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Abstract

We propose to discuss a new technique to derive an good approximated solution for the price of a European Vanilla options, in a market model with stochastic volatility. In particular, the models that we have considered are the Heston and SABR(for $\beta = 1$). These models allow arbitrary correlation between volatility and spot asset returns. We are able to write the price of European call and put, in the same form in which one can see in the Black-Scholes model. The solution technique is based upon coordinate transformations that reduce the initial PDE in a straightforward one-dimensional heat equation.

1 Volatility Risk

Let us consider the simple case of a stock price model. The underlying variable is today's observed stock price. The most popular market model is the Black-Scholes model. It assumes for the underlying process, a geometric Brownian motion with constant volatility, that is

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$$

$$dB_t = rB_t dt$$

where r is the constant risk-free rate, S_t is the stock and σ is the constant volatility of the stock. Under these assumptions, closed form solutions for the values of European call and put options, are derived by use the PDE method.

The assumption of constant volatility is not reasonable, since we require different values for the volatility parameter for different strikes and different expiries to match market prices. The volatility parameter that is required in the Black-Scholes formula to reproduce market prices is called the implied volatility. This is a critical internal inconsistency, since the implied volatility of the underlying should not be dependent on the specifications of the contract. Thus to obtain market prices of options maturing at a certain date, volatility needs to be a function of the strike. This function is the so called volatility skew or smile. Furthermore for a fixed strike we also need different volatility parameters to match the market prices of options maturing on different dates written on the same underlying, hence volatility is a function of both the strike and the expiry date of the derivative security. This bivariate function is called the volatility surface. There are two prominent ways of working around this problem, namely, local volatility models and stochastic volatility models. For local volatility models the assumption of constant volatility made in Black and Scholes [1973] is relaxed. The underlying risk-neutral stochastic process becomes

$$dS_t = r(t)S_t dt + \sigma(t, S_t)S_t d\tilde{W}_t$$

where $r(t)$ is the instantaneous forward rate of maturity t implied by the yield curve and the function $\sigma(S_t, t)$ is chosen (calibrated) such that the model is consistent with market data, see Dupire [1994], Derman and Kani [1994] and [Wilmott, 2000, x25.6]. It is claimed in Hagan et al. [2002] that local volatility models predict that the smile shifts to higher prices (resp. lower prices) when the price of the underlying decreases (resp. increases). This is in contrast to the market behavior where the smile shifts to higher prices (resp. lower prices) when the price of the underlying increases (resp. decreases). Another way of working around the inconsistency introduced by constant volatility is by introducing a stochastic process for the volatility itself; such models are called stochastic volatility models. The major advances in stochastic volatility models are Hull and White [1987], Heston [1993] and Hagan et al. [2002]. Such models have the following general form

$$
\begin{aligned}
&dS_t = \mu_t S_t dt + \sigma_t^\delta a_2(S_t) dW_t^{(1)} \\
&d\sigma_t^j = b_1(\sigma, t)dt + \alpha\sigma_t^\delta dW_t^2 \\
&dW_t^{(1)} dW_t^{(2)} = \rho dt. \\
&dB_r = rB_t dt
\end{aligned}
$$

and varying its parameters we obtain different models::
• for $\delta = 1$, $\mu_t = 0$, $j = 1$ $a_2(S) = S^\beta$ and $b_1 = 0$, we get the SABR model, by Hagan;

• for $\delta = 1$, $j = 2$, $a_2(S) = S$ and $b_1 = k(\theta - \sigma_t)$, we get Heston model, by Heston;

• for $\delta = 1$, $\alpha = 0$, we get Black-Scholes model time dependent volatility, by Black-Scholes-Merton;

• for $\delta = 1$, $\alpha = 0$ and $b_1 = 0$ we get Black-Scholes model with constant volatility, by Black-Scholes-Merton;

where the tradeable security S_t and its volatility σ_t are correlated, i.e. $d\tilde{W}_{S_t} d\tilde{W}_{\sigma_t} = \rho dt$.

We are going to use some geometrical transformations in order to simplify the pricing PDE; our method can be used whenever it is possible to write the second derivative term as following: $\frac{\partial^2 f_1}{\partial x^2} + 2\rho \frac{\partial^2 f_1}{\partial x \partial \tilde{\nu}} + \frac{\partial^2 f_1}{\partial \tilde{\nu}^2}$, but this will be clear later.

2 Heston's Model and Pricing Options

The stochastic volatility model proposed by Heston (1993), assumes that the asset price S satisfies

$$dS_t = S_t(\mu_t dt + \sqrt{\nu}_t) dW_t^{(1)} \qquad S \in [0, \infty) \tag{1}$$

with the instantaneous variance ν governed by the SDE

$$d\nu_t = k(\theta - \nu_t)dt + \alpha\sqrt{\nu_t} dW_t^{(2)}, \qquad \nu \in (0, \infty); \quad k, \theta, \alpha \in \mathbb{R} \tag{2}$$

where $W^{(1)}$ and $W^{(2)}$ are standard one-dimensional Brownian motions defined on filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, which the cross-variation $\langle W^{(1)}, W^{(2)} \rangle = \rho t$ for some constant $\rho \in (-1, 1)$. In this case, it is more convenient to express the pricing function f and the market price of volatility risk λ in terms of variables (S, ν, t), rather than (S, σ, t). We now make a judicious choice of the market price of volatility risk; specifically, we set $\lambda(\nu_t, t) = \lambda\sqrt{\nu_t}$ for some constant λ such that $\lambda\alpha \neq k$. Hence, under a martingale measure $\mathbb{Q}$, equations (1) (2) become

$$dS_t = S_t(rdt + \sqrt{\nu_t})dW^{(1)}_{t,(Q)} \tag{3}$$

and

$$d\nu_t = \kappa(\Theta - \nu_t)dt + \alpha\sqrt{\nu_t}dW^{(2)}_{t,(Q)} \tag{4}$$

where we set

$$\kappa = (k - \lambda\alpha), \qquad \Theta = \theta k(k - \lambda\alpha)^{-1}, \tag{5}$$

and where $W^{(1)}_{(Q)}$ and $W^{(2)}_{(Q)}$ are standard one-dimensional Brownian motions such that $\langle dW^{(1)}, dW^{(2)} \rangle = \rho dt$. It is now easy to see that the pricing PDE for European derivatives in Heston model, by Itô's lemma, has the following form:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 f}{\partial S^2} + \rho\nu\alpha S \frac{\partial^2 f}{\partial S \partial \nu} + \frac{1}{2}\nu\alpha^2 \frac{\partial^2 f}{\partial \nu^2} + \kappa(\Theta - \nu)\frac{\partial f}{\partial \nu} + rS\frac{\partial f}{\partial S} - rf = 0 \tag{6}$$

with the terminal condition $f(S, \nu, T) = \phi(S)$ for every $S \in \mathbb{R}_+$, $\nu \in \mathbb{R}_+$ and $t \in [0, T]$. We take here for granted the existence and uniqueness of (nonnegative) solutions S and ν to Heston's SDE. It is common to assume $2K\Theta/\alpha^2 > 1$, so that, the solution ν is strictly positive if $\nu_0 > 0$.

3 Numerical methods for Option Valuation

For the Heston model we are able to compute the solution by numerical techniques, as:

- Finite Difference method (Crank Nicolson);
- Monte-Carlo simulation method combined with a variance reduction technique:
- Fourier Transform Technique.
- Geometrical Approximation.

Here, we want to highlight some important aspects. The PDE method is a flexible method which can be used for many pay-offs: European Options or certain path dependent derivatives; in this case, the drawback is that we have to approximate the option prices on a grid. Accurate pricing requires a substantial amount of grid points. The PDE method is somewhat expensive.

The Monte-Carlo method is the most general, but it has long computation times.

The Fourier transformation technique has been used to evaluate the model option prices. This method is both fast and accurate. Its major technical difficulty lies in the derivation of a characteristic function, i.e., the Fourier transform of the risk-neutral density function. See Carr and Madan for further details. The Fourier transformation technique can take advantage of a very numerical algorithm called the Fast Fourier Transform (FFT) technique, which drastically improves the numerical efficiency of the calibration.

Now, we focus on proposed method, that we have called ”Geometrical Approximation”, it is based only on considerations about the pay-off function. For suitable values of ρ, ν, α, where $\epsilon = \frac{\rho\nu}{\alpha} << 1$, we have a closed form solution of the exact PDE, but with modified Cauchy condition, in which we consider the following pay-off function $(S_T e^{-\frac{\rho\nu}{\alpha}} - E)^+$, instead of, the standard pay-off function $(S_T - E)^+$. It is clear that the former goes to the latter for ϵ that goes to zero.

$$e^{-\epsilon} \simeq (1 - \epsilon)$$

thus

$$f(T, S, \nu) = \left(S_T e^{-\epsilon} - E\right)^+ \simeq \left(S_T\,(1 - \epsilon) - E\right)^+$$

$$\lim_{\epsilon \to 0} \left(S_T\,(1 - \epsilon) - E\right)^+ = (S_T - E)^+$$

In order to evaluate a European call option, first we simplify the PDE (6) at hand. To this end, let us introduce a new variable x and a new function f_1:

$$S = e^x, \quad \nu = \tilde{\nu}\alpha, \qquad x \in (-\infty, \infty), \qquad \nu \in [0, \infty), \qquad t \in [0, T]]$$
$$f(t, S, \nu) = e^{-r(T-t)} f_1(t, x, \tilde{\nu})$$

so that we have a new PDE

$$\frac{\partial f_1}{\partial t}+\frac{1}{2}\tilde{\nu}\alpha\left(\frac{\partial^2 f_1}{\partial x^2}+2\rho\frac{\partial^2 f_1}{\partial x\partial\tilde{\nu}}+\frac{\partial^2 f_1}{\partial\tilde{\nu}^2}\right)+\frac{\kappa}{\alpha}\left(\Theta-\tilde{\nu}\alpha\right)\frac{\partial f_1}{\partial\tilde{\nu}}+\left(r-\frac{1}{2}\tilde{\nu}\alpha\right)\frac{\partial f_1}{\partial x}=0, \tag{7}$$

now we consider only the terms that have derivatives of the second order and after that, we try a new set of coordinates that transform the PDE in its canonical form. It is important to remember that our PDE, is of parabolic kind and its canonical form is the heat equation, and we want to transform the above PDE in a heat equation. First step, we write the characteristic equation associated to the second order terms of our PDE (7) , thus we compute its roots:

$$\frac{\partial^2 f_1}{\partial x^2}+2\rho\frac{\partial^2 f_1}{\partial x\partial\tilde{\nu}}+\frac{\partial^2 f_1}{\partial\tilde{\nu}^2}=0.$$

The characteristic equation results to be

$$\left(\frac{dx}{d\tilde{\nu}}\right)^2-2\rho\left(\frac{dx}{d\tilde{\nu}}\right)+1=0,$$

$$\Delta=4^2(1-\rho^2)\leq 0,\qquad \rho\in(-1,1)$$

so that the squared term is of elliptic kind, and the roots belong at the set of complex numbers

$$\left(\frac{dx}{d\tilde{\nu}}\right)_{1/2}=\rho\pm\imath\sqrt{1-\rho^2}.$$

At this point we can define the characteristic lines (these are also defined like geodesics) as follows

$$x-(\rho+\imath\sqrt{1-\rho^2})\tilde{\nu}=z$$
$$x-(\rho-\imath\sqrt{1-\rho^2})\tilde{\nu}=w.$$

Through another change of variable, that we show hereafter, we obtain a linear system easy to solve

$$z=\xi+\imath\eta;\qquad w=\xi-\imath\eta;$$

so that results $w=\overline{z}$

$$\begin{aligned}\tilde{\nu}=-\frac{\eta}{\sqrt{1-\rho^2}}\qquad & x=\frac{\xi\sqrt{1-\rho^2}-\rho\eta}{\sqrt{1-\rho^2}}\\ \eta=-\tilde{\nu}\sqrt{1-\rho^2}\qquad & \xi=x-\rho\tilde{\nu}\end{aligned} \tag{8}$$

where $\eta \in (-\infty, 0)$ and $\xi \in (-\infty, \infty)$ and it is clear that our function f_1 must be transformed in another f_2. At this point is fundamental to make the following geometrical consideration, in order to understand our method. We have defined a new system of coordinates, where $\vec{e}_\eta, \vec{e}_\xi, \vec{e}_t$, are orthogonal directions; we can think of x, ν as vectors, whose projections on the axes are respectively given by

$$\vec{x} = (0)\vec{e}_\eta + (x)\vec{e}_\xi \qquad \vec{\tilde{\nu}} = (\tilde{\nu}\cos\theta_\rho)\vec{e}_\eta + (\tilde{\nu}\sin\theta_\rho)\vec{e}_\xi$$

where, we have supposed $\rho = \sin\theta_\rho$ and $\sqrt{1-\rho^2} = \cos\theta_\rho$, $\theta_\rho \in (-\pi/2, \pi/2)$. Now we can define a new vector, that we call $\vec{V}$, whose projections are

$$\vec{V} \equiv (V_\eta, V_\xi) \qquad V_\eta = -\tilde{\nu}\cos\theta_\rho \qquad V_\xi = x - \tilde{\nu}\sin\theta_\rho$$

by which, we can show the vectorial relation that exists between the variables $(x, \tilde{\nu})$.

Now, from the Cauchy's condition, we are able to write the new function f_2, like function of variables t and $V_\xi(x, \tilde{\nu})$, because, the function f depends, at the time T, only on the projection terms upon the axis ξ,

$$f(T, S, \nu) = (S_T - E)^+ = \left(e^{x'} - E\right)^+ = \left(e^{\left(\vec{V}' + \vec{\tilde{\nu}}\right)\cdot\vec{e}_\xi} - E\right)^+$$
$$= \left(S' e^{-\frac{\rho\nu'}{\alpha}} - E\right)^+$$

(where with the apex ($'$) we indicate the variables at the time $t = T$), therefore, because of the continuity properties of the Feynman-Kač formula, we can suppose that is true at any time t.

$$f_1(t, x, \tilde{\nu}) = f_2(t, V_\xi(x, \tilde{\nu})); \qquad t \in [0, T]$$

now we may substitute them in the old squared term

$$\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x \partial\tilde{\nu}} + \frac{\partial^2 f_1}{\partial\tilde{\nu}^2} = (1-\rho^2)\nabla^2_{V_\xi} f_2(t, V_\xi(x, \tilde{\nu})).$$

Thus, the new Black-Scholes PDE of Heston's model has become

$$\frac{\partial f_2}{\partial t} - \frac{\alpha V_\eta}{\sqrt{1-\rho^2}}\left[\frac{(1-\rho^2)}{2}\frac{\partial^2 f_2}{\partial V_\xi^2} + \left(\frac{1}{2} - \frac{\kappa}{\alpha}\rho\theta\right)\frac{\partial f_2}{\partial V_\xi}\right] + \left(r - \frac{\kappa}{\alpha}\rho\theta\right)\frac{\partial f_2}{\partial V_\xi} = 0 \qquad (9)$$

where we have changed the final condition $(S_T - E)^+$, in

$$\left(S' e^{-\frac{\rho\nu'}{\alpha}} - E\right)^+ = \left(e^{V'_\xi} - E\right)^+$$

Now, we can compute the solution of PDE (9) in a closed form, that is an approximation of the original problem for $\frac{\rho\nu}{\alpha} << 1$.

By another change of coordinates is sufficient to simplify last PDE. We may define a new transformation of coordinates; and the new function f_3, as follows

$$\begin{aligned}
&\gamma = V_\xi + \left(r - \frac{k}{\alpha}\rho\theta\right)(T-t), \qquad \gamma \in (-\infty, \infty);\\
&\tau = -\int_t^T ds \frac{\alpha V_\eta}{\sqrt{1-\rho^2}} = \int_t^T ds\nu(s), \qquad \tau \in \left[0, \int_0^T ds\nu(s)\right];\\
&f_2(t, V_\xi) = f_3(\tau(t, V_\eta), \gamma(t, V_\xi));
\end{aligned}$$

for $t = T$ we have

$$f_3(0, \gamma') = \left(e^{\gamma'} - E\right)^+.$$

Substituting what we have just found, in the previous equation, we finally have a very easy partial differential equation

$$\begin{aligned}
&\frac{\partial f_3}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla^2_\gamma f_3 + \left(\frac{1}{2} - \frac{\kappa\rho}{\alpha}\right)\frac{\partial f_3}{\partial \gamma}\\
&\gamma \in (-\infty, \infty), \quad \tau \in \left[0, \int_0^T ds\nu(s)\right];
\end{aligned} \tag{10}$$

Now we can rewrite the function f_3 as follows, in order to obtain the one-dimensional heat equation:

$$f_3(\tau, \gamma) = e^{\lambda\tau + \beta\gamma} f_4(\tau, \gamma);$$

where

$$\lambda = -\frac{(1/2 - \kappa\rho/\alpha)^2}{2(1-\rho^2)}, \qquad \beta = -\frac{(1/2 - \kappa\rho/\alpha)}{(1-\rho^2)};$$

so we have

$$\frac{\partial f_4}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla^2_\gamma f_4$$

At this point we have another problem that has an easier solution:

$$\frac{\partial f_4}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla^2_\gamma f_4 \qquad \gamma \in (-\infty, +\infty), \tau \in \left[0, \int_0^T ds\nu(s)\right]$$

$$f_4(0,\gamma') = \left(e^{\gamma'} - E\right)^+$$

Now, we are able to write the solution, that is

$$\begin{aligned} f_4(\tau,\gamma) &= \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}}\int_{-\infty}^{+\infty} d\gamma' f_4(0,\gamma') \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right] \\ &= \int_{-\infty}^{\infty} d\gamma' f_4(0,\gamma') G(\gamma', 0|\gamma,\tau) \end{aligned} \quad (11)$$

where

$$G(\gamma', 0|\gamma,\tau) = \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}} \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

where

$$f(t,S,\nu) = e^{-r(T-t)+\lambda\tau+\beta\gamma} f_4(\tau,\gamma)$$

$$f(T,S,\nu) = e^{\beta\gamma'} f_4(0,\gamma')$$

$$f_4(0,\gamma') = e^{-\beta\gamma'}\left(e^{\gamma'} - E\right)^+$$

At this point we have

$$f_4(\tau,\gamma) = \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}} \int_{-\infty}^{+\infty} d\gamma' e^{-\beta\gamma'} \left(e^{\gamma'} - E\right)^+ \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}} \int_{\ln E}^{+\infty} d\gamma' e^{-\beta\gamma'} \left(e^{\gamma'} - E\right) \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

Thus we can write the price of a European Call option in Heston's market model as follows

$$f(t,S,\nu) = \frac{e^{-r(T-t)+\lambda\tau+\beta\gamma}}{\sqrt{2\pi(1-\rho^2)\tau}} \int_{\ln E}^{+\infty} d\gamma' e^{-\beta\gamma'} \left(e^{\gamma'} - E\right) \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

$$= \left(S_t e^{-\frac{\rho\nu}{\alpha}}\right) e^{\delta_1^\rho} \mathbf{N}(d_1^\rho) - E e^{\delta_2^\rho} \mathbf{N}(d_2^\rho) \quad (12)$$

where

$$\delta_1^\rho = -\left[\frac{\kappa}{\alpha}\rho\Theta - \left(\lambda + \frac{(1-\beta)^2}{2}(1-\rho^2)\overline{\nu}\right)\right](T-t);$$

$$\delta_2^\rho = -\left[r - \left(\lambda + \frac{\beta^2}{2}(1-\rho^2)\right)\overline{\nu}\right](T-t);$$

$$\overline{\nu}_\rho = \frac{1}{(T-t)} \int_t^T ds(1-\rho^2)\nu(s)$$

$$\overline{\nu}_{\rho=0} = \overline{\nu} = \frac{1}{(T-t)} \int_t^T ds\nu(s)$$

$$d_1^\rho = \frac{\ln(S/E) - \frac{\rho}{\alpha}\nu + \left[\left(r - \frac{\kappa}{\alpha}\rho\Theta\right) + (1-\beta)\overline{\nu}_\rho\right](T-t)}{\sqrt{\overline{\nu}_\rho(T-t)}}$$

$$d_2^\rho = \frac{\ln(S/E) - \frac{\rho}{\alpha}\nu + \left[\left(r - \frac{\kappa}{\alpha}\rho\Theta\right) - \beta\overline{\nu}_\rho\right](T-t)}{\sqrt{\overline{\nu}_\rho(T-t)}}$$

$$d_2^\rho = d_1^\rho - \sqrt{\overline{\nu}_\rho(T-t)}$$

Thus for $\epsilon = \frac{\rho\nu}{\alpha} << 1$, the final value of Call option is given by:

$$C_{\rho,\alpha,\Theta,\kappa}(t, S_t, \nu_t) = S_t\,(1-\epsilon)\,e^{\delta_1^\rho}\mathbf{N}(d_1^\rho) - Ee^{\delta_2^\rho}\mathbf{N}(d_2^\rho), \tag{13}$$

and for a Put, the final value is

$$P_{\rho,\alpha,\Theta,\kappa}(t, S_t, \nu_t) = Ee^{\delta_2^\rho}\mathbf{N}(-d_2^\rho) - S_t\,(1-\epsilon)\,e^{\delta_1^\rho}\mathbf{N}(-d_1^\rho); \tag{14}$$

3.1 Hedging and Put-Call-Parity

In order to find the better hedging strategy, we use a replicant portfolio. So that we need to know the value of the first and second, derivative of the price, with respect to S_t, that we respectively call Δ and Γ strategies for a European call option and European put option, where $\epsilon << 1$:

$$\Delta_{call} = \frac{\partial C_{\rho,\alpha,\Theta,\kappa}}{\partial S} = (1-\epsilon)\,e^{\delta_1^\rho}N(d_1^\rho)$$

$$\Gamma_{call} = \frac{Ee^{\delta_1^\rho - (d_1^\rho)^2/2}}{S\sqrt{2\pi\overline{\nu}_\rho(T-t)}} \tag{15}$$

and

$$\Delta_{put} = \frac{\partial P_{\rho,\alpha,\Theta,\kappa}}{\partial S} = -\left(1-\epsilon\right) e^{\delta_1^\rho} N(-d_1^\rho)$$

$$\Gamma_{put} = \frac{E e^{\delta_1^\rho - (d_1^\rho)^2/2}}{S\sqrt{2\pi\overline{\nu}_\rho(T-t)}} \tag{16}$$

Thus we have

$$\Gamma_{put} = \Gamma_{call}$$

It is necessary to highlight that, in the Heston's model, the Put-Call-Parity condition is verified, and this proves that we are in a free arbitrage market.

4 SABR Model

Another popular stochastic volatility market model proposed and analyzed by Hagan et al. (2002) is the SABR model. The latter is specified as follows: under a martingale measure $\mathbb{Q}$ the forward price is assumed to obey the SDE

$$dF_t^T = \sigma_t^F (F_t^T)^\beta dW_{t,(Q)}^{(1)} \qquad \beta \in (0,1] \tag{17}$$

where

$$d\sigma_t^F = \alpha\sigma_t^F dW_{t,(Q)}^{(2)} \qquad \alpha \in \mathbb{R} \tag{18}$$

where $W_Q^{(1)}$ and $W_Q^{(2)}$ are Brownian motions with respect to a common filtration $\mathbb{F}^W$, with a constant correlation coefficient $\rho \in (-1,1)$. The model given by (17)-(18) is known like the SABR model. It can be seen as a natural extension of the classical CEV model, proposed by Cox(1975). The model can be accurately fitted to the observed implied volatility curve for a single maturity T. A more complicated version of the model is needed if we wish to fit volatility smiles at several different maturities. More importantly, the model seems to predict the correct dynamics of the implied volatility skews (as opposed to the CEV model or any model based on the concept of a local volatility function). To support this claim, Hagan et al. (2002) derive and study the approximate formulas for the implied Black and Bachelier volatilities in the SABR model. It appears that the Black implied volatility $\hat{\sigma}(K,T)$, in this model can be represented as follows:

$$\hat{\sigma}(K,T) = \frac{\sigma_0}{(S_0/K)^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}\ln^2(S_0/K)+\frac{(1-\beta)^4}{1920}\ln^4(S_0/K)+.....\right)}\times$$
$$\frac{z}{x(z)}\left\{1+\left[\frac{(1-\beta)^2\sigma_0^2}{24(S_0K)^{(1-\beta)}}+\frac{\rho\beta\sigma_0\nu}{4(S_0K)^{(1-\beta)/2}}+\frac{(2-3\rho^2)\nu^2}{24}\right]T+.......\right\}, \quad (19)$$

where K is the strike price, S_0 is the underlying asset value at the time $t=0$ and σ_0 is the value of the volatility at time $t=0$,

$$z = \frac{\nu}{\sigma_0}(S_0/K)^{\hat{\beta}/2}\ln(S_0/K),$$

and

$$x(z) = \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

In the case of at-the-money option, the formula above reduces to

$$\hat{\sigma}(S_0,T) = \frac{\sigma_0}{S_0^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2\sigma_0^2}{24(S_0)^{2(1-\beta)}}+\frac{\rho\beta\sigma_0\nu}{4(S_0)^{(1-\beta)}}+\frac{(2-3\rho^2)\nu^2}{24}\right]T+.......\right\}.$$

5 SABR Model for $\beta = 1$

5.1 SABR Model and Option Pricing

It is our intention to use the method proposed in previous section also for SABR market model for $\beta = 1$.

Be given the following market, where $\beta \in (0,1]$, under natural measure $\mathbb{P}$

$$\begin{aligned}
dS_t &= \mu_t^{(S)}S_t dt + \sigma_t S_t^{\beta} dW_{t,(\mathbb{P})}^{(1)}\\
d\sigma_t &= \mu_t^{(\sigma)}\sigma_t dt + \alpha\sigma_t dW_{t,(\mathbb{P})}^{(2)}\\
dB_t &= rB_t dt\\
dW_{t,(\mathbb{P})}^{(1)}dW_{t,(\mathbb{P})}^{(2)} &= \rho dt.
\end{aligned} \quad (20)$$

in which S_t is the underlying asset value at the time t, σ_t is the stochastic volatility, ρ is the correlation factor between $W^{(1)}$, $W^{(2)}$, that are Brownian motions, and for last B_t is a zero coupon bond with borrowing interest rate r. The market price risk for S_t is given by

$$\lambda_t^{(S)}(S_t, \sigma_t, t) = \frac{r - \mu_t^{(S)}}{S_t^{\beta-1}\sigma_t}. \tag{21}$$

Now we choose the market price of volatility risk, in order to have the SABR model, as follows

$$\lambda_t^{(\sigma)}(\sigma_t, t) = \frac{r(1-\beta) - \mu_t^{(\sigma)}}{\alpha} \tag{22}$$

Under the martingale measure $\mathbb{Q}$, the forward price is assumed to obey the SDE:

$$\begin{aligned}
&dF_t^T = \sigma_t^{(F)}(F_t^T)^\beta dW_{t,(\mathbb{Q})}^{(1)}, \qquad F_t^T \in [0,\infty), t \in [0,T], \beta \in (0,1)\\
&d\sigma_t^{(F)} = \alpha\sigma_t^{(F)} dW_{t,(\mathbb{Q})}^{(2)}, \qquad \alpha \in \mathbb{R}\\
&dW_{t,(\mathbb{Q})}^{(1)} dW_{t,(\mathbb{Q})}^{(2)} = \rho dt, \qquad \rho \in (-1,1)\\
&B_t = rB_t dt\\
&f(F_T^T = S_T, \sigma_T^F, T) = \phi(S_T)
\end{aligned}$$

where F_t^T is the forward price of S_t,

$$F_t^T = e^{r(T-t)}S_t$$

and $\phi(S_T)$ is the generic pay off of contracts of some derivatives.

The pricing PDE for European derivatives in SABR model is given by:

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{1}{2}(\sigma^F)^2\left((F_t^T)^{2\beta}\frac{\partial^2 f}{\partial (F_t^T)^2} + 2\rho(F_t^T)^\beta\alpha\frac{\partial^2 f}{\partial F_t^T\partial\sigma^F} + \alpha^2\frac{\partial^2 f}{\partial(\sigma^F)^2}\right) - rf = 0;\\
&F_t^T \in [0,\infty), \qquad \sigma^F \in [0,\infty), \qquad t \in [0,T];\\
&f(T, F_{t=T}^T = S_T, \sigma_T^F) = \phi(S(T)).
\end{aligned} \tag{23}$$

Suppose being in a market equal to the one seen before, obtained for $\beta = 1$, under the martingale measure $\mathbb{Q}$, the forward price is assumed to obey the following SDE:

$$\begin{aligned}
&dF_t^T = \sigma_t^{(F)}(F_t^T)dW_{t,(\mathbb{Q})}^{(1)}, \qquad F_t^T \in [0,\infty), t \in [0,T], \beta \in (0,1)\\
&d\sigma_t^{(F)} = \alpha\sigma_t^{(F)}dW_{t,(\mathbb{Q})}^{(2)}, \qquad \alpha \in \mathbb{R}\\
&dW_{t,(\mathbb{Q})}^{(1)}dW_{t,(\mathbb{Q})}^{(2)} = \rho dt, \qquad \rho \in (-1,1)\\
&B_t = rB_t dt\\
&f(T, F_T^T, \sigma^F) = \phi(S_T)
\end{aligned} \tag{24}$$

where F_t^T is the forward price of S_t, and $\sigma_t^F = \sigma_t$

$$F_t^T = e^{r(T-t)}S_t$$

and $\phi(S_T)$ is the generic pay off of contracts of some derivatives. The price of the risk market of S_t is given by

$$\lambda_t^{(S)}(S_t, \sigma_t, t) = \frac{r - \mu_t^{(S)}}{\sigma_t}.$$

and we choose the market price of volatility risk, in order to have the SABR model, as follows

$$\lambda_t^{(\sigma)}(\sigma_t, t) = -\frac{\mu_t^{(\sigma)}}{\alpha}$$

Also in this case is possible to use our method, that we have called as Geometrical Approximation. The pricing PDE, is

$$\frac{\partial f}{\partial t} + \frac{1}{2}(\sigma)^2\left((F_t^T)^2\frac{\partial^2 f}{\partial (F_t^T)^2} + 2\rho F_t^T\alpha\frac{\partial^2 f}{\partial F_t^T\partial\sigma} + \alpha^2\frac{\partial^2 f}{\partial(\sigma)^2}\right) - rf = 0; \tag{25}$$

In order to simplify the eq. (25), we change some variables:

$$\begin{aligned}
&x = \ln F_t^T, \qquad x \in (-\infty,\infty) \quad t \in [0,T]\\
&\tilde{\sigma}_t^F = \frac{\sigma_t^F}{\alpha}, \qquad \alpha \in \mathbb{R} \quad \tilde{\sigma}_t^F \in [0,\infty);\\
&f(F_t^T, \sigma_t^F, t) = e^{-r(T-t)}f_1(x, \tilde{\sigma}_t^F, t)
\end{aligned}$$

$$\frac{\partial f_1}{\partial t} + \frac{1}{2}(\tilde{\sigma})^2\alpha^2\left(\frac{\partial^2 f}{\partial x^2} + 2\rho\frac{\partial^2 f}{\partial x\partial\tilde{\sigma}} + \frac{\partial^2 f}{\partial(\tilde{\sigma})^2}\right) - \frac{1}{2}(\tilde{\sigma})^2\alpha^2\frac{\partial f_1}{\partial x} = 0;$$

Using the same method that we have used in the previous sections, we have

$$
\begin{aligned}
&V_\xi = x - \rho\tilde{\sigma}, \qquad V_\xi \in (-\infty, +\infty) \\
&\tau = \int_t^T ds \frac{V_\eta^2 \alpha^2}{2(1-\rho^2)} = \int_t^T ds \frac{(\sigma^F)^2}{2}, \qquad \tau \in \left[0, \int_0^T ds \frac{(\sigma^F(s))^2}{2}\right]; \\
&f_1(t, x, \tilde{\sigma}, t) = f_2(\tau(t, V_\eta), V_\xi(x, \tilde{\sigma}))
\end{aligned}
$$

where we have considered the pay-off function, as we have made in the previous cases; i.e.

$$
(F_t^T e^{-\frac{\rho}{\alpha}\sigma} - E)^+ = (F_t^T e^{-\epsilon} - E)^+
$$

where $\epsilon = \rho\sigma/\alpha << 1$, for suitable values of our parameters, we have that the PDE to solve is

$$
\frac{\partial f_2}{\partial \tau} = (1-\rho^2)\frac{\partial^2 f_2}{\partial V_\xi^2} - \frac{\partial f_2}{\partial V_\xi}.
$$

Now in order to eliminate the linear term, we make the following transformation

$$
f_2(\tau, V_\xi) = e^{\frac{V_\xi}{1-\rho^2}} f_3(\tau, V_\xi),
$$

and we obtain

$$
\begin{aligned}
&\frac{\partial f_3}{\partial \tau} = (1-\rho^2)\frac{\partial^2 f_3}{\partial V_\xi^2}, \qquad V_\xi \in (-\infty, +\infty), \quad \tau \in \left[0, \int_0^T ds \frac{1}{2}(\sigma^F(s))^2\right] \\
&f_3(0, V_\xi') = e^{-\frac{V_\xi'}{1-\rho^2}} \left(e^{V_\xi'} - E\right)^+
\end{aligned}
\tag{26}
$$

Thus, the solution of the PDE (26) is given by

$$
\begin{aligned}
f(t, F_t^T, \sigma) &= \frac{e^{-r(T-t)+\frac{V_\xi}{1-\rho^2}}}{2\sqrt{\pi(1-\rho^2)\tau}} \int_{-\infty}^{+\infty} dV_\xi' e^{-\frac{V_\xi'}{1-\rho^2}} (e^{V_\xi'} - E)^+ exp\left[-\frac{(V_\xi' - V_\xi)^2}{4(1-\rho^2)\tau}\right] \\
&= \left(S_t e^{-\frac{\rho\sigma}{\alpha}}\right) e^{\left(\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2(T-t)}{2}\right)} \mathbf{N}(d_1^\rho) - E e^{-\left(r - \frac{\overline{\sigma}^2}{2(1-\rho^2)}\right)(T-t)} \mathbf{N}(d_2^\rho)
\end{aligned}
$$

for $\epsilon = \frac{\rho\sigma}{\alpha} << 1$ we can write

$$f(t, F_t^T, \sigma) \simeq S_t\,(1-\epsilon)\, e^{\left(\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2(T-t)}{2}\right)} \mathbf{N}(d_1^\rho) - Ee^{-\left(r-\frac{\overline{\sigma}^2}{2(1-\rho^2)}\right)(T-t)} \mathbf{N}(d_2^\rho)$$

Again, we can write that the price of a Call option, in a SABR market model, for $\beta = 1$ and $\epsilon = \frac{\rho\sigma}{\alpha} << 1$, is given by

$$C(t, S_t, \sigma_t) = S_t\,(1-\epsilon)\, e^{\left(\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2(T-t)}{2}\right)} \mathbf{N}(d_1^\rho) - Ee^{-\left(r-\frac{\overline{\sigma}^2}{2(1-\rho^2)}\right)(T-t)} \mathbf{N}(d_2^\rho)$$

where

$$d_1^\rho = \frac{\ln\left(Se^{-\frac{\rho\sigma}{\alpha}}/E\right) + (r - \rho^2\overline{\sigma}^2)(T-t)}{\sqrt{(1-\rho^2)\overline{\sigma}^2(T-t)}}$$

$$d_2^\rho = d_1^\rho - \sqrt{(1-\rho^2)\overline{\sigma}^2(T-t)}$$

$$\overline{\sigma} = \frac{1}{T-t}\int_t^T ds\sigma^2(s)$$

For a Put option we have:

$$P(t, S_t, \sigma_t) = Ee^{-\left(r-\frac{\overline{\sigma}^2}{2(1-\rho^2)}\right)(T-t)} \mathbf{N}(d_2^\rho) - S_t\,(1-\epsilon)\, e^{\left(\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2(T-t)}{2}\right)} \mathbf{N}(d_1^\rho)$$

5.2 Hedging and Put-Call-Parity

Exactly like in the Heston's model, also in the SABR model, in order to find the better hedging strategy, we use a replicant portfolio. So that we need to know the value of the first and second derivative of the price, with respect to S, that we respectively call Δ and

Γ strategies:

$$\Delta_{call} = \frac{\partial C(t,s,\sigma)}{\partial S} = (1-\epsilon)\, e^{\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2}{2}(T-t)}\mathbf{N}(d_1^\rho)$$

$$\Gamma_{call} = \frac{Ee^{\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2}{2}(T-t)-\frac{\left(d_1^\rho\right)^2}{2}}}{S\sqrt{2\pi\overline{\sigma}^2(T-t)}} \tag{27}$$

and

$$\Delta_{put} = -\,(1-\epsilon)\, e^{\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2}{2}(T-t)}\mathbf{N}(-d_1^\rho)$$

$$\Gamma_{put} = \frac{Ee^{\frac{\rho^4}{1-\rho^2}\frac{\overline{\sigma}^2}{2}(T-t)-\frac{\left(d_1^\rho\right)^2}{2}}}{S\sqrt{2\pi\overline{\sigma}^2(T-t)}} \tag{28}$$

Thus we have

$$\Gamma_{put} = \Gamma_{call} \tag{29}$$

Also in the SABR model, the Put-Call-Parity condition is verified, and this proves that we are in a free arbitrage market.

6 Numerical Experiments

Now, we can compare options prices calculated according to techniques described above, with our approximation method. The Monte-Carlo algorithm was implemented in $C++$ code, while other algorithms are implemented in $MatLab$ code. For $\rho = 0$, we obtain the Black-Scholes solution with averaged volatility, as in Hull-White formula. This proves that our approach, even if only an approximation, is correct.

For values of $\rho = -1, +1$ we have two degenerate cases, and they are not interesting. In order to have an idea of the derivatives price, we compute Vanilla Call Option value in Black-Scholes market model; and after that, one can see the price of Vanilla Call Option

Table 1: Black-Scholes price $S(0) = 100$, $E = 100$

σ_t	r	T	Value
0.1	0.03	0.5	3.6065
0.1	0.05	0.5	4.1923
0.3	0.03	1	13.2833
0.5	0.05	1	21.7926
0.5	0.05	5	49.5965

for Heston market model. Here, we have compared our method, G.A., with others obtained by Heston and Lipton, Fourier transform method, and by finite difference method, f.d.m.(Crank Nicolson). Our results are suitable, and this proves in analytical way, the goodness of method proposed. It is interesting that our prices go to heston prices, by increasing maturity date T, unlike that for f.d. method. We compare also our results with those obtained by Monte Carlo method, for different values of parameters.

Table 2: Heston price $S(0) = 100$, $E = 100$, $Err = \|(Heston)_{value} - (G.A.)_{value}\|$

r	ρ	κ	α	Θ	ν_t	T	G.A. Value	H. Value	f.d.m. Value	Err
0.03	0.1	1.0	0.2	0.01	0.01	0.5	3.2992	3.4386	3.4376	0.139
0.03	0.1	1.0	0.2	0.01	0.01	1	5.2461	5.2953	5.2840	0.049
0.03	0.1	1.0	0.2	0.01	0.01	2	8.4954	8.4583	8.5943	0.037
0.05	0.1	1.0	0.2	0.01	0.01	1	6.4339	6.5025	6.5223	0.0686
0.05	0.1	1.0	0.2	0.01	0.01	2	10.9954	11.0196	11.2186	0.0242
0.03	0.4	1.0	0.6	0.01	0.01	2	7.4459	7.3439	7.7829	0.102

Table 3: Heston price $S(0) = 100$, $E = 50$ $Err = \|(Heston)_{value} - (G.A.)_{value}\|$

r	ρ	κ	α	Θ	ν_t	T	G.A. Value	H. Value	f.d.m. Value	Err
0.03	0.1	1.0	0.7	0.04	0.01	0.5	50.7421	50.7341	50.8215	0.08
0.03	0.2	1.0	0.5	0.0225	0.01	0.5	50.1853	50.7336	50.7756	0.548
0.03	0.1	1.0	0.5	0.0225	0.01	1	50.7597	51.4585	51.8893	0.698
0.05	0.1	1.0	0.5	0.0225	0.01	2	53.7232	54.6672	55.9912	0.994
0.03	0.1	1.0	0.5	0.0225	0.01	0.5	50.6919	50.7352	51.0340	0.043
0.03	0.1	1.0	0.5	0.0225	0.01	1	51.5730	51.5830	56.3770	0.01

Table 4: Heston price for a Call with $S(0) = 100$, $E = 100$, $Err = \|(M.C._{value} - (G.A.)_{value}\|$ for Monte Carlo method we used day pass $(1/250)$ and 10^6 trajectories

r	ρ	κ	α	Θ	ν_t	T	G.A. Value	M.C. Value	S.S.E.	Err
0.03	0.1	1.0	0.2	0.01	0.01	0.5	3.2992	3.4591	0.0022	0.1599
0.03	0.1	1.0	0.2	0.01	0.01	1	5.2461	5.3417	0.0031	0.0956
0.03	0.1	1.0	0.2	0.01	0.01	2	8.4954	8.5857	0.0042	0.0903
0.05	0.1	1.0	0.2	0.01	0.01	5	22.9333	23.4234	0.0039	0.4901

Generally the SABR model is used as market model for derivatives whose underlying is the interest rate, but here we have used the SABR model, to evaluate European Call and Put options, upon an asset using its forward price.

As our tables show, we can be satisfied. The Geometrical Approximation method does work when the following condition is verified:

$$\left(S_T e^{-\frac{\rho \nu_T}{\alpha}} - E\right)^+ \simeq \left(S_T - E\right),^+ \qquad t \in [0, T]; \tag{30}$$

where

$$\|1 - e^{-\frac{\rho \nu_T}{\alpha}}\| \sim \|10^{-2}\|. \tag{31}$$

So that, before using the G.A. method is necessary to estimate the value of volatility, or variance, at maturity date T.

Table 5: Call Options value in SABR market model for $\beta = 1$, $F_t^T = 100$ (where F_t^T is forward price), $E = 100$, $Err = \|\epsilon\| = \|\frac{\rho\sigma_T}{\alpha}\|$

r	ρ	α	σ_0	T	G.A.Value	Err
0.03	0.1	0.2	0.1	1	4.0565	0.05
0.03	-0.1	0.2	0.	1	10.4849	0.05
0.05	0.15	1	0.3	1	4.6426	0.09
0.03	-0.3	1	0.5	1	26.3879	0.15
0.05	-0.3	10	0.7	1	24.0808	0.021

Table 6: Call Options value in SABR market model for $\beta = 1$, $F_t^T = 100$ (where F_t^T is forward price), $E = 50$, $Err = \|\epsilon\| = \|\frac{\rho\sigma_T}{\alpha}\|$

r	ρ	α	σ_0	T	G.A.Value	Err
0.03	0.1	0.2	0.1	1	48.6796	0.05
0.03	-0.1	0.2	0.1	1	59.1993	0.05
0.05	0.15	1	0.3	1	49.1608	0.075
0.03	-0.3	1	0.5	1	63.7801	0.15
0.05	-0.3	10	0.7	1	50.0696	0.021

7 Conclusions

The G.A. method is a good technique, for suitable values of the parameters α, ρ, K, θ; and it is less expensive than the other numerical methods F.F.T(inverse Fourier transform), Monte-Carlo and F.D.M. The proposed method can be used for every market model in which the associated PDE has the second derivative term, by Ito's lemma, of the form:

$$\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x \partial \tilde{\nu}} + \frac{\partial^2 f_1}{\partial \tilde{\nu}^2} = (1-\rho^2)\nabla^2_{V_\xi} f_2(t, V_\xi(x,\tilde{\nu})), \tag{32}$$

we can call the latter condition as necessary condition. We want to remark that our idea is to approximate a closed form solution obtained by using a different Cauchy's condition, to that obtained by above indicated numerical methods, in this case using the correct Cauchy's condition. The proposed method has the advantage to compute a solution in closed form, therefore, we do not have the problems that there are using numerical methods. For example, one can consider the inverse Fourier transform method, in which we have to compute an integral between zero and infinity. In this case in fact, there is always some problem in order to define the correct domain of integration; or equivalently, considering also the finite difference method, in which we have to define a suitable grid, in other words we have some problems about the choice of the grid's meshes. Thus we can conclude that our method is easier, from the algorithmic point of view.

Another important aspect of our method is to compute in an explicit way the greeks (Δ, Γ). This is very interesting when we want to use the **VaR** technique in **Risk Management**. In fact we need to know the values of (Δ, Γ) if our portfolio is composed even by derivative securities. In this case we have to know the sensibility of first and second order with respect to underlying asset, to evaluate how difference our distribution is compared to Normal-distribution of yields, and by using the proposed method we are able to accomplish this.

The G.A. method can be used to price derivative as Digital options, options in American style and some Asian options. Besides we can extend G.A. technique also when we add jump processes in our market model. Therefore if the necessary condition (32) is verified, we can say that our methodology is a general technique.

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