A Proposal of Portfolio Choice for Infinitely Divisible Distributions of Assets Returns

Kliber, Pawel

Poznan University of Economics

2008

A PROPOSAL OF PORTFOLIO CHOICE FOR INFINITELY DIVISIBLE DISTRIBUTIONS OF ASSET RETURNS

Paweł KLIBER*

Abstract. In the paper we present a proposal of augmenting portfolio analysis for the infinitely divisible distributions of returns - so that the prices of assets can follow Lévy processes. In the classical portfolio analysis (by Markovitz or Sharp) the portfolio is evaluated according to two criteria: mean return and variance of returns. Such an approach is cumbersome second moments of assets' returns do not exist or if the interdependence between the returns of different assets can not be described only by covariation. In this article we propose a model in which asset prices follow multidimensional Lévy process and the interdependence between assets are described by covariance (Gaussian part) and multidimensional jump measure (Poisson part). Then we propose to choose the optimal portfolio based on three criteria: mean return, total variance of diffusion and a measure of jump risk. We also consider augmenting this multi-criteria choice setup for the costs of possible portfolio adjustments.

Keywords: portfolio analysis, Lévy processes, jump-diffusion models.

1. Introduction

Classical portfolio analysis (as proposed in [14] or [19]) is based on the assumption that returns are normally distributed. Although this assumption is not explicit, it is hidden in the fact that the distributions of returns are given by means and variances only. The empirical research however reveals that distribution of stocks' returns is vary from being normal (see [4], [12], [13]). It is believed that the stock prices can be better described using Lévy processes (and infinitely divisible distributions) instead of Wiener processes (and Gaussian distributions). Recently more and more papers have been appearing that uses this new method in modelling stock prices (see for example [1], [4], [5], [6], [8], [10], [11], [16], [18]).

There are two main problems connected with augmenting portfolio analysis for the infinitely divisible distributions of returns. Firstly, the criteria of the classical portfolio

* Poznań University of Economics, Department of Mathematical Economics, al. Niepodległości 10, 60-967 Poznań, Poland.

analysis are not adequate now. The criteria are based on moments (first and second) and they can be undefined in the case of Lévy processes. Thus the problem arises how to measure the risk of the portfolio. Secondly, the covariances do not suffice to describe interdependences between returns of different assets. For example the covariance matrix for several Lévy processes can be diagonal although the processes are not independent (because the jumps of these processes are dependent).

The article consists of five sections. In the section two we remind some basic information about Lévy processes and jump-diffusion model. In particular we present there Lévy-Itô decomposition. In the section three we deal with the problem of modelling interdependences between asset returns and present some usually used solutions. In the section four we present our proposal how to deal with the jump-diffusion models in portfolio analysis. The section five contains exemplary computations for generalized portfolio analysis.

2. Lévy processes and jump-diffusion models

Lévy process L_t is a stochastic *cadlag*[1] process which starts at zero ($L_0 = 0$) and fulfils the following conditions.

1. Its increments are independent and stationary, i.e. for any $t_1 < t_2 < ... < t_n$ the variables $L_{t_2} - L_{t_1}$, ..., $L_{t_n} - L_{t_{n-1}}$ are independent and the distribution of $L_{t+h} - L_t$ depends only on h (not on t).
2. The process is stochastically continuous, that is $\forall \varepsilon > 0 \ \lim_{h \to 0} P(|L_{t+h} - L_t| \geq \varepsilon) = 0$, which means that the jumps of the process are random – the probability that the process jumps at any given moment t equals 0.

The Lévy processes are closely connected with infinitely divisible distributions, i.e. with distributions that can be represented as a sum n of identically distributed random variables for all n . The infinitely divisible distributions are the broadest class of distributions that can appear in limit theorems for the sum of independent variables[2]. It is true that the distribution of Lévy process at any moment of time is infinitely divisible[3]. On the other hand – for any infinitely divisible distribution f there is such a Lévy process L_t that $X_1 \sim f$ (the distribution of random variable X_1 is f). Thus the Lévy processes are the widest class of processes which can be interpreted as a result of many small and independent random increments.

[1] That is its trajectories are right-continuous and have left limits (fr. - *continue à droite, limite à gauche*, see for example [20]).
[2] See [7], chapter XVII.
[3] See for example [17].

2.1. Lévy-Khinchin representation

According to Lévy-Khnitchin theorem (see [2], [4], [11]) any Lévy process L_t is completely described by its characteristic exponent, that is by the logarithm of the characteristic function of L_1. We have

$$E[e^{iuL_t}] = e^{t\psi(u)}, \tag{1}$$

where the function ψ (characteristic exponent) is given by

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\mu u + \int_R \left(e^{iux} - 1 - iux1_{|x|\le 1}\right) dv(x), \tag{2}$$

where $\sigma^2 \in R_+$, $\mu \in R$, and v is a measure on R (so called Lévy measure) which fulfils

$$\int_{|x|\le 1} x^2 dv(x) < \infty \quad \text{and} \quad v\big([1,\infty) \cup (-\infty,-1]\big) < \infty. \tag{3}$$

The measure v describes jumps of the process – the value $v(R)$ is the number of jumps in the unit of time. The value $v([c,d])$ denotes relative frequency of jumps in the size between c and d. If the v fulfils

$$\int_{|x|\le 1} |x| dv(x) < \infty, \tag{4}$$

then (2) can be reformulated as

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\mu u + \int_R \left(e^{iux} - 1\right) dv(x) \tag{5}$$

and μ denotes drift of the process.

2.2. Lévy-Itô decomposition

According to Lévy-Itô theorem (see [2], [4], [11]) any Lévy process can be decomposed into a sum of a linear trend, a Wiener process, a Poisson process of large jumps and a completely discontinuous martingale:

$$L_t = \mu t + \sigma W_t + P_t + M_t^d, \tag{6}$$

where W is a standard Wiener process, P is a Poisson process with jumps in $(-\infty,1] \cup [1,\infty)$ and M^d is a completely discontinuous martingale with jumps in (-1,1) If the Lévy measure fulfils (4), then we can rewrite (6) as

$$L_t = \mu t + \sigma W_t + \sum_{s\le t} \Delta L_t, \tag{7}$$

where $\Delta L_t = L_t - \lim_{s\to t^-} L_s$.

2.3. Jump-diffusion models

We assume that a asset return process is a Lévy process. Thus the asset price at the moment t equals $S_t = S_0 \exp(L_t)$. Alternatively, one can assume that the asset price is stochastic exponent of Lévy process and fulfils stochastic differential equation $dS_t = S_{t-} dL_t$ (where $S_{t-} = \lim_{u \to t^-} S_u$). It was shown in [9] that both approaches are equivalent. In both cases logarithmic returns of asset are infinitely distributed.

Both approaches are referred to as jump-diffusion models. Although in the literature this term denotes most often models with finite measure of jumps ($\nu(R) < \infty$), we understand this term more broadly. Examples of such models are Merton model (see [15]) or Kou model (see [10]). In jump-diffusion models the returns of asset are described by three parameters: mean μ, variance of Gaussian part σ^2 and jump measure ν. The method of portfolio analysis proposed in this paper can be applied also if distributions of returns are α-stable or are Student-distributed or belong to generalized hyperbolic family of distributions (these assumptions are frequent in financial literature and models based on them fit to data very good, see for example [12] or [13]).

3. Interdependence between assets' returns

In portfolio analysis one has to take into account interdependences between returns of assets. In classical approach it suffices to take into account covariances between returns of assets. However if there are jumps in the processes of returns, one should also model interdependences of jumps.

To describe interdependences between n Lévy processes we should specify covariance matrix Ω and joint measure of jumps ν. Matrix Ω contains covariances for Gaussian parts of processes and ν is a measure on R^n which describe intensity of jumps for multidimensional process $(L_1, L_2, ..., L_n)$. The margins of ν are the jump measure for one-dimensional processes L_1, …, L_n. There are two methods of specifying such measure.

In the first method (see [4]) one decomposes jumps of assets into "market" and idiosyncratic parts. The jump processes are thus given by

$$\begin{aligned} L_1^d &= R^d + S_1^d, \\ &\ldots\ldots \\ L_n^d &= R^d + S_n^d, \end{aligned} \tag{8}$$

where L_i^d is discontinuous part of returns for asset i, R^d describes "market" jumps and S_i^d describes jumps connected with specific asset i (idiosyncratic jumps).

The second method consists on application of Lévy copulas (see [4]). If $U_1(x)$ and $U_2(x)$ are upper tails of jump measures ν_1 and ν_2 (that is $U_i(x)=\int_x^{\infty}\nu(dy)$) and $U(x_1,x_2)$ is upper tail of joint measure ν ($U(x_1,x_2)=\int_{x_1}^{\infty}\int_{x_2}^{\infty}\nu(dy,dz)$) then there exists a function F^{++} (Lévy copula) such that $U(x_1,x_2)=F^{++}\left(U_1(x_1),U_2(x_2)\right)$. To fully describe interdependence of jumps for two processes we need to specify four copulas – for positive tails (F^{++}), for negative tails (F^{--}) and for "mixed" tails (F^{+-}, F^{-+}). If there are more processes we have to specify more copulas and the method becomes rather cumbersome.

4. Portfolio analysis

Let $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ denote the structure of the portfolio, that is the value α_i denotes the parts of investor's wealth invested in asset i. Of course $\sum_{i=1}^{n}\alpha_i=1$.

In the classical portfolio analysis by Markowitz or Sharpe ([14], [19]) the portfolio is evaluated according to two criteria: mean return and variance of return. We propose to introduce third, additional, criterion connected with possible jumps of portfolio's value. Let the measure ν on R^n describes intensity of jumps for all assets. For a given portfolio structure α let us define the mapping $F^{\alpha}:R^n\to R^n$ as follows: $F^{\alpha}(x_1,x_2,...,x_n)=\left(\alpha_1x_1,\alpha_2x_2,...,\alpha_nx_n\right)$. By ν^{α} we denote a measure on R^n defined as

$$\nu^{\alpha}(B)=\nu\left(\left(F^{\alpha}\right)^{-1}(B)\right)=\nu\left(\left\{x\in R^n : F^{\alpha}(x)\in B\right\}\right).$$

The jump measure for the returns of the whole portfolio is a measure η^{α} on R defined as follows:

$$\eta^{\alpha}(B)=\int_B dx\underbrace{\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\ldots\int_{-\infty}^{\infty}dx_{n-1}\nu^{\alpha}(dx_1,dx_2,...,dx_{n-1},dx-dx_1-...-dx_{n-1})}_{n-1}$$

Let $u:R\to R_+$ be a function which describes investor's attitude toward sudden changes of asset prices. We interpret it so that the higher the value of u the worst it is for investor. We propose that investor should rate his or her portfolio according to following three criteria.

1. Mean return:

$$K_1(\alpha)=\sum_{i=1}^{n}\alpha_i\mu_i\ .$$

2. Variance of return:

$$K_2(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} .$$

3. Jumps' risk:

$$K_3(\alpha) = \int_{-\infty}^{\infty} u(x) \eta^{\alpha}(dx) . \tag{9}$$

Portfolio optimization starts with finding the set of effective portfolios (that is the portfolios for which one cannot improve any criterion K_1, K_2, K_3 without worsen some other criterion – we try to maximise criterion K_1 and to minimize criteria K_2 and K_3). Then the investor has to choose portfolio from this set according to his or her preferences (which include his or her attitude toward risk and/or desirable mean return).

Alternatively, we can search the solution to the problem

$$\max_{\alpha} K_1(\alpha) - \lambda_2 K_2(\alpha) - \lambda_3 K_3(\alpha) , \tag{10}$$

subject to

$$\sum_{i=1}^{n} \alpha_i = 1 , \tag{11}$$

where constants λ_2 and λ_3 describe investor's attitude toward risk of diffusion and risk of jumps. (If short sale is not allowed, then we should add $\alpha \geq 0$ to the constrain (11)).

The third criterion K_3 can be sometimes hard in computation when using the formula (9). We can propose two possible solutions. Sometimes it is possible to find the analytical formula for K_3. In other cases one can use Monte Carlo simulations.

Let us consider for example generalized Merton model, in which jump measure has multidimensional normal distribution, $\nu \sim N(0,W)$, where W is covariance matrix. The measure η^{α} is also Gaussian: $\eta^{\alpha} \sim N(0,\sigma_{\alpha}^2)$ where

$$\sigma_{\alpha}^2 = \sum_{i,j=1}^{n} \alpha_i \alpha_j w_{ij} ,$$

(w_{ij} are the elements of the matrix W). If we appropriately choose the function u, it is easy to compute K_3. For example taking $u(x) = x^2$ we obtain $K_3(\alpha) = \sigma_{\alpha}^2$. The problem (10) is then a problem of quadratic programming and can be solved using standard methods.

We can also compute K_3 using Monte Carlo simulation. If we know jump measures for all assets and interdependence between them, then we can generate multidimensional process $(\tilde{x}_1,...,\tilde{x}_n)$, which simulates the jumps of assets. The simulation of the jump for the whole portfolio is $\tilde{x} = \alpha_1 \tilde{x}_1 + \alpha_2 \tilde{x}_2 + ... + \alpha_n \tilde{x}_n$. Then we compute the value $u(\tilde{x})$. We repeat this many times and obtain numerical approximation for the true value of K_3:

$$K_3(\alpha) \approx \frac{1}{m}\sum u(\widetilde{x}),$$

where we sum all m simulated values.

5. Examples

We give two examples of portfolio analysis with the new method. The first one concerns multidimensional Merton model, in which jump measure has multidimensional normal distribution $N(0,E)$. The function measuring jumps' risk is $u(x) = x^2$, so that the criterion K_3 is given by formula $K_3 = \alpha' W \alpha$.

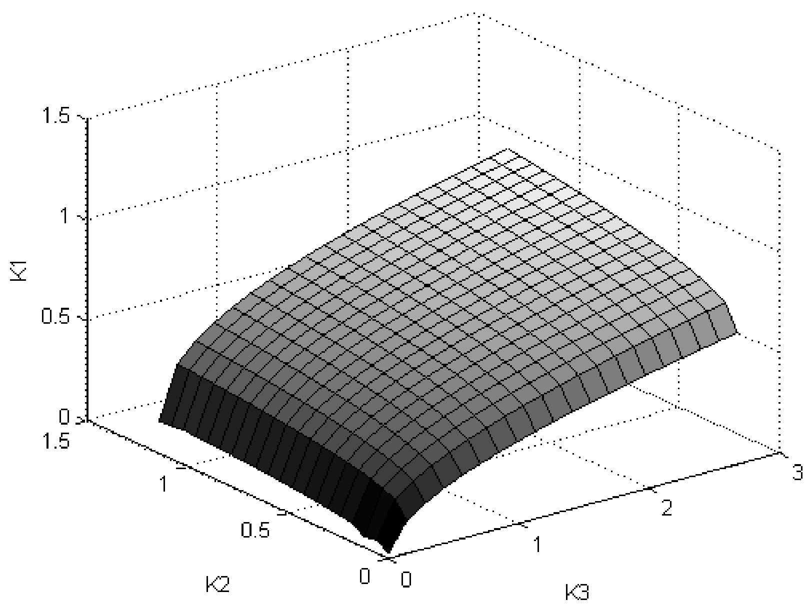

Figure 1. The surface of effective portfolios in space of criteria – an example for multidimensional Merton model

The figure 1 presents the surface of efficient portfolios in the space (K_1, K_2, K_3). The portfolios consist of five assets and the mean returns μ, covariance matrix Ω and matrix W were chosen randomly. The investor chooses the optimal portfolio from the set of efficient portfolios according to his or her attitude to risk and gain. For example if he or she wants to have mean return no lower then 0.378 with variance of diffusion part (K_2) no greater then 0.241, then according to the figure 1 mean square of jumps of the portfolio (K_3) cannot be lower then 0.281. The computations were performed in Excel with package Solver.

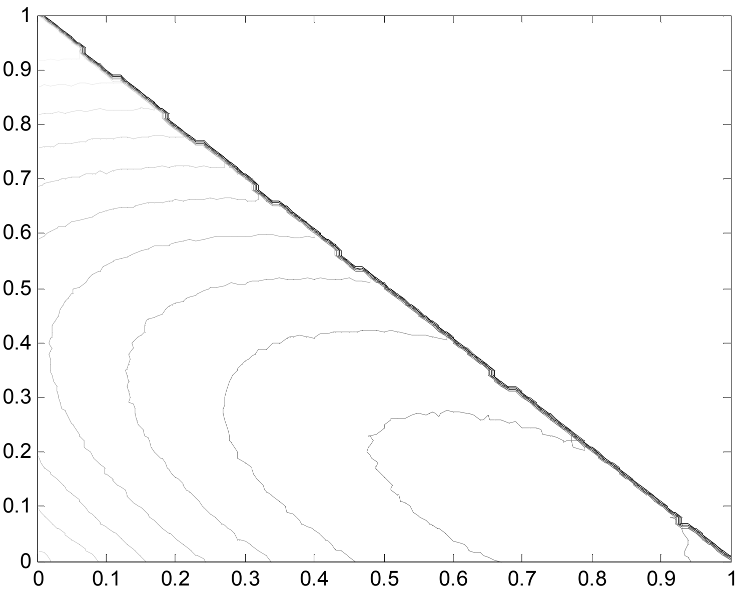

Figure 2. An exemplary isoquants of K_3 (jumps' risk) in the Kou model

The second example concerns multidimensional Kou model. We assume that jump measure for each asset is exponentially distributed. The interdependences between assets' jumps are described using decomposition into "market" jumps and idiosyncratic jumps as in (8). We have analyzed portfolios of three assets. The means of their idiosyncratic jumps were 0.5, 0.2 and 0.4 respectively and the mean of "market" jump was 0.4. The function measuring jumps' risk was $u(x) = x^2$. The derivation of analytical formulas for K_3 can be very complicated (although possible) in this model. The values of K_3 can be easily computed numerically with Monte Carlo method. Figure 2 presents isoquants of K_3 for different portfolio structures (horizontal axis represents the share of the first asset and vertical axis – of the second). The computations were performed in Matlab (ver. 7.0). This results can be used to calculate the optimal portfolio. For example if the mean returns of the assets are μ_1=0.3, μ_2=0.2, μ_3=0.1, the standard deviations of the diffusion parts of returns are σ_1=0.1, σ_2=0.2, σ_3=0.3 and correlations between the diffusion parts of returns are ρ_{12}=0.7, ρ_{13}=-0.5, ρ_{23}=-0.3, then we can calculate the optimal portfolio by solving the problem (10) with λ_2 and λ_3 chosen by investor according to his or her preferences. For example if λ_2 =1 and λ_3 =0.1, then the optimal portfolio is (0.11, 0.73, 0.16).

6. Conclusions

In the article we presented the method of choosing the optimal portfolio according to three criteria: mean return, variance of return and jumps' risk. The choice of the portfolio can be made either by obtaining the set of effective portfolios (and then choosing the portfolio from this set according to investor's preferences) or by solving the problem (10) with appropriate weights put to all the criteria.

While the computations of the criteria K_1 and K_2 are easy (the first one is linear form and the second one – quadratic form of the structure of the portfolio), the calculation of the third criterion K_3 is more problematic. With some assumption about model and the function u one can derive analytical formulae for K_3 – this is the case of the Merton model with quadratic disutility function. Alternatively one can compute K_3 numerically using Monte Carlo method. However calculating the set of the effective portfolios using this second method of computing K_3 can be very time-consuming as the required time grows exponentially with the number of assets. With the three assets and with grid 0.01 (i.e. we assume that the share of any asset can be multiple of 0.01) we had to consider 50000 possible portfolios (performing Monte Carlo simulation for each of them). If we add fourth asset then the number of possible portfolios grows to more then 170000, etc. Because of these computational difficulties we recommend rather to use the first method (with analytical formulae).

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