Robustness of Bayes decisions for normal and lognormal distributions under hierarchical priors

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30 April 2010

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Abstract
In this paper we derive the Bayes estimates of the location parameter of normal and lognormal distribution under the hierarchical priors for the vector parameter, $\underset{\sim}{\theta} \in \mathbb{R}^n$. The ML-II ε-contaminated class of priors are employed at the second stage of hierarchical priors to examine the robustness of Bayes estimates with respect to possible misspecification at the second stage. The simulation studies for both normal and lognormal distributions confirm Berger's (1985) assertion that form of the second stage prior does not affect the Bayes decisions.

1. Introduction

The paper attempts to examines the assertion made by Berger (1985, page 232) that choice of a form for the second stage of hierarchical prior seems to have relatively little effect on Bayes estimates. The hierarchical priors are employed to contain the structural and subjective prior information at the same time, which is convenient to model in stages. Hierarchical priors are employed when vector parameter $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ is considered and it is assumed that $\theta_i \ (i = 1, 2, \ldots, n)$ are distributed independently with common prior distribution $g(\theta_i \mid \lambda)$. In general $g(\theta_i \mid \lambda)$ is assumed to be member of class

$$\Gamma_1 = \left\{ g(\underset{\sim}{\theta} \mid \lambda) : \text{ g is of given functional form and } \lambda \in \Lambda \right\},$$

and on the hyper parameter λ we define a second stage prior, say, $h(\lambda)$.

From Bayesian viewpoint investigation of robustness of priors is vital both at the first and second stage. In the study of hierarchical Bayes estimators of the normal mean, Berger (1985) considered a normal second stage prior for the mean and non-informative prior for variance of first stage normal prior. Since the second stage prior is based on only subjective prior information, a conjugate prior at the second stage is considered for mathematical convenience. The robustness study of Bayes procedures with respect to a possible misspecification of the prior has three possible concerns in case of hierarchical priors:
(a) $\theta_i \ (i = 1, 2, \ldots, n)$ are independent and identically distributed, (b) First stage prior $g(\theta_i \mid \lambda)$ belongs to Γ_1, and (c) Second stage $h(\lambda)$ is specified correctly.

Berger and Berliner (1986) used ε-contaminated class of priors to represent the uncertainty both in $h(\lambda)$ and $g(\theta_i \mid \lambda)$ in order to study the robustness with respect to misspecification in the hierarchical priors. Deeley and Lindley (1981) consider the difference between an empirical Bayes model and a Bayes empirical Bayes model. Berger and Berliner (1984) study empirical Bayes Type-II likelihood prior methods to study the relationship between Stein estimation of multivariate normal mean and Bayesian analysis. Moreno and Pericchi (1993) examined the hierarchical ε-contaminated class of priors with different contaminating classes when the true prior belongs to the location-scale family of distributions. Sivaganeshan (2000) discussed the uses and limitations of global and local robustness approaches.

We restrict our robustness study when second stage prior, $h(\lambda)$, is considered uncertain. An ε-contaminated model for h would be

$$h(\lambda) = (1-\varepsilon)h_o(\lambda) + \varepsilon s(\lambda), \ s \in S$$

Here h_o is the true assessed prior and s, being a contamination, belongs to the class S of all distributions. S determines the allowed contaminations that are mixed with h_o, and $\varepsilon \in [0, 1]$ reflects the amount of probabilistic deviation from h_o .

Let $Q = \{q : s \in S\}$, the uncertainty in first stage can be expressed by

$$\Gamma = \left\{\pi \ : \ \pi = (1-\varepsilon)\pi_o + \varepsilon q, \ q \in Q \ \right\}$$

Type II Maximum Likelihood (ML-II) technique is used to select a robust prior from ε-contaminated class of priors having the above form. This technique naturally selects a prior with a large tail which will be robust against all plausible deviations.

For selecting a ML-II prior, we choose a robust prior π in the class Γ of priors which maximizes the marginal $m(\underset{\sim}{x} \mid \pi)$. Thus for

$$\pi(\underset{\sim}{\theta}) = (1-\varepsilon)\pi_o(\underset{\sim}{\theta}) + \varepsilon q(\underset{\sim}{\theta})$$

where $\pi_o(\underset{\sim}{\theta}) = \int g(\underset{\sim}{\theta} \mid \lambda) h_o(\lambda) d\lambda$ and $q(\underset{\sim}{\theta}) = \int g(\underset{\sim}{\theta} \mid \lambda) s(\lambda)\, d\lambda$.

The marginal of $\underset{\sim}{x}$

$$m(\underset{\sim}{x} \mid \pi) = (1-\varepsilon) m(\underset{\sim}{x} \mid \pi_o) + \varepsilon m(\underset{\sim}{x} \mid q)$$

where $m(\underset{\sim}{x} \mid q) = \int m(\underset{\sim}{x} \mid \lambda) s(\lambda) d\lambda$ and $m(\underset{\sim}{x} \mid \lambda) = \int f(\underset{\sim}{x} \mid \underset{\sim}{\theta}) g(\underset{\sim}{\theta} \mid \lambda) d\underset{\sim}{\theta}$

can be maximized by maximizing $m(\underset{\sim}{x} \mid \lambda)$ over Q. Let the maximum be attained at unique $\hat{s} \in Q$. Thus an estimated ML-II prior $\hat{\pi}(\underset{\sim}{\theta})$ is given by

$$\hat{\pi}(\underset{\sim}{\theta}) = (1-\varepsilon)\hat{\pi}_o(\underset{\sim}{\theta}) + \varepsilon \hat{q}(\underset{\sim}{\theta}) \tag{1}$$

2. Robustness under second stage prior misspecification for Normal distribution

Suppose $\underset{\sim}{x}$ consists of independent components $\{x_1, x_2, \ldots, x_n\}$, where each x_i has density $f(x_i \mid \theta_i)$ independently from $N(\theta_i, r)$; with common known precision r. Assume θ_i's are exchangeable and their prior distribution are staged as follows

Stage I: $\theta_i \ (i = 1, 2, \ldots, n)$ are independent $N(\mu, \tau)$; known precision with pdf

$$g(\theta_i \mid \mu) = \sqrt{\frac{\tau}{2\pi}} \exp\left[-\frac{\tau}{2}(\theta_i - \mu)^2\right]$$

Here we use the fact that the sample mean is the sufficient statistic for the unknown mean of the related normal population. Hence we take $\bar{\theta} = \sum_{i=1}^{n} \theta_i / n$ which gives $g(\bar{\theta} \mid \mu) \sim N(\mu, n\tau)$.

Stage II: The hyper parameter μ belongs to the ML-II ε-contaminated class of priors. Following Berger and Berliner (1986), we have $h_o(\mu)$ as $N(\mu_o, b)$, known *b*, with pdf

$$h_o(\mu) = \sqrt{\frac{b}{2\pi}} \exp\left[-\frac{b}{2}(\mu - \mu_o)^2\right]$$

and $\hat{s}(\mu)$ as $uniform(\mu_o - \hat{a}, \mu_o + \hat{a})$, $\hat{a}$ being the value of $'a'$ which maximizes

$$m(\underset{\sim}{x} \mid a) = \begin{cases} \dfrac{1}{2a} \displaystyle\int_{\mu_o - a}^{\mu_o + a} m(\underset{\sim}{x} \mid \mu)\, d\mu = \int_{-\infty}^{\infty} \int_{\mu_o - a}^{\mu_o + a} L(\bar{\theta} \mid \underset{\sim}{x})\, g(\bar{\theta} \mid \mu)\, d\mu\, d\bar{\theta} & a > 0 \\ m(\underset{\sim}{x} \mid \mu_o) & a = 0 \end{cases}$$

$m(\underset{\sim}{x} \mid \hat{a})$ is an upper bound on $m(\underset{\sim}{x} \mid q)$.

$$m(\underset{\sim}{x} \mid a) = \left(\frac{r}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{r}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) \sqrt{\frac{2\pi}{nr}}\, \frac{1}{2a} \int_{\mu_o - a}^{\mu_o + a} \sqrt{\frac{\tau'}{2\pi}} \exp\left[-\frac{\tau'}{2}(\mu - \bar{x})^2\right] d\mu$$

$$= \frac{C}{2a}\left\{\Phi\left[\sqrt{\tau'}(\mu_o + a - \bar{x})\right] - \Phi\left[\sqrt{\tau'}(\mu_o - a - \bar{x})\right]\right\} \tag{2}$$

where $C = \left(\frac{r}{2\pi}\right)^{\frac{n}{2}} \sqrt{\frac{2\pi}{nr}} \exp\left(-\frac{r}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$, $\tau' = \frac{n\tau r}{n\tau + r}$ and $\Phi(\cdot)$ denotes standard normal cdf.

On differentiating above equation with respect to $'a'$, we get

$$\frac{d}{da} m(\underset{\sim}{x} \mid a) = -\frac{C}{2a^2}\left\{\Phi\left[\sqrt{\tau'}(\mu_o + a - \bar{x})\right] - \Phi\left[\sqrt{\tau'}(\mu_o - a - \bar{x})\right]\right\} + \frac{C\sqrt{\tau'}}{2a}\left\{\phi\left[\sqrt{\tau'}(\mu_o + a - \bar{x})\right] + \phi\left[\sqrt{\tau'}(\mu_o - a - \bar{x})\right]\right\} \tag{3}$$

where $\phi(\cdot)$ denotes standard normal pdf.

Now we substitute $z = \sqrt{\tau'}\left|\bar{x} - \mu_o\right|$ and $a^* = a\sqrt{\tau'}$ in equation (3) and equate to zero. The equation becomes

$$\Phi(a^* - z) - \Phi\left[-(a^* + z)\right] = a^*\left\{\phi(a^* - z) + \phi\left[-(a^* + z)\right]\right\}$$

which can be written as

$$a^* = z + \left\{-2\log_e\left[\sqrt{2\pi}\left(\frac{1}{a^*}\left\{\Phi(a^* - z) - \Phi\left[-(a^* + z)\right]\right\} - \phi\left[-(a^* + z)\right]\right)\right]\right\}^{\frac{1}{2}} \tag{4}$$

We solve a^* by standard fixed-point iteration, set $a^* = z$ on the right-hand side of (4), which gives

$$\hat{a} = \begin{cases} 0 & if\ \ z \le 1.65 \\ \dfrac{a^*}{\sqrt{\tau'}} & if\ \ z > 1.65 \end{cases}$$

The posterior distribution of parameter $\bar{\theta}$ with respect to prior $\pi\left(\bar{\theta}\right)$ is given by

$$\pi\left(\bar{\theta} \mid \underset{\sim}{x}\right) = \frac{L\left(\bar{\theta} \mid \underset{\sim}{x}\right)\pi\left(\bar{\theta}\right)}{\lambda(\underset{\sim}{x})\int_{\Theta} L\left(\bar{\theta} \mid \underset{\sim}{x}\right)\pi_o\left(\bar{\theta}\right)d\bar{\theta} + (1-\lambda(\underset{\sim}{x}))\int_{\Theta} L\left(\bar{\theta} \mid \underset{\sim}{x}\right)q\left(\bar{\theta}\right)d\bar{\theta}}$$

$$= \frac{L\left(\bar{\theta} \mid \underset{\sim}{x}\right)\pi\left(\bar{\theta}\right)}{\lambda(\underset{\sim}{x})m\left(\underset{\sim}{x} \mid \pi_o\right) + (1-\lambda(\underset{\sim}{x}))m\left(\underset{\sim}{x} \mid q\right)} = \lambda(\underset{\sim}{x})\pi_o\left(\bar{\theta} \mid \underset{\sim}{x}\right) + (1-\lambda(\underset{\sim}{x}))q\left(\bar{\theta} \mid \underset{\sim}{x}\right) \tag{5}$$

Here

$$\pi_o\left(\bar{\theta} \mid \underset{\sim}{x}\right) = \frac{L\left(\bar{\theta} \mid \underset{\sim}{x}\right)\pi_o\left(\bar{\theta}\right)}{m\left(\underset{\sim}{x} \mid \pi_o\right)} = \sqrt{\frac{\tau_2}{2\pi}}\exp\left[-\frac{\tau_2}{2}\left(\bar{\theta} - t_3\right)^2\right] \tag{6}$$

where

$$\pi_o\left(\bar{\theta}\right) = \int_{-\infty}^{\infty} g\left(\bar{\theta} \mid \mu\right)h_o\left(\mu\right)d\mu = \sqrt{\frac{\tau_1}{2\pi}}\exp\left[-\frac{\tau_1}{2}\left(\bar{\theta} - \mu_o\right)^2\right]$$

$$m\left(\underset{\sim}{x} \mid \pi_o\right) = \int_{-\infty}^{\infty} m\left(\underset{\sim}{x} \mid \mu\right)\, h_o\left(\mu\right)d\mu = \left(\frac{r}{2\pi}\right)^{\frac{n}{2}}\sqrt{\frac{\tau b}{(r+\tau)(\tau'+b)}}e^{-\beta}$$

$\tau_2 = nr + \tau_1,\ t_3 = \dfrac{nr\bar{x} + \tau_1\mu_o}{nr + \tau_1},\quad \tau_1 = \dfrac{n\tau b}{n\tau + b},\ \beta = \beta' + \dfrac{r}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2$ and $\beta' = \dfrac{\tau' b}{2(\tau'+b)}(\mu_o - \bar{x})^2$,

and

$$q\left(\bar{\theta} \mid \underset{\sim}{x}\right) = \frac{L\left(\bar{\theta} \mid \underset{\sim}{x}\right)\hat{q}\left(\bar{\theta}\right)}{m\left(\underset{\sim}{x} \mid q\right)} = \sqrt{\frac{nr}{2\pi}}\exp\left[-\frac{nr}{2}\left(\bar{\theta} - \bar{x}\right)^2\right]\frac{\varphi}{\hat{\varphi}_1} \tag{7}$$

where

$$\hat{q}\left(\bar{\theta}\right) = \int_{\mu_o-\hat{a}}^{\mu_o-\hat{a}} g\left(\bar{\theta} \mid \mu\right)\hat{s}\left(\mu\right)d\mu = \frac{1}{2\hat{a}}\int_{\sqrt{n\tau}\left(\mu_o-\hat{a}-\bar{\theta}\right)}^{\sqrt{n\tau}\left(\mu_o+\hat{a}-\bar{\theta}\right)} \phi(u)du = \frac{\varphi}{2\hat{a}}$$

$$m\left(\underset{\sim}{x} \mid q\right) = \int_{\mu_o-\hat{a}}^{\mu_o+\hat{a}} m\left(\underset{\sim}{x} \mid \mu\right)\hat{s}\left(\mu\right)d\mu = C\frac{\hat{\varphi}_1}{2\hat{a}}\ ,\ \hat{\varphi}_1 = \Phi\left[\sqrt{\tau'}\left(\mu_o + \hat{a} - \bar{x}\right)\right] - \Phi\left[\sqrt{\tau'}\left(\mu_o - \hat{a} - \bar{x}\right)\right],$$

and
$$\lambda(\underset{\sim}{x}) = \left[1 + \frac{\varepsilon m\left(\underset{\sim}{x} \mid q\right)}{(1-\varepsilon)m\left(\underset{\sim}{x} \mid \pi_o\right)}\right]^{-1} = \left[1 + \frac{\varepsilon}{(1-\varepsilon)}\sqrt{2\pi}\left(\frac{1}{\tau_1} + \frac{1}{nr}\right)^{\frac{1}{2}}\frac{\hat{\varphi}_1 e^{\beta'}}{2\hat{a}}\right]^{-1}. \tag{8}$$

2.1. Bayes Estimator and Bayes Risk

Under the quadratic loss function, $L(\hat{\bar{\theta}}, \bar{\theta}) = (\hat{\bar{\theta}} - \bar{\theta})^2$, the Bayes estimator $\xi(\underset{\sim}{x})$ and Bayes risk $\delta(\underset{\sim}{x})$ for $\bar{\theta}$ are given as

$$\xi(\underset{\sim}{x}) = \int_{\Theta} \bar{\theta}\ \pi\left(\bar{\theta} \mid \underset{\sim}{x}\right)d\bar{\theta} = E_o^{\pi_o(\bar{\theta}|\underset{\sim}{x})}\left(\bar{\theta}\right) - E_q^{q(\bar{\theta}|\underset{\sim}{x})}\left(\bar{\theta}\right)$$

$$= \lambda(\underset{\sim}{x})t_3 + (1-\lambda(\underset{\sim}{x}))\left(\frac{\tau}{(r+\tau)\sqrt{\tau'}}\frac{\phi(v) - \phi(v')}{\hat{\varphi}_1} + \bar{x}\right) \tag{9}$$

$$\delta(\underset{\sim}{x}) = \int_{\Theta} \overline{\theta}^2 \pi(\overline{\theta} \mid \underset{\sim}{x}) d\overline{\theta} - (\xi(\underset{\sim}{x}))^2$$

$$= \lambda(\underset{\sim}{x})\left(\frac{1}{\tau_2} + t_3^2\right) + (1-\lambda(\underset{\sim}{x}))\left\{\frac{\tau^2}{(r+\tau)^2}\left(\frac{1}{\tau'} + \overline{x}^2 + \frac{w\phi(v) - w'\phi(v')}{\widehat{\varphi}_1}\right)\right.$$

$$\left. + \frac{2r\tau\overline{x}}{(r+\tau)^2}\left(\frac{\phi(v)-\phi(v')}{\widehat{\varphi}_1\sqrt{\tau'}} + \overline{x}\right) + \frac{1}{n(r+\tau)} + \left(\frac{r\overline{x}}{r+\tau}\right)^2\right\} - (\xi(\underset{\sim}{x}))^2 \qquad (10)$$

$w = \dfrac{\mu_o - \widehat{a} + \overline{x}}{\sqrt{\tau'}}$, $w' = \dfrac{\mu_o + \widehat{a} + \overline{x}}{\sqrt{\tau'}}$, $v = \sqrt{\tau'}(\mu_o - \widehat{a} - \overline{x})$ and $v' = \sqrt{\tau'}(\mu_o + \widehat{a} - \overline{x})$

2.2. Conditional Density of $x_{n+1} \mid \underset{\sim}{x}$

Let x_{n+1} be an independent potential future observation from $N(\theta_{n+1}, r)$ population. The conditional density function of x_{n+1}, given $\underset{\sim}{x}$ is defined as

$$p(x_{n+1} \mid \underset{\sim}{x}) = \int_{\mu} p(x_{n+1} \mid \mu)\pi(\mu \mid \underset{\sim}{x}) d\mu \qquad (11)$$

Here $\pi(\mu \mid \underset{\sim}{x}) = \lambda(\underset{\sim}{x})\pi_o(\mu \mid \underset{\sim}{x}) + (1-\lambda(\underset{\sim}{x}))\pi_q(\mu \mid \underset{\sim}{x})$

where

$$\pi_o(\mu \mid \underset{\sim}{x}) = \frac{m(\underset{\sim}{x} \mid \mu)h_o(\mu)}{m(\underset{\sim}{x} \mid \pi_o)} = \sqrt{\frac{\tau' + b}{2\pi}} \exp\left[-\frac{\tau' + b}{2}(\mu - t_1')^2\right], \; t_1' = \frac{b\mu_o + \tau'\overline{x}}{b + \tau'}$$

$$\pi_q(\mu \mid \underset{\sim}{x}) = \frac{m(\underset{\sim}{x} \mid \mu)\widehat{s}(\mu)}{m(\underset{\sim}{x} \mid q)} = \sqrt{\frac{\tau'}{2\pi}} \exp\left[-\frac{\tau'}{2}(\mu - \overline{x})^2\right]\frac{1}{\widehat{\varphi}_1}.$$

Therefore equation (11) becomes

$$p(x_{n+1} \mid \underset{\sim}{x}) = \lambda(\underset{\sim}{x})p_o(x_{n+1} \mid \underset{\sim}{x}) + (1-\lambda(\underset{\sim}{x}))q(x_{n+1} \mid \underset{\sim}{x})$$

where

$$p(x_{n+1} \mid \mu) = \int_{\Theta} f(x_{n+1} \mid \theta_{n+1})g(\theta_{n+1} \mid \mu)d\theta_{n+1} = \sqrt{\frac{\tau_p}{2\pi}} \exp\left[-\frac{\tau_p}{2}(x_{n+1} - \mu)^2\right]; \; \tau_p = \frac{r\tau}{r+\tau}$$

$$p_o(x_{n+1} \mid \underset{\sim}{x}) = \int_{-\infty}^{\infty} p(x_{n+1} \mid \mu)\pi_o(\mu \mid \underset{\sim}{x})d\mu = \sqrt{\frac{\tau_{p1}}{2\pi}} \exp\left[-\frac{\tau_{p1}}{2}(x_{n+1} - t_1')^2\right]; \; \tau_{p1} = \frac{\tau_p(\tau' + b)}{\tau_p + \tau' + b}$$

$$q(x_{n+1} \mid \underset{\sim}{x}) = \int_{\mu_o - \widehat{a}}^{\mu_o + \widehat{a}} p(x_{n+1} \mid \mu)\pi_q(\mu \mid \underset{\sim}{x})d\mu = \sqrt{\frac{\tau_{p2}}{2\pi}} \exp\left[-\frac{\tau_{p2}}{2}(x_{n+1} - \overline{x})^2\right] \frac{\varphi_3}{\widehat{\varphi}_1}$$

$$\varphi_3 = \int_{\sqrt{\tau_p + \tau'}(\mu_o - \widehat{a} - t_3')}^{\sqrt{\tau_p + \tau'}(\mu_o + \widehat{a} - t_3')} \phi(u)du, \; \tau_{p2} = \frac{\tau_p\tau'}{\tau_p + \tau'}, \; t_3' = \frac{\tau_p x_{n+1} + \tau'\overline{x}}{\tau_p + \tau'}$$

In order to study the changes in the conditional density $x_{n+1} \mid \underset{\sim}{x}$ due to varying ε and parameter values in the second stage, we compute the following tail probabilities

$$P\left(x_{n+1} > l \mid \underset{\sim}{x}\right) = \int_{l}^{\infty} p\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1}$$

$$= \lambda(\underset{\sim}{x}) \int_{l}^{\infty} p_o\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1} + (1-\lambda(\underset{\sim}{x})) \int_{l}^{\infty} q\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1}$$

where l varies from $(-\infty,\ \infty)$.

3. Robustness under second stage prior misspecification for Lognormal Distribution

Here again $\underset{\sim}{x}$ consists of independent components $\{x_1, x_2, \ldots, x_n\}$, each x_i has density $f(x_i \mid \theta_i)$ independently from $Lognormal(\theta_i, r) / LN(\theta_i, r)$; with common known precision r. Assume $\theta_i\,'s$ are exchangeable and their prior distribution are similarly staged as follows

Stage I: $\theta_i \ (i = 1, 2, \ldots, n)$ are independent $N(\mu, \tau)$; known precision with pdf

$$g(\theta_i \mid \mu) = \sqrt{\frac{\tau}{2\pi}} \exp\left[-\frac{\tau}{2}(\theta_i - \mu)^2 \right]$$

Here we use the fact that sample mean is the sufficient statistics for the unknown mean of the related normal population. Hence we let $\bar{\theta} = \sum_{i=1}^{n} \theta_i / n$ which gives $g\left(\bar{\theta} \mid \mu\right) \sim N(\mu, n\tau)$.

Stage II: The hyper parameter μ belongs to the ML-II ε-contaminated class of priors. Following Berger and Berliner (1986), we have $h_o(\mu)$ as $N(\mu_o, b)$, known b, with pdf

$$h_o(\mu) = \sqrt{\frac{b}{2\pi}} \exp\left[-\frac{b}{2}(\mu - \mu_o)^2 \right]$$

and $\hat{s}(\mu)$ as $uniform(\mu_o - \hat{a}, \mu_o + \hat{a})$, $\hat{a}$ being the value of $'a'$ which maximizes

$$m\left(\underset{\sim}{x} \mid a\right) = \begin{cases} \dfrac{1}{2a} \displaystyle\int_{\mu_o - a}^{\mu_o + a} m\left(\underset{\sim}{x} \mid \mu\right) d\mu = \int_{-\infty}^{\infty} \int_{\mu_o - a}^{\mu_o + a} L\left(\bar{\theta} \mid \underset{\sim}{x}\right) g\left(\bar{\theta} \mid \mu\right) d\mu\, d\bar{\theta} & a > 0 \\ m\left(\underset{\sim}{x} \mid \mu_o\right) & a = 0 \end{cases}$$

$m\left(\underset{\sim}{x} \mid \hat{a}\right)$ ia an upper bound on $m\left(\underset{\sim}{x} \mid q\right)$.

$$m\left(\underset{\sim}{x} \mid a\right) = \left(\frac{r}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{r}{2} \sum_{i=1}^{n} \left[\log_e(x_i) - \bar{x}\right]^2 \right) \sqrt{\frac{2\pi}{nr}} \frac{1}{2a} \int_{\mu_o - a}^{\mu_o + a} \sqrt{\frac{\tau'}{2\pi}} \exp\left[-\frac{\tau'}{2}(\mu - \bar{x})^2 \right] d\mu$$

$$= \frac{C}{2a} \left\{ \Phi\left[\sqrt{\tau'}\left(\mu_o + a - \bar{x}\right) \right] - \Phi\left[\sqrt{\tau'}\left(\mu_o - a - \bar{x}\right) \right] \right\} \tag{12}$$

where $C = \left(\dfrac{r}{2\pi}\right)^{\frac{n}{2}} \sqrt{\dfrac{2\pi}{nr}} \exp\left(-\dfrac{r}{2} \displaystyle\sum_{i=1}^{n} \left[\log_e(x_i) - \bar{x}\right]^2 \right)$, $\bar{x} = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \log_e(x_i)$, $\tau' = \dfrac{n\tau r}{n\tau + r}$ and $\Phi(\cdot)$ denotes standard normal cdf.

On differentiating equation (12) with respect to $'a'$, we have

$$\frac{d}{da}m(\underset{\sim}{x}|a)=-\frac{C}{2a^2}\left\{\Phi\left[\sqrt{\tau'}(\mu_o+a-\bar{x})\right]-\Phi\left[\sqrt{\tau'}(\mu_o-a-\bar{x})\right]\right\}+\frac{C\sqrt{\tau'}}{2a}\left\{\phi\left[\sqrt{\tau'}(\mu_o+a-\bar{x})\right]+\phi\left[\sqrt{\tau'}(\mu_o-a-\bar{x})\right]\right\} \tag{13}$$

Now we substitute $z=\sqrt{\tau'}\left|\bar{x}-\mu_o\right|$ and $a^*=a\sqrt{\tau'}$ in (13) and equate to zero. The equation becomes

$$a^*=z+\left\{-2\log_e\left[\sqrt{2\pi}\left(\frac{1}{a^*}\left\{\Phi\left(a^*-z\right)-\Phi\left[-\left(a^*+z\right)\right]\right\}-\phi\left[-\left(a^*+z\right)\right]\right)\right]\right\}^{\frac{1}{2}} \tag{14}$$

We solve a^* by standard fixed-point iteration, set $a^*=z$ on the right-hand side, which gives

$$\hat{a}=\begin{cases}0 & \text{if } z\le 1.65\\ \dfrac{a^*}{\sqrt{\tau'}} & \text{if } z>1.65\end{cases}$$

The posterior distribution of parameter $\bar{\theta}$ with respect to prior $\pi\left(\bar{\theta}\right)$ is given by

$$\pi\left(\bar{\theta}\mid\underset{\sim}{x}\right)=\lambda(\underset{\sim}{x})\pi_o\left(\bar{\theta}\mid\underset{\sim}{x}\right)+(1-\lambda(\underset{\sim}{x}))q\left(\bar{\theta}\mid\underset{\sim}{x}\right) \tag{15}$$

Here

$$\pi_o\left(\bar{\theta}\mid\underset{\sim}{x}\right)=\sqrt{\frac{\tau_2}{2\pi}}\exp\left[-\frac{\tau_2}{2}\left(\bar{\theta}-t_3\right)^2\right] \tag{16}$$

where

$$\pi_o\left(\bar{\theta}\right)=\sqrt{\frac{\tau_1}{2\pi}}\exp\left[-\frac{\tau_1}{2}\left(\bar{\theta}-\mu_o\right)^2\right],\ m\left(\underset{\sim}{x}\mid\pi_o\right)=\left(\frac{r}{2\pi}\right)^{\frac{n}{2}}\sqrt{\frac{\tau b}{(r+\tau)(\tau'+b)}}e^{-\beta}$$

$$\tau_2=nr+\tau_1,\ t_3=\frac{nr\bar{x}+\tau_1\mu_o}{nr+\tau_1},\quad \tau_1=\frac{n\tau b}{n\tau+b},\ \beta=\beta'+\frac{r}{2}\sum_{i=1}^{n}\left[\log_e(x_i)-\bar{x}\right]^2 \text{ and } \beta'=\frac{\tau' b}{2(\tau'+b)}(\mu_o-\bar{x})^2,$$

and

$$q\left(\bar{\theta}\mid\underset{\sim}{x}\right)=\sqrt{\frac{nr}{2\pi}}\exp\left[-\frac{nr}{2}\left(\bar{\theta}-\bar{x}\right)^2\right]\frac{\varphi}{\hat{\varphi}_1} \tag{17}$$

where

$$\hat{q}\left(\bar{\theta}\right)=\frac{1}{2\hat{a}}\int_{\sqrt{n\tau}\left(\mu_o-\hat{a}-\bar{\theta}\right)}^{\sqrt{n\tau}\left(\mu_o+\hat{a}-\bar{\theta}\right)}\phi(u)du=\frac{\varphi}{2\hat{a}},\ m(\underset{\sim}{x}|q)=C\frac{\hat{\varphi}_1}{2\hat{a}};\ \hat{\varphi}_1=\Phi\left[\sqrt{\tau'}\left(\mu_o+\hat{a}-\bar{x}\right)\right]-\Phi\left[\sqrt{\tau'}\left(\mu_o-\hat{a}-\bar{x}\right)\right],$$

and $\lambda(\underset{\sim}{x})=\left[1+\dfrac{\varepsilon}{(1-\varepsilon)}\sqrt{2\pi}\left(\dfrac{1}{\tau_1}+\dfrac{1}{nr}\right)^{\frac{1}{2}}\dfrac{\hat{\varphi}_1 e^{\beta'}}{2\hat{a}}\right]^{-1}$.

3.1. Bayes Estimator and Bayes Risk

Under the quadratic loss function, $L(\hat{\bar{\theta}},\bar{\theta})=(\hat{\bar{\theta}}-\bar{\theta})^2$, the Bayes estimator $\xi(\underset{\sim}{x})$ and Bayes risk $\delta(\underset{\sim}{x})$ for $\bar{\theta}$ are given as

$$\xi(\underset{\sim}{x})=E_o^{\pi_o(\bar{\theta}|\underset{\sim}{x})}\left(\bar{\theta}\right)-E_q^{q(\bar{\theta}|\underset{\sim}{x})}\left(\bar{\theta}\right)==\lambda(\underset{\sim}{x})t_3+(1-\lambda(\underset{\sim}{x}))\left(\frac{\tau}{(r+\tau)\sqrt{\tau'}}\frac{\phi(v)-\phi(v')}{\hat{\varphi}_1}+\bar{x}\right) \tag{18}$$

$$\delta(\underset{\sim}{x}) = \lambda(\underset{\sim}{x})\left(\frac{1}{\tau_2} + t_3^2\right) + (1-\lambda(\underset{\sim}{x}))\left\{\frac{\tau^2}{(r+\tau)^2}\left(\frac{1}{\tau'} + \bar{x}^2 + \frac{w\phi(v) - w'\phi(v')}{\hat{\varphi}_1}\right)\right.$$
$$\left. + \frac{2r\tau\bar{x}}{(r+\tau)^2}\left(\frac{\phi(v)-\phi(v')}{\hat{\varphi}_1\sqrt{\tau'}} + \bar{x}\right) + \frac{1}{n(r+\tau)} + \left(\frac{r\bar{x}}{r+\tau}\right)^2\right\} - \left(\xi(\underset{\sim}{x})\right)^2 \qquad (19)$$

$w = \dfrac{\mu_o - \hat{a} + \bar{x}}{\sqrt{\tau'}}$, $w' = \dfrac{\mu_o + \hat{a} + \bar{x}}{\sqrt{\tau'}}$, $v = \sqrt{\tau'}(\mu_o - \hat{a} - \bar{x})$ and $v' = \sqrt{\tau'}(\mu_o + \hat{a} - \bar{x})$

3.2. Conditional Density of $x_{n+1} \mid \underset{\sim}{x}$

Here x_{n+1} is an independent potential future observation from $LN(\theta_{n+1}, r)$ population. The conditional density function of x_{n+1}, given $\underset{\sim}{x}$ is defined as

$$p(x_{n+1} \mid \underset{\sim}{x}) = \lambda(\underset{\sim}{x}) p_o(x_{n+1} \mid \underset{\sim}{x}) + (1-\lambda(\underset{\sim}{x})) q(x_{n+1} \mid \underset{\sim}{x}) \qquad (20)$$

where

$$\pi_o(\mu \mid \underset{\sim}{x}) = \sqrt{\frac{\tau'+b}{2\pi}} \exp\left[-\frac{\tau'+b}{2}(\mu - t_1')^2\right], \quad t_1' = \frac{b\mu_o + \tau'\bar{x}}{b+\tau'}$$

$$\pi_q(\mu \mid \underset{\sim}{x}) = \sqrt{\frac{\tau'}{2\pi}} \exp\left[-\frac{\tau'}{2}(\mu - \bar{x})^2\right] \frac{1}{\hat{\varphi}_1}$$

$$p(x_{n+1} \mid \mu) = \sqrt{\frac{\tau_p}{2\pi}} \exp\left[-\frac{\tau_p}{2}(x_{n+1} - \mu)^2\right]; \quad \tau_p = \frac{r\tau}{r+\tau}$$

$$p_o(x_{n+1} \mid \underset{\sim}{x}) = \sqrt{\frac{\tau_{p1}}{2\pi}} \exp\left[-\frac{\tau_{p1}}{2}(x_{n+1} - t_1')^2\right]; \quad \tau_{p1} = \frac{\tau_p(\tau'+b)}{\tau_p + \tau' + b}$$

$$q(x_{n+1} \mid \underset{\sim}{x}) = \sqrt{\frac{\tau_{p2}}{2\pi}} \exp\left[-\frac{\tau_{p2}}{2}(x_{n+1} - \bar{x})^2\right] \frac{\varphi_3}{\hat{\varphi}_1};$$

$$\varphi_3 = \int_{\sqrt{\tau_p+\tau'}(\mu_o - \hat{a} - t_3')}^{\sqrt{\tau_p+\tau'}(\mu_o + \hat{a} - t_3')} \phi(u)\,du, \quad \tau_{p2} = \frac{\tau_p \tau'}{\tau_p + \tau'}, \quad t_3' = \frac{\tau_p x_{n+1} + \tau'\bar{x}}{\tau_p + \tau'}$$

Similarly as in the case of normal distribution in order to study the changes in the conditional density $x_{n+1} \mid \underset{\sim}{x}$ for lognormal case due to varying ε and parameter values in the second stage, we compute tail probabilities

$$P(x_{n+1} > l \mid \underset{\sim}{x}) = \int_l^\infty p(x_{n+1} \mid \underset{\sim}{x})\,dx_{n+1},$$

where l varies from $(0, \infty)$.

4. Illustration

In order to study sensitivity of the Bayes estimator and risk to misspecification in the second stage prior distribution, we consider two simulated data sets for normal (data-sets 1, 2) and lognormal distributions (data-sets 3, 4). The data is obtained by generating 20 independent population components x_i^j $(i = 1, 2, \ldots, n,\ \ j = 1, 2, \ldots, m)$ (*'n'* being the number of population and *'m'* being number of observations in the population). Independence of the data is preserved by considering unique mean and fixed precision for each population. The final population used for analysis is the mean of each of the independent component i.e. $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ which we for convenience denote by $\{x_1, x_2, \ldots, x_n\}$. Simulation is carried out using the Box-Muller technique.

Data-set for Normal population

Data-Set 1 (n=20)
99.95, 103.41, 106.34, 108.63, 109.12, 110.41, 111.36, 112.73, 113.57, 116.68, 117.02, 117.45, 118.3, 119.93, 120.56, 122.45, 124.47, 124.61, 126.16, 130.01
Data-Set 2 (n=30)
7.91, 8.59, 8.88, 9.38, 10.44, 11.64, 12.05, 12.13, 12.19, 12.23, 12.36, 12.59, 12.64, 12.93, 12.98, 13.39, 13.54, 14.24, 14.45, 15.28, 15.46, 16.30, 16.96, 16.99, 17.11, 17.52, 18.25, 18.48, 20.11, 21.58

Data-set for Lognormal population

Data-Set 3 (n=20)
0.41, 0.42, 0.8, 1.13, 1.27, 1.78, 1.8, 2.63, 4.32, 5.68, 6.57, 6.88, 8.76, 9.01, 12.21, 20.76, 25.11, 30.17, 41.26, 48.02
Data-Set 4 (n=30)
4, 5, 6, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 51, 62, 70, 71, 82, 91, 95, 120, 120, 220, 245, 258

The Kolmogorov-Smirnov test statistic for the above four data-sets and the graphs of empirical and the theoretical curves are given in Appendix 1. The results show that normal distribution is a good fit for data-sets 1, 2 and lognormal distribution is a fair fit for data-sets 3, 4.

In case of Normal distribution the sample precision is estimated by $\hat{r} = \left\{ \frac{1}{m(m-1)n} \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i^j - \bar{x}_i)^2 \right\}^{-1}$ and first stage prior precision τ is estimated by $\hat{\tau} = \left[\max\left\{ 0, \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 - \frac{1}{\hat{r}} \right\} \right]^{-1}$. In case of lognormal distribution both $\hat{r}$ and $\hat{\tau}$ are estimated using the above formulas by replacing x_i by $\log_e(x_i)$ and $\bar{x}$ by $\sum_{i=1}^{n} \log_e(x_i)/n$. Further for the hyper parameter values $(\mu_o,\ b)$ at the second stage prior we take various guess values as per subjective beliefs.

Bayesian Results for Normal Distribution

Data Set -1

Table-1
Comparative values of Bayes estimate and risk (underlined) for varying (μ_o, b), ε

μ_o	b \ ε	0	0.05	0.2	0.5	0.9
50	0.0033	115.59941190 0.27029718	115.65151232 0.26933586	115.65355463 0.26918760	115.65398245 0.26915548	115.65411052 0.26914580
100	0.0044	115.63943569 0.27021783	115.63976565 0.26992639	115.64064071 0.26915241	115.64200177 0.26794554	115.64329251 0.26679760
150	0.0056	115.70871838 0.27013904	115.69278624 0.26988916	115.67678715 0.26912736	115.66876237 0.26855247	115.66554773 0.26828605

Comparative values of $P\left(x_{n+1} > l \mid \underset{\sim}{x}\right) = \int_{l}^{\infty} p\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1}$ for varying ε, l

Table 2
$\mu_o = 50,\ b = 0.0033$

l \ ε	0	0.05	0.2	0.5	0.9
60	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
90	0.9988703283	0.9991274198	0.9991374977	0.9991396088	0.9991402407
110	0.7282952186	0.7528366942	0.7537987052	0.7540002294	0.7540605533
115	0.4984529340	0.5284639785	0.5296403933	0.5298868313	0.5299605996
120	0.2691383513	0.2943313489	0.2953188992	0.2955257734	0.2955876988
130	0.0329938473	0.0387239827	0.0389486006	0.0389956540	0.0390097389
150	9.1501e-006	1.2301e-005	1.2425e-005	1.2451e-005	1.2459e-005
170	8.4336e-012	1.2261e-011	1.2411e-011	1.2442e-011	1.2451e-011

Table 3
$\mu_o = 100,\ b = 0.0044$

l \ ε	0	0.05	0.2	0.5	0.9
60	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
90	0.9990699992	0.9990735902	0.9990831135	0.9990979259	0.9991119730
110	0.7470858485	0.7473076688	0.7478959406	0.7488109317	0.7496786474
115	0.5214344267	0.5216439699	0.5221996825	0.5230640314	0.5238837216
120	0.2884770295	0.2885903668	0.2888909394	0.2893584466	0.2898017989
130	0.0374540840	0.0374427328	0.0374126292	0.0373658063	0.0373214027
150	1.1820e-005	1.1747e-005	1.1554e-005	1.1253e-005	1.0968e-005
170	1.2465e-011	1.2223e-011	1.1581e-011	1.0583e-011	9.6358e-012

Table 4

$\mu_o = 150,\ b = 0.0056$

l \ ε	0	0.05	0.2	0.5	0.9
60	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
90	0.9993403476	0.9992913573	0.9992421611	0.9992174854	0.9992076006
110	0.7779158934	0.7710601779	0.7641756510	0.7607225265	0.7593392420
115	0.5610098975	0.5519089032	0.5427696616	0.5381856227	0.5363493061
120	0.3234355534	0.3151626484	0.3068549763	0.3026880340	0.3010188019
130	0.0463426260	0.0441410238	0.0419301693	0.0408212542	0.0403770348
150	1.8299e-005	1.6594e-005	1.4883e-005	1.4024e-005	2.4401e-011
170	2.4401e-011	2.1137e-011	1.7860e-011	1.6216e-011	1.5557e-011

Data Set -2

Table 5

Comparative values of Bayes estimate and risk (underlined) for varying (μ_o, b), ε

ε		0	0.05	0.2	0.5	0.9
μ_o	b					
10	0.2	13.93985572 <u>0.01830518</u>	13.94093645 <u>0.01830096</u>	13.94348146 <u>0.01828178</u>	13.94670938 <u>0.01823882</u>	13.94916506 <u>0.01819217</u>
15	0.25	13.95771484 <u>0.01829091</u>	13.95831859 <u>0.01827108</u>	13.95995894 <u>0.01821352</u>	13.96262950 <u>0.01810829</u>	13.96530625 <u>0.01798850</u>
18	0.1	13.96048872 <u>0.01833532</u>	13.96020678 <u>0.01832414</u>	13.95946031 <u>0.01829377</u>	13.95830279 <u>0.01824446</u>	13.95720905 <u>0.01819542</u>

Comparative values of $P\left(x_{n+1} > l \mid \underset{\sim}{x}\right) = \int_l^\infty p\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1}$ for varying $\varepsilon,\ l$

Table 6

$\mu_o = 10,\ b = 0.2$

l \ ε	0	0.05	0.2	0.5	0.9
-9	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0	0.9999604477	0.9999614985	0.9999639731	0.9999671118	0.9999694995
14	0.4619052071	0.4645349463	0.4707276906	0.4785821712	0.4845575598
18	0.1055217489	0.1067078085	0.1095008474	0.1130433592	0.1157383668
20	0.0337571956	0.0342365995	0.0353655425	0.0367974216	0.0378867405
28	1.7489e-005	1.7916e-005	1.8920e-005	2.0194e-005	2.1163e-005
34	2.1663e-009	2.2263e-009	2.3676e-009	2.5468e-009	2.6832e-009
44	9.8818e-019	1.0079e-018	1.0543e-018	1.1132e-018	1.1580e-018

Table 7

$\mu_o = 15,\ b = 0.25$

l \ ε	0	0.05	0.2	0.5	0.9
-9	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0	0.9999751509	0.9999756850	0.9999771362	0.9999794989	0.9999818670
14	0.5052807748	0.5067514882	0.5107473147	0.5172526856	0.5237731483
18	0.1266989011	0.1273771711	0.1292199840	0.1322201594	0.1352272951
20	0.0427324421	0.0430104471	0.0437657674	0.0449954601	0.0462280055
28	2.7821e-005	2.8095e-005	2.8837e-005	3.0048e-005	3.1259e-005
34	4.1163e-009	4.1616e-009	4.2848e-009	4.4853e-009	4.6863e-009
44	2.5290e-018	2.5583e-018	2.6380e-018	2.7678e-018	2.8979e-018

Table 8

$\mu_o = 18,\ b = 0.1$

l \ ε	0	0.05	0.2	0.5	0.9
-9	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0	0.9999765535	0.9999765056	0.9999763788	0.9999761823	0.9999759966
14	0.5120169019	0.5113244729	0.5094911604	0.5066483435	0.5039621400
18	0.1304442927	0.1300326292	0.1289426868	0.1272525729	0.1256555689
20	0.0444242591	0.0442346630	0.0437326781	0.0429542778	0.0422187602
28	3.0315e-005	3.0031e-005	2.9278e-005	2.8111e-005	2.7008e-005
34	4.6790e-009	4.6165e-009	4.4511e-009	4.1945e-009	3.9521e-009
44	3.1211e-018	3.0603e-018	2.8994e-018	2.6499e-018	2.4141e-018

Tables 1(results using data-set 1) and 5(results using data-set 2) suggests that the increase in the contamination in the second stage prior does not affect the Bayes estimate and risk for normal population. Further we observe insignificant variation in the Bayes estimate and risk with varying $(\mu_o,\ b)$.

Tables 2, 3, 4 (results using data-set 1) and 6, 7, 8 (results using data-set 2) suggest that the probability $P\left(x_{n+1} > s \mid \underset{\sim}{x}\right)$, is not sensitive to both increasing contamination and varying $(\mu_o,\ b)$ in the second stage. The graphs (6 to 9) for data-set 1 and (10 to 14) for data-set 2 in Appendix 1 validate the above findings.

Bayesian Results for Lognormal Distribution

Data Set -3

Table 9

Comparative values of Bayes estimate and risk (underlined) for varying (μ_o, b), ε

μ_o	b \ ε	0	0.05	0.2	0.5	0.9
4	0.5	1.58575802 <u>0.00296213</u>	1.58549577 <u>0.00296143</u>	1.58487477 <u>0.00295923</u>	1.58408014 <u>0.00295528</u>	1.58347031 <u>0.00295139</u>
10	0.2	1.58724498 <u>0.00296458</u>	1.58294270 <u>0.00296119</u>	1.58273960 <u>0.00296011</u>	1.58269665 <u>0.00295988</u>	1.58268377 <u>0.00295980</u>
14	0.1	1.58600145 <u>0.00296543</u>	1.58268516 <u>0.00296195</u>	1.58260467 <u>0.00296160</u>	1.58258810 <u>0.00296152</u>	1.58258316 <u>0.00296150</u>

Comparative values of $P\left(x_{n+1} > l \mid \underset{\sim}{x}\right) = \int_l^\infty p\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1}$ for varying ε, l

Table 10

$\mu_o = 4,\ b = 0.5$

l \ ε	0	0.05	0.2	0.5	0.9
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
10	0.3468928987	0.3444917218	0.3388058262	0.3315301414	0.3259464893
20	0.1970373936	0.1952205018	0.1909181711	0.1854128990	0.1811879326
100	0.0276239940	0.0272066895	0.0262185281	0.0249540746	0.0239836823
160	0.0129457256	0.0127270572	0.0122092589	0.0115466829	0.0110381956
300	4.1006e-002	4.0215e-002	3.8343e-002	3.5947e-002	3.41078e-002
500	1.4338e-003	1.4033e-003	1.3313e-003	1.2391e-003	1.1684e-003
1000	2.9073e-004	2.8380e-004	2.6741e-004	2.4643e-004	2.3032e-004

Table 11

$\mu_o = 10,\ b = 0.2$

l \ ε	0	0.05	0.2	0.5	0.9
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
10	0.3604373971	0.3218533423	0.3200318331	0.3196466966	0.3195311681
20	0.2073970276	0.1785246464	0.1771616142	0.1768734173	0.1767869675
100	0.0300959643	0.0235592696	0.0232506798	0.0231658599	0.0231658599
160	0.0142593934	0.0108394745	0.0106780240	0.0106438872	0.0106336473
300	4.5856e-002	3.4931e-002	3.2910e-002	3.2786e-002	3.2749e-002
500	1.6238e-003	1.1477e-003	1.1252e-003	1.1204e-003	1.1190e-003
1000	3.3511e-004	2.2643e-004	2.2130e-004	2.2022e-004	2.1989e-004

Table 12

$\mu_o = 14,\ b = 0.1$

ε / l	0	0.05	0.2	0.5	0.9
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
10	0.3492218049	0.3196634939	0.3189460819	0.3187984002	0.3187543655
20	0.1989149414	0.1769498737	0.1764167579	0.1763070141	0.1762742916
100	0.0281223696	0.0232290851	0.0231103198	0.0230858716	0.0230785818
160	0.0132179933	0.0106699077	0.0106080630	0.0105953320	0.0105915360
300	4.2049e-002	3.2894e-002	3.2672e-002	3.2626e-002	3.2612e-002
500	1.4756e-003	1.1249e-003	1.1164e-003	1.1147e-003	1.1142e-003
1000	3.0085e-004	2.2135e-004	2.1942e-004	2.1902e-004	2.1891e-004

Data Set -4

Table 13

Comparative values of Bayes estimate and risk (underlined) for varying $(\mu_o,\ b),\ \varepsilon$

μ_o	b	ε = 0	0.05	0.2	0.5	0.9
7	0.5	3.42702448 <u>0.00206539</u>	3.42595923 <u>0.00206476</u>	3.42481437 <u>0.00206154</u>	3.42420924 <u>0.00205879</u>	3.42396068 <u>0.00205744</u>
12	0.2	3.42692479 <u>0.00206663</u>	3.42377181 <u>0.00206283</u>	3.42366746 <u>0.00206236</u>	3.42364576 <u>0.00206226</u>	3.42363927 <u>0.00206223</u>
15	0.1	3.42579338 <u>0.00206705</u>	3.42373395 <u>0.00206393</u>	3.42361555 <u>0.00206349</u>	3.42359022 <u>0.00206339</u>	3.42358260 <u>0.00206336</u>

Comparative values of $P\left(x_{n+1} > l \mid \underset{\sim}{x}\right) = \int_l^\infty p\left(x_{n+1} \mid \underset{\sim}{x}\right) dx_{n+1}$ for varying $\varepsilon,\ l$

Table 14

$\mu_o = 7,\ b = 0.5$

ε / l	0	0.05	0.2	0.5	0.9
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
30	0.5343273854	0.5263705008	0.5178189072	0.5132988620	0.5114422766
60	0.3112797275	0.3042325218	0.2966585978	0.2926553115	0.2910109821
180	0.0794272485	0.0765107190	0.0733762038	0.0717194191	0.0710389032
300	0.0332395273	0.0317883637	0.0302287381	0.0294043797	0.0290657787
500	1.1866e-002	1.1265e-002	1.0618e-002	1.0277e-002	1.0136e-002
1000	2.2571e-003	2.1210e-003	1.9748e-003	1.8975e-003	1.8658e-003
1500	7.4113e-004	6.9233e-004	6.3989e-004	6.1217e-004	6.0078e-004

Table 15

$\mu_o = 12,\ b = 0.2$

l \ ε	0	0.05	0.2	0.5	0.9
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
30	0.5335785830	0.5100479689	0.5092692286	0.5091072540	0.5090588461
60	0.3106586405	0.2899932837	0.2893093683	0.2891671169	0.2891246036
180	0.0791970074	0.0707370006	0.0704570186	0.0703987835	0.0703813794
300	0.0331310847	0.0289390043	0.0288002683	0.0287714119	0.0287627878
500	1.1824e-002	1.0092e-002	1.0035e-00	1.0023e-002	1.0020e-002
1000	2.2483e-003	1.8583e-003	1.8454e-003	1.8427e-003	1.8419e-003
1500	7.3814e-004	5.9853e-004	5.9391e-004	5.9295e-004	5.9266e-004

Table 16

$\mu_o = 15,\ b = 0.1$

l \ ε	0	0.05	0.2	0.5	0.9
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
30	0.5251554728	0.5097720077	0.5088875540	0.5086983171	0.5086413985
60	0.3032370233	0.2897919192	0.2890189092	0.2888535167	0.2888037700
180	0.0761356611	0.0706759371	0.0703620369	0.0702948751	0.0702746742
300	0.0316075398	0.0289127407	0.0287578065	0.0287246569	0.0287146862
500	1.1192e-002	1.0083e-002	1.0019e-002	1.0006e-002	1.0002e-002
1000	2.1048e-003	1.8565e-003	1.8423e-003	1.8392e-003	1.8383e-003
1500	6.8657e-004	5.9797e-004	5.9288e-004	5.9179e-004	5.9146e-004

Tables 9(results using data-set 3) and 13(results using data-set 4) suggest that the increase in the contamination in the second stage prior does not affect the Bayes estimate and risk for lognormal population. Further we observe insignificant variation in the Bayes estimate and risk with varying $(\mu_o,\ b)$.

Tables 10-12 (data-set 3) and 14-16 (data-set 4) suggest that the probability $P\left(x_{n+1} > s \mid \underset{\sim}{x}\right)$ is not sensitive to both contamination and varying $(\mu_o,\ b)$ in the second stage. The graphs (15 to 19) for data-set 3 and (20 to 24) for data-set 4 in Appendix 1 confirm the above findings.

5. Conclusion

Above illustrations suggest that the Bayes estimate and risk are little affected by the misspecification in the second stage prior for both normal and lognormal distribution. Further the probability $P\left(x_{n+1} > s \mid \underset{\sim}{x}\right)$ is also little affected by the presence of contamination in the second stage. Thus the predictive decision problems based on percentiles may allow moderate contamination of second stage without significantly changing the decisions. These conclusions agree with Berger (1985) where he asserts that form of the second stage prior does not affect the Bayes decisions.

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Appendix 1

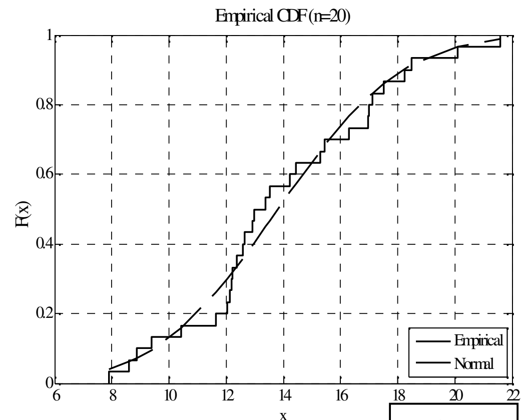


Fig. 1

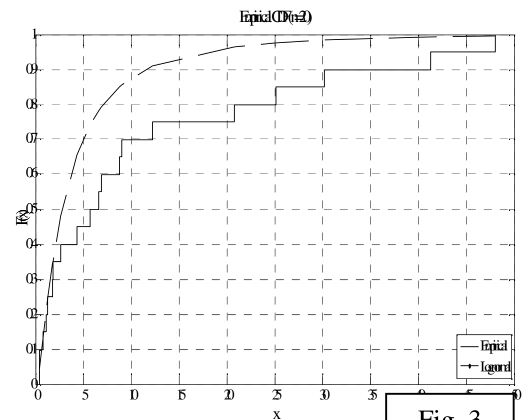


Fig. 3

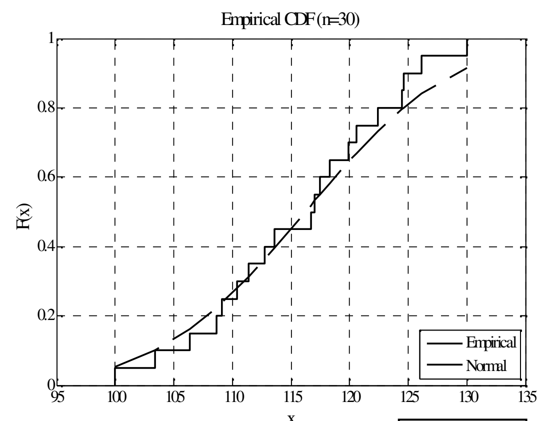


Fig. 2

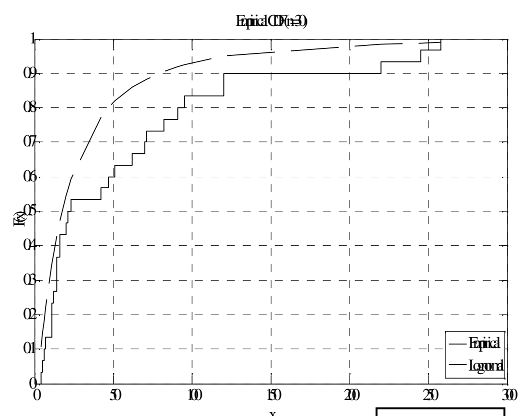


Fig. 4

n	Kolmogorov –Smirnov Test and p sig. values		Decision at 5%
	k-s	p	0.05
20	0.1099	0.9475	Data fits Normal
30	0.1010	0.8895	Data fits Normal

n	Kolmogorov –Smirnov Test and p sig. values		Decision at 5%
	k-s	p	0.05
20	0.2914	0.0534	Data fits lognormal
30	0.2388	0.0546	Data fits lognormal

n=20 Normal Distribution n=30

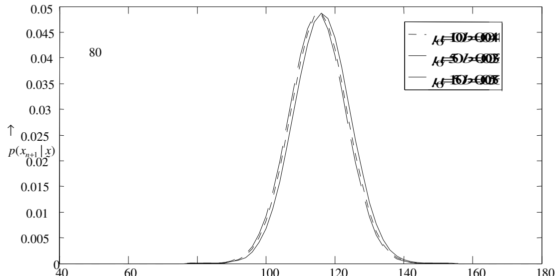


Fig. 5

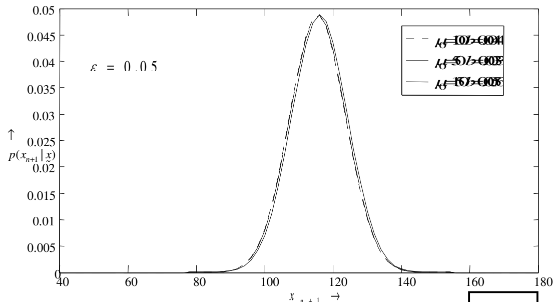


Fig. 6

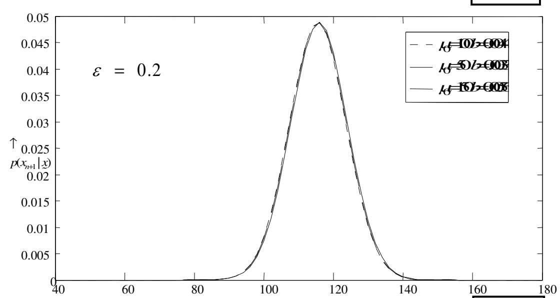


Fig. 7

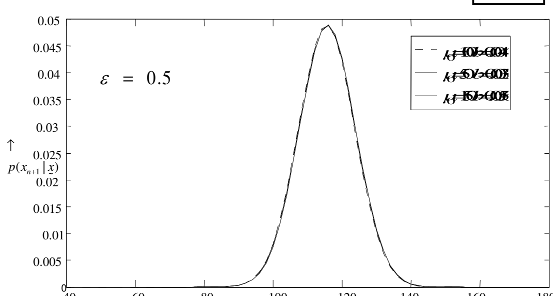


Fig. 8

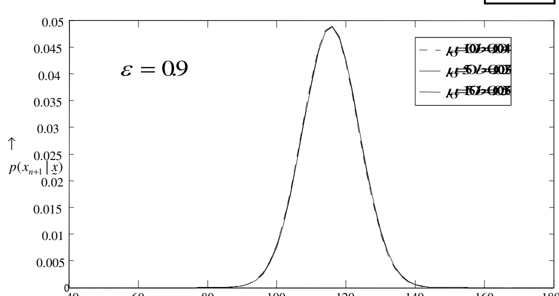


Fig. 9

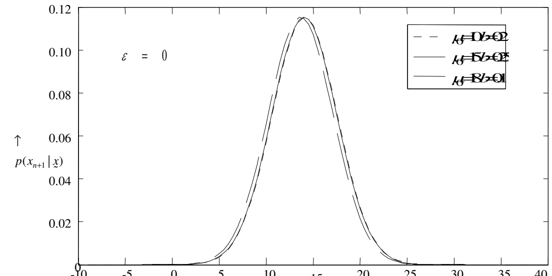


Fig. 10

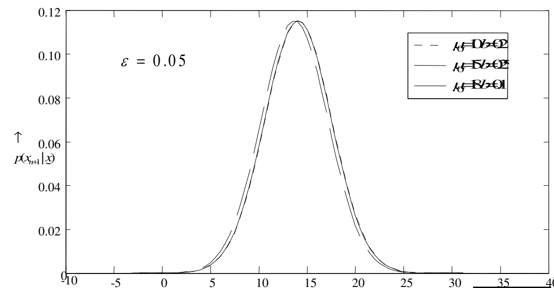


Fig. 11

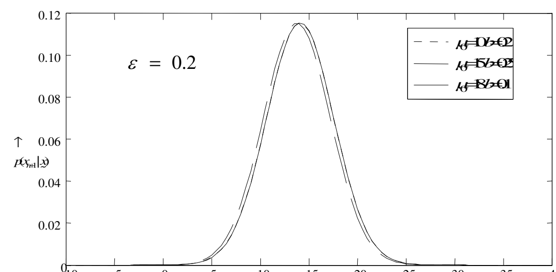


Fig. 12

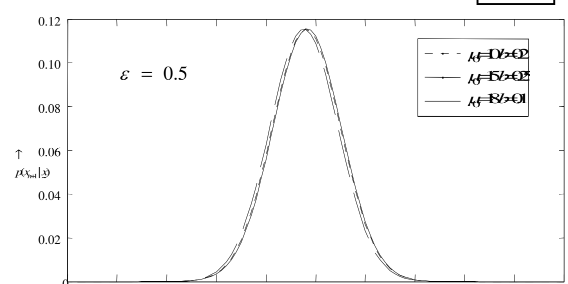


Fig. 13

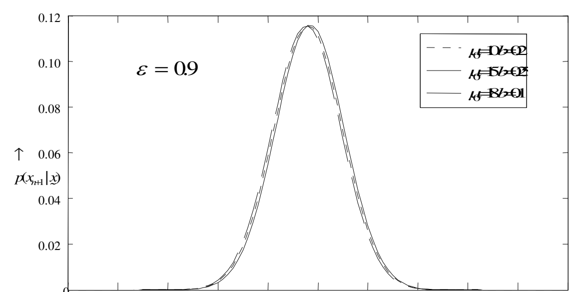


Fig. 14

n=20 Lognormal n=30

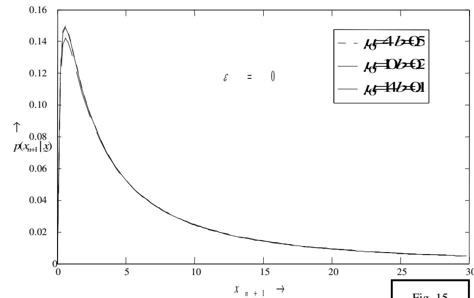


Fig. 15

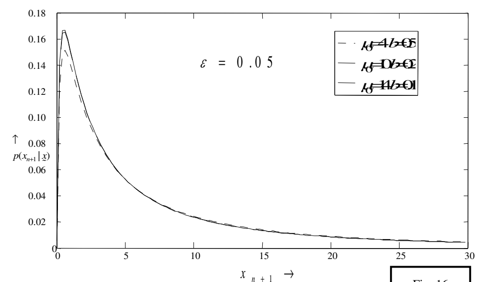


Fig. 16

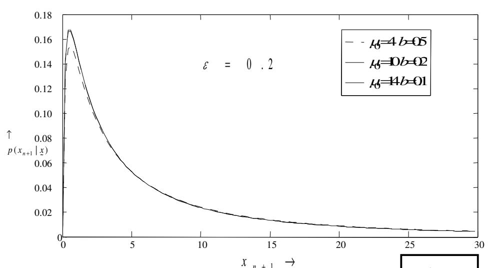


Fig. 17

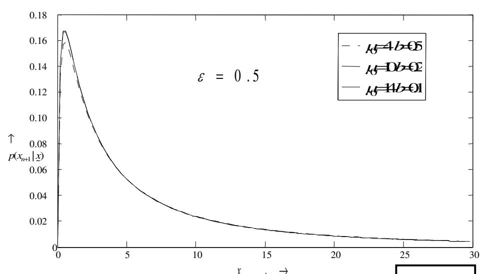


Fig. 18

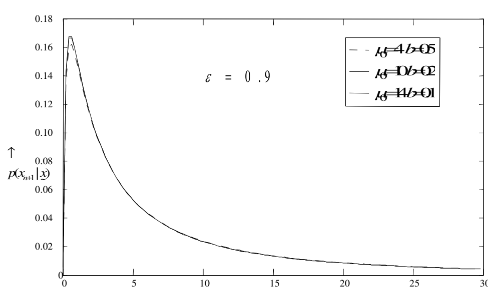


Fig. 19

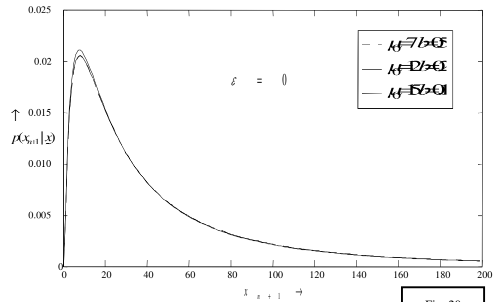


Fig. 20

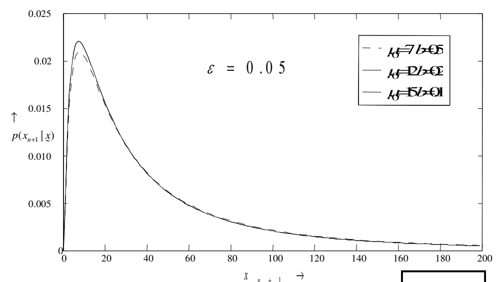
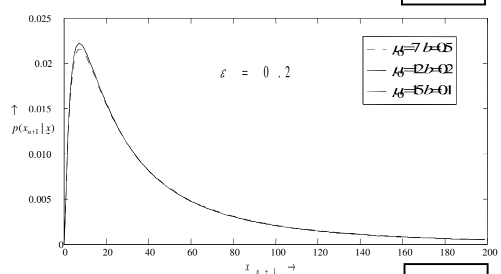


Fig. 21



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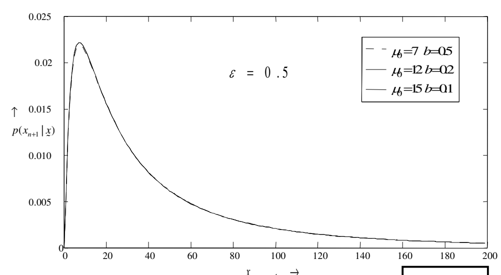

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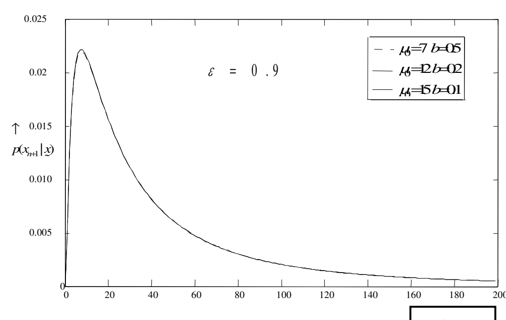

Fig. 24