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Pivato, Marcus and Nehring, Klaus

Department of Mathematics, Trent University, Department of Economics, University of California, Davis

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The McGarvey problem in judgement aggregation

Klaus Nehring[a], Marcus Pivato[b,*]

[a] *Department of Economics, University of California at Davis, U.S.A.*
[b] *Department of Mathematics, Trent University, Canada.*

Abstract

Judgement aggregation is a model of social choice where the space of social alternatives is the set of consistent truth-valuations ('judgements') on a family of logically interconnected propositions. It is well-known that propositionwise majority voting can yield logically inconsistent judgements. We show that, for a variety of spaces, propositionwise majority voting can yield *any* possible judgement. By considering the geometry of sub-polytopes of the Hamming cube, we also estimate the number of voters required to achieve all possible judgements. These results generalize the classic results of McGarvey (1953) and Stearns (1959).

Keywords: judgement aggregation; majority voting; McGarvey; Stearns; 0/1 polytope;

Let $\mathcal{K}$ be a finite set of propositions or 'properties'. An element $\mathbf{x} = (x_k)_{k\in\mathcal{K}} \in \{\pm1\}^{\mathcal{K}}$ is called a *judgement*, and interpreted as an assignment of a truth value of 'true' (+1) or 'false' (-1) to each proposition. Not all judgements are feasible, because there will be logical constraints between the propositions (determined by the structure of the underlying decision problem faced by the voters). Let $\mathcal{X} \subseteq \{\pm1\}^{\mathcal{K}}$ be the set of 'admissible' judgements —we refer to $\mathcal{X}$ as a *property space*. An *anonymous profile* is a probability measure on $\mathcal{X}$ —that is, a function $\mu : \mathcal{X} \longrightarrow [0,1]$ such that $\sum_{\mathbf{x}\in\mathcal{X}} \mu(\mathbf{x}) = 1$. (Interpretation: for all $\mathbf{x} \in \mathcal{X}$, $\mu(\mathbf{x})$ is the proportion of the voters who hold the judgement $\mathbf{x}$). *Judgement aggregation* is the problem of converting the profile $\mu \in \Delta^*(\mathcal{X})$ into the element $\mathbf{x} \in \mathcal{X}$ which best represents the 'collective will' of the voters. This problem (with different terminology) was originally posed by Guilbaud [1], and later investigated by Wilson [2], Rubinstein and Fishburn [3], and Barthelémy and Janowitz [4]. Since the work of List and Pettit [5], there has been an explosion of interest in this area; see List and Puppe [6] for a recent survey of judgement aggregation research.

For example, let $\mathcal{A}$ be a finite set of social alternatives. A *tournament* on $\mathcal{A}$ is a complete antisymmetric relation "$\prec$" over $\mathcal{A}$. A *preference order* is a transitive tournament (i.e. a linear ordering) on $\mathcal{A}$. Let $\mathcal{K} \subset \mathcal{A} \times \mathcal{A}$ contain exactly one of the pairs (a, b) or (b, a) for each distinct $a, b \in \mathcal{A}$. Any $\mathbf{x} \in \{\pm1\}^{\mathcal{K}}$ represents a tournament "$\prec$", where $a \prec b$ iff $x_{a,b} = 1$. Every tournament on $\mathcal{A}$ corresponds to a unique element of $\{\pm1\}^{\mathcal{K}}$. Let $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$ denote the subset of all elements of $\{\pm1\}^{\mathcal{K}}$ which correspond to preference orders. Thus, a profile $\mu \in \Delta^*(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}})$ represents a group of voters who each assert some preference order over $\mathcal{A}$. In this case, the goal of judgement aggregation is to distill μ into some 'collective' preference order on $\mathcal{A}$ —this is the familiar Arrovian model of preference aggregation.

Let $\Delta(\mathcal{X})$ be the set of all anonymous profiles. *Propositionwise majority vote* is defined as follows. For any $\mu \in \Delta(\mathcal{X})$, any $k \in \mathcal{K}$, let

$$\widetilde{\mu}_k \quad := \quad \sum_{\mathbf{x}\in\mathcal{X}} \mu(\mathbf{x}) \cdot x_k \tag{1}$$

be the μ-expected value of coordinate x_k. Thus, $\widetilde{\mu}_k > 0$ if and only if a strict majority of voters assert '$x_k = 1$'; whereas $\widetilde{\mu}_k < 0$ if and only if a strict majority of voters assert '$x_k = -1$'. Let $\Delta^*(\mathcal{X}) := \{\mu \in \Delta(\mathcal{X});$

*Corresponding address: Department of Mathematics, Trent University, 1600 West Bank Drive, Peterborough, Ontario, K9J 7B8, Canada.

Email addresses: kdnehring@ucdavis.edu (Klaus Nehring), marcuspivato@trentu.ca (Marcus Pivato)

$\widetilde{\mu}_k \neq 0$, $\forall k \in \mathcal{K}\}$ be the set of anonymous profiles where there is a strict majority supporting either +1 or -1 in each coordinate.[1] For any $\mu \in \Delta^*(\mathcal{X})$, define $\mathrm{maj}(\mu) \in \{\pm1\}^{\mathcal{K}}$ as follows:

$$\text{for all } k \in \mathcal{K}, \qquad \mathrm{maj}_k(\mu) \quad := \quad \begin{cases} 1 & \text{if} \quad \widetilde{\mu}_k > 0; \\ -1 & \text{if} \quad \widetilde{\mu}_k < 0. \end{cases} \tag{2}$$

Unfortunately, it is quite common to find that $\mathrm{maj}(\mu) \notin \mathcal{X}$ —the 'majority will' can be inconsistent with the underlying logical constraints faced by the voters. (In the case of aggregation over $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$, this problem was first observed by Condorcet [7].) Let $\mathrm{maj}(\mathcal{X}) := \{\mathrm{maj}(\mu)\ ;\ \mu \in \Delta^*(\mathcal{X})\}$; this describes the set of all majoritarian voting patterns that can result from some possible profile of judgements. Following McGarvey [8], we think of $\mathrm{maj}(\mathcal{X}) \setminus \mathcal{X}$ as the range of possible 'voting paradoxes' which can occur under propositionwise majority vote.

Clearly $\mathcal{X} \subseteq \mathrm{maj}(\mathcal{X})$. We say that $\mathcal{X}$ is *majority consistent* if $\mathrm{maj}(\mathcal{X}) = \mathcal{X}$. This occurs only when $\mathcal{X}$ satisfies a strong combinatorial/geometric condition, as we new explain. For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}$, we define $\mathrm{med}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathrm{maj}(\mu)$, where $\mu \in \Delta^*(\mathcal{X})$ is defined by $\mu(\mathbf{x}_j) = \frac{1}{3}$ for $j = 1, 2, 3$; this defines a ternary operator on $\{\pm1\}^{\mathcal{K}}$, called the *median operator*. Let $\mathrm{med}^1(\mathcal{X}) := \{\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z})\ ;\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}\}$. For all $n \in \mathbb{N}$, we inductively define $\mathrm{med}^{n+1}(\mathcal{X}) := \{\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z});\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{med}^n(\mathcal{X})\}$. This yields an ascending chain $\mathcal{X} \subseteq \mathrm{med}^1(\mathcal{X}) \subseteq \mathrm{med}^2(\mathcal{X}) \subseteq \cdots$. Let $\mathrm{med}^\infty(\mathcal{X}) := \bigcup_{n=1}^{\infty} \mathrm{med}^n(\mathcal{X})$ be the *median closure* of $\mathcal{X}$. We say that $\mathcal{X}$ is a *median space* if $\mathrm{med}^1(\mathcal{X}) = \mathcal{X}$ (equivalently: $\mathrm{med}^\infty(\mathcal{X}) = \mathcal{X}$). At the opposite extreme, was say $\mathcal{X}$ is *median-saturating* if $\mathrm{med}^\infty(\mathcal{X}) = \{\pm1\}^{\mathcal{K}}$. For any $\mathcal{X} \subseteq \{\pm1\}^{\mathcal{K}}$, we have:

$$\mathcal{X} \quad \subseteq \quad \mathrm{med}^1(\mathcal{X}) \quad \subseteq \quad \mathrm{maj}(\mathcal{X}) \quad \subseteq \quad \mathrm{med}^\infty(\mathcal{X}). \tag{3}$$

The first two inclusions are obvious by definition. The last inclusion is due to Nehring and Puppe [9]; see also [10].[2] It follows that $\mathcal{X}$ is majority consistent if and only if $\mathcal{X}$ is a median space. If $\mathcal{X}$ is *not* a median space, then eqn.(3) is is useful because it is relatively easy to compute $\mathrm{med}^\infty(\mathcal{X})$, as we now explain.

Let $\mathcal{J} \subseteq \mathcal{K}$ and let $\mathbf{w} \in \{\pm1\}^{\mathcal{J}}$; we say that $\mathbf{w}$ is a *word*, and call $\mathcal{J}$ the *support* of $\mathbf{w}$, denoted $\mathrm{supp}(\mathbf{w})$. If $\mathcal{I} \subseteq \mathcal{J}$ and $\mathbf{v} \in \{\pm1\}^{\mathcal{I}}$, then we write $\mathbf{v} \sqsubseteq \mathbf{w}$ if $v_i = w_i$ for all $i \in \mathcal{I}$. We define $|\mathbf{w}| := |\mathcal{J}|$. We say $\mathbf{w}$ is an *$\mathcal{X}$-forbidden word* if, for all $\mathbf{x} \in \mathcal{X}$, we have $\mathbf{w} \not\sqsubseteq \mathbf{x}$. Let $\mathcal{W}_2(\mathcal{X})$ be the set all $\mathcal{X}$-forbidden words of length 2. We have:

Proposition 1. *Let $\mathcal{X} \subseteq \{\pm1\}^{\mathcal{K}}$.*

- **(a)** $\mathrm{med}^\infty(\mathcal{X}) := \{\mathbf{x} \in \{\pm1\}^{\mathcal{K}}\ ;\ \mathbf{w} \not\sqsubseteq \mathbf{x},\ \forall\, \mathbf{w} \in \mathcal{W}_2(\mathcal{X})\}$.
- **(b)** *In particular, $\mathcal{X}$ is median-saturating if and only if $\mathcal{W}_2(\mathcal{X}) = \emptyset$.*

(The proof of this and all other results are in Appendix A at the end of the paper.)

Example 2. Let $\mathcal{N}$ be a set and let $\mathcal{K} := \{(n, m) \in \mathcal{N} \times \mathcal{N}\ ;\ n \neq m\}$; then any $\mathbf{x} \in \{\pm1\}^{\mathcal{K}}$ represents a binary relation "$\preceq$" on $\mathcal{N}$ such that $n \preceq m$ if and only if $x_{n,m} = 1$. Let $\mathcal{X} \subset \{\pm1\}^{\mathcal{K}}$ be any space of *complete* binary relations. Then $\mathcal{W}_2(\mathcal{X}) \neq \emptyset$, because for any $\mathbf{x} \in \mathcal{X}^*$ and $(n, m) \in \mathcal{K}$, we cannot have both $x_{n,m} = -1$ and $x_{m,n} = -1$ (by completeness). Thus, $\mathrm{med}^\infty(\mathcal{X}) \neq \{\pm1\}^{\mathcal{K}}$. $\diamondsuit$

Given a property space $\mathcal{X} \subseteq \{\pm1\}^{\mathcal{K}}$, Proposition 1 and eqn.(3) raise the question: is

$$\mathrm{maj}(\mathcal{X}) \quad = \quad \mathrm{med}^\infty(\mathcal{X})? \tag{4}$$

[1] Usually, judgement aggregation is considered on *all* of $\Delta(\mathcal{X})$. However, we will confine our attention to profiles in $\Delta^*(\mathcal{X})$ for expositional simplicity. (If the set of voters is large (respectively odd), then a profile in $\Delta(\mathcal{X}) \setminus \Delta^*(\mathcal{X})$ is highly unlikely (respectively impossible) anyways.)

[2] The close relationship between the median operator and majoritarian consensus on median graphs and median lattices had earlier been explored by [1, 4, 11] and others.

Clearly, if $\mathcal{X}$ is a median space, then eqn.(3) implies that $\mathrm{maj}(\mathcal{X}) = \mathrm{med}^\infty(\mathcal{X})$. At the opposite extreme, McGarvey [8] showed that $\mathrm{maj}(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) = \{\pm1\}^{\mathcal{K}}$; this automatically implies that $\mathrm{maj}(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) = \mathrm{med}^\infty(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}})$.

Question (4) appears to be difficult to answer in full generality. We will thus focus on the special case when equation (4) holds and $\mathcal{X}$ is median-saturating —in other words, when $\mathrm{maj}(\mathcal{X}) = \{\pm1\}^{\mathcal{K}}$. In this case, we say that $\mathcal{X}$ is *McGarvey*.

If $\mathcal{X}$ is McGarvey, then every conceivable 'voting paradox' can be obtained through propositionwise majority voting on $\mathcal{X}$. The McGarvey property is also useful in establishing other results about $\mathcal{X}$. For example, Nehring, Pivato and Puppe [12] consider judgement aggregation rules based on 'Condorcet efficiency' (a generalization of the 'Condorcet principle' of preference aggregation). The McGarvey property of certain property spaces implies that Condorcet efficient judgement aggregation can be quite indeterminate on those spaces. The central question of this paper is: *What property spaces are McGarvey?*

Let $\mathrm{conv}(\mathcal{X})$ denote the convex hull of $\mathcal{X}$ (seen as a subset of $\mathbb{R}^{\mathcal{K}}$), and let $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$ denote its topological interior. Let $\mathbf{0} := (0, 0, \ldots, 0) \in \mathbb{R}^{\mathcal{K}}$. For any $\mathbf{x} \in \{\pm1\}^{\mathcal{K}}$, the *open orthant* of $\mathbf{x}$ is the open set $\mathcal{O}_{\mathbf{x}} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}}\ ;\ \mathrm{sign}(r_k) = x_k,\ \forall\ k \in \mathcal{K}\}$. Most of the results in this paper are based on the following key result:

Theorem 3. *Let $\mathcal{X} \subset \{\pm1\}^{\mathcal{K}}$. Then*

(a) $\mathrm{maj}(\mathcal{X}) := \{\mathbf{x} \in \{\pm1\}^{\mathcal{K}}\ ;\ \mathcal{O}_{\mathbf{x}} \cap \mathrm{conv}(\mathcal{X}) \neq \emptyset\}$.

(b) *The following are equivalent:* (1) *$\mathcal{X}$ is McGarvey;* (2) $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$*;* (3) *For every nonzero $\mathbf{z} \in \mathbb{R}^{\mathcal{K}}$, there exists $\mathbf{x} \in \mathcal{X}$ with $\mathbf{z} \bullet \mathbf{x} > 0$.*

The rest of this paper is organized as follows. In §1, we ask how small $\mathcal{X}$ can be while still being McGarvey, or how large it can be without being McGarvey. In §2, we characterize the McGarvey property for judgement aggregation spaces with many symmetries; this includes spaces of preference relations, equivalence relations, and connected graphs. In Sections 3, 4 and 5 we consider the McGarvey problem for comprehensive spaces, truth-functional aggregation spaces, and convexity spaces, respectively. Finally, in §6, we consider a problem originally studied by Stearns [13]: how *many* voters are required to realize the McGarvey property of a space $\mathcal{X}$? We show that several important families of aggregation spaces only require around $2K$ voters. However, using a result of Alon and Vũ [14], we also show that the required number of voters can be extremely large for some McGarvey spaces.

1. Minimal McGarvey spaces and maximal non-McGarvey spaces

If $\mathcal{X} \subseteq \mathcal{Y} \subseteq \{\pm1\}^{\mathcal{K}}$, and $\mathcal{X}$ is McGarvey, then clearly $\mathcal{Y}$ is also McGarvey. We say that $\mathcal{X}$ is *minimal McGarvey* if $\mathcal{X}$ is McGarvey, but no proper subset of $\mathcal{X}$ is McGarvey. For the next result and the rest of the paper, we define $K := |\mathcal{K}|$.

Proposition 4. **(a)** *Suppose $K \geq 3$. Then* $\min\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm1\}^{\mathcal{K}}$ *is McGarvey*$\} = K + 1$.

(b) $\max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm1\}^{\mathcal{K}}$ *is minimal McGarvey*$\} = 2K$.

Example 5. Suppose $K \geq 3$. For all $j \in \mathcal{K}$, define $\boldsymbol{\chi}^j \in \{\pm1\}^{\mathcal{K}}$ by $\chi^j_j := 1$, while $\chi^j_k := -1$ for all $k \in \mathcal{K} \setminus \{j\}$. Define $\mathcal{X} := \{\pm\boldsymbol{\chi}^j\}_{j\in\mathcal{K}}$. Then $|\mathcal{X}| = 2K$. In Appendix A, we show that $\mathcal{X}$ is a minimal McGarvey space. In particular, if $K = 3$, then $\mathcal{X} = \{(1,1,-1),\ (1,-1,1),\ (-1,1,1),\ (-1,-1,1),\ (-1,1,-1),\ (1,-1,-1)\}$ is a minimal McGarvey set with six elements. Let $\mathcal{A} := \{a, b, c\}$ and identify $\mathcal{K}$ with the set $\{(a,b),(b,c),(c,a)\}$; then $\mathcal{X} = \mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$.

(Another class of minimal McGarvey spaces is described in Appendix B.) ◇

By comparison, Carathéodory's theorem says that if $\mathcal{Y} \subset \{\pm1\}^{\mathcal{K}}$ is a minimal set with $\mathbf{0} \in \mathrm{conv}(\mathcal{Y})$, then $2 \leq |\mathcal{Y}| \leq K + 1$. The requirement that $\mathbf{0}$ be in the *interior* of $\mathrm{conv}(\mathcal{Y})$ instead entails $K + 1 \leq |\mathcal{Y}| \leq 2K$; this shows that the interiority condition is quite substantive.

Proposition 6. **(a)** $\max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *is* not *McGarvey*$\}\ =\ \frac{3}{4}2^K$.

(b) $\max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *is* not *median-saturating*$\}\ =\ \frac{3}{4}2^K$.

Example 7. Let $\mathcal{K} = \{1, 2, \ldots, K\}$ and let $\mathcal{X} := \left\{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}\ ;\ (x_1, x_2) \neq (0, 0)\right\}$. Then $\mathcal{X}$ is a median space (hence, neither McGarvey nor median-saturating) but $|\mathcal{X}| = \frac{3}{4}2^K$. ♢

Propositions 4 and 6 show that the McGarvey property places only very loose constraints on the cardinality of $\mathcal{X}$. Much more important is how 'dispersed' $\mathcal{X}$ is as a subset of $\{\pm 1\}^{\mathcal{K}}$.

2. Symmetric property spaces

For any $\mathcal{X} \subset \mathbb{R}^{\mathcal{K}}$, the *symmetry group* of $\mathcal{X}$ is the set $\Gamma_{\mathcal{X}}$ of all invertible linear transformations $\gamma : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R}^{\mathcal{K}}$ such that $\gamma(\mathcal{X}) = \mathcal{X}$. Let $\mathrm{Fix}\,(\Gamma_{\mathcal{X}}) := \left\{\mathbf{r} \in \mathbb{R}^{\mathcal{K}}\ ;\ \gamma(\mathbf{r}) = \mathbf{r},\ \forall\ \gamma \in \Gamma\right\}$. For example, $\mathbf{0} \in \mathrm{Fix}\,(\Gamma_{\mathcal{X}})$, because $\gamma(\mathbf{0}) = \mathbf{0}$ for any linear transformation $\gamma : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R}^{\mathcal{K}}$.

Proposition 8. *Let* $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *and suppose* $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$.

(a) *If* $\mathrm{Fix}\,(\Gamma_{\mathcal{X}}) = \{\mathbf{0}\}$, *then* $\mathcal{X}$ *is McGarvey.*

(b) *In particular, if* $-\mathcal{X} = \mathcal{X}$, *then* $\mathcal{X}$ *is McGarvey.*

Clearly, $\mathcal{X}$ cannot be McGarvey unless $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$ —or equivalently, unless the set $(\mathcal{X} - \mathcal{X}) := \{\mathbf{x} - \mathbf{y};\ \mathbf{x}, \mathbf{y} \in \mathcal{X}\}$ spans $\mathbb{R}^{\mathcal{K}}$. One advantage of Proposition 11 over Theorem 3(b) is that it is generally easier to verify that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$ than it is to verify that $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$. For instance, the next result is often sufficient.

Lemma 9. *Let* $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$. *Suppose that, for every* $j \in \mathcal{K}$, *there exist* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *such that* $x_j \neq y_j$, *but* $x_k = y_k$ *for all* $k \in \mathcal{K} \setminus \{j\}$. *Then* $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$.

Example 10. (*Preference aggregation*) Let $\mathcal{A}$ be a set, and let $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ be the space of preference orders on $\mathcal{A}$, as discussed in the introduction. For any $(a, b) \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{pr}}$ such that $x_{a,b} \neq y_{a,b}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. (For example: let $\mathbf{x}$ represent an ordering of the form $a \prec b \prec c_3 \prec c_4 \prec \cdots \prec c_N$, and let $\mathbf{y}$ represent the ordering $b \prec a \prec c_3 \prec c_4 \prec \cdots \prec c_N$.) Thus, Lemma 9 implies that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}})] \neq \emptyset$.

Clearly, $-\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}} = \mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ (if $\mathbf{x}$ represents the ordering $a_1 \prec a_2 \prec \cdots \prec a_N$, then $-\mathbf{x}$ represents the ordering $a_1 \succ a_2 \succ \cdots \succ a_N$). Thus, Proposition 8(b) implies McGarvey's original result that $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is McGarvey. ♢

Let $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^{\mathcal{K}}$, and let $\mathbb{R}\mathbf{1} \subset \mathbb{R}^{\mathcal{K}}$ be the linear subspace it generates.

Proposition 11. *Let* $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *and suppose* $\mathrm{Fix}\,(\Gamma_{\mathcal{X}}) \subseteq \mathbb{R}\mathbf{1}$. *Then* $\mathcal{X}$ *is McGarvey if and only if* $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$ *and there exist* $r < 0 < t \in \mathbb{R}$ *such that* $r\mathbf{1}, t\mathbf{1} \in \mathrm{conv}(\mathcal{X})$.

A *coordinate permutation* of $\mathbb{R}^{\mathcal{K}}$ is a linear map $\gamma : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R}^{\mathcal{K}}$ which maps any vector $(r_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$ to the vector $(r_{\pi(k)})_{k \in \mathcal{K}}$, for some fixed permutation $\pi : \mathcal{K} \longrightarrow \mathcal{K}$. The set of all coordinate permutations in $\Gamma_{\mathcal{X}}$ forms a subgroup, which is isomorphic to a group $\Pi_{\mathcal{X}}$ of permutations on $\mathcal{K}$ in the obvious fashion. We say that $\Pi_{\mathcal{X}}$ is *transitive* if, for any $j, k \in \mathcal{K}$, there is some $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(j) = k$. For any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, let $\#(\mathbf{x}) := \#\{k \in \mathcal{K}\ ;\ x_k = 1\}$.

Corollary 12. *Let* $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *and suppose* $\Pi_{\mathcal{X}}$ *is transitive. Then* $\mathcal{X}$ *is McGarvey if and only if* $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$ *and there exist* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *with* $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$.

Example 13. (*Symmetric binary relations*) Let $\mathcal{N}$ be a set, and let $\mathcal{K}$ be the set of all subsets $\{n,m\} \subseteq \mathcal{N}$ containing exactly two elements. Interpret each element of $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ as encoding a symmetric, reflexive binary relation "$\sim$" (i.e. for any $\{n,m\} \in \mathcal{K}$, we have $n \sim m$ if $x_{n,m} = 1$ and $n \not\sim m$ if $x_{n,m} = -1$). For any permutation $\pi : \mathcal{N} \longrightarrow \mathcal{N}$, define $\pi_* : \mathcal{K} \longrightarrow \mathcal{K}$ by $\pi\{n,m\} := \{\pi(n), \pi(m)\}$ for all $\{n,m\} \in \mathcal{K}$. Let Π_* be the set of all such permutations; then Π_* acts transitively on $\mathcal{K}$ (for any $\{n_1, m_1\} \in \mathcal{K}$ and $\{n_2, m_2\} \in \mathcal{K}$, let $\pi : \mathcal{N} \longrightarrow \mathcal{N}$ be any permutation such that $\pi(n_1) = n_2$ and $\pi(m_1) = m_2$; then $\pi_*\{n_1, m_1\} = \{n_2, m_2\}$).

(a) (*Equivalence relations*) Let $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of equivalence relations. Then $\Pi_{\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}}$ is transitive because it contains Π_*.

For any $\{n,m\} \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ such that $x_{n,m} \neq y_{n,m}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. (For example: let $\mathbf{x}$ represent an equivalence relation where n and m are both in singleton equivalence classes, and let $\mathbf{y}$ represent the relation obtained from $\mathbf{x}$ by joining n and m together into one doubleton equivalence class). Thus, Lemma 9 implies that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}})] \neq \emptyset$.

Note that $\pm\mathbf{1} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ ($\mathbf{1}$ represents the 'complete' relation "$\sim$" such that $n \sim m$ for all $n, m \in \mathcal{N}$, whereas $-\mathbf{1}$ represents the 'trivial' relation such that $n \not\sim m$ for any $n \neq m \in \mathcal{N}$). Thus, Corollary 12 implies that $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ is McGarvey.

This result (and Example 10) do not really require Corollary 12; in fact, we can obtain more refined results about $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ and $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ by using special structural properties of these spaces which have nothing to do with symmetry *per se* (see Example 24 below). However, the next four examples do make essential use of symmetry.

(b) (*Restricted Equivalence Relations*) For any $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$, let $\mathrm{rank}(\mathbf{x})$ be the number of distinct equivalence classes of the relation defined by $\mathbf{x}$. Let $1 \leq r \leq R \leq N$ and let $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r,R)$ be the set of all $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ with $r \leq \mathrm{rank}(\mathbf{x}) \leq R$. If $R = 1$, then also $r = 1$, and clearly $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(1,1) = \{\mathbf{1}\}$. So assume $R \geq 2$. Clearly $\Pi_{\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r,R)} = \Pi_{\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}} \supseteq \Pi_*$, so it is transitive. Through a very similar argument to example (a), one can show $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r,R))] \neq \emptyset$. Thus, we can apply Corollary 12. Define

$$\overline{r}(N) \quad := \quad N + 1 \; - \; \frac{1 + \sqrt{2N^2 - 2N + 1}}{2}.$$

(Thus, if N is large, then $\overline{r}(N) \approx N - N/\sqrt{2}$.) In Appendix A, we show that $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r,R)$ is McGarvey if and only if $r \; < \; \overline{r}(N)$.

(c) (*Connected graphs*) We can also interpret any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ as encoding a ***graph***. Let $\mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of all elements of $\{\pm 1\}^{\mathcal{K}}$ representing connected graphs on $\mathcal{N}$. Then $\Pi_{\mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}}$ is transitive because it contains Π_*.

For any $\{n,m\} \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}$ such that $x_{n,m} \neq y_{n,m}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. (For example: let $\mathbf{x}$ represent a connected graph where vertices n and m are *not* linked. Let $\mathbf{y}$ represent the graph obtained from $\mathbf{x}$ by adding a link from n to m). Thus, Lemma 9 implies that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}})] \neq \emptyset$.

There exists $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}$ with $\#(\mathbf{x}) < K/2$ (for example, let $\mathbf{x}$ represent a graph where the elements of $\mathcal{N}$ are arranged in a loop —then $\#(\mathbf{x}) = |\mathcal{N}| < K/2$). There also exists $\mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}$ with $\#(\mathbf{y}) > K/2$ (for example: $\mathbf{1} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}$). Thus, Corollary 12 says that $\mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}$ is McGarvey.

(d) (*Trees*) A graph is a ***tree*** if it is connected but contains no loops. Let $\mathcal{X}_{\mathcal{N}}^{\mathrm{tree}} \subset \mathcal{X}_{\mathcal{N}}^{\mathrm{cnct}}$ be the space of all trees. Let $N := |\mathcal{N}|$; then $\#(\mathbf{x}) = N - 1$ for every $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{tree}}$ (because every tree has exactly $N - 1$ activated edges). Thus, Corollary 12 implies that $\mathcal{X}_{\mathcal{N}}^{\mathrm{tree}}$ is *not* McGarvey.

Interestingly, however, $\mathcal{X}_{\mathcal{N}}^{\mathrm{tree}}$ *is* median-saturating. To see this, note that any loop in a graph must involve at least three activated edges, and if $|\mathcal{N}| \geq 4$, then any disconnected graph must have at least three deactivated edges. Thus, $\mathcal{W}_2(\mathcal{X}_{\mathcal{N}}^{\mathrm{tree}}) = \emptyset$; hence Proposition 1(b) implies that $\mathrm{med}^{\infty}(\mathcal{X}_{\mathcal{N}}^{\mathrm{tree}}) = \{\pm 1\}^{\mathcal{K}}$. Thus, equation (4) is false for $\mathcal{X}_{\mathcal{N}}^{\mathrm{tree}}$.

(Two more examples of symmetric McGarvey spaces are described in Appendix B.) $\diamondsuit$

3. Comprehensive property spaces

For any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{K}}$, write $\mathbf{r} \leq \mathbf{s}$ if $r_k \leq s_k$ for all $k \in \mathcal{K}$. Write $\mathbf{r} \ll \mathbf{s}$ if $r_k < s_k$ for all $k \in \mathcal{K}$. The space $\mathcal{X}$ is *comprehensive* if, for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{y} \in \{\pm 1\}^{\mathcal{K}}$, if $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y} \in \mathcal{X}$ also.

Proposition 14. *Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be comprehensive. Then $\mathcal{X}$ is McGarvey if and only if there exists $\mathbf{c} \in \operatorname{conv}(\mathcal{X})$ with $\mathbf{c} \ll \mathbf{0}$.*

For example, suppose $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is comprehensive and there is a subset $\mathcal{Y} \subseteq \mathcal{X}$ such that, for each $k \in \mathcal{K}$, we have $\#\{\mathbf{y} \in \mathcal{Y};\ y_k = 1\} < |\mathcal{Y}|/2$. Let $\mathbf{c} := \frac{1}{|\mathcal{Y}|} \sum_{\mathbf{y} \in \mathcal{Y}} \mathbf{y}$; then $\mathbf{c} \in \operatorname{conv}(\mathcal{X})$ and $\mathbf{c} \ll \mathbf{0}$; hence $\mathcal{X}$ is McGarvey.

Proposition 15. *Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be comprehensive. Then $\mathcal{X}$ is median-saturating if and only if, for every $j, k \in \mathcal{K}$, there exists $\mathbf{x} \in \mathcal{X}$ with $x_j = 0 = x_k$.*

For example, let $K/2 \leq M \leq K - 2$, and let $\mathcal{X}^{\mathrm{com}}_{\geq M} := \left\{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}\ ;\ \#(\mathbf{x}) \geq M\right\}$. Then $\mathcal{X}^{\mathrm{com}}_{\geq M}$ is median-saturating (by Proposition 15) but not McGarvey (by Corollary 12); thus, eqn.(4) is false for $\mathcal{X}^{\mathrm{com}}_{\geq M}$.

4. Truth-functional aggregation

Let $\mathcal{J}$ be a set of logically independent propositions, and let $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be some function. Let $\mathcal{K} := \mathcal{J} \sqcup \{0\}$, and define $\mathcal{X}_f := \{(\mathbf{x}, y);\ \mathbf{x} \in \{\pm 1\}^{\mathcal{J}}$ and $y = f(\mathbf{x})\}$; a subset of $\{\pm 1\}^{\mathcal{K}}$; this is called a *truth-functional space*; see [15, 16].

Many truth-functional spaces are not McGarvey. For example, let $\& : \{\pm 1\}^2 \longrightarrow \mathcal{X}$ be the Boolean 'and' operation (i.e. $\&(x_1, x_2) = 1$ if and only if $x_1 = 1 = x_2$; otherwise $\&(x_1, x_2) = -1$), and let $\mathcal{X}_\& \subset \{\pm 1\}^3$ be the corresponding truth-functional space. Then $\mathcal{X}_\&$ is not McGarvey. Indeed, $\mathcal{X}_\&$ is not even median-saturating (this follows from Proposition 1(b), because $\mathcal{W}_2(\mathcal{X}_\&)$ contains the forbidden word $(*, 0; 1)$).

Proposition 16. *Suppose $|\mathcal{J}| \geq 2$, and suppose $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ depends nontrivially on more than one $\mathcal{J}$-coordinate. If $\displaystyle\sum_{\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}} f(\mathbf{x}) = 0$, then $\mathcal{X}_f$ is McGarvey.*

For example, let $\oplus : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be the J-ary 'exclusive or' function. That is: $\oplus(\mathbf{x}) = 1$ if and only if $\#\{j \in \mathcal{J}\ ;\ x_j = 1\}$ is *odd*. Then $\mathcal{X}_\oplus$ is McGarvey.

Proposition 17. *Let $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be a truth function. Suppose $f^{-1}\{1\}$ and $f^{-1}\{-1\}$ are both McGarvey, when seen as subsets of $\{\pm 1\}^{\mathcal{J}}$. Then $\mathcal{X}_f$ is McGarvey.*

A truth-function $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ is *monotone* if, for all $\mathbf{x}, \mathbf{y} \in \{\pm 1\}^{\mathcal{J}}$,

$$\Big(f(\mathbf{x}) = 1 \text{ and } \mathbf{x} \leq \mathbf{y}\Big) \quad \Longrightarrow \quad \Big(f(\mathbf{y}) = 1\Big).$$

Proposition 18. *Let $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be monotone. Suppose that:*

1. *there exists $\mathcal{Y}_+ \subseteq f^{-1}\{1\}$ such that for each $j \in \mathcal{J}$, we have $\#\{\mathbf{y} \in \mathcal{Y}_+;\ y_j = 1\} < |\mathcal{Y}_+|/2$; and*
2. *there exists $\mathcal{Y}_- \subseteq f^{-1}\{-1\}$ such that for each $j \in \mathcal{J}$, we have $\#\{\mathbf{y} \in \mathcal{Y}_-;\ y_j = -1\} < |\mathcal{Y}_-|/2$.*

Then $\mathcal{X}_f$ is McGarvey.

For example, let J be odd, and let $I := (J-1)/2$. Let $\mathcal{J} := [1...J]$. For any $n \in \mathbb{N}$, let $[n]$ be the unique element of $\mathcal{J}$ which is congruent to n, mod J. For all $j \in \mathcal{J}$, define $\mathbf{y}^j \in \{\pm 1\}^{\mathcal{K}}$ by $y^j_{[j+i]} = 1$ for all $i \in [1...I]$, and $y^j_k = -1$ for all other $k \in \mathcal{J}$. Then define $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ as follows: $f(\mathbf{x}) = 1$ if and only if $\mathbf{x} \geq \mathbf{y}^j$ for some $j \in \mathcal{J}$. Then f is monotone, and the set $\mathcal{Y}_+ := \left\{\mathbf{y}^j\ ;\ j \in \mathcal{J}\right\}$ satisfies hypothesis #1 of Proposition 18. On the other hand, let $\mathbf{z}^1 := (1, 1, -1, 1, 1, -1, 1, 1, -1, \ldots)$, let $\mathbf{z}^2 := (1, -1, 1, 1, -1, 1, 1, -1, 1, \ldots)$, and let $\mathbf{z}^3 := (-1, 1, 1, -1, 1, 1, -1, 1, 1, \ldots)$. Then $\mathcal{Y}_- := \{\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3\}$ satisfies hypothesis #2 of Proposition 18. Thus, $\mathcal{X}_f$ is McGarvey.

5. Convexities

A *convexity structure* on $\mathcal{K}$ is a collection $\mathfrak{C}$ of subsets of $\mathcal{K}$ such that $\emptyset \in \mathfrak{C}$, $\mathcal{K} \in \mathfrak{C}$, and $\mathfrak{C}$ is closed under intersections [17]. Convexity structures often represent the 'convex' subsets of some geometry on $\mathcal{K}$.

Example 19. A *metric graph* is a graph where each edge is assigned a positive real number specifying its 'length'. Let $\mathcal{K}$ be the vertices of a metric graph. For any $j, k \in \mathcal{K}$, a *geodesic* between j and k is a minimal-length path from j to k. A subset $\mathcal{C} \subseteq \mathcal{K}$ is *convex* if it contains all the geodesics between any pair of points in $\mathcal{C}$. The set $\mathfrak{C}$ of all convex subsets of $\mathcal{K}$ is then a convexity structure on $\mathcal{K}$. $\diamondsuit$

For any $\mathcal{J} \subseteq \mathcal{K}$, define $\boldsymbol{\chi}^{\mathcal{J}} \in \{\pm 1\}^{\mathcal{K}}$ by $\chi_j^{\mathcal{J}} := 1$ for all $j \in \mathcal{J}$ and $\chi_k^{\mathcal{J}} := -1$ for all $k \in \mathcal{K} \setminus \mathcal{J}$. Given a convexity structure $\mathfrak{C}$ on $\mathcal{K}$, let $\mathcal{X}_{\mathfrak{C}} := \left\{\boldsymbol{\chi}^{\mathcal{C}} \,;\, \mathcal{C} \in \mathfrak{C}\right\}$. Thus, judgement aggregation on $\mathcal{X}_{\mathfrak{C}}$ is the problem of democratically selecting a convex subset of $\mathcal{K}$. (This problem arises, for example, when a jury wishes to award prizes to some selected subset of contestants according to some 'quality metric', or when an expert committee tries to classify an unfamiliar entity within a taxonomic hierarchy.)

Proposition 20. *Let $\mathfrak{C}$ be a convexity on $\mathcal{K}$, and let $\mathcal{X}_{\mathfrak{C}}$ be as above.*

(a) *For any $\mathcal{J} \subseteq \mathcal{K}$,* $\left(\boldsymbol{\chi}^{\mathcal{J}} \in \mathrm{maj}(\mathcal{X}_{\mathfrak{C}})\right) \iff \left(\mathcal{J} \text{ is a union of elements of } \mathfrak{C}\right)$.

(b) *The following are equivalent:*

[i] *$\mathcal{X}_{\mathfrak{C}}$ is McGarvey.*

[ii] *$\mathcal{X}_{\mathfrak{C}}$ is median-saturating.*

[iii] *$\mathfrak{C}$ includes all the singleton subsets of $\mathcal{K}$.*

For example, the metric graph convexity in Example 19 is McGarvey.

6. Stearns numbers

Even if $\mathcal{X}$ is McGarvey, the hypothesis of Theorem 3(b) leaves the possibility that we can only realize this McGarvey property using very precisely engineered profiles involving an astronomically large number of voters. This would greatly diminish the practical relevance of the McGarvey property. So we now ask: what is the smallest number of voters required to realize the McGarvey property of $\mathcal{X}$? This question was first studied by Stearns [13] for preference-aggregation on $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$. For any $N \in \mathbb{N}$, let

$$\Delta_N^*(\mathcal{X}) \quad := \quad \left\{\mu \in \Delta^*(\mathcal{X}) \;;\; \forall\, \mathbf{x} \in \mathcal{X},\ \mu(\mathbf{x}) = \frac{n}{N} \text{ for some } n \in [0 \ldots N]\right\}.$$

In other words, $\Delta_N^*(\mathcal{X})$ is the set of profiles which can be generated by a population of exactly N voters. Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be McGarvey. We define the *Stearns number* $S(\mathcal{X})$ to be the smallest integer such that, for any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, there exists some $N \leq S(\mathcal{X})$ and $\mu \in \Delta_N^*(\mathcal{X})$ with $\mathrm{maj}(\mu) = \mathbf{x}$. (Define $S(\mathcal{X}) := \infty$ if $\mathcal{X}$ is not McGarvey). For example, if $A := |\mathcal{A}|$, then Stearns [13] showed that $0.55 \cdot A/\log(A) \leq S(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}) \leq A + 2$. Erdös and Moser [18] refined Stearn's estimate by showing that $S(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}) = \Theta(A/\log(A))$. We now investigate the Stearns numbers of other McGarvey spaces. For any $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$, let $\|\mathbf{r}\|_\infty := \sup_{k \in \mathcal{K}} |r_k|$. For any $\epsilon > 0$, let $\mathcal{B}(\epsilon) := \left\{\mathbf{r} \in \mathbb{R}^{\mathcal{K}} \,;\, \|\mathbf{r}\|_\infty \leq \epsilon\right\}$. The next result can be seen as a 'quantitative' refinement of Theorem 3.

Theorem 21. *Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ and let $N(\mathcal{X}) := \min\{N \in \mathbb{N}; \mathcal{B}(\frac{1}{N}) \subseteq \mathrm{conv}(\mathcal{X})\}$. Then $N(\mathcal{X}) \leq S(\mathcal{X}) \leq 4(K+1)\,N(\mathcal{X})$.*

The upper bound in Theorem 21 is an overestimate, in general. For example, Alon [19] has shown that $N(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}) = \Theta(\sqrt{A})$; and in the case of $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$, we have $K := A(A-1)/2$; thus Theorem 21 yields $S(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}) \leq \mathcal{O}(A^{5/2})$, which is much worse than the estimate of $\Theta(A/\log(A))$ obtained by Erdös and Moser [18]. Nevertheless, it may not be possible to improve the estimate in Theorem 21, without making further assumptions about the structure of $\mathcal{X}$. The next result provides some bounds on the size of $N(\mathcal{X})$ and $S(\mathcal{X})$. For any $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \{\pm 1\}^{\mathcal{K}}$, let $\delta(\mathbf{x}_1, \ldots, \mathbf{x}_K) := \min\{\|\mathbf{c}\|_\infty; \mathbf{c} \in \mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_K)\}$.

Proposition 22. *Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$.*

(a) *Let $\delta(\mathcal{X}) := \min\{\delta(\mathbf{x}_1, \ldots, \mathbf{x}_K);\ \mathbf{x}_1, \ldots, \mathbf{x}_K \in \mathcal{X}$ and $\mathbf{0} \notin \mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_K)\}$. If $\mathcal{X}$ is McGarvey, then $N(\mathcal{X}) \leq \lceil 1/\delta(\mathcal{X}) \rceil$.*

(b) *Let $\delta(K) := \delta(\{\pm 1\}^K)$. Then $S(\mathcal{X}) \leq 4(K+1)\lceil 1/\delta(K) \rceil$ for every McGarvey $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$. However, there exist McGarvey $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ with $S(\mathcal{X}) \geq 1/\delta(K)$.*

(c) $\dfrac{K^{K/2}}{2^{3K+o(K)}} \leq \dfrac{1}{\delta(K)} \leq \dfrac{K^{2+K/2}}{2^{2K-1}}$.

The inequalities in Proposition 22(c) are derived from inequalities obtained by Alon and Vũ [14] for the inverses of $\{0,1\}$-matrices; these inequalities have many implications for the geometry of sub-polytopes of $\{\pm 1\}^{\mathcal{K}}$ [20, §5.2]. Proposition 22(b,c) imply that the Stearns numbers of some McGarvey spaces can be extremely large. However, for the McGarvey spaces typically encountered in practice, the Stearns numbers are often much smaller, as shown by the next result and following examples.

Proposition 23. **(a)** *If $\mathbf{1} \in \mathcal{X}$, and $\boldsymbol{\chi}^k \in \mathcal{X}$ for all $k \in \mathcal{K}$, then $S(\mathcal{X}) \leq 2K - 1$.*

(b) *Suppose that $-\mathbf{1} \in \mathcal{X}$, and suppose that, for all $k \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ such that $x_k = 1 = y_k$, but $\mathbf{x}$ and $\mathbf{y}$ differ in every other coordinate. Then $S(\mathcal{X}) \leq 2K + 1$.*

(c) *Suppose $-\mathcal{X} = \mathcal{X}$ and suppose that, for all $k \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ such that $x_k \neq y_k$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. Then $S(\mathcal{X}) \leq 2K$.*

Example 24. (a) (*Convexities*) Let $\mathfrak{C}$ be a convexity on $\mathcal{K}$. Then $\mathbf{1} \in \mathcal{X}_{\mathfrak{C}}$ (because $\mathcal{K} \in \mathfrak{C}$). If $\mathcal{X}_{\mathfrak{C}}$ is McGarvey, then Proposition 20(b) says $\boldsymbol{\chi}^k \in \mathcal{X}$ for all $k \in \mathcal{K}$; thus, Proposition 23(a) says $S(\mathcal{X}_{\mathfrak{C}}) \leq 2K - 1$.

(b) (*Equivalence Relations*) Let $\mathcal{N}$ be a set, and let $\mathcal{K}$ and $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}} \subset \{\pm 1\}^{\mathcal{K}}$ be as in Example 13(a). Observe that $\mathbf{1} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ (it represents the 'complete equivalence' relation such that $n \sim m$ for all $n, m \in \mathcal{N}$). Also, for all $\{n, m\} \in \mathcal{N}$, $\boldsymbol{\chi}^{n,m} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ (it represents the equivalence relation such that $n \sim m$, but no other pair of elements are equivalent). Thus, Proposition 23(a) implies that $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ is McGarvey, and $S(\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}) \leq N(N-1) - 1$.

(c) (*Preorders*) Let $\mathcal{K} := \{(n, m) \in \mathcal{N} \times \mathcal{N};\ n \neq m\}$. Thus, an element of $\{\pm 1\}^{\mathcal{K}}$ can represent a reflexive binary relation "$\preceq$" on $\mathcal{N}$. A *preorder* is a reflexive, transitive binary relation on $\mathcal{N}$ (note that we do not assume preorders are complete). Let $\mathcal{X}_{\mathcal{N}}^{\mathrm{preo}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of all preorders on $\mathcal{N}$. Thus, $\mathbf{1} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{preo}}$ (it represents the relation of total indifference). Also, for all $(n, m) \in \mathcal{N}$, $\boldsymbol{\chi}^{n,m} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ (it represents the preorder such that $n \preceq m$, but no other pair of elements are comparable). Thus, Proposition 23(a) implies $\mathcal{X}_{\mathcal{N}}^{\mathrm{preo}}$ is McGarvey, and $S(\mathcal{X}_{\mathcal{N}}^{\mathrm{preo}}) \leq 2N(N-1) - 1$.

(d) (*Complete preorders*) Now let $\mathcal{X}^* \subset \mathcal{X}_{\mathcal{N}}^{\mathrm{preo}}$ be the set of all *complete* preorders. Then $\mathcal{X}^*$ is *not* McGarvey. Indeed, Example 2 shows that $\mathcal{X}^*$ is not even median-saturating.

(e) (*Committees*) Let $\mathcal{K}$ be a set of candidates; then any element of $\{\pm 1\}^{\mathcal{K}}$ represents a 'committee' formed from these candidates. Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_L \subseteq \mathcal{K}$ be (possibly overlapping) subsets with cardinalities $K_1, K_2, \ldots, K_L$, respectively. Fix $I, J \in \mathbb{N}$ with $0 \leq I < K/2 < J \leq K$. Likewise, for all $\ell \in [1...L]$, fix $I_\ell, J_\ell \in \mathbb{N}$ with $0 \leq I_\ell < K_\ell/2 < J_\ell \leq K_\ell$. For any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ and $\ell \in [1...L]$, define $\#_\ell(\mathbf{x}) := \#\{k \in \mathcal{K}_\ell\ ;\ x_k = 1\}$. Consider the set:

$$\mathcal{X}^{\mathrm{com}} \quad := \quad \left\{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}\ ;\ I \leq \#(\mathbf{x}) \leq J \text{ and } I_\ell \leq \#_\ell(\mathbf{x}) \leq J_\ell,\ \forall\ \ell \in [1...L]\right\}.$$

Thus, $\mathcal{X}^{\mathrm{com}}$ represents the set of all committees formed from the candidates in $\mathcal{K}$, with upper and lower bounds on the size of the whole committee, and also upper/lower bounds on the level of representation from various 'constituencies' $\mathcal{K}_1, \ldots, \mathcal{K}_L$.

We claim $S(\mathcal{X}^{\mathrm{com}}) \leq 2K$. To see this, suppose first that $I = K - J$ and $I_\ell = K_\ell - J_\ell$ for all $\ell \in [1...L]$. Then $-\mathcal{X}^{\mathrm{com}} = \mathcal{X}^{\mathrm{com}}$. For all $k \in \mathcal{X}$, let $\mathbf{x}^k \in \mathcal{X}^{\mathrm{com}}$ be an admissible committee of minimal size not involving k. Thus, $I \leq \#(\mathbf{x}^k) < J$ and $I_\ell \leq \#_\ell(\mathbf{x}^k) < J_\ell$ for all $\ell \in [1...L]$. Let $\mathbf{y}^k$ be the committee obtained from

$\mathbf{x}^k$ by adding k; then $I < \#(\mathbf{y}^k) \leq J$ and $I_\ell \leq \#_\ell(\mathbf{y}^k) \leq J_\ell$ for all $\ell \in [1...L]$, so $\mathbf{y}^k \in \mathcal{X}^{\rm com}$. Thus, the hypotheses of Proposition 23(c) are satisfied, so $S(\mathcal{X}^{\rm com}) \leq 2K$.

Now consider the general case. Let $I' := \max\{I, K-J\}$ and $J' := \min\{J, K-I\}$, and for all $\ell \in [1...L]$, let $I'_\ell := \max\{I_\ell, K_\ell - J_\ell\}$ and $J' := \min\{J_\ell, K_\ell - I_\ell\}$. Let $\mathcal{X}'$ be the resulting committee space. Then $\mathcal{X}'$ satisfies the hypotheses of the previous paragraph, so $S(\mathcal{X}') \leq 2K$. But $\mathcal{X}' \subseteq \mathcal{X}^{\rm com}$; thus, $S(\mathcal{X}^{\rm com}) \leq 2K$ also. $\diamondsuit$

Appendix A: Proofs

Proof of Proposition 1. Part (b) follows immediately from (a). Part (a) follows (after some decryption) from Lemma I.6.20(1) on p.130 of [17]. We will give another proof of part (a), using 'critical words'. For any $\mathcal{Y} \subseteq \{\pm 1\}^{\mathcal{K}}$, let $\mathcal{W}(\mathcal{Y})$ be the set of all $\mathcal{Y}$-forbidden words. A word $\mathbf{w} \in \mathcal{W}(\mathcal{Y})$ is *$\mathcal{Y}$-critical* if no proper subword of $\mathbf{w}$ is in $\mathcal{W}(\mathcal{Y})$. Let $\mathcal{W}^*(\mathcal{Y})$ be the set of $\mathcal{Y}$-critical words. Observe that $(\mathcal{X} \subseteq \mathcal{Y}) \iff \Big(\mathcal{W}(\mathcal{Y}) \subseteq \mathcal{W}(\mathcal{X})\Big) \iff \Big(\mathcal{W}^*(\mathcal{Y}) \subseteq \mathcal{W}^*(\mathcal{X})\Big)$. Proposition 4.1 of [9] states:

$$\Big(\mathcal{Y} \text{ is a median space}\Big) \iff \Big(\text{All } \mathcal{Y}\text{-critical words have order } 2\Big). \tag{5}$$

Let $\mathcal{Y} := \big\{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}} \,;\, \mathbf{w} \not\sqsubset \mathbf{x},\ \forall\, \mathbf{w} \in \mathcal{W}_2(\mathcal{X})\big\}$. We must show that $\mathrm{med}^\infty(\mathcal{X}) = \mathcal{Y}$. By construction, $\mathcal{W}^*(\mathcal{Y}) = \mathcal{W}_2(\mathcal{X})$. Thus, every $\mathcal{Y}$-critical word has order 2, so statement (5) says $\mathcal{Y}$ is a median space. Clearly $\mathcal{X} \subseteq \mathcal{Y}$ (because $\mathcal{W}(\mathcal{Y}) \subseteq \mathcal{W}(\mathcal{X})$). But by definition, $\mathrm{med}^\infty(\mathcal{X})$ is the smallest median space containing $\mathcal{X}$. Thus, $\mathrm{med}^\infty(\mathcal{X}) \subseteq \mathcal{Y}$.

To see the reverse inclusion, note that $\mathrm{med}^\infty(\mathcal{X})$ is a median space; thus, statement (5) says every $\mathrm{med}^\infty(\mathcal{X})$-critical word has order 2. However, the $\mathcal{W}[\mathrm{med}^\infty(\mathcal{X})] \subseteq \mathcal{W}(\mathcal{X})$ (because $\mathcal{X} \subseteq \mathrm{med}^\infty(\mathcal{X})$). Thus, $\mathcal{W}^*[\mathrm{med}^\infty(\mathcal{X})] \subseteq \mathcal{W}_2(\mathcal{X}) = \mathcal{W}^*(\mathcal{Y})$. Thus, $\mathcal{Y} \subseteq \mathrm{med}^\infty(\mathcal{X})$. Thus, $\mathcal{Y} = \mathrm{med}^\infty(\mathcal{X})$. $\Box$

Proof of Theorem 3. (a) Let $\mu \in \Delta^*(\mathcal{X})$. For all $k \in \mathcal{K}$, define $\widetilde{\mu}_k$ as in eqn.(1), and let $\widetilde{\mu} := (\widetilde{\mu}_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$. Let $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ be the unique element such that $\widetilde{\mu} \in \mathcal{O}_{\mathbf{x}}$; then eqn.(2) implies that $\mathrm{maj}(\mu) = \mathbf{x}$.

If we treat $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ as a subset of $\mathbb{R}^{\mathcal{K}}$, then $\widetilde{\mu} := \sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x})\mathbf{x}$; thus, $\mu \in \mathrm{conv}(\mathcal{X})$. Furthermore, every element of $\mathrm{conv}(\mathcal{X})$ can be represented in this way. Thus, for any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$,

$$\Big(\mathbf{x} \in \mathrm{maj}(\mathcal{X})\Big) \iff \Big(\exists\, \mu \in \Delta^*(\mathcal{X}) \text{ such that } \widetilde{\mu} \in \mathcal{O}_{\mathbf{x}}\Big) \iff \Big(\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}} \neq \emptyset\Big).$$

(b) "(2) $\iff$ (3)" The Separating Hyperplane Theorem says that $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$ if and only if, for all nonzero $\mathbf{z} \in \mathbb{R}^{\mathcal{K}}$, there exists $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ such that $\mathbf{z} \bullet \mathbf{c} > 0$. This, in turn, occurs if and only if there exists $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{z} \bullet \mathbf{x} > 0$ (because $\mathcal{X}$ is the set of extreme points of $\mathrm{conv}(\mathcal{X})$).

"(1) $\Longleftarrow$ (2)" If $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$, then $\mathrm{conv}(\mathcal{X})$ intersects every open orthant of $\mathbb{R}^{\mathcal{K}}$, so (a) implies that $\mathrm{maj}(\mathcal{X}) = \{\pm 1\}^{\mathcal{K}}$.

"(2) $\Longrightarrow$ (1)" (by contrapositive) $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$ is an open convex subset of $\mathbb{R}^{\mathcal{K}}$. Suppose $\mathbf{0} \notin \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$. Then the Separating Hyperplane Theorem says there is some vector $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ such that $\langle \mathbf{r}, \mathbf{c} \rangle < 0$ for all $\mathbf{c} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$. Pick $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ such that the open orthant $\mathcal{O}_{\mathbf{x}}$ contains $\mathbf{r}$ (if $\mathbf{r}$ sits on a boundary between two or more orthants, then pick one). Then we must have $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \cap \mathcal{O}_{\mathbf{x}} = \emptyset$. Thus, $\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}} = \emptyset$ (because $\mathrm{conv}(\mathcal{X})$ is the closure of $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$, and $\mathcal{O}_{\mathbf{x}}$ is an open set). Thus, part (a) implies that $\mathbf{x} \notin \mathrm{maj}(\mathcal{X})$; hence $\mathcal{X}$ is not McGarvey. $\Box$

Proof of Proposition 4. (a) Let $M := \min\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is McGarvey$\}$.

"$M \geq K+1$": Suppose $|\mathcal{X}| = J \leq K$. Let $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^J\}$. Define $\mathbf{y}^j := \mathbf{x}^j - \mathbf{x}^J$ for all $j \in [1 \ldots J-1]$, and let $\mathcal{Y}$ be the linear subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\{\mathbf{y}^1, \ldots, \mathbf{y}^{J-1}\}$. Then $\dim(\mathcal{Y}) \leq J-1 < K$. However, $\mathrm{conv}(\mathcal{X}) \subset \mathcal{Y} + \mathbf{x}^J$; thus, $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] = \emptyset$, so $\mathcal{X}$ is not McGarvey.

"$M \leq K+1$": Let $\mathbf{1} := (1, 1, \ldots, 1)$. For all $k \in \mathcal{K}$, define $\boldsymbol{\chi}^k \in \{\pm 1\}^{\mathcal{K}}$ as we did prior to Proposition 20. Let $\mathcal{X} := \{\boldsymbol{\chi}^k\}_{k \in \mathcal{K}} \sqcup \{\mathbf{1}\}$. Then $|\mathcal{X}| = K+1$. Observe that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$, because the collection $\{\boldsymbol{\chi}^k - \mathbf{1}\}_{k \in \mathcal{K}}$ spans $\mathbb{R}^{\mathcal{K}}$. Furthermore,

$$\left(\frac{K-2}{2K-2}\right)\mathbf{1} + \left(\frac{1}{2K-2}\right)\sum_{k \in \mathcal{K}} \boldsymbol{\chi}^k \quad = \quad \left(\frac{K-2}{2K-2}\right)\mathbf{1} - \left(\frac{K-2}{2K-2}\right)\mathbf{1} \quad = \quad \mathbf{0},$$

so $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$. Thus, Theorem 3(b) says $\mathcal{X}$ is McGarvey.

(b) Let $M := \max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is minimal McGarvey$\}$.

"$M \geq 2K$" follows from Example 5. To see "$M \leq 2K$", let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be McGarvey. Then Theorem 3(b) says $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$.

Claim 1: *There exists some $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| \leq 2K$ such that $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{Y})]$.*

Proof: For any nonzero $\mathbf{v} \in \mathbb{R}^{\mathcal{K}}$, consider the line $\mathcal{L}_{\mathbf{v}} := \{r\mathbf{v}\ ;\ r \in \mathbb{R}\}$. This line intersects the boundary of $\mathrm{conv}(\mathcal{X})$ in exactly two places —say at $\mathbf{u} = -s\mathbf{v}$ and $\mathbf{w} = t\mathbf{v}$, for some $-s < 0 < t$. For a generic choice of $\mathbf{v} \in \mathbb{R}^{\mathcal{K}}$, the points $\mathbf{u}$ and $\mathbf{w}$ are each contained in the relative interior of some $(K-1)$-dimensional face of $\mathrm{conv}(\mathcal{X})$ —that is, there are sets $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_K\} \subseteq \mathcal{X}$ and $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_K\} \subseteq \mathcal{X}$, such that $\mathrm{conv}(\mathcal{U})$ and $\mathrm{conv}(\mathcal{W})$ each have dimension $(K-1)$, and such that $\mathbf{u} = \sum_{k=1}^K q_k \mathbf{u}_k$ and $\mathbf{w} = \sum_{k=1}^K r_k \mathbf{w}_k$, for some $q_1, \ldots, q_K, r_1, \ldots, r_K > 0$ with $\sum_{k=1}^K q_k = 1 = \sum_{k=1}^K r_k$.

Let $\mathcal{Y} := \mathcal{U} \cup \mathcal{W}$. Then $\mathrm{conv}(\mathcal{Y})$ contains the $(K-1)$-dimensional sets $\mathrm{conv}(\mathcal{U})$ and $\mathrm{conv}(\mathcal{W})$, and it also contains two different points on the line $\mathcal{L}$ transversal to these sets (because $\mathrm{conv}(\mathcal{U})$ and $\mathrm{conv}(\mathcal{W})$ intersect $\mathcal{L}$ at two different points). Thus $\mathrm{conv}(\mathcal{Y})$ must have dimension K (hence, nonempty interior). Furthermore, $|\mathcal{Y}| \leq |\mathcal{U}| + |\mathcal{W}| = 2K$. Let $R := \frac{1}{s} + \frac{1}{t}$, let $S := \frac{1}{sR} > 0$ and let $T := \frac{1}{tR} > 0$. Then $S + T = 1$, and

$$\sum_{k=1}^K S q_k\, \mathbf{u}_k + \sum_{k=1}^K T r_k\, \mathbf{u}_k \quad = \quad S \sum_{k=1}^K q_k\, \mathbf{u}_k \ + \ T \sum_{k=1}^K r_k\, \mathbf{u}_k \quad = \quad \frac{-s\mathbf{v}}{sR} + \frac{t\mathbf{v}}{tR} \quad = \quad \frac{-\mathbf{v}}{R} + \frac{\mathbf{v}}{R} \quad = \quad \mathbf{0}.$$

By construction, we have $Sq_1, \ldots, Sq_K, Tr_1, \ldots, Tr_K > 0$, and $\sum_{k=1}^K Sq_k + \sum_{k=1}^K Tr_k = 1$. Thus, $\mathbf{0}$ is a strictly positive convex combination of the elements of $\mathcal{Y}$, so $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{Y})]$, as claimed. ◇ `Claim 1`

If $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{Y})]$, then Theorem 3(b) implies that $\mathcal{Y}$ is McGarvey. But if $\mathcal{X}$ is *minimal* McGarvey, then this means that $\mathcal{Y} = \mathcal{X}$. Thus, $|\mathcal{X}| \leq 2K$, as claimed. □

Remark. The proof of Claim 1 in Proposition 4(b) easily generalizes to prove the following 'relative interior' version of Carathéodory's theorem: *Let $\mathcal{X} \subset \mathbb{R}^K$ be finite, let* $\dim(\mathrm{conv}(\mathcal{X})) = D \leq K$, *and let $\mathbf{x}$ be in the relative interior of* $\mathrm{conv}(\mathcal{X})$. *Then there exists some $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| \leq 2D$ such that $\mathbf{x}$ is in the relative interior of* $\mathrm{conv}(\mathcal{Y})$.

Proof of Example 5. We must show that $\mathcal{X}$ is McGarvey, but no proper subset of $\mathcal{X}$ is McGarvey.

$\mathcal{X}$ is McGarvey: Clearly, $2\boldsymbol{\chi}^j \in (\mathcal{X} - \mathcal{X})$ for all $j \in \mathcal{K}$. Thus, $\mathrm{span}(\mathcal{X} - \mathcal{X}) = \mathbb{R}^{\mathcal{K}}$, so $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$.

Recall from §2 that $\Pi_{\mathcal{X}}$ is the set of coordinate permutation symmetries of $\mathcal{X}$. In this case, $\Pi_{\mathcal{X}}$ contains every possible permutation of $\mathcal{K}$, so $\Pi_{\mathcal{X}}$ is transitive. Clearly $\#(\boldsymbol{\chi}^j) = 1 < K/2$, whereas $\#(-\boldsymbol{\chi}^j) = K-1 > K/2$. Thus, Corollary 12 implies that $\mathcal{X}$ is McGarvey.

No proper subset of $\mathcal{X}$ is McGarvey: Suppose $\mathcal{K} := [1...K]$. Let $\mathcal{Y} := \mathcal{X} \setminus \{\boldsymbol{\chi}^1\}$. To see that $\mathcal{Y}$ is not McGarvey, let $\mathbf{z} := (K-3;\ -1,-1,\ldots,-1)$; then $\mathbf{z} \bullet \mathbf{y} \leq 0$ for all $\mathbf{y} \in \mathcal{Y}$. Thus, Theorem 3(b) implies that $\mathcal{Y}$ is not McGarvey.

A similar argument shows that the sets $\mathcal{X} \setminus \{\boldsymbol{\chi}^k\}$ and $\mathcal{X} \setminus \{-\boldsymbol{\chi}^k\}$ are not McGarvey, for any $k \in \mathcal{K}$. $\square$

Lemma 25. *Let $\mathcal{S} \subset \mathbb{R}^{\mathcal{K}}$ be an affine subspace of dimension $D \leq K$. Then $|\mathcal{S} \cap \{\pm 1\}^{\mathcal{K}}| \leq 2^D$.*

Proof: Suppose $\mathcal{K} = [1...K]$, and identify $\mathbb{R}^{\mathcal{K}}$ with $\mathbb{R}^D \times \mathbb{R}^{K-D}$ in the obvious way. If $\dim(\mathcal{S}) = D$, then there exists some affine function $\phi : \mathbb{R}^D \longrightarrow \mathbb{R}^{K-D}$ such that (after some permutation of $\mathcal{K}$), we have $\mathcal{S} = \{(\mathbf{r}, \phi(\mathbf{r}));\ \mathbf{r} \in \mathbb{R}^D\}$. This means that $\mathcal{S} \cap \{\pm 1\}^{\mathcal{K}} = \{(\mathbf{x}, \phi(\mathbf{x}));\ \mathbf{x} \in \{\pm 1\}^D$ and $\phi(\mathbf{x}) \in \{\pm 1\}^{K-D}\}$. Thus, $\left|\mathcal{S} \cap \{\pm 1\}^{\mathcal{K}}\right| \leq \left|\{\pm 1\}^D\right| = 2^D$. $\square$

Proof of Proposition 6. (a) Let $M_0 := \max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is not McGarvey$\}$.

"$M_0 \geq \frac{3}{4}2^K$" follows immediately from Example 7. To see "$M_0 \leq \frac{3}{4}2^K$", suppose $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is *not* McGarvey. Then Theorem 3(b) says there exists nonzero $\mathbf{z} \in \mathbb{R}^{\mathcal{K}}$, such that $\mathbf{z} \bullet \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathcal{X}$. Let $\mathcal{Y}_+ := \{\mathbf{y} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{z} \bullet \mathbf{y} > 0\}$, let $\mathcal{Y}_- := \{\mathbf{y} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{z} \bullet \mathbf{y} < 0\}$, and let $\mathcal{Y}_0 := \{\mathbf{y} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{z} \bullet \mathbf{y} = 0\}$. Now, $|\mathcal{Y}_-| = |\mathcal{Y}_+|$ (because these sets are images of one another under negation). Thus,

$$|\mathcal{Y}_-| \quad = \quad \frac{1}{2}\left|\{\pm 1\}^{\mathcal{K}} \setminus \mathcal{Y}_0\right| \quad = \quad \frac{1}{2}\left(2^K - |\mathcal{Y}_0|\right) \quad = \quad 2^{K-1} - \frac{1}{2}|\mathcal{Y}_0|. \tag{6}$$

$$\text{Also,} \quad \mathcal{X} \quad \subseteq \quad \mathcal{Y}_- \sqcup \mathcal{Y}_0.$$

$$\text{Thus,} \quad |\mathcal{X}| \quad \leq \quad |\mathcal{Y}_- \sqcup \mathcal{Y}_0| \quad = \quad |\mathcal{Y}_-| + |\mathcal{Y}_0| \underset{(\dagger)}{=} 2^{K-1} - \frac{1}{2}|\mathcal{Y}_0| + |\mathcal{Y}_0| \quad = \quad 2^{K-1} + \frac{1}{2}|\mathcal{Y}_0|$$
$$\underset{(*)}{\leq} \quad 2^{K-1} + \frac{1}{2}2^{K-1} \quad = \quad \frac{3}{4}2^K,$$

as claimed. Here, $(\dagger)$ is by eqn.(6), and $(*)$ is because $|\mathcal{Y}_0| \leq 2^{K-1}$ by Lemma 25.

(b) Let $M_1 := \max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is not median-saturating$\}$.

"$M_1 \geq \frac{3}{4}2^K$" follows immediately from Example 7. To see "$M_1 \leq \frac{3}{4}2^K$", observe that $\{\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}};$ $\mathcal{X}$ is not median-saturating$\} \subseteq \{\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}};\ \mathcal{X}$ is not McGarvey$\}$ (because McGarvey implies median-saturating). Thus, $M_1 \leq M_0$, and we have already verified that $M_0 \leq \frac{3}{4}2^K$. $\square$

Lemma 26. *Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$. If $\operatorname{int}[\operatorname{conv}(\mathcal{X})] \neq \emptyset$, and $\sum_{\mathbf{x} \in \mathcal{X}} \mathbf{x} = \mathbf{0}$, then $\mathcal{X}$ is McGarvey.*

Proof: Let $\mathcal{Y} := \{\mathbf{x} - \mathbf{x}'\ ;\ \mathbf{x}, \mathbf{x}' \in \mathcal{X}\}$. For any $\epsilon > 0$, let $\mathcal{B}(\epsilon) := \{\sum_{\mathbf{y} \in \mathcal{Y}} r_{\mathbf{y}} \mathbf{y};\ \ r_{\mathbf{y}} \in \mathbb{R}$ for all $\mathbf{y} \in \mathcal{Y}$, and $\sum_{\mathbf{y} \in \mathcal{Y}} |r_{\mathbf{y}}| < \epsilon\}$.

Suppose $\epsilon < \dfrac{1}{|\mathcal{X}|}$; then for any $\mathbf{b} \in \mathcal{B}(\epsilon)$, we have $\mathbf{b} + \dfrac{1}{|\mathcal{X}|} \displaystyle\sum_{\mathbf{x} \in \mathcal{X}} \mathbf{x} \in \operatorname{conv}(\mathcal{X})$. Thus, if $\mathbf{0} = \dfrac{1}{|\mathcal{X}|} \displaystyle\sum_{\mathbf{x} \in \mathcal{X}} \mathbf{x}$, then $\mathcal{B}(\epsilon) \subseteq \operatorname{conv}(\mathcal{X})$.

If $\operatorname{int}[\operatorname{conv}(\mathcal{X})] \neq \emptyset$, then $\mathcal{Y}$ spans $\mathbb{R}^{\mathcal{K}}$. Thus, $\mathcal{B}(\epsilon)$ is an open neighbourhood around $\mathbf{0}$, for any $\epsilon > 0$. Thus, Theorem 3(b) says $\mathcal{X}$ is McGarvey. $\square$

Proof of Proposition 8. (a) Let $\mathbf{z} := \displaystyle\sum_{\mathbf{x} \in \mathcal{X}} \mathbf{x}$. Then $\gamma(\mathbf{z}) = \mathbf{z}$ for all $\gamma \in \Gamma_{\mathcal{X}}$; hence $\mathbf{z} \in \operatorname{Fix}(\Gamma_{\mathcal{X}})$, which means $\mathbf{z} = \mathbf{0}$ (by hypothesis). Thus, $\displaystyle\sum_{\mathbf{x} \in \mathcal{X}} \mathbf{x} = \mathbf{0}$, so Lemma 26 says $\mathcal{X}$ is McGarvey.

(b) If $-\mathcal{X} = \mathcal{X}$, then $-\mathbf{I} \in \Gamma_{\mathcal{X}}$. Thus, for any $\mathbf{r} \in \operatorname{Fix}(\Gamma_{\mathcal{X}})$, we have $-\mathbf{r} = \mathbf{r}$, which means $\mathbf{r} = \mathbf{0}$. Thus, $\operatorname{Fix}(\Gamma_{\mathcal{X}}) = \{\mathbf{0}\}$. Thus, part (a) says $\mathcal{X}$ is McGarvey. $\square$

Proof of Proposition 11. "$\Longrightarrow$" (by contrapositive) Suppose there do *not* exist $r < 0 < t \in \mathbb{R}$ such that $r\mathbf{1}, t\mathbf{1} \in \mathrm{conv}(\mathcal{X})$. Then $\mathbf{0} \notin \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$. Thus, Theorem 3(b) says $\mathcal{X}$ is not McGarvey.

"$\Longleftarrow$" Let $\mathbf{y} := \dfrac{1}{|\mathcal{X}|} \sum_{\mathbf{x}\in\mathcal{X}} \mathbf{x}$. Then $\mathbf{y} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$ (same argument as Lemma 26). However, $\mathbf{y} \in \mathrm{Fix}\,(\Gamma_{\mathcal{X}})$, as in part (a). Thus, $\mathbf{y} = s\mathbf{1}$ for some $s \in \mathbb{R}$ (by hypothesis). If $s = 0$, then $\mathbf{y} = \mathbf{0}$, so Lemma 26 says $\mathcal{X}$ is McGarvey. So suppose $s \neq 0$.

By hypothesis, there exist $r < 0 < t \in \mathbb{R}$ such that $r\mathbf{1}, t\mathbf{1} \in \mathrm{conv}(\mathcal{X})$. If $s < 0$, then $\mathbf{0} = \left(\frac{-s}{t-s}\right) t\mathbf{1} + \left(\frac{t}{t-s}\right)\mathbf{y}$ is also in $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$, so Theorem 3(b) says $\mathcal{X}$ is McGarvey. If $s > 0$, then $\mathbf{0} = \left(\frac{s}{s-r}\right) r\mathbf{1} + \left(\frac{-r}{s-r}\right)\mathbf{y}$ is again in $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$, so again Theorem 3(b) says $\mathcal{X}$ is McGarvey. □

Proof of Corollary 12 "$\Longrightarrow$" (by contrapositive) Suppose there does *not* exist any $\mathbf{x} \in \mathcal{X}$ with $\#(\mathbf{x}) < K/2$. Then $\#(\mathbf{x}) \geq K/2$ for all $\mathbf{x} \in \mathcal{X}$. This means $\sum_{k\in\mathcal{K}} x_k \geq 0$ for all $\mathbf{x} \in \mathcal{X}$ —i.e. $\mathbf{1} \bullet \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathcal{X}$. Thus, Theorem 3(b) says $\mathcal{X}$ is not McGarvey.

Similarly, if $\#(\mathbf{y}) \leq K/2$ for all $\mathbf{y} \in \mathcal{X}$, then $\mathcal{X}$ cannot be McGarvey.

"$\Longleftarrow$" First note that $\mathrm{Fix}\,(\Pi_{\mathcal{X}}) \subseteq \mathbb{R}\mathbf{1}$. To see this, let $\mathbf{r} \in \mathrm{Fix}\,(\Pi_{\mathcal{X}})$; then $\pi(\mathbf{r}) = \mathbf{r}$ for all $\pi \in \Pi_{\mathcal{X}}$. If $\Pi_{\mathcal{X}}$ is transitive, then all coordinates of $\mathbf{r}$ must be equal; hence $\mathbf{r} \in \mathbb{R}\mathbf{1}$.

By hypothesis, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$. Observe that $\#[\pi(\mathbf{x})] = \#(\mathbf{x})$ and $\#[\pi(\mathbf{y})] = \#(\mathbf{y})$ for all $\pi \in \Pi_{\mathcal{X}}$. Let

$$\mathbf{x}^* \quad := \quad \frac{1}{|\Pi_{\mathcal{X}}|} \sum_{\pi\in\Pi_{\mathcal{X}}} \pi(\mathbf{x}) \quad \text{and} \quad \mathbf{y}^* \quad := \quad \frac{1}{|\Pi_{\mathcal{X}}|} \sum_{\pi\in\Pi_{\mathcal{X}}} \pi(\mathbf{y});$$

Then $\mathbf{x}^*, \mathbf{y}^* \in \mathrm{Fix}\,(\Pi_{\mathcal{X}})$, so $\mathbf{x}^* = r\mathbf{1}$ and $\mathbf{y}^* = t\mathbf{1}$, where $r := 2\#(\mathbf{x})/K - 1 < 0$ and $t := 2\#(\mathbf{y})/K - 1 > 0$.

Finally, $\Gamma_{\mathcal{X}} \supseteq \Pi_{\mathcal{X}}$, so $\mathrm{Fix}\,(\Gamma_{\mathcal{X}}) \subseteq \mathrm{Fix}\,(\Pi_{\mathcal{X}}) \subseteq \mathbb{R}\mathbf{1}$. At this point, all hypotheses of Proposition 11 are verified; thus, $\mathcal{X}$ is McGarvey. □

Proof of Lemma 9. Let $\mathcal{Y} := \{\mathbf{x} - \mathbf{y}\ ;\ \mathbf{x}, \mathbf{y} \in \mathcal{X}\}$. For all $j \in \mathcal{K}$, let $\mathbf{e}^j := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the '1' appears in the jth coordinate. If $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ are such that $x_j \neq y_j$, but $x_k = y_k$ for all $k \in \mathcal{K} \setminus \{j\}$, then $\mathbf{x} - \mathbf{y} = \pm\mathbf{e}^j$. Thus, by hypothesis, $\mathcal{Y}$ contains $\{\pm\mathbf{e}^j\}_{j\in\mathcal{K}}$. Thus, $\mathrm{span}(\mathcal{Y}) = \mathbb{R}^{\mathcal{K}}$. Thus, $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$. □

Proof of Example 13(b). Clearly $\Pi_{\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r,R)} = \Pi_{\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}} \supseteq \Pi_*$, so it is transitive. Thus, Corollary 12 says that $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)$ is McGarvey if and only if there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)$ with $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$.

Claim 1: *There always exists $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)$ with $\#(\mathbf{x}) < K/2$.*

Proof: Recall that $R \geq 2$. There are two cases.

Case 1. ($R > 2$). Let $r' := \max\{r, 2\}$; then $r \leq r' < R$ (since $R > 2$ by hypothesis). In fact, we will construct $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r', R)$.

Let $L := \left\lfloor \dfrac{N + 2 - r'}{2} \right\rfloor$. If $(N + 2 - r')$ is *even*, then let $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ describe an equivalence relation where $\mathcal{N}$ splits into two equivalence classes of sizes L, along with $r' - 2$ singleton classes. If $(N + 2 - r')$ is *odd*, then let $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ describe an equivalence relation where $\mathcal{N}$ splits into two equivalence classes of sizes L, and $r' - 1$ singleton classes. In either case, $\mathrm{rank}(\mathbf{x}) = r'$ or $r' + 1$, so $\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r', R)$. We have

$$\#(\mathbf{x}) \quad = \quad 2\left(\frac{L(L-1)}{2}\right) \quad = \quad L(L-1) \quad < \quad L(L - \tfrac{1}{2}) \quad \underset{(*)}{\leq} \quad \frac{N(N-1)}{4} \quad = \quad \frac{K}{2},$$

as desired. Here $(*)$ is because $L \leq N/2$ because $r' \geq 2$.

Case 2. $(R = 2)$. Suppose N is *even* —say $N = 2L$. Then $K/2 = L^2 - L/2$. Let $\mathbf{x}$ represent an equivalence relation which divides $\mathcal{N}$ into two equivalence classes of size L. Then $\#(\mathbf{x}) = 2L(L-1)/2 = L^2 - L < K/2$.

Now suppose N is *odd* —say $N = 2L + 1$. Then $K/2 = L^2 + L/2$. Let $\mathbf{x}$ represent an equivalence relation which divides $\mathcal{N}$ into one equivalence class of size L, and one of size $L + 1$. Then $\#(\mathbf{x}) = L(L-1)/2 + (L+1)L/2 = L^2 < K/2$. ◇ `Claim 1`

Claim 1 and Corollary 12 imply that $\mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)$ is McGarvey if and only if there exists $\mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)$ with $\#(\mathbf{y}) > K/2$. We must show this occurs if and only if $r < \overline{r}(N)$.

Let $M := N - r + 1$, and let $\mathcal{M} \subset \mathcal{N}$ be a subset of cardinality M, so that $|\mathcal{N} \setminus \mathcal{M}| = N - M = r - 1$. Let $\mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}$ describe the equivalence relation where $\mathcal{M}$ forms one equivalence class, and each element of $\mathcal{N} \setminus \mathcal{M}$ forms a singleton equivalence class, for r equivalence classes in total. Thus, $\mathrm{rank}(\mathbf{y}) = r$, so $\mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)$. It is easy to see that $\#(\mathbf{y}) = \max\{\#(\mathbf{x});\ \mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{eq}}(r, R)\}$. Thus, it suffices to show that $\#(\mathbf{y}) > K/2$ if and only if $r < \overline{r}(N)$. To see this, let

$$\overline{M} \quad := \quad N - \overline{r}(N) + 1 \quad = \quad \frac{1 + \sqrt{2N^2 - 2N + 1}}{2}.$$

Then $\overline{M}$ is the positive root of the polynomial $f(M) = M^2 - M - (N^2 - N)/2$. Thus, for any $M \in \mathbb{N}$, we have

$$\Big(r < \overline{r}(N)\Big) \iff \Big(M > \overline{M}\Big) \iff \Big(f(M) > 0\Big) \iff \left(M^2 - M \ > \ \frac{N^2 - N}{2}\right)$$
$$\underset{(*)}{\iff} \left(\frac{M(M-1)}{2} \ > \ \frac{K}{2}\right) \underset{(\dagger)}{\iff} \left(\#(\mathbf{y}) \ > \ \frac{K}{2}\right),$$

as claimed. Here, $(*)$ is because $K = N(N-1)/2$, and $(\dagger)$ is because $\#(\mathbf{y}) = M(M-1)/2$. □

Proof of Proposition 14. "$\Longrightarrow$" (by contrapositive) Let $\mathcal{O}_{-\mathbf{1}}$ be the open orthant containing $-\mathbf{1}$. If there is no $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ with $\mathbf{c} \ll \mathbf{0}$, then $\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{-\mathbf{1}}$; thus, Theorem 3(a) says $\mathcal{X}$ is not McGarvey.

"$\Longleftarrow$" If $\mathcal{X}$ is comprehensive, then $\mathrm{conv}(\mathcal{X})$ is also comprehensive. That is, for all $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ and $\mathbf{r} \in [-1, 1]^{\mathcal{K}}$, if $\mathbf{c} \leq \mathbf{r}$, then $\mathbf{r} \in \mathrm{conv}(\mathcal{X})$ also. If $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ and $\mathbf{c} \ll \mathbf{0}$, then the set $\left\{\mathbf{r} \in [-1, 1]^{\mathcal{K}}\ ;\ \mathbf{r} \gg \mathbf{c}\right\} \subseteq \mathrm{conv}(\mathcal{X})$ is an open neighbourhood of $\mathbf{0}$; thus, Theorem 3(b) says $\mathcal{X}$ is McGarvey. □

Proof of Proposition 15. Proposition 1(b) says $\mathcal{X}$ is median-saturating if and only if $\mathcal{W}_2(\mathcal{X}) = \emptyset$. If $\mathcal{X}$ is comprehensive, then any $\mathcal{X}$-forbidden word must be all zeros. Thus, any element of $\mathcal{W}_2(\mathcal{X})$ has the form $(0_j, 0_k)$ for some $j, k \in \mathcal{K}$. Thus, $\mathcal{W}_2(\mathcal{X}) = \emptyset$ if and only if, for all $j, k \in \mathcal{K}$, there exists $\mathbf{x} \in \mathcal{X}$ with $x_j = 0 = x_k$. □

Proof of Proposition 16. First we must show that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_f)] \neq \emptyset$.

Claim 1: *If* $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_f)] = \emptyset$*, then there is some* $j \in \mathcal{J}$ *and* $s_j \in \{\pm 1\}$ *such that* $f(\mathbf{x}) = s_j x_j$ *for all* $\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}$.

Proof: If $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}_f)] = \emptyset$, then for all $(\mathbf{x}, y) \in \mathcal{X}_f$, the coordinate y must be an affine function of $\mathbf{x}$; in other words, f must be an affine function. Thus, there are constants $s_j \in \mathbb{R}$ for all $j \in \mathcal{J}$, and another constant $r \in \mathbb{R}$ such that $f(\mathbf{x}) = r + \sum_{j \in \mathcal{J}} s_j x_j$ for all $\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}$.

Claim 1.1: *For all* $j \in \mathcal{J}$*, we have* $s_j \in \{-1, 0, 1\}$.

Proof: Let $\mathcal{I} := \mathcal{J} \setminus \{j\}$, Fix $\mathbf{x}_\mathcal{I} \in \{\pm 1\}^\mathcal{I}$. Then either $f(\mathbf{x}_\mathcal{I}, -1_j) = f(\mathbf{x}_\mathcal{I}, 1_j)$, or $f(\mathbf{x}_\mathcal{I}, -1_j) = -f(\mathbf{x}_\mathcal{I}, 1_j)$. But clearly,

$$f(\mathbf{x}_\mathcal{I}, 1_j) - f(\mathbf{x}_\mathcal{I}, -1_j) \quad = \quad r + \sum_{i \in \mathcal{I}} s_i x_i + s_j(+1) \ - r - \sum_{i \in \mathcal{I}} s_i x_i - s_j(-1) \quad = \quad 2s_j.$$

Thus, if $f(\mathbf{x}_\mathcal{I}, -1_j) = f(\mathbf{x}_\mathcal{I}, 1_j)$, then $s_j = 0$. If $f(\mathbf{x}_\mathcal{I}, -1_j) = -f(\mathbf{x}_\mathcal{I}, 1_j)$, then $s_j = \pm 1$. $\triangledown$ `Claim 1.1`

Claim 1.2: *There is at most one $j \in \mathcal{J}$ such that $s_j \neq 0$.*

Proof: (by contradiction) Suppose $s_j \neq 0 \neq s_k$ for some $j \neq k \in \mathcal{J}$. Let $\mathcal{I} := \mathcal{J} \setminus \{j, k\}$.
Fix $\mathbf{x}_\mathcal{I} \in \{\pm 1\}^\mathcal{I}$. If $s_j = s_k$, then $f(\mathbf{x}_\mathcal{I}, 1_j, 1_k) - f(\mathbf{x}_\mathcal{I}, -1_j, -1_k) = s_j(1 + 1 - (-1 - 1)) = 4s_j$, which is impossible because $f(\{\pm 1\}^\mathcal{J}) \subseteq \{\pm 1\}$ while $s_j = \pm 1$ (by Claim 1.1).
If $s_j = -s_k$, then $f(\mathbf{x}_\mathcal{I}, -1_j, 1_k) - f(\mathbf{x}_\mathcal{I}, 1_j, -1_k) = s_k(-(-1) + 1 - (-1 - 1)) = 4s_k$, which is again impossible because $f(\{\pm 1\}^\mathcal{J}) \subseteq \{\pm 1\}$ while $s_k = \pm 1$ (by Claim 1.1).
Either way, we have a contradiction. Thus, either $s_j = 0$ or $s_k = 0$. $\triangledown$ `Claim 1.2`

Claim 1.2 implies that $f(\mathbf{x}) = s_j x_j + r$ for all $\mathbf{x} \in \{\pm 1\}^\mathcal{J}$. Claim 1.1 says that $s_j = \pm 1$, while $f(\mathbf{x}) = \pm 1$ and $x_j = \pm 1$ by definition. Thus, $r = 0$; hence $f(\mathbf{x}) = s_j\, x_j$. $\diamond$ `Claim 1`

Thus, if $f(\mathbf{x})$ depends nontrivially on more than one coordinate of $\mathbf{x}$, then the conclusion of Claim 1 is contradicted; hence $\operatorname{int}[\operatorname{conv}(\mathcal{X}_f)] \neq \emptyset$. Now,

$$\sum_{\mathbf{y} \in \mathcal{X}_f} \mathbf{y} \;=\; \sum_{\mathbf{x} \in \{\pm 1\}^\mathcal{J}} (\mathbf{x}, f(\mathbf{x})) \;=\; (\mathbf{0}_\mathcal{J}, 0) \;=\; \mathbf{0}_\mathcal{K},$$

because $\sum_{\mathbf{x} \in \{\pm 1\}^\mathcal{J}} f(\mathbf{x}) = 0$ by hypothesis, and clearly $\sum_{\mathbf{x} \in \{\pm 1\}^\mathcal{J}} \mathbf{x} = \mathbf{0}_\mathcal{J}$. Thus Lemma 26 implies that $\mathcal{X}_f$ is McGarvey. $\Box$

Proof of Proposition 17. Let $\mathbf{x} \in \{\pm 1\}^\mathcal{K}$; we want $\mu \in \Delta(\mathcal{X}_f)$ such that $\operatorname{maj}(\mu) = \mathbf{x}$. Recall $\mathcal{K} = \mathcal{J} \sqcup \{0\}$; write $\mathbf{x} = (\mathbf{x}_\mathcal{J}, x_0)$ for some $\mathbf{x}_\mathcal{J} \in \{\pm 1\}^\mathcal{J}$. Let $\mathcal{Y}_+ := f^{-1}\{1\}$ and $\mathcal{Y}_- := f^{-1}\{-1\}$; by hypothesis, both these spaces are McGarvey.

If $x_0 = 1$, then find some $\mu_\mathcal{J} \in \Delta(\mathcal{Y}_+)$ such that $\operatorname{maj}(\mu) = \mathbf{x}_\mathcal{J}$. Define $\mu \in \Delta(\mathcal{X})$ by $\mu(\mathbf{y}, 1) = \mu_\mathcal{J}(\mathbf{y})$ for all $\mathbf{y} \in \mathcal{Y}_+$. Then $\operatorname{maj}(\mu) = \mathbf{x}$. If $x_0 = -1$, then perform a similar construction using some $\mu_\mathcal{J} \in \Delta(\mathcal{Y}_-)$. $\Box$

Proof of Proposition 18. If f is monotone, then $f^{-1}\{1\}$ is a comprehensive subset of $\{\pm 1\}^\mathcal{J}$. Thus, hypothesis #1 and Proposition 14 imply that $f^{-1}\{1\}$ is McGarvey.

If f is monotone, then $-f^{-1}\{-1\}$ is also a comprehensive subset of $\{\pm 1\}^\mathcal{J}$. Thus, hypothesis #2 and Proposition 14 imply that $f^{-1}\{-1\}$ is McGarvey.

At this point, Proposition 17 implies that $\mathcal{X}_f$ is McGarvey. $\Box$

Proof of Proposition 20. (a) "$\Longrightarrow$" It suffices to show that, for any $j \in \mathcal{J}$, there is some $\mathcal{C}_j^* \in \mathfrak{C}$ such that $j \in \mathcal{C}_j^* \subseteq \mathcal{J}$; it follows that $\mathcal{J}$ is a union of $\mathfrak{C}$-elements.

Let $\mu \in \Delta^*(\mathcal{X}_\mathfrak{C})$ be such that $\operatorname{maj}(\mu) = \boldsymbol{\chi}^\mathcal{J}$. Let $j \in \mathcal{J}$. Then $\operatorname{maj}_j(\mu) = 1$, so $\widetilde{\mu}_j > \frac{1}{2}$. Let $\mathfrak{C}_j := \{\mathcal{C} \in \mathfrak{C}\ ;\ j \in \mathcal{C}\}$; then $\widetilde{\mu}_j = \sum_{\mathcal{C} \in \mathfrak{C}_j} \mu(\boldsymbol{\chi}^\mathcal{C}) - \sum_{\mathcal{C} \in \mathfrak{C} \setminus \mathfrak{C}_j} \mu(\boldsymbol{\chi}^\mathcal{C})$. Let $\mathcal{C}_j^* = \bigcap_{\mathcal{C} \in \mathfrak{C}_j} \mathcal{C}$; then $\mathcal{C}_j^* \in \mathfrak{C}$, and for all $k \in \mathcal{C}_j^*$, we have $\widetilde{\mu}_k \geq \sum_{\mathcal{C} \in \mathfrak{C}_j} \mu(\boldsymbol{\chi}^\mathcal{C}) - \sum_{\mathcal{C} \in \mathfrak{C} \setminus \mathfrak{C}_j} \mu(\boldsymbol{\chi}^\mathcal{C}) = \widetilde{\mu}_j > \frac{1}{2}$; hence $\operatorname{maj}_k(\mu) = 1$, which means $k \in \mathcal{J}$. Thus, $\mathcal{C}_j^* \subseteq \mathcal{J}$, as claimed.

"$\Longleftarrow$" Let $\mathcal{C}_1, \ldots, \mathcal{C}_N \in \mathfrak{C}$, and let $\mathcal{J} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_N$; we will construct $\mu \in \Delta^*(\mathcal{X}_\mathfrak{C})$ such that $\operatorname{maj}(\mu) = \boldsymbol{\chi}^\mathcal{J}$. Define $\mu \in \Delta^*(\mathcal{X}_\mathfrak{C})$ as follows:

- Set $\mu[\mathbf{1}] := \dfrac{N-1}{2N-1}$.
- For all $n \in [1...N]$, set $\mu[\boldsymbol{\chi}^{\mathcal{C}_n}] := \dfrac{1}{2N-1}$.

Thus, for all $n \in [1...N]$ and $j \in \mathcal{C}_n$, we have $\widetilde{\mu}_j \geq 2\left(\frac{N-1}{2N-1} + \frac{1}{2N-1}\right) - 1 = \frac{1}{2N-1} > 0$, whereas for all $k \in \mathcal{K} \setminus \mathcal{J}$, we have $\widetilde{\mu}_j = 2\left(\frac{N-1}{2N-1}\right) - 1 = \frac{-1}{2N-1} < 0$. Thus, $\mathrm{maj}(\mu) = \boldsymbol{\chi}^{\mathcal{J}}$.

(b) "[i]$\Longrightarrow$[ii]" is immediate because equation (3) asserts $\mathrm{maj}(\mathcal{X}) \subseteq \mathrm{med}^{\infty}(\mathcal{X})$.

"[ii]$\Longrightarrow$[iii]" (by contrapositive) Let $k \in \mathcal{K}$, but suppose $\{k\} \notin \mathfrak{C}$. Define $\mathcal{C}_k^*$ as in part (a); then $k \in \mathcal{C}_k^*$ and $\mathcal{C}_k^*$ is the smallest element of $\mathfrak{C}$ which contains k. Now, $\mathcal{C}_k^* \neq \{k\}$, because $\{k\} \notin \mathfrak{C}$. Thus, there exists $j \in \mathcal{C}_k^* \setminus \{k\}$. Define the word $\mathbf{w} \in \{\pm 1\}^{\{k,j\}}$ by $w_k = 1$ and $w_j = -1$; then $\mathbf{w}$ is $\mathcal{X}_{\mathfrak{C}}$-forbidden. Thus, $\mathcal{W}_2(\mathcal{X}_{\mathfrak{C}}) \neq \emptyset$; thus, Proposition 1(b) implies that $\mathcal{X}_{\mathfrak{C}}$ is not median-saturating.

"[iii]$\Longrightarrow$[i]" follows immediately from part (a), because any subset of $\mathcal{K}$ can be written as a union of singleton sets. □

Proof of Theorem 21. "$S(\mathcal{X}) \leq 4(K+1)\, N(\mathcal{X})$" Let $\mathcal{U} \subset \mathrm{conv}(\mathcal{X})$, and let $\epsilon > 0$. We say that $\mathcal{U}$ is *ϵ-dense* in $\mathrm{conv}(\mathcal{X})$ if, for all $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$, there exists some $\mathbf{u} \in \mathcal{U}$ with $\|\mathbf{u} - \mathbf{c}\|_{\infty} < \epsilon$.

Claim 1: *For any $M \in \mathbb{N}$, let $\mathcal{C}_M := \{\widetilde{\mu}\ ;\ \mu \in \Delta_M^*(\mathcal{X})\}$. Then $\mathcal{C}_M$ is a $\left(\frac{2(K+1)}{M}\right)$-dense subset of* $\mathrm{conv}(\mathcal{X})$.

Proof: Let $\mathbb{Q}_M := \left\{\frac{n}{M}\ ;\ n \in \mathbb{N}\right\}$, and let $\mathbb{Q}_M^{\mathcal{X}}$ be the set of all functions $\mu : \mathcal{X} \longrightarrow \mathbb{Q}_M$ (thus, $\Delta_M^*(\mathcal{X}) \subset \mathbb{Q}_M$). For any $r \in \mathbb{R}_+$, we define $\lfloor r \rfloor_M := \dfrac{\lfloor M\, r \rfloor}{M}$; this is the largest element of the set $\mathbb{Q}_M$ which is no greater than r. Note that $0 \leq r - \lfloor r \rfloor_M \leq 1/M$.
Let $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$; we must find some $\mu \in \Delta_M^*(\mathcal{X})$ such that $\|\widetilde{\mu} - \mathbf{c}\|_{\infty} < 2(K+1)/M$.
Carathéodory's theorem says there exists some subset $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| = K+1$, and some $\nu \in \Delta(\mathcal{Y})$, such that $\widetilde{\nu} = \mathbf{c}$. Now define $\lambda \in \mathbb{Q}_M^{\mathcal{Y}}$ by $\lambda(\mathbf{y}) := \lfloor \nu(\mathbf{y}) \rfloor_M$ for all $\mathbf{y} \in \mathcal{Y}$. Let

$$q \quad := \quad \sum_{\mathbf{y} \in \mathcal{Y}} \Big|\nu(\mathbf{y}) - \lambda(\mathbf{y})\Big| \quad \leq \quad \frac{|\mathcal{Y}|}{M} \quad = \quad \frac{K+1}{M}. \tag{7}$$

Then

$$\left\|\widetilde{\lambda} - \mathbf{c}\right\|_{\infty} \quad = \quad \left\|\widetilde{\lambda} - \widetilde{\nu}\right\|_{\infty} \quad \leq \quad q. \tag{8}$$

Observe that

$$\begin{aligned} 1 - \sum_{\mathbf{y} \in \mathcal{Y}} \lambda(\mathbf{y}) \quad &= \quad \sum_{\mathbf{y} \in \mathcal{Y}} \nu(\mathbf{y}) - \sum_{\mathbf{y} \in \mathcal{Y}} \lambda(\mathbf{y}) \quad = \quad \sum_{\mathbf{y} \in \mathcal{Y}} \Big(\nu(\mathbf{y}) - \lambda(\mathbf{y})\Big) \\ &= \quad \sum_{\mathbf{y} \in \mathcal{Y}} \Big|\nu(\mathbf{y}) - \lambda(\mathbf{y})\Big| \quad = \quad q. \end{aligned} \tag{9}$$

Thus, $q \in \mathbb{Q}_M$ (because $\lambda \in \mathbb{Q}_M^{\mathcal{Y}}$). However, in general $q > 0$, so $\lambda \notin \Delta^*(\mathcal{X})$. Fix some $\mathbf{y}_0 \in \mathcal{Y}$, and define $\mu \in \Delta_M^*(\mathcal{X})$ as follows: $\mu(\mathbf{y}_0) := \lambda(\mathbf{y}_0) + q \in \mathbb{Q}_M$, and $\mu(\mathbf{y}) := \lambda(\mathbf{y})$ for all other $\mathbf{y} \in \mathcal{Y} \setminus \{\mathbf{y}_0\}$ (and of course $\mu(\mathbf{x}) := 0$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{Y}$). Then equation (9) implies that $\sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{Y}} \mu(\mathbf{y}) = 1$, so $\mu \in \Delta_M^*(\mathcal{X})$. Furthermore,

$$\left\|\widetilde{\mu} - \widetilde{\lambda}\right\|_{\infty} \quad \leq \quad |\mu(\mathbf{y}_0) - \lambda(\mathbf{y}_0)| \quad = \quad q. \tag{10}$$

Combining equations (7), (8), and (10), we have $\|\widetilde{\mu} - \mathbf{c}\|_{\infty} \leq \left\|\widetilde{\mu} - \widetilde{\lambda}\right\|_{\infty} + \left\|\widetilde{\lambda} - \mathbf{c}\right\|_{\infty} \leq q + q \leq 2(K+1)/M$, as desired. ◇ Claim 1

Now, let $M := 4(K+1)\,N(\mathcal{X})$; Then $\text{conv}(\mathcal{X})$ contains the ball $\mathcal{B}\left(\frac{4(K+1)}{M}\right)$. Given $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, let $\mathbf{x}' := \frac{2(K+1)}{M}\mathbf{x}$; then $\text{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}}$ must contain the ball $\mathcal{B}' := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}};\ \|\mathbf{r} - \mathbf{x}'\|_\infty \le \frac{2(K+1)}{M}\}$. But $\mathcal{C}_M$ is $(\frac{2(K+1)}{M})$-dense in $\text{conv}(\mathcal{X})$ (by Claim 1), so $\mathcal{C}_M$ must intersect $\mathcal{B}'$. Thus, $\mathcal{C}_M$ intersects $\text{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}}$; thus, there is some $\mu \in \Delta^*_M(\mathcal{X})$ with $\text{maj}(\mu) = \mathbf{x}$.

"$N(\mathcal{X}) \le S(\mathcal{X})$" For every $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, there exists $N \le S(\mathcal{X})$ and some $\mu^{\mathbf{x}} \in \Delta^*_N(\mathcal{X})$ such that $\text{maj}(\mu^{\mathbf{x}}) = \mathbf{x}$. This means that $\widetilde{\mu}^{\mathbf{x}} \in \mathcal{O}_{\mathbf{x}}$. However, if $\mu \in \Delta^*_N(\mathcal{X})$, then every coordinate of $\widetilde{\mu}$ is an integer multiple of $1/N$. Thus, if $\widetilde{\mu} \in \mathcal{O}_{\mathbf{x}}$, then $\widetilde{\mu}_k \ge x_k/N \ge x_k/S(\mathcal{X})$ for all $k \in \mathcal{K}$ (and recall $x_k = \pm 1$). Thus, if $\mathcal{C} = \text{conv}\{\widetilde{\mu}^{\mathbf{x}};\ \mathbf{x} \in \{\pm 1\}^{\mathcal{K}}\}$, then $\mathcal{B}\left(\frac{1}{S(\mathcal{X})}\right) \subseteq \mathcal{C} \subseteq \text{conv}(\mathcal{X})$. Thus, $S(\mathcal{X}) \ge N(\mathcal{X})$. □

Proof of Proposition 22. (a) If $\mathcal{X}$ is McGarvey, then $\mathbf{0} \in \text{int}\,[\text{conv}\,(\mathcal{X})]$. Thus, the boundary of $\text{conv}(\mathcal{X})$ does not include $\mathbf{0}$. The boundary of $\text{conv}(\mathcal{X})$ is a union of faces, each of the form $\text{conv}(\mathbf{x}^1, \ldots, \mathbf{x}^K)$ for some $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \mathcal{X}$.

Now, if $M := \lceil 1/\delta(\mathcal{X}) \rceil$, then $\frac{1}{M} \le \delta(\mathcal{X})$. Thus, $\mathcal{B}(\frac{1}{M})$ is disjoint from every boundary face of $\mathcal{X}$. Thus, $\mathcal{B}(\frac{1}{M}) \subseteq \text{conv}(\mathcal{X})$. Thus, $M \ge N(\mathcal{X})$.

(b) Let $\delta := \delta(K)$. For all McGarvey $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$, we have

$$S(\mathcal{X}) \quad \underset{(\dagger)}{\le} \quad 4(K+1)\,N(\mathcal{X}) \quad \underset{(@)}{\le} \quad 4(K+1)\,\lceil 1/\delta(\mathcal{X}) \rceil \quad \underset{(*)}{\le} \quad 4(K+1)\,\lceil 1/\delta \rceil,$$

where (†) is by Theorem 21, (@) is by part (a), and (∗) is because $\delta(\mathcal{X}) \ge \delta$ for any $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ (by their definitions).

Now, find $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$ such that $\delta(\mathbf{x}^1, \ldots, \mathbf{x}^K) = \delta$, and let $\mathbf{y} \in \text{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ be such that $\|\mathbf{y}\|_\infty = \delta$. Let $\mathbf{z} \in \{\pm 1\}^{\mathcal{K}}$ be such that $\mathbf{y} \in \mathcal{O}_{\mathbf{z}}$. Let $\mathcal{P} \subset \mathbb{R}^{\mathcal{K}}$ be the hyperplane containing $\text{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$; then $\mathcal{P}$ cuts $\mathbb{R}^{\mathcal{K}}$ into two open halfspaces, $\mathcal{H}^+$ and $\mathcal{H}^-$, where $\mathbf{z} \in \mathcal{H}^+$ and $\mathbf{0} \in \mathcal{H}^-$. Let $\mathcal{X}' := \{\pm 1\}^{\mathcal{K}} \cap (\mathcal{H}^- \cup \mathcal{P})$. Then $\mathcal{X}'$ is McGarvey (because $\mathbf{0} \in \text{int}\,[\text{conv}\,(\mathcal{X})]$). Also, $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \mathcal{X}'$, and $\text{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ is one of the boundary faces of $\text{conv}(\mathcal{X})$ (because $\text{conv}(\mathcal{X}) \subset \mathcal{H}^- \cup \mathcal{P}$). Thus, $N(\mathcal{X}') \ge 1/\delta$ (because $\mathbf{y} \in \text{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$). Thus $S(\mathcal{X}') \ge 1/\delta$, by Theorem 21.

(c) Without loss of generality, let $\mathcal{K} = [1...K]$. If $\mathbf{B} := [b_{jk}]_{j,k \in \mathcal{K}}$ is a $K \times K$ matrix, then let $\|\mathbf{B}\|_\infty := \max_{j,k \in \mathcal{K}} |b_{j,k}|$. We then define $\chi(K) := \max\{\|\mathbf{A}^{-1}\|_\infty;$ any invertible matrix $\mathbf{A} \in \{\pm 1\}^{K \times K}\}$. We will use a result of Alon and Vũ [14], which says that

$$\frac{K^{K/2}}{2^{3K+o(K)}} \quad \le \quad \chi(K) \quad \le \quad \frac{K^{K/2}}{2^{2K-1}}. \tag{11}$$

Left-hand inequality. Let $\mathbf{A} \in \{\pm 1\}^{K \times K}$ be such that $\|\mathbf{A}^{-1}\|_\infty = \chi(K)$. Let $\mathbf{B} := \mathbf{A}^{-1}$, and find $\ell, m \in [1...K]$ such that $|b_{\ell m}| = \chi(K)$.

Let $\mathbf{y} := \mathbf{B} \cdot \mathbf{1}$. For any $k \in [1...K]$, if $\mathbf{A}'$ is obtained by negating the kth row of $\mathbf{A}$, then $(\mathbf{A}')^{-1}$ is obtained by negating the kth column of $\mathbf{B}$, which in particular negates $b_{\ell k}$. By negating the rows of $\mathbf{A}$ and columns of $\mathbf{B}$ as required, we can assume that $b_{\ell k} \ge 0$ for all $k \in [1...K]$. Thus, $y_\ell = \sum_{k=1}^{K} b_{\ell k} \ge b_{\ell m} = \chi(K)$.

For any $k \in [1...K]$, if $\mathbf{A}'$ is obtained by negating the kth column of $\mathbf{A}$, then $(\mathbf{A}')^{-1}$ is obtained by negating the kth row of $\mathbf{B}$, and hence, the kth entry in $\mathbf{y}$. By negating the columns of $\mathbf{A}$ and rows of $\mathbf{B}$ as required, we can assume that $\mathbf{y} \in \mathbb{R}^K_{\not-}$. Thus, if $Y := \sum_{j=1}^{K} y_j$, then $Y \ge y_\ell \ge \chi(K)$.

Let $\mathbf{s} := \frac{1}{Y}\,\mathbf{y}$; then $\mathbf{s} \in \mathbb{R}^K_{\not-}$ and $\sum_{k=1}^{K} s_k = 1$. Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$ be the column vectors of $\mathbf{A}$; then $\mathbf{0} \notin \text{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$, because $\mathbf{A}$ is invertible. Now, $\mathbf{A}\,\mathbf{s} = \sum_{k=1}^{K} s_k \mathbf{x}^k$, so $\mathbf{A}\,\mathbf{s} \in \text{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$.

However, $\mathbf{A}\,\mathbf{s} = \frac{1}{Y}\,\mathbf{1}$, so $\delta(\mathbf{x}^1, \ldots, \mathbf{x}^K) \leq \|\mathbf{A}\mathbf{s}\|_\infty = \frac{1}{Y}$. Thus,

$$\frac{1}{\delta(K)} \quad \geq \quad \frac{1}{\delta(\mathbf{x}^1, \ldots, \mathbf{x}^K)} \quad \geq \quad Y \quad \geq \quad \chi(K) \quad \underset{(*)}{\geq} \quad \frac{K^{K/2}}{2^{3K+o(K)}},$$

where $(*)$ is by the left-hand Alon-Vũ inequality (11).

Right-hand inequality. Let $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$ be any points such that $\mathbf{0} \notin \operatorname{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$, and let $\mathbf{c} \in \operatorname{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ be such that $\|\mathbf{c}\|_\infty = \delta(\mathbf{x}^1, \ldots, \mathbf{x}^K)$. Let $\mathbf{A}$ be the $K \times K$ matrix whose columns are $\mathbf{x}^1, \ldots, \mathbf{x}^K$; then $\mathbf{c} = \mathbf{A}\,\mathbf{s}$ for some $\mathbf{s} \in \mathbb{R}^K_{\not-}$ with $\sum_{k=1}^K s_k = 1$. By negating the rows of $\mathbf{A}$ if necessary, we can assume $\mathbf{c} \in \mathbb{R}^K_{\not-}$; then $\mathbf{c} = \delta\,\mathbf{1}$, where $\delta := \delta(\mathbf{x}^1, \ldots, \mathbf{x}^K)$. If $\mathbf{B} = \mathbf{A}^{-1}$, then $\mathbf{s} = \delta\,\mathbf{B}\,\mathbf{1}$. Thus,

$$1 \quad = \quad \sum_{j=1}^K s_j \quad = \quad \delta \sum_{j=1}^K \sum_{k=1}^K b_{jk} \quad \leq \quad \delta\, K^2 \cdot \chi(K).$$

$$\text{Thus,} \qquad \frac{1}{\delta} \quad \leq \quad K^2 \cdot \chi(K) \quad \underset{(*)}{\leq} \quad \frac{K^{2+K/2}}{2^{2K-1}},$$

where $(*)$ is by the right-hand Alon-Vũ inequality (11). Since this holds for all $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$, we conclude that $\dfrac{1}{\delta(K)} \leq \dfrac{K^{2+K/2}}{2^{2K-1}}$, as claimed. □

Proof of Proposition 23. (a) (Similar to the proof of Proposition 20(a) "$\Longleftarrow$".) Given $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, let $\mathcal{J} := \{j \in \mathcal{K}\ ;\ x_j = 1\}$ and let $J := |\mathcal{J}|$. Define $\mu \in \Delta^*_{2J-1}(\mathcal{X})$ as follows:

- Set $\mu[\mathbf{1}] := \dfrac{J-1}{2J-1}$.
- For all $j \in \mathcal{J}$, set $\mu[\boldsymbol{\chi}^j] := \dfrac{1}{2J-1}$.

Thus, for all $j \in \mathcal{J}$ we have $\widetilde{\mu}_j = \frac{1}{2J-1}$, whereas for all $k \in \mathcal{K} \setminus \mathcal{J}$, we have $\widetilde{\mu}_j = \frac{-1}{2J-1}$. Thus, $\operatorname{maj}(\mu) = \mathbf{x}$. This works for any $\mathbf{x} \in \mathcal{X}$;

(b) Suppose without loss of generality that $\mathcal{K} = [1 \ldots K]$. For all $k \in \mathcal{K}$, let $\mathbf{e}^k := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the "1" appears in the kth coordinate. By hypothesis, there exist $\mathbf{x}^k, \mathbf{y}^k \in \mathcal{X}$ such that $x^k_k = 1 = y^k_k$, but $\mathbf{x}^k$ and $\mathbf{y}^k$ differ in every other coordinate. Thus, $\frac{1}{2}(\mathbf{x}^k + \mathbf{y}^k) = \mathbf{e}^k$.

Now, let $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ be arbitrary. Let $\mathcal{J} := \{j \in \mathcal{K}\ ;\ x_j = 1\}$ and let $J := |\mathcal{J}|$. Define $\mu \in \Delta_{2J+1}(\mathcal{X})$ as follows:

- $\mu(\mathbf{x}^j) = \mu(\mathbf{y}^j) = 1/(2J+1)$ for all $j \in \mathcal{J}$.
- $\mu(-\mathbf{1}) = 1/(2J+1)$

Thus, for all $j \in \mathcal{J}$, we have $\widetilde{\mu}_j = 2/(2J+1) - 1/(2J+1) = 1/(2J+1) > 0$. Meanwhile for all $k \in \mathcal{K} \setminus \mathcal{J}$, we have $\widetilde{\mu}_k = -1/(2J+1) < 0$. Thus, $\operatorname{maj}(\mu) = \mathbf{x}$, as desired.

(c) For all $k \in \mathcal{K}$, let $\mathbf{e}^k$ be as in part (b). By hypothesis, there exist $\mathbf{x}^k, \mathbf{y}^k \in \mathcal{X}$ such that $x^k_k \neq y^k_k$, but $\mathbf{x}^k$ and $\mathbf{y}^k$ agree in every other coordinate. Now $-\mathcal{X} = \mathcal{X}$, so $-\mathbf{y}^k \in \mathcal{X}$ also. Note that $x^k_k = -y^k_k$, and $\mathbf{x}^k$ and $-\mathbf{y}^k$ differ in every other coordinate. Thus, $\frac{1}{2}(\mathbf{x}^k - \mathbf{y}^k) = s_k\,\mathbf{e}^k$, for some $s_k \in \{\pm 1\}$. Likewise, $-\mathbf{x}^k \in \mathcal{X}$, and $\frac{1}{2}(\mathbf{y}^k - \mathbf{x}^k) = -s_k\,\mathbf{e}^k$.

Now, given any $\mathbf{z} \in \{\pm 1\}^{\mathcal{K}}$, define $\mu \in \Delta^*_{2K}(\mathcal{X})$ as follows. For all $k \in \mathcal{K}$,

- Set $\mu[\mathbf{x}^k] := \mu[-\mathbf{y}^k] := \frac{1}{2K}$ if $z_k = s_k$.
- Set $\mu[-\mathbf{x}^k] := \mu[\mathbf{y}^k] := \frac{1}{2K}$ if $z_k = -s_k$.

Thus, for every $k \in \mathcal{K}$, we have $\widetilde{\mu}_k = \frac{z_k}{K}$, so $\operatorname{maj}(\mu) = \mathbf{z}$, as desired. □

Appendix B: More examples.

This appendix contains further examples of some of the themes of this paper. First, here is another class of 'minimal' McGarvey spaces, somewhat different to the class presented in Example 5.

Example 27. Let $K = 2N+1$, and let $\mathcal{K} = [0\ldots 2N]$. Define $\mathbf{x}^0 := (+1;\ +1,-1,+1,-1,+1,-1,\ldots,+1,-1)$. In other words, set $x_0^0 := 1$, and for all $k \in [1\ldots 2N]$, set $x_k^0 := 1$ if k is odd, while $x_k^0 := -1$ if k is even. Define $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{2N}$ by cyclically permuting the coordinates of $\mathbf{x}^0$ (i.e. identify $\mathcal{K}$ with the group $\mathbb{Z}_{/K}$). Let $\mathcal{X} := \{\pm\mathbf{x}^0, \pm\mathbf{x}^1, \ldots, \pm\mathbf{x}^{2N}\}$. Then $|\mathcal{X}| = 2K$.

Claim 1: *$\mathcal{X}$ is minimal McGarvey.*

Proof: $\mathcal{X}$ is McGarvey: It can be checked that $\mathrm{span}(\mathcal{X} - \mathcal{X}) = \mathbb{R}^{\mathcal{K}}$, so $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})] \neq \emptyset$.

Recall that $\mathcal{K} := [0\ldots 2N]$. In this case, $\Pi_{\mathcal{X}}$ consists of all cyclic permutations of $\mathcal{K}$ (obtained by identifying $\mathcal{K}$ with the group $\mathbb{Z}_{/K}$); Thus, $\Pi_{\mathcal{X}}$ is transitive. Clearly $\#(\mathbf{x}^0) = N+1 > K/2$, whereas $\#(-\mathbf{x}^0) = N < K/2$. Thus, Corollary 12 implies that $\mathcal{X}$ is McGarvey.

No proper subset of $\mathcal{X}$ is McGarvey: Let $\mathcal{Y} := \mathcal{X} \setminus \{\mathbf{x}^0\}$. To see that $\mathcal{Y}$ is not McGarvey, let $\mathbf{z} := (1,1,0,0,\ldots,0)$. Then $\mathbf{z} \bullet \mathbf{y} \leq 0$ for all $\mathbf{y} \in \mathcal{Y}$. Thus, Theorem 3(b) implies that $\mathcal{Y}$ is not McGarvey.

A similar argument shows that the sets $\mathcal{X} \setminus \{\mathbf{x}^k\}$ and $\mathcal{X} \setminus \{-\mathbf{x}^k\}$ are not McGarvey, for any $k \in \mathcal{K}$. $\diamondsuit$ `Claim 1`

In particular, if $K = 3$, then once again, $\mathcal{X} = \{(1,1,-1),\ (1,-1,1),\ (-1,1,1),\ (-1,-1,1),\ (-1,1,-1),\ (1,-1,-1)\}$ is a minimal McGarvey set with six elements. Let $\mathcal{A} := \{a,b,c\}$ and identify $\mathcal{K}$ with the set $\{(a,b),(b,c),(c,a)\}$; then $\mathcal{X} = \mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$. $\diamondsuit$

Next, here are two more applications of Corollary 12.

Example 28. (*Connected digraphs*) Let $\mathcal{N}$ be a finite set, and let $\mathcal{K} := \{(n,m) \in \mathcal{N} \times \mathcal{N};\ n \neq m\}$. Thus, an element of $\{\pm 1\}^{\mathcal{K}}$ can represent a directed graph (***digraph***) with vertex set $\mathcal{N}$. For any permutation $\pi : \mathcal{N} \longrightarrow \mathcal{N}$, define $\pi_* : \mathcal{K} \longrightarrow \mathcal{K}$ by $\pi(n,m) := (\pi(n), \pi(m))$ for all $(n,m) \in \mathcal{K}$. Let Π_* be the set of all such permutations; then Π_* acts transitively on $\mathcal{K}$ (for any $(n_1,m_1) \in \mathcal{K}$ and $(n_2,m_2) \in \mathcal{K}$, let $\pi : \mathcal{N} \longrightarrow \mathcal{N}$ be any permutation such that $\pi(n_1) = n_2$ and $\pi(m_1) = m_2$; then $\pi_*(n_1,m_1) = (n_2,m_2)$).

A digraph is ***connected*** if any two vertices can be connected with a directed path. Let $\mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of connected digraphs. Then $\Pi_{\mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}}}$ is transitive, because it contains Π_*.

Through a similar argument to Example 13(c), one can show that $\mathrm{int}\left[\mathrm{conv}\left(\mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}}\right)\right] \neq \emptyset$. There exists $\mathbf{x} \in \mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}}$ with $\#(\mathbf{x}) < K/2$ (for example, let $\mathbf{x}$ represent a digraph where the elements of $\mathcal{N}$ are arranged in a directed loop —then $\#(\mathbf{x}) = |\mathcal{N}| < K/2$). There also exists $\mathbf{y} \in \mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}}$ with $\#(\mathbf{y}) > K/2$ (for example: $\mathbf{1} \in \mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}}$). Thus, Corollary 12 says that $\mathcal{X}^{\vec{\mathrm{cnct}}}_{\mathcal{N}}$ is McGarvey. $\diamondsuit$

Example 29. (*Committee Selection*) As in Example 24(e), let $\mathcal{K}$ be a set of 'candidates', so that any element of $\{\pm 1\}^{\mathcal{K}}$ represents a 'committee' formed from these candidates. Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_L$ be disjoint subsets of $\mathcal{K}$, with cardinalities $K_1, K_2, \ldots, K_L$, respectively. Let $\mathcal{N} \subseteq [0...K]$, and for all $\ell \in [1...L]$, let $\mathcal{N}_\ell \subseteq [0...K_\ell]$. For any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ and $\ell \in [1...L]$, recall that $\#_\ell(\mathbf{x}) := \#\{k \in \mathcal{K}_\ell\ ;\ x_k = 1\}$. Consider the set:

$$\mathcal{X}^{\mathrm{com}} \quad := \quad \left\{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}\ ;\ \#(\mathbf{x}) \in \mathcal{N} \text{ and } \#_\ell(\mathbf{x}) \in \mathcal{N}_\ell,\ \forall\ \ell \in [1...L]\right\}.$$

Thus, $\mathcal{X}^{\mathrm{com}}$ represents the set of all committees formed from the candidates in $\mathcal{K}$, with certain restrictions on the size of the whole committee, and also certain restrictions on the level of representation from various 'constituencies' $\mathcal{K}_1, \ldots, \mathcal{K}_L$. For example, $\mathcal{N}$ might be a subinterval of $[0...K]$, encoding a minimum and/or maximum size for the whole committee. Also, we might restrict $\mathcal{N}$ to contain only odd values (e.g. to reduce the likelihood of tied votes). Meanwhile, $\mathcal{N}_\ell$ might be a subinterval of $[0...K_\ell]$, encoding minimum and/or maximum admissible levels of representation from constituency $\mathcal{K}_\ell$.

(a) Suppose that $\mathrm{int}\,[\mathrm{conv}\,(\mathcal{X}^{\mathrm{com}})] \neq \emptyset$, and also that:

(a1) $\mathcal{K} = \bigsqcup_{\ell=1}^{L} \mathcal{K}_\ell$; (a2) $K_1 = K_2 = \cdots = K_L = \dfrac{K}{L}$;

(a3) $\mathcal{N}_1 = \cdots = \mathcal{N}_L = \mathcal{N}_*$ for some subset $\mathcal{N}_* \subseteq \left[0 \ldots \frac{K}{L}\right]$; and

(a4) If $\mathcal{N}_\dagger := \mathcal{N} \cap \{n_1 + \cdots + n_L \; ; \; n_1, \ldots, n_L \in \mathcal{N}_*\}$, then $\min(\mathcal{N}_\dagger) < K/2 < \max(\mathcal{N}_\dagger)$.

Then $\mathcal{X}^{\rm com}$ is McGarvey. To see this, let $\pi : \mathcal{K} \longrightarrow \mathcal{K}$ be any permutation. Suppose that, for all $\ell \in [1...L]$, there is some $i \in [1...L]$ such that $\pi(\mathcal{K}_\ell) = \mathcal{K}_i$. Then $\pi \in \Pi_{\mathcal{X}^{\rm com}}$ by (a3). The set of all such permutations is transitive (by (a1) and (a2)). Thus, $\Pi_{\mathcal{X}^{\rm com}}$ is transitive. Meanwhile, (a4) means that there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\rm com}$ such that $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$. Thus, Corollary 12 implies that $\mathcal{X}^{\rm com}$ is McGarvey.

(b) More generally, let K_* be the largest divisor of K which is no greater than $\min\{K_1, \ldots, K_L\}$. Let $\mathcal{N}_* := \mathcal{N}_1 \cap \cdots \cap \mathcal{N}_L \cap [0...K_*]$. Suppose that $\mathcal{N}_* \neq \emptyset$, and suppose condition (a4) holds (in particular, we suppose $\mathcal{N}_\dagger \neq \emptyset$). Then $\mathcal{X}^{\rm com}$ is McGarvey.

To see this, for each $\ell \in [1...L]$, let $\mathcal{K}'_\ell \subseteq \mathcal{K}_\ell$ be a subset with $|\mathcal{K}'_\ell| = K_*$. Let $Q := K/K_*$ (an integer), and find $Q - L$ further disjoint subsets $\mathcal{K}'_{L+1}, \ldots, \mathcal{K}'_Q \subset \mathcal{K}$ such that $\mathcal{K} = \bigsqcup_{q=1}^{Q} \mathcal{K}'_q$. Define $\mathcal{N}'_1 = \cdots = \mathcal{N}'_Q := \mathcal{N}_*$. Let $\mathcal{X}'$ be the committee space constructed using the constituencies $\mathcal{K}'_1, \ldots, \mathcal{K}'_Q$ and the cardinality constraint sets $\mathcal{N}, \mathcal{N}'_1, \ldots, \mathcal{N}'_Q$. Then $\mathcal{X}' \neq \emptyset$ (because $\mathcal{N}_\dagger \neq \emptyset$), and $\mathcal{X}'$ satisfies the hypotheses of Example (a), so $\mathcal{X}'$ is McGarvey. But $\mathcal{X}' \subseteq \mathcal{X}^{\rm com}$; hence $\mathcal{X}^{\rm com}$ is also McGarvey. $\diamondsuit$

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