beta}. \tag{27-2}$$

Again, we consider the NSS $\left(\overline{k}, \overline{l}\right) = (1, 1)$. We have the following result.

Proposition 4. *Let $V(B) = v'(1/B)/B$. Under Assumptions 1-2 and the assumption of gross substitutability $\gamma > 1$, $\left(\overline{k}, \overline{l}\right) = (1, 1)$ is a normalized steady state (NSS) of the dynamic system (26) if and only if $lim_{A\to+\infty}\left(1-\eta(aA-g)\right)bA > 1$. The scaling parameters are set at the levels of $A^* > 0$, $B^* > 0$ that satisfy the following equations:*

$$1 = \left(1-\eta(aA-g)\right)bA, \tag{28-1}$$

$$B = V^{-1}\{u_1[1, h(aA-g)]bA\}. \tag{28-2}$$

Proof. See Appendix 3. ■

Again, in the rest of Section 4, we assume that the conditions of Proposition 4 hold in order to ensure the existence of the NSS.

Assumption 5. *$lim_{A \to +\infty} (1 - \eta(aA - g))\, bA > 1$, $A = A^*$* and *$B = B^*$.*

4.2. Local dynamics

Let us linearize the dynamical system (26) around the NSS. We can have the following result.

Proposition 5. *The two-dimensional system (26) defines uniquely a local dynamics near the NSS $(\overline{k}, \overline{l}) = (1, 1)$. Let $\Phi \equiv \frac{1+\varepsilon_l}{\varepsilon_l} > 1$. The linearized dynamics for the deviations $dk_t = k_t - \overline{k}$, $dl_t = l_t - \overline{l}$ are determined by the determinant D_C and the trace T_C of the Jacobian matrix. And the expressions of D_C and T_C are given by*

$$D_C = \frac{\alpha\Phi}{\beta}\left[\frac{1}{1-\eta} - \frac{b}{a}g\right], \tag{29-1}$$

$$T_C = \left[\frac{b}{a}g - \frac{1}{1-\eta}\right]\left[\eta\gamma - \frac{\Phi}{\beta}\left(\eta\left(\gamma - 1\right) + 1\right)\right] + \Phi\frac{\alpha\eta(1-\gamma)}{\beta\left(1-\eta\right)}. \tag{29-2}$$

Proof. See Appendix 4. ■

Depending on the degree of increasing returns to scale, local dynamics will be affected by a change in government expenditure. As before, we need figure out the range of g. Taking $g = aA^*\tau_r^*$ into Eq. (28-1) gives rise to $1 = bA^*\left[1 - \eta(aA^*\left(1 - \tau_r^*\right))\right]$. The homogeneity condition implies that η is independent of $aA - g$. Thus, we have that $A^* = \left[b\left(1-\eta\right)\right]^{-1}$, which means that A^* is independent of τ_r^* and $g = \frac{a\tau_r^*}{b(1-\eta)}$. Let $g^{\max} \equiv a/\left[b\left(1-\eta\right)\right]$. Since $\tau_r^* \in (0,1)$, the range of g is $(0, g^{\max})$ under the case with capital income taxes. Then we can have the following result.

Lemma 3. *Let $\overline{S}_C \equiv \frac{\alpha\Phi}{\beta}\frac{-1}{\eta\gamma - \frac{\Phi}{\beta}[\eta(\gamma-1)+1]} > 0$. Then (T_C, D_C) describes a half-line Δ_C as g goes from 0 to $g^{\max}$,*

$$D_C = \overline{S}_C T_C - \overline{S}_C \Phi\frac{\alpha\eta(1-\gamma)}{\beta\left(1-\eta\right)}. \tag{30}$$

Proof. Under the assumption $\frac{\Phi}{\beta} > 1$, we have that $\frac{\Phi}{\beta}\left[\eta\left(\gamma-1\right)+1\right] - \eta\gamma > \eta\left(\gamma - 1\right) + 1 - \eta\gamma = 1 - \eta > 0$, which implies that $0 < \overline{S}_C$. ■

Only one part of Δ_C is relevant as $g \in (0, g^{\max})$. First, we observe the starting and end points of the pair $(T_C(g), D_C(g))$ in order to locate the half line Δ_C. It is easy to derive that

$$\begin{aligned}
\lim_{g \to 0} D_C &= D^0 > 0, \ \lim_{g \to 0} T_C = T^0, \\
\lim_{g \to g^{\max}} D_C &= 0, \text{ and } T_C^m \equiv \lim_{g \to g^{\max}} T_C = \Phi \frac{\alpha\eta(1-\gamma)}{\beta(1-\eta)} < 0.
\end{aligned}$$

As a result, the relevant part of Δ_C describes a half-line which starts in $\left(T^0, D^0\right)$ for $g = 0$ and points downwards to the left as g lies in $(0, g^{\max})$. Again, we impose the same conditions: $D^0 < 1$, $D^0 > T^0 - 1$, and $D^0 > -T^0 - 1$. Similar to the analysis shown in Section 3, we require the following restrictions:

$$1 < \gamma < \frac{1+\eta}{\eta} \text{ and } 1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}.$$

These conditions imply that $|T_C^m| = \Phi \frac{\alpha\eta(\gamma-1)}{\beta(1-\eta)} < \eta(\gamma-1) < 1$ and $0 < \overline{S}_C < 1$, which in turn implies that the whole half-line Δ_C lies inside the triangle ABC as g lies in $(0, g^{\max})$. Therefore, when $\eta < 1-\alpha$, $1 < \gamma < \frac{1+\eta}{\eta}$ and $1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}$ hold, the equilibrium paths are indeterminate for $g \in (0, g^{\max})$.

We can summarize the local dynamics in the following proposition.

Proposition 6. *Under Assumptions 1, 2, 4 and 5, when $\eta < 1-\alpha$, $1 < \gamma < \frac{1+\eta}{\eta}$ and $1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}$ hold, as g goes up from 0 to $g^{\max}$, (T_C, D_C) moves from the point $\left(T^0, D^0\right)$ downwards to the left and stops in the T_C axis. For any $g \in (0, g^{\max})$, the half-line Δ_C lies inside the triangle ABC, and the economy always exhibits local indeterminacy.*

In contrast to the case with labor income taxes, the dynamic system here only exhibits equilibrium indeterminacy and there do not exist flip/Hopf bifurcations. The local dynamics here are less complicated than those in the former case. In other words, for any level of government expenditures financed by capital income taxes, local indeterminacy always exists for mild externalities. In the case with labor income taxes, local indeterminacy occurs only for small level of government expenditures. Therefore, local indeterminacy occurs more easily in the case with capital income taxes.

5. Interpretation

5.1. The case with labor income taxes

To gain the insights behind the indeterminacy result, first, let us consider the labor market. Following Lloyd-Braga et al. (2007), it is easy to get the following equations:

$$\frac{r_{t+1}dl_t}{l_t dr_{t+1}} = \frac{1-\eta}{1/\varepsilon_l - (1-\eta)\, g_L}, \; \frac{k_{t+1}dr_{t+1}}{r_{t+1}dk_{t+1}} = \alpha - 1, \; \frac{l_{t+1}dr_{t+1}}{r_{t+1}dl_{t+1}} = \beta > 0.$$

For small values of government expenditures g_L, we have that $\frac{dl_t}{dr_{t+1}} \frac{r_{t+1}}{l_t} = \frac{1-\eta}{1/\varepsilon_l - (1-\eta) g_L} > 0$. When we substitute $r_{t+1} = aAk_{t+1}^{\alpha-1} l_{t+1}^{\beta}$ and $w_t = bAk_t^{\alpha} l_t^{\beta-1}$ into the system (17) and log-linearize it, simple computations give rise to

$$(1-\eta)\, \beta \widehat{l}_{t+1} = M_2 \widehat{l}_t + M_3 \widehat{k}_t, \tag{31}$$

$$\widehat{k}_{t+1} = \widehat{l}_t \left\{ \frac{\eta\,(\gamma-1)\,\Phi}{1-\eta} + \beta M_1 \frac{1-\eta\gamma}{1-\eta} \right\} + \alpha M_1 \frac{1-\eta\gamma}{1-\eta} \widehat{k}_t, \tag{32}$$

where $M_1 = 1 + (1-\eta)\, g_L$, $M_2 = \Phi\,[1 + (1-\alpha)\,\eta\,(\gamma-1)] - \beta M_1\,[1 - (1-\alpha)\,(1-\eta\gamma)]$, and $M_3 = -\alpha M_1\,[1 - (1-\alpha)\,(1-\eta\gamma)]$.

When the government expenditure g_L is not large, we will have that $\Phi - \beta M_1 > 0$, $M_2 > 0$ and $\frac{r_{t+1}dl_t}{l_t dr_{t+1}} > 0$ since $\Phi/\beta > 1$. The expectations on an increase of the future real interest rate r_{t+1} will lead to an increase of the labor supply l_t since $\frac{r_{t+1}dl_t}{l_t dr_{t+1}} > 0$ and k_t is predetermined. Hence an increase in tomorrow's capital stock k_{t+1} through Eq. (32) follows. In addition, $M_2 > 0$ implies that the rise in the current labor supply must be sustained by an increase in the future hours worked l_{t+1}

from Eq. (31). An increase in the capital stock k_{t+1} can decrease the real interest rate r_{t+1} since $\frac{k_{t+1}dr_{t+1}}{r_{t+1}dk_{t+1}} = \alpha - 1 < 0$. $\frac{l_{t+1}dr_{t+1}}{r_{t+1}dl_{t+1}} = \beta > 0$ implies that an increase in the labor supply l_{t+1} will raise the real interest rate r_{t+1}. When the latter effect dominates the former one, the expectations can be self-fulfilling.

Notice that from (32), we have that

$$\widehat{k}_{t+n+1} - \widehat{k}_{t+n} = \frac{1}{1-\eta}\left[M_4\widehat{l}_{t+n} - M_5\widehat{k}_{t+n}\right] - g_L\left(\eta\gamma - 1\right)\left[\beta\widehat{l}_{t+n} + \alpha\widehat{k}_{t+n}\right], \tag{33}$$

with $M_4 = \eta\left(\gamma - 1\right)\Phi - \beta\left(\eta\gamma - 1\right)$ and $M_5 = \alpha\left(\eta\gamma - 1\right) + \left(1 - \eta\right) > 0$. It is easy to find that $M_4 > 0$ and $M_5 > 0$ hold since $\Phi/\beta > 1$ and $1 - \eta > \alpha$.

First, suppose $g_L = 0$. From Eq. (33), when the capital stock grows faster than hours worked at time $t+n$, the deviation of the capital stock from the NSS will be reduced, which makes indeterminacy occur. Now we consider the case: $g_L > 0$. The introduction of government expenditures can instead increase the deviation of the capital stock from its steady state since $g_L\left(\eta\gamma - 1\right)$ can be less than zero (for instance, $\gamma = 1.1$ and $\eta = 0.51$ as in Lloyd-Braga et al. (2007)). To summarize, indeterminacy can arise for small value of government expenditures g_L when the latter is financed by labor income taxes.[6]

5.2. The case with capital income taxes

In the case with capital income taxes, we can obtain the following derivatives

$$\begin{aligned}
\frac{\widetilde{R}_{t+1}dl_t}{l_t d\widetilde{R}_{t+1}} &= \left(1-\eta\right)\varepsilon_l > 0, \\
\frac{k_{t+1}d\widetilde{R}_{t+1}}{\widetilde{R}_{t+1}dk_{t+1}} &= \frac{\alpha r}{\widetilde{R}} - 1 \equiv c_1, \; \frac{l_{t+1}dr_{t+1}}{r_{t+1}dl_{t+1}} = \frac{\beta r}{\widetilde{R}} \equiv c_2 > 0.
\end{aligned}$$

[6] As the government expenditure g_L is large, the deviation of the capital stock from the NSS will increase a lot, which makes indeterminacy hard to arise.

After we substitute $\widetilde{R}_{t+1} = r_{t+1} - g/k_{t+1} = aAk_{t+1}^{\alpha-1}l_{t+1}^{\beta} - g/k_{t+1}$ and $w_t = bAk_t^{\alpha}l_t^{\beta-1}$ into the dynamic system (26) and log-linearize it, simple computations give rise to

$$(1-\eta)\, c_2 \widehat{l}_{t+1} = M_6 \widehat{l}_t + M_7 \widehat{k}_t, \tag{34}$$

$$\widehat{k}_{t+1} = \alpha \frac{1-\eta\gamma}{1-\eta} \widehat{k}_t + \left[\beta \frac{1-\eta\gamma}{1-\eta} + \frac{\eta(\gamma-1)\Phi}{1-\eta} \right] \widehat{l}_t, \tag{35}$$

where $M_6 = \Phi - \beta - c_1 [\Phi\eta(\gamma-1) + \beta(1-\eta\gamma)]$ and $M_7 = -\alpha[1 + c_1(1-\eta\gamma)]$.

For small values of g_C, we have that $c_1 < 0$, $c_2 > 0$, and $M_6 > 0$. The expectations on an increase of the future real interest rate $\widetilde{R}_{t+1}$ will lead to an increase of the current labor supply, since $\frac{\widetilde{R}_{t+1} dl_t}{l_t d\widetilde{R}_{t+1}} = (1-\eta)\varepsilon_l > 0$. Since $k_{t+1} = (1-\eta)w_t l_t$, it follows a higher capital stock in the next period. In addition, from Eq. (34), the rise in the current labor must be sustained by an increase in the next period labor supply l_{t+1}. Although an increase in the capital stock k_{t+1} can decrease the real interest rate $\widetilde{R}_{t+1}$ since $\frac{k_{t+1} d\widetilde{R}_{t+1}}{\widetilde{R}_{t+1} dk_{t+1}} = \alpha r/\widetilde{R} - 1 < 0$, an increase in the labor supply l_{t+1} will raise the real interest rate $\widetilde{R}_{t+1}$ because $\frac{l_{t+1} dr_{t+1}}{r_{t+1} dl_{t+1}} = \beta r/\widetilde{R} > 0$. When the latter effect dominates the former one, expectations can be self-fulfilling.

But for large values of g_C, we can have that $c_1 > 0$, $c_2 > 0$, and $M_6 < 0$. $c_1 > 0$ implies that the after-tax real interest rate is increasing with respect to the capital stock. Therefore, an increase of the labor supply l_t can raise the future capital stock from Eq. (35), then increase the after-tax real interest rate in the next period. However, $M_6 < 0$ implies that the rise in the current labor must be sustained by an decrease in the next period labor supply l_{t+1}, which in turn decreases the after-tax real interest rate. When the former effect dominates the latter one, the after-tax real interest rate in the next period can increase, and the expectations become self-fulfilling.

Notice that

$$(1-\eta)\left[\widehat{k}_{t+n+1}-\widehat{k}_{t+n}\right]=\left[\beta\left(1-\eta\gamma\right)+\eta\left(\gamma-1\right)\Phi\right]\widehat{l}_{t+n}-\left[\alpha\left(\eta\gamma-1\right)+\left(1-\eta\right)\right]\widehat{k}_{t+n}. \tag{36}$$

It is easy to find that $\beta\left(1-\eta\gamma\right)+\eta\left(\gamma-1\right)\Phi>0$ and $\alpha\left(\eta\gamma-1\right)+\left(1-\eta\right)>0$ can hold when $\Phi/\beta>1$ and $\eta\gamma<1$. When the capital stock grows faster than the hours worked at time $t+n$, the deviation of the capital stock from the NSS will be reduced, which makes indeterminacy occur.

6. Constant income taxes

In previous sections, we consider that either labor or capital income tax rate is endogenously adjusted to satisfy the budget constraint for a given value of government expenditure. In this section, we consider another kind of fiscal policy specification in which government expenditure is endogenously determined for fixed tax rates on labor and capital income. Then the intertemporal competitive equilibrium paths become

$$k_{t+1} = (1-\eta(\widetilde{R}_{t+1}))\left(1-\tau_w\right)w_t l_t, \tag{37-1}$$

$$v'(l_t/B)/B = u_1(1,h(\widetilde{R}_{t+1}))\left(1-\tau_w\right)w_t, \tag{37-2}$$

where $\widetilde{R}_{t+1}=1-\delta+\left(1-\tau_r\right)r_{t+1}$, $w_t=bAk_t^{\alpha}l_t^{\beta-1}$, $r_t=aAk_t^{\alpha-1}l_t^{\beta}$, and $\tau_w w_t l_t+\tau_r r_t k_t=g_t$.

As for the existence of the normalized steady state, we have the following result.

Proposition 7. *Under Assumptions 1-2, let $V(B)=v'(1/B)/B$. Then $\left(\bar{k},\bar{l}\right)=(1,1)$ is a normalized steady state of the dynamic system (37) if and only if $lim_{A\to+\infty}\left[1-\eta\left(aA\left(1-\tau_r\right)\right)\right]\left(1-\tau_w\right)bA>1$. The scaling parameters A, B are set at the levels $A^{**}>0$, $B^{**}>0$ that satisfy the following* *equations:*

$$1 = [1 - \eta(aA(1-\tau_r))](1-\tau_w)bA, \qquad (38\text{-}1)$$

$$B = V^{-1}\{u_1[1, h(aA(1-\tau_r))](1-\tau_w)bA. \qquad (38\text{-}2)$$

Proof. See Appendix 5. ■

First, we assume that the conditions in Proposition 7 hold in order to ensure the existence of the NSS in the rest of Section 6.

Assumption 6. $lim_{A\to+\infty}[1-\eta(aA(1-\tau_r))](1-\tau_w)bA > 1$, $A = A^{**}$ and $B = B^{**}$.

Second, we linearize the dynamic system (37) around the NSS to study the local dynamics. After tedious algebra, we can get

$$\begin{bmatrix} dk_{t+1} \\ dl_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\eta(1-\gamma)(\alpha-1) & \eta(1-\gamma)\beta \\ (1-\eta)(\alpha-1) & \beta(1-\eta) \end{bmatrix}^{-1} \begin{bmatrix} \alpha & \beta \\ -\alpha & \frac{1}{\varepsilon_l}-(\beta-1) \end{bmatrix} \begin{bmatrix} dk_t \\ dl_t \end{bmatrix} \qquad (39)$$

Moreover, the trace and the determinant can be written as follows

$$Tr = \frac{1}{(1-\eta)\beta}\left[-\eta\gamma\beta + \frac{1+\varepsilon_l}{\varepsilon_l}(1+\eta(\gamma-1)(1-\alpha))\right], \qquad (40\text{-}1)$$

$$Det = \frac{1+\varepsilon_l}{\varepsilon_l}\frac{\alpha}{(1-\eta)\beta}. \qquad (40\text{-}2)$$

From Eq. (40), we can find that the trace and the determinant do not depend on the constant labor and capital income tax rates. In other words, the stability of the NSS is not affected by the presence of constant labor and capital income tax rates. We can summarize these results as follows.

Proposition 8. *Unlike the case of endogenous income taxes, the constant labor and capital income tax rates have no impact on the range of values of increasing returns in production.*

Therefore, we conclude that the local dynamic property is the same as that obtained in Lloyd-Braga et al. (2007). In other words, the range of values of increasing returns is independent of the constant tax rates on labor as well as capital income, when we consider an extended dynamic model with consumptions in two periods and observe the relation between fiscal policy and the occurrence of multiple equilibria. Guo and Lansing (2002) suggest that (in a Ramsey model) the minimum level of increasing returns leading to indeterminacy is raised by increasing constant capital income tax rate. But this property does *not* hold in the OLG framework.

7. Concluding Remarks

In this paper, we explore how both alternative government financing methods and increasing returns influence aggregate fluctuations driven by self-fulfilling expectations in an OLG model with consumption in both periods of life, homothetic preferences and productive external effects as studied in Lloyd-Braga et al. (2007). We find that (1) when constant government expenditures are financed by labor income taxes, local indeterminacy arises for small government expenditures and mild externalities and; (2) when constant government expenditures are financed by capital income taxes, local indeterminacy always occurs for mild externalities. In addition, local indeterminacy in both cases occurs under a large enough share of first period consumption over the wage income. Therefore, indeterminacy is more likely to occur if the government uses capital income taxes to finance its expenditure. Moreover, we consider the case where government expenditure is endogenously determined for fixed rates on labor and capital income under a balanced-budget rule. In contrast to the previous results, we show that the constant tax rates on labor and capital income have no impact on the range of values of increasing returns leading to local indeterminacy.

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Appendix:

A.1. Proof of Proposition 2

Let $\Phi = (1+\varepsilon_l)/\varepsilon_l$, and $M_1 \equiv 1+(1-\eta)g$. Under the case with labor income taxes, the linearization of Eq. (17) around the NSS is

$$\begin{bmatrix} dk_{t+1} \\ dl_{t+1} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} dk_t \\ dl_t \end{bmatrix} \tag{A1}$$

where $J_{11} = \frac{\alpha(1-\eta\gamma)M_1}{1-\eta}$, $J_{21} = \frac{\alpha M_1}{\beta}\frac{(1-\alpha)(1-\eta\gamma)-1}{1-\eta}$, $J_{12} = \frac{\beta(1-\eta\gamma)M_1}{1-\eta} - \Phi\frac{\eta(1-\gamma)}{1-\eta}$, and $J_{22} = \frac{[(1-\alpha)(1-\eta\gamma)-1]M_1}{1-\eta} - \Phi\frac{\eta(1-\gamma)(1-\alpha)-1}{\beta(1-\eta)}$.

A.2. Proof of Lemma 2

$D^0 < 1$ requires that $\alpha\Phi/\beta < 1-\eta < 1$, or $\Phi/\beta < (1-\eta)/\alpha$. Since $\Phi/\beta > 1$, therefore $1-\eta > \alpha$ holds. To ensure that (T^0, D^0) lies inside the triangle ABC, we need that $D^0 - T^0 + 1 > 0$ and $D^0 + T^0 + 1 > 0$. It is easy to get $D^0 - T^0 + 1 = \frac{1+\eta(\gamma-1)}{1-\eta}\left[\frac{\Phi}{\beta}(\alpha-1)+1\right]$. Therefore $D^0 - T^0 + 1 > 0$ holds if $\frac{\Phi}{\beta} < \frac{1}{1-\alpha}$. $D^0 + T^0 + 1 = \frac{\Phi}{\beta}\left[\frac{\alpha(1-\eta\gamma+\eta)}{1-\eta} + 1 + \frac{\beta}{\Phi}\right] + \frac{\eta\gamma}{1-\eta}\left[\frac{\Phi}{\beta} - 1\right]$ holds. We assume that $1 < \gamma < \frac{1+\eta}{\eta}$. Therefore, $D^0 + T^0 + 1 > 0$ holds since $\frac{\Phi}{\beta} > 1$. When $D^0 < 1$, $\eta > \frac{1}{2}$, and $1 < \gamma < \frac{1+\eta}{\eta}$ hold, $\left|\overline{\mathcal{S}}_W\right| = \frac{\alpha\Phi}{\beta\eta\gamma} < 1$ holds.

Thus, when

$$1 < \gamma < \frac{1+\eta}{\eta} \text{ and } 1 < \frac{\Phi}{\beta} < \min\left\{\frac{1-\eta}{\alpha}, \frac{1}{1-\alpha}\right\}, \tag{A2}$$

we can find that $D^0 < 1$, the point (T^0, D^0) lies inside the triangle ABC and $\left|\overline{\mathcal{S}}_W\right| < 1$.

A.3. Proof of Proposition 4

If $(\bar{k}, \bar{l}) = (1,1)$ is a normalized steady state, the dynamic system (26) becomes

$$\begin{aligned} 1 &= (1 - \eta(aA - g))\, bA, && \text{(A3)} \\ v'(1/B)/B &= u_1[1, h(aA-g)]bA. && \text{(A4)} \end{aligned}$$

Since $V'(B) < 0$, $V(B) = v'(1/B)/B$ is invertible. $aA - g = aA(1-\tau_k^*) > 0$ holds since $\tau_k^* \in (0,1)$ is the steady state capital income tax rate. Let $G(A) = (1-\eta(aA-g))\,bA$. We can easily get $\frac{G'(A)A}{G(A)} = 1 - \eta(1-\gamma)\frac{aA}{aA-g}$, since $\eta'\left(\widetilde{R}\right)\widetilde{R}/\eta\left(\widetilde{R}\right) = \left(1-\eta\left(\widetilde{R}\right)\right)\left(1-\gamma\left(\widetilde{R}\right)\right)$ where $\widetilde{R} = aA - g$. With gross substitutability $\gamma > 1$, $G'(A)A/G(A) > 0$ always holds. Since $\eta \in (0,1)$, we have that $lim_{A\to 0}(1-\eta(aA-g))\,bA = 0$. Then we can obtain a unique $A^* > 0$ from (A3) iff $lim_{A\to+\infty}(1-\eta(aA-g))\,bA > 1$. $B^* > 0$ can be easily derived from (A4) after the unique A^* is pinned down.

A.4. Proof of Proposition 5

In the case with capital income taxes, linearizing Eq. (26) around the NSS yields

$$\begin{bmatrix} dk_{t+1} \\ dl_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\eta(1-\gamma)[a(\alpha-1)+(1-\eta)bg]}{a-(1-\eta)bg} & \frac{a\beta\eta(1-\gamma)}{a-(1-\eta)bg} \\ -\frac{a(\alpha-1)+(1-\eta)bg}{a-(1-\eta)bg} & -\frac{a\beta\eta(1-\gamma)}{a-(1-\eta)bg} \end{bmatrix}^{-1} \times \begin{bmatrix} \alpha & \beta \\ \alpha & \beta-\Phi \end{bmatrix} \begin{bmatrix} dk_t \\ dl_t \end{bmatrix}. \tag{A5}$$

A.5. Proof of Proposition 7

Let $V(B) = v'(1/B)/B$. Assumption 1 implies that $V'(B) < 0$. Therefore, $V(B)$ is invertible. If $(\bar{k}, \bar{l}) = (1, 1)$ is a normalized steady state, then system (37) becomes

$$1 = [1 - \eta (aA(1 - \tau_r))] (1 - \tau_w) bA \equiv G(A), \tag{A6}$$

$$v'(1/B)/B = u_1[1, h(aA(1 - \tau_r))] (1 - \tau_w) bA. \tag{A7}$$

We can easily get $G'(A)A/G(A) = 1 - (1-\gamma)\eta > 0$, since $\eta'\left(\widetilde{R}\right)\widetilde{R}/\eta\left(\widetilde{R}\right) = \left(1 - \eta\left(\widetilde{R}\right)\right)\left(1 - \gamma\left(\widetilde{R}\right)\right)$ where $\widetilde{R} = aA(1 - \tau_r)$. It follows that $G(A)$ is monotonic with respect to A for any $\gamma > 0$. For any $\widetilde{R} \geq 0$, $\eta\left(\widetilde{R}\right) \in (0, 1)$ holds. Therefore, we have that $lim_{A \to 0} [1 - \eta(aA(1 - \tau_r))] (1 - \tau_w) bA = 0$. There exists a unique $A^{**} > 0$ which satisfies (A6) iff $lim_{A \to +\infty} [1 - \eta (aA(1 - \tau_r))] (1 - \tau_w) bA > 1$. $B^{**} > 0$ can be solved from (A7) after the unique A^{**} is pinned down.

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Tables and Figures

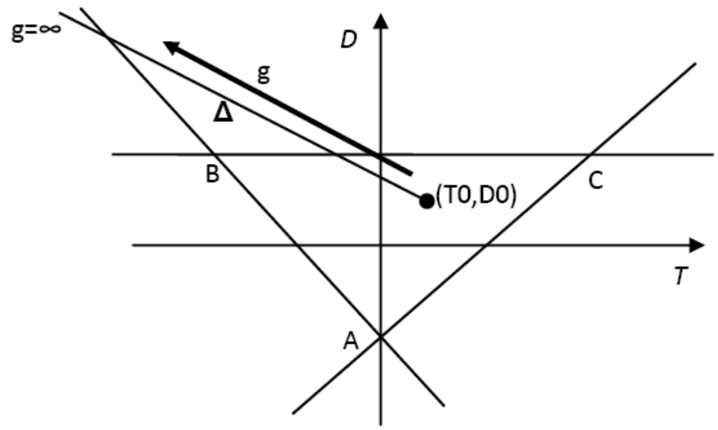

Figure 1. Labor income taxes: Case 1.1. The line Δ_W intersects the line AB and the segment BC, both Hopf and flip bifurcations can occur.

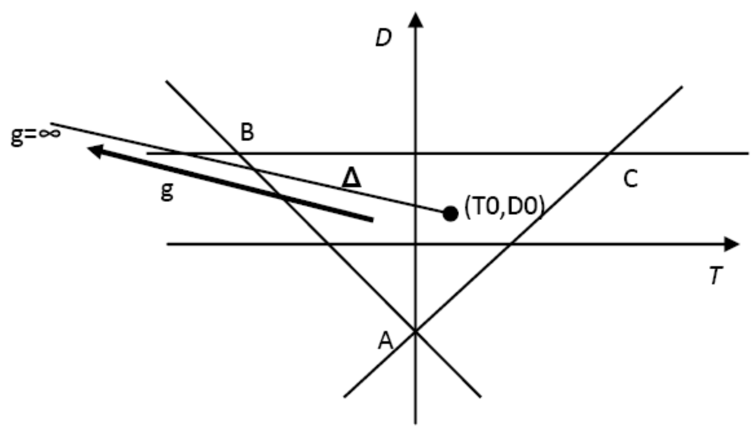

Figure 2. Labor income taxes: Case 1.2. The line Δ_W intersects the line AB, only flip bifurcations can occur.

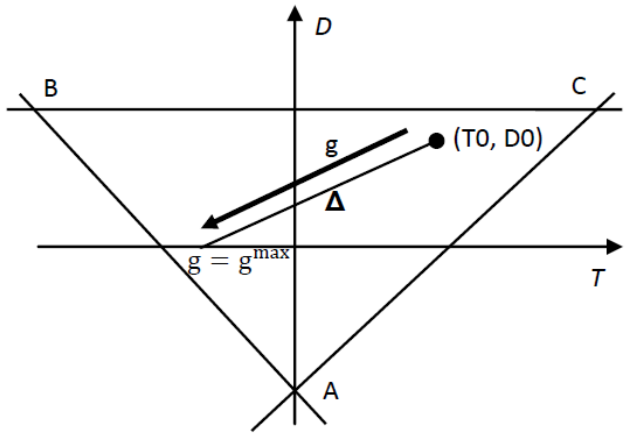

Figure 3. Capital income taxes.