Goodness-of-fit testing for regime-switching models

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Abstract

In this paper we propose a novel goodness-of-fit testing scheme for regime-switching models. We consider models with an observable, as well as, a latent state process. The test is based on the Kolmogorov-Smirnov supremum-distance statistic and the concept of the weighted empirical distribution function. We apply the proposed scheme to test whether a 2-state Markov regime-switching model fits electricity spot price data.

Keywords: Regime-switching, Goodness-of-fit, Weighted empirical distribution function, Kolmogorov-Smirnov test.

1. Introduction

Regime-switching models have attracted a lot of attention in recent years. A flexible specification allowing for abrupt changes in model dynamics has led to its popularity in many fields including economics (Hamilton, 1990), population dynamics (Luo and Mao, 2007), speech recognition (Juang and Rabiner, 1985), river flow analysis (Vasas et al., 2007) and traffic modeling (Cetin and Comert, 2006). Yet despite this popularity, the statistical verification of regime-switching models is often neglected. But a statistical model cannot be reliable, if it does not fit empirical data. Derivation of appropriate goodness-of-fit testing techniques is needed.

Recent work concerning the statistical fit of regime-switching models has been mainly devoted to testing parameter stability versus regime-switching hypothesis. There have been several tests developed for verification of the number of regimes. Most of them are based on the likelihood ratio (LR) technique (Garcia, 1998; Cho and White, 2007), but there are also approaches related to recurrence times (Sen and Hsieh, 2009) or the information matrix (Carrasco et al., 2004). Hamilton (1996) applied the score function technique for different tests of model misspecification, like omitted autocorrelation or omitted explanatory variables. However, to our best knowledge, appropriate procedures for goodness-of-fit testing of the distribution of regime-switching models have not been derived to date. With this paper we want to fill the gap. We propose an edf-based testing technique build on the Kolmogorov-Smirnov test. The testing procedure is developed for regime-switching models with an observable, as well as, a latent state process. The later involves application of the weighted empirical distribution function concept.

The paper is structured as follows. In section 2 we describe the structure of the analyzed regime-switching models and briefly explain the estimation process. In section 3 we introduce goodness-of-fit testing procedures appropriate for regime-switching models both with observable and latent state processes. Next, in section 4 we provide a simulation study and check the performance of the proposed technique. In section 5 we show how Markov regime-switching models and the described goodness-of-fit procedure can be applied to electricity spot prices. Finally, in section 6 we conclude.

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2. Regime-switching models

2.1. Model definition

Assume that the process X_t may be in one of L states (regimes) at time t, driven by an independent state process R_t. The possible specifications of the process R_t may be divided into two classes: those where the current state of the process is observable (like threshold models, e.g. TAR, STAR, SETAR) and those where it is latent. The most prominent specifications of the second group are the Markov regime-switching models (MRS), in which R_t is assumed to be a Markov chain. It is governed by the transition matrix $\mathbf{P}$ containing the probabilities p_{ij} of switching from regime i at time t to regime j at time $t+1$, for $i, j = \{1, 2, ..., L\}$:

$$\mathbf{P} = (p_{ij}) = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1L} \\ p_{21} & p_{22} & \cdots & p_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L1} & p_{L2} & \cdots & p_{LL} \end{pmatrix}, \text{ with } p_{ii} = 1 - \sum_{j \neq i} p_{ij}. \tag{1}$$

The state process R_t follows the Markov property. Therefore the current state R_t at time t depends on the past only through the most recent value R_{t-1}. The probability of being in regime j at time $t+m$ starting from regime i at time t is given by

$$P(R_{t+m} = j \mid R_t = i) = (\mathbf{P}')^m \cdot e_i, \tag{2}$$

where $\mathbf{P}'$ denotes the transpose of $\mathbf{P}$ and e_i denotes the ith column of the identity matrix.

Definitions of the separate regimes can be arbitrarily chosen depending on the modeling needs. However, in this paper we will focus on two commonly used specifications. The first one assumes that the process X_t is driven by independent regimes defined in one of two ways: as a mean-reverting AR(1) process or an i.i.d. sample with a specified distribution. In the second specification X_t is described by an AR(1) model with only parameters changing between different regimes.

Recall that the AR(1) time series model is defined as:

$$X_{t+1} = \alpha + (1-\beta)X_t + \sigma\epsilon_t, \tag{3}$$

where $\epsilon_t \sim N(0,1)$, i.e. it is a standard Gaussian random variable. Note, that (3) is a discrete time version of a continuous process given by the stochastic differential equation (SDE)

$$dX_t = (\alpha - \beta X_t)dt + \sigma dW_t \tag{4}$$

known as the Vasiček (1977) model.

2.2. Calibration

Calibration of regime-switching models with an observable state process simplifies to the problem of estimating separate regime's parameters. In case of MRS models, though, it is not straightforward, since the state process is latent and not directly observable. In this paper we use the Expectation-Maximization (EM) algorithm that was first applied to MRS models by Hamilton (1990) and later refined by Kim (1994). It is a two-step iterative procedure, reaching a local maximum of the likelihood function. The steps are as follows

- **Step 1** For a parameter vector θ compute the conditional probabilities $P(R_t = j | x_1, ..., x_T; \theta)$ - the so called 'smoothed inferences' - for the process being in regime j at time t.
- **Step 2** Calculate new and more exact maximum likelihood estimates of θ using the likelihood function weighted with the smoothed inferences from step 1.

For a detailed description of the estimation algorithm see Kim (1994) and Janczura and Weron (2010).

3. Goodness-of-fit testing

In this section we provide a goodness-of-fit technique, that can be applied to evaluate the fit of regime-switching models. It is based on the Kolmogorov-Smirnov (K-S) goodness-of-fit test and verifies whether the null hypothesis H_0 that observations come from the distribution specified by the model cannot be rejected. The procedure can be easily adapted to other empirical distribution function (edf) type tests, e.g. Anderson-Darling. Note, that proofs of all lemmas and theorems given in this section are moved to the Appendix.

3.1. Testing in case of observable state process

First, we focus on the independent regimes specification. Provided that the values of the state process R_t are known, observations can be split into separate subsamples related to each of the regimes. Namely, subsample j consists of all values X_t satisfying $R_t = j$. The regimes are independent from each other, but still the i.i.d. condition among subsamples must be ensured. Therefore the mean-reverting regime observations are exchanged with their respective residuals. Precisely, the following transformation is applied to each pair of consecutive AR(1) observations

$$h(x, y, k) = \frac{x - (1-\beta)^k y - \alpha \frac{1-(1-\beta)^k}{\beta}}{\sigma \sqrt{\frac{1-(1-\beta)^{2k}}{1-(1-\beta)^2}}}, \tag{5}$$

where α,β and σ are the model parameters, see (3).

Lemma 3.1 *If H_0 is true (i.e. the sample is generated from the theoretical distribution) transformation $h(X_{t+k}, X_t, k)$ applied to consecutive observations from the mean-reverting AR(1) regime leads to a sample of independent and $N(0, 1)$ distributed random variables.*

Observe that transformation $h(X_{t+k}, X_t, k)$ is based on subtracting the conditional mean from X_{t+k} and standardizing it with the conditional variance. Indeed, $(1-\beta)^k X_t + \alpha \frac{1-(1-\beta)^k}{\beta}$ is the conditional expected value of X_{t+k} given $(X_1, X_2, ..., X_t)$ and $\sigma^2 \frac{1-(1-\beta)^{2k}}{1-(1-\beta)^2}$ is the respective conditional variance.

Note, that for models described by a more general SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{6}$$

a transformation similar to (5) can be derived. Using the Euler scheme and rearranging terms of formula (6), we get that

$$\epsilon_{t-\Delta t,t} = \frac{X_t - X_{t-\Delta t} - \mu(X_{t-\Delta t})\Delta t}{\sqrt{\Delta t}\sigma(X_{t-\Delta t})} \tag{7}$$

has the standard Gaussian distribution. However, since the Euler scheme is an approximation of a continuous process, (7) is valid only for small Δt (for details on errors of the Euler scheme see e.g. Bally and Talay, 1996). In contrast, transformation (5) is exact.

Transformation (5) ensures that the subsample containing observations from the mean-reverting regime is i.i.d. Since other regimes are i.i.d. by definition, standard edf tests can be applied. Moreover, combining all subsamples yields an i.i.d. sample coming from a distribution being a mixture of normal and model-specified laws. The cumulative distribution function is given by

$$F(x) = \sum_{j=1}^{L} P(R = j)F_j(x), \tag{8}$$

where $P(R = j)$ is the probability of the process being in regime j and $F_j(x)$ is the cumulative distribution function (cdf) related to regime j. Note, that for the mean-reverting AR(1) regime $F_j(x)$ is the standard Gaussian cdf. Therefore, not only for separate regimes, but also for the whole model the goodness-of-fit can be tested.

Now, we focus on the case when the model dynamics is described by the AR(1) process with only parameters changing between regimes. Namely, given that the process X_t is in the jth regime at time t, we have that

$$X_t = \alpha_j + (1-\beta_j)X_{t-1} + \sigma_j \epsilon_t. \tag{9}$$

Similarly, as in the independent regimes case, the testing procedure is based on extracting the residuals of the AR(1) process (9). Indeed, observe that the transformation $h(X_t, X_{t-1}, 1)$, (5), with parameters α_{R_t},β_{R_t} and σ_{R_t} corresponding to the current value of the state process R_t, yields an i.i.d. $N(0, 1)$ distributed sample. Thus, the standard edf type tests can be applied.

3.2. Testing in case of latent state process

In the standard goodness-of-fit testing based on the empirical distribution function (edf) each observation is taken into account with weight $\frac{1}{n}$ (i.e. proportionally to the size of the sample). However, in MRS models the state process is latent. The estimation procedure (the EM algorithm) only yields the probabilities that a certain observation comes from a given, say jth, regime. Moreover, in the resulting model distribution each observation is, in fact, weighted with the corresponding probability. Therefore, similar approach should be used in a testing procedure. In the following we introduce a weighted empirical distribution function (wedf) concept and employ it to goodness-of-fit testing.

Definition 3.1 *For a sample of observations $X_1, X_2, ..., X_n$ and corresponding weights $w_1, ..., w_n$, such that $0 \leq w_i \leq M$, $\forall_{i=1,...,n}$, the weighted empirical distribution function (wedf) is defined as:*

$$F_n(t) = \sum_{i=1}^{n} \frac{w_i \mathbb{I}_{\{X_i<t\}}}{\sum_{i=1}^{n} w_i}, \tag{10}$$

where $\mathbb{I}$ is the indicator function.

The idea of the weighted empirical distribution function appears in literature in different contexts. Maiboroda (1996, 2000) applied it to the problem of estimation and testing for homogeneity of components of mixtures with varying coefficients. Withers and Nadarajah (2010) investigated properties of distributions of smooth functionals of $F_n(t)$. In both approaches weights were assumed to fulfill the condition $\sum_{i=1}^{n} w_i = n$. Different choice of weights was given by Huang and Brill (2004). They proposed the level-crossing method to find weights improving efficiency of the edf in the distribution tails. Yet another approach employing weighted distribution is the generalized (weighted) bootstrap technique, see e.g. Haeusler et al. (1991), where a specified random weights are used to improve resampling method. In this paper the weighted empirical distribution function is applied to testing goodness-of-fit of regime-switching models in case when observations cannot be unambiguously classified to one of the regimes. The only restrictions imposed on the choice of weights are the ones guarantying that $F_n(t)$ is an unbiased and consistent estimator of $F(t)$, as stated in the following lemma.

Lemma 3.2 *If $\forall_{i\in\mathbb{N}}$ $0 \leq w_i \leq M$ and $\lim_{n\to\infty} \sum_{i=1}^{n} w_i = \infty$, then the weighted empirical distribution function $F_n(t)$ is an unbiased and consistent estimator of the theoretical cumulative distribution function $F(t)$.*

The following theorem yields a generalization of the K-S test to the case of the weighted empirical distribution function (wedf).

Theorem 3.1 *If $X_1, X_2, ...$ are independent, $\forall_{i\in\mathbb{N}}$ $Var(X_i) < \infty$, $0 \leq w_i \leq M$, $\lim_{n\to\infty} \sum_{i=1}^{n} w_i^2 = \infty$, and the theoretical distribution $F(t)$ is continuous then $\frac{\sum_{i=1}^{n} w_i}{\sqrt{\sum_{i=1}^{n} w_i^2}} \sup_{t\in\mathbb{R}} |F_n(t) - F(t)|$ converges (weekly) to the Kolmogorov-Smirnov distribution as $n \to \infty$.*

The proof of Theorem 3.1 is given in the Appendix. Note, that if each $w_i \equiv 1$, Theorem 3.1 simplifies to the result for the standard Kolmogorov-Smirnov test (Lehmann and Romano, 2005, p. 584).

If hypothesis H_0 is true than, by Theorem 3.1, the statistic

$$D_n = \frac{\sum_{i=1}^{n} w_i}{\sqrt{\sum_{i=1}^{n} w_i^2}} \sup_{t\in\mathbb{R}} |F_n(t) - F(t)| \tag{11}$$

asymptotically has the Kolmogorov-Smirnov distribution KS. Therefore if n is large enough, the following approximation holds

$$P(D_n \geq c|H_0) \approx P(\kappa \geq c), \tag{12}$$

where $\kappa \sim KS$, and c is the critical value. Hence, the p-value for the analyzed sample $(x_1, x_2, ..., x_n)$ can be approximated by $P(\kappa \geq d_n)$, where

$$d_n = \frac{\sum_{i=1}^n w_i}{\sqrt{\sum_{i=1}^n w_i^2}} \max_{1 \leq j \leq n} \left|F_n(x_j) - F(x_j)\right| \tag{13}$$

is the test statistic. Note that, for a given value of d_n, $P(\kappa > d_n)$ is the standard Kolmogorov-Smirnov test p-value, so the K-S test tables can be easily applied in the wedf approach.

Theorem 3.1 is especially useful in case of MRS models. Note, that if the state process R_t is a Markov chain with no transient states and $w_i = P(R_i = j)$, the assumptions of Theorem 3.1 are satisfied. Goodness-of-fit of the individual regimes, as well as, of the whole model can be verified. Again, the mean-reverting AR(1) regime is subjected to a similar transformation as (5). If only parameters change between regimes, see (9), the transformation (5) applies directly with $k = 1$ and parameters $\alpha_{R_t}, \beta_{R_t}$, and σ_{R_t} corresponding to the current value of the state process R_t. However, in case of independent regimes and latent state process the calculation of the conditional mean and variance is not straightforward and, hence, transformation (5) has to be modified. Denote the mean reverting regime observation at time t by $X_{t,MR}$. Observe that, from (3), $X_{t,MR}$ has a Gaussian distribution. Its conditional mean and variance, given the previous observations $\mathbf{x}_{t-1} = (x_1, x_2, ..., x_{t-1})$ are equal to $\alpha + (1-\beta)E(X_{t-1,MR}|\mathbf{x}_{t-1})$ and $(1-\beta)^2 Var(X_{t-1,MR}|\mathbf{x}_{t-1}) + \sigma^2$, respectively. Therefore

$$g(X_{t,MR}, \mathbf{x}_{t-1}) = \frac{X_{t,MR} - \alpha - (1-\beta)E(X_{t-1,MR}|\mathbf{x}_{t-1})}{\sqrt{(1-\beta)^2 Var(X_{t-1,MR}|\mathbf{x}_{t-1}) + \sigma^2}} \tag{14}$$

has the standard Gaussian distribution. The values $E(X_{t-1,MR}|\mathbf{x}_{t-1})$ and $Var(X_{t-1,MR}|\mathbf{x}_{t-1})$ can be calculated as stated in the following lemma.

Lemma 3.3 *For the mean-reverting AR(1) regime observations $X_{t,MR}$ the following equalities hold*

$$\begin{aligned} E(X_{t,MR}|\mathbf{x}_t) &= P(R_t = MR|\mathbf{x}_t)x_t + P(R_t \neq MR|\mathbf{x}_t)\left[\alpha + (1-\beta)E(x_{t-1,MR}|\mathbf{x}_{t-1})\right], & (15)\\ E(X_{t,MR}^2|\mathbf{x}_t) &= P(R_t = MR|\mathbf{x}_t)x_t^2 + P(R_t \neq MR|\mathbf{x}_t)\Big[\alpha^2 + 2\alpha(1-\beta)E(X_{t-1,MR}|\mathbf{x}_{t-1}) + \\ &\quad + (1-\beta)^2 E(X_{t-1,MR}^2|\mathbf{x}_{t-1}) + \sigma^2\Big]. & (16) \end{aligned}$$

Note, that if k is such a number that $P(R_{t-1} = MR|\mathbf{x}_{t-1}) = P(R_{t-2} = MR|\mathbf{x}_{t-2}) = ... = P(R_{t-k+1} = MR|\mathbf{x}_{t-k+1}) = 0$ and $P(R_{t-k} = MR|\mathbf{x}_{t-k}) = 1$, then g leads to the transformation (5), i.e. $g(X_{t,MR}, \mathbf{x}_{t-1}) = h(X_{t,j}, x_{t-k,j}, k)$.

The values $P(R_t = j|\mathbf{x}_t)$ are calculated during the EM estimation procedure. To test, if observations $(x_1, x_2, ..., x_n)$ come from the distribution F_j (Gaussian for mean-reverting regime and model-specified distributions for the other regimes), it is enough to calculate d_n according to formula (13), with $w_i^j = P(R_i = j)$, $i = 1, 2, ..., n$, and apply approximation (12). In the mixture case (8) the procedure is similar, but the tested sample consists of L sequences $(x_1^j, x_2^j, ..., x_n^j)$ and corresponding weights $(w_1^j, w_2^j, ...w_n^j)$, $j = 1, ..., L$, where x_i^j is the value related to regime j (i.e. transformed data for the mean-reverting regime and sample observations for the other regimes).

4. Simulations

In this section we check the performance of the procedure introduced in section 3.2. We generate 10000 trajectories of the MRS model with two independent regimes – one driven by an AR(1) process (3) and a second described by an independent sample following the Gaussian law with mean m and variance s^2, $N(m, s^2)$. The length of each trajectory is 2000 observations. The simulation study is performed for two different sets of parameters, see Table 1 for details. Observe that regimes of MRS models are not directly observable and, hence, the standard edf approach can be used only if some identification of the state process is performed. A natural choice is to relate each observation with the most probable regime by letting $R_t = j$ if $P(R_t = j) > 0.5$. We call this approach ewedf (equally-weighted distribution function). We apply the ewedf, as well as, the wedf-based goodness-of-fit test and calculate the percentage of rejected

Table 1: Parameters of simulated trajectories of a MRS model with AR(1) base regime dynamics and an independent $N(m, s^2)$ distributed regime.

	Parameters					Probabilities	
	α	β	σ^2	m	s^2	p_{11}	p_{22}
Sim #1	10.0	0.3	10.00	42	2.0	0.7	0.4
Sim #2	0.9	0.2	0.01	3	0.5	0.5	0.5

Table 2: Percentage of rejected hypotheses H_0 at the 5% significance level calculated from 10000 simulated trajectories with parameters given in Table 1. The results of the K-S test based on the ewedf, as well as, the wedf approach are reported independently for the two regimes and the whole model.

	ewedf			wedf		
Regime	AR(1)	$N(m, s^2)$	Model	AR(1)	$N(m, s^2)$	Model
Sim #1	0.2522	0.0831	0.7727	0.0553	0.0557	0.0410
Sim #2	0.1004	0.5180	0.3452	0.0535	0.0539	0.0441

hypotheses H_0 at the 5% significance level. The results are reported in Table 2. Clearly all of the obtained values are close to the significance level only in case of the wedf test. The values obtained for the ewedf-based test are far from the expected level of 5%. This simple example clearly shows that in case of MRS models the wedf approach should be used.

In Figure 1 we illustrate different types of empirical distribution functions. The wedf and ewedf functions are compared with the true edf. Note, that the edf can be calculated only when the simulated state process is known. However, when dealing with the real data, the state process is latent and, hence, the standard edf cannot be calculated. The distribution functions are calculated separately for the two regimes of the trajectory of the MRS model, see Sim #1 in Table 1 for parameter details. Observe that, while the wedf function replicates the true edf quite well, the ewedf approximation is not that good. This is in compliance with the rejection percentage given in Table 2.

5. Application to electricity spot prices

Now, we are ready to apply the new goodness-of-fit technique to electricity price models. We analyze the mean daily (baseload) day-ahead spot prices from two major power markets: the PJM Interconnection (PJM; U.S.) and the European Energy Exchange (EEX; Germany). For each market the sample totals 1827 daily observations (or 261 full weeks) and covers the 5-year period January 5, 2004 - January 4, 2009.

It is well known that electricity prices show strong seasonality (on the annual, weekly and daily level), mean reversion, high volatility and abrupt short-lived price changes called spikes (Eydeland and Wolyniec, 2003; Weron, 2006). Therefore we assume that the electricity price, P_t, is represented by a sum of two independent parts: a predictable (seasonal) component f_t and a stochastic component Y_t , i.e. $P_t = f_t + Y_t$. Moreover, as in Huisman and de Jong (2003), we model log-prices instead of prices themselves and let $X_t = \log(Y_t)$ be driven by a Markov regime-switching model with mean-reverting, see (3), base regime ($R_t = 1$) and i.i.d. Gaussian distributed spikes ($R_t = 2$).

Following Weron (2009) and Janczura and Weron (2010) the deseasonalization is conducted in three steps. First, the long term trend T_t is estimated from daily spot prices P_t using a wavelet filtering-smoothing technique (for details see Trück et al., 2007; Weron, 2006). The price series without the long-term seasonal trend is obtained by subtracting the T_t approximation from P_t. Next, the weekly periodicity s_t is removed by subtracting the 'average week' calculated as the arithmetic mean of prices corresponding to each day of the week (U.S. and German national holidays are treated as the eight day of the week). Finally, the deseasonalized prices, i.e. $P_t - T_t - s_t$, are shifted so that the minimum of the new process is the same as the minimum of P_t. The resulting deseasonalized time series $X_t = \log(P_t - T_t - s_t)$ can be seen in Figure 2. The estimated model parameters are presented in Table 3.

For both analyzed datasets the K-S test based on the wedf approach is performed. Since the state process is latent, the standard edf-type goodness-of-fit techniques are not applicable. The obtained p-values are reported in Table 4. For the PJM market the model yields a satisfactory fit only for the spike distribution (regime). Hypothesis about the

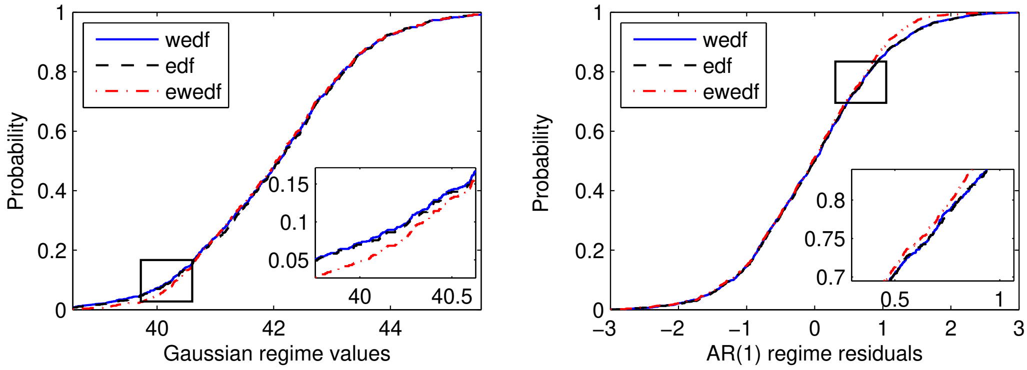

Figure 1: Comparison of the weighted empirical distribution function (wedf), the equally-weighted empirical distribution function (ewedf) and the standard empirical distribution function (edf) calculated for a sample trajectory of a MRS model with two independent regimes. Distribution functions of the i.i.d. Gaussian regime are given in the left panel, while of the residuals of the AR(1) regime in the right panel.

Table 3: Parameters of the 2-regime model with mean reverting base regime and independent Gaussian distributed spikes fitted to PJM and EEX log-prices.

	Base regime			Spike regime		Probabilities			
	α	β	σ^2	μ	s^2	p_{11}	p_{22}	$P(R_t = 1)$	$P(R_t = 2)$
PJM	0.60	0.15	0.01	4.26	0.14	0.9808	0.8349	0.8958	0.1042
EEX	0.99	0.26	0.02	3.79	0.38	0.9816	0.8321	0.9015	0.0985

base regime and the model distribution can be rejected at the 5% significance level. The EEX log-prices yield a better fit, as none of the tests can be rejected at the 5% significance level.

6. Conclusions

In this paper we have proposed a goodness-of-fit testing scheme for regime-switching models. We have analyzed two different classes of models – with an observable and a latent state process. For both specifications we described the testing procedure. The latent state process case involved introduction of the weighted empirical distribution function (wedf) concept and a generalization of the Kolmogorov-Smirnov test.

We have focused on two commonly used specifications of regime-switching models – one with dependent autoregressive states and a second with independent autoregressive or i.i.d. regimes. Nevertheless, the proposed approach can be easily applied to other specifications of regime-switching models. The performed simulation study has confirmed the good performance of the wedf approach. Moreover, we have applied the wedf testing technique to verify the statistical fit of a sample Markov regime-switching model to electricity spot price data.

Table 4: p-values of the K-S test based on the wedf approach for both datasets.

	Base regime	Spike regime	Model
PJM	0.0160	0.3066	0.0401
EEX	0.0558	0.1030	0.0687

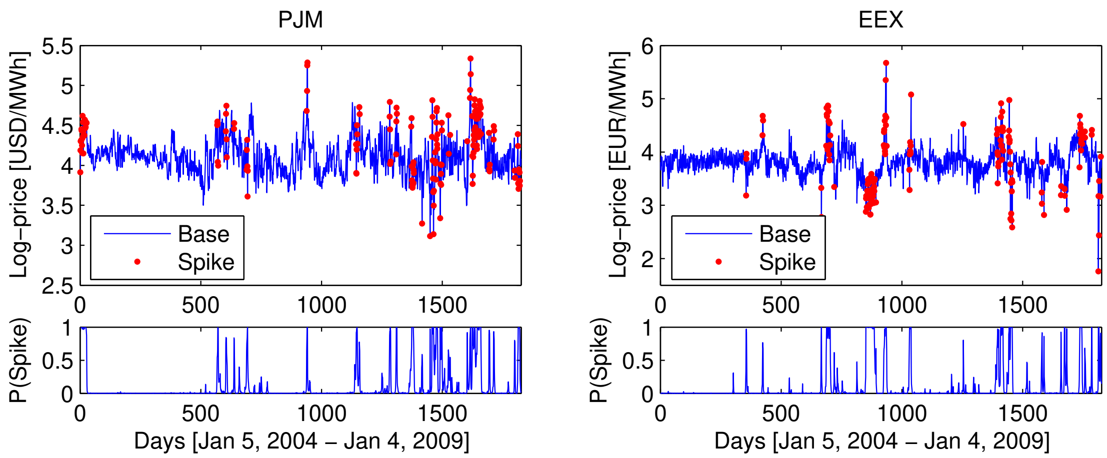

Figure 2: Calibration results for the 2-regime model with mean reverting base regime and independent Gaussian distributed spikes fitted to PJM (*left panel*) and EEX (*right panel*) log-prices. Observations with $P(R_t = 2) > 0.5$, i.e. spikes, are denoted by dots. The lower panels display the probability $P(R_t = 2)$.

Appendix A.

Proof of Lemma 3.1. Assume, that $R_{t+k} = j, R_{t+k-1} \neq j, ..., R_{t+1} \neq j, R_t = j$, where j stands for the mean-reverting regime. From (3) we have

$$X_{t+k} = \alpha \frac{1-(1-\beta)^k}{\beta} + (1-\beta)^k X_t + \sigma \left[\epsilon_{t+k} + (1-\beta)\epsilon_{t+k-1} + ... + (1-\beta)^{k-1}\epsilon_t \right]. \tag{A.1}$$

Since $\epsilon_t, ..., \epsilon_{t+k}$ are independent and normally distributed, the linear combination $\sigma[\epsilon_{t+k} + (1-\beta)\epsilon_{t+k-1} + ... + (1-\beta)^{k-1}\epsilon_t]$ is also Gaussian. Moreover

$$\sigma[\epsilon_{t+k} + (1-\beta)\epsilon_{t+k-1} + ... + (1-\beta)^{k-1}\epsilon_t] \stackrel{d}{=} \sigma \sqrt{\frac{1-(1-\beta)^{2k}}{1-(1-\beta)^2}} \epsilon_{t,t+k}, \tag{A.2}$$

where $\epsilon_{t,t+k} \sim N(0,1)$ and $\stackrel{d}{=}$ denotes equality of distributions. The subscript $(t, t+k)$ means that $\epsilon_{t,t+k}$ is a combination of $\epsilon_t, ..., \epsilon_{t+k}$. Thus

$$X_{t+k} \stackrel{d}{=} \alpha \frac{1-(1-\beta)^k}{\beta} + (1-\beta)^k X_t + \sigma \sqrt{\frac{1-(1-\beta)^{2k}}{1-(1-\beta)^2}} \epsilon_{t,t+k}. \tag{A.3}$$

Rearranging the terms of (A.3) we get that $h(X_{t+k}, X_t, k)$ has the standard Gaussian distribution. Moreover, independence of $h(X_{t+k}, X_t, k)$ and $h(X_{s+l}, X_s, l)$, for $s < s + j < t < t + k$ is implied by the independence of $\epsilon_{t,t+k}$ and $\epsilon_{s,s+l}$. □

Proof of Lemma 3.2. First, observe that

$$E(\mathbb{I}_{\{X<t\}}) = P(X < t) = F(t). \tag{A.4}$$

Thus, from the definition of $F_n(t)$ we have that

$$E[F_n(t)] = \sum_{i=1}^{n} \frac{w_i E(\mathbb{I}_{\{X_i<t\}})}{\sum_{i=1}^{n} w_i} = \sum_{i=1}^{n} \frac{w_i F(t)}{\sum_{i=1}^{n} w_i} = F(t) \tag{A.5}$$

and $F_n(t)$ is an unbiased estimator of $F(t)$. Moreover,

$$Var(\mathbb{I}_{\{X<t\}}) = E(\mathbb{I}_{\{X<t\}}) - [E(\mathbb{I}_{\{X<t\}})]^2 = F(t)[1 - F(t)], \tag{A.6}$$

implying that

$$Var[F_n(t)] = \frac{\sum_{i=1}^n w_i^2 Var(\mathbb{I}_{\{X_i<t\}})}{\left(\sum_{i=1}^n w_i\right)^2} = \frac{\sum_{i=1}^n w_i^2}{\left(\sum_{i=1}^n w_i\right)^2} F(t)[1-F(t)]. \tag{A.7}$$

Finally, from the Chebyshev's inequality (Billingsley, 1986, p. 65), for any $\epsilon > 0$ we have

$$\begin{aligned} P(|F_n(t) - E[F_n(t)]| > \epsilon) &\leq \frac{Var[F_n(t)]}{\epsilon^2} = \frac{F(t)[1-F(t)]\sum_{i=1}^n w_i^2}{\epsilon^2\left(\sum_{i=1}^n w_i\right)^2} \leq \frac{F(t)[1-F(t)]\sum_{i=1}^n Mw_i}{\epsilon^2\left(\sum_{i=1}^n w_i\right)^2} = \\ &= \frac{F(t)[1-F(t)]M}{\epsilon^2\sum_{i=1}^n w_i} \end{aligned} \tag{A.8}$$

and $F_n(t)$ converges in probability to $F(t)$, if $\lim_{n\to\infty}\sum_{i=1}^n w_i = \infty$. Therefore $F_n(t)$ is a consistent estimator of $F(t)$. $\square$

Proof of Theorem 3.1. First, note that $F(t) \in \{0,1\}$ implies $F_n(t) = F(t)$ and $\sup_{t\in\mathbb{R}} |F_n(t)-F(t)| = \sup_{t\in D} |F_n(t)-F(t)|$, where $D = \mathbb{R}\backslash\{t : F(t) = 0 \vee F(t) = 1\}$. Therefore in the following we will limit ourselves to the case $0 < F(t) < 1$.

Second, observe that the distribution of $\sup_{t\in\mathbb{R}} |F_n(t) - F(t)|$ does not depend on F. Indeed, since $U_i = F(X_i)$ has the uniform distribution, $P(\sup_{t\in\mathbb{R}} |F_n(t) - F(t)| \leq x) = P(\sup_{t\in\mathbb{R}} |\sum_{i=1}^n \frac{w_i\mathbb{I}_{\{U_i<y\}}}{\sum_{i=1}^n w_i} - y| \leq x)$, where $y = F(t)$.

Next, note that the sequence of random variables $Y_i = w_i\mathbb{I}_{\{X_i<t\}}$ satisfies the Lindeberg condition (Billingsley, 1986, p.369). Let $S_n^2 = \sum_{i=1}^n Var(Y_i)$ and $\mu_i = E(Y_i)$, where $Var(Y_i) = F(t)[1-F(t)]w_i^2$ and $E(Y_i) = w_iF(t)$, see equations (A.6) and (A.4), respectively. The Lindeberg condition yields

$$\begin{aligned} \frac{1}{S_n^2}\sum_{i=1}^n \int_{\{|Y_i-\mu_i|>\delta S_n\}} (Y_i-\mu_i)^2 dP &= \frac{1}{S_n^2}\sum_{i=1}^n w_i^2 \int_{\{w_i|\mathbb{I}_{\{X_i<t\}}-F(t)|>\delta S_n\}} [\mathbb{I}_{\{X_i<t\}} - F(t)]^2\, dP \leq \\ &\leq \frac{1}{S_n^2}\sum_{i=1}^n w_i^2 P\left(w_i|\mathbb{I}_{\{X_i<t\}} - F(t)| > \delta S_n\right) \max\left\{F(t)^2, [1-F(t)]^2\right\} \leq \\ &\leq \frac{\max\left\{F(t)^2, [1-F(t)]^2\right\}}{[1-F(t)]\,F(t)} \max_{1\leq i\leq n} P\left(w_i|\mathbb{I}_{\{X_i<t\}} - F(t)| > \delta S_n\right). \end{aligned}$$

Since $w_i|\mathbb{I}_{\{X_i<t\}} - F(t)| \leq M$ and the fact that $\lim_{n\to\infty}\sum_{i=1}^n w_i^2 = \infty$ implies $\lim_{n\to\infty} S_n = \infty$ we have

$$\exists_{n_0}\forall_{n>n_0}\forall_{1\leq i\leq n} \quad P\left(w_i|\mathbb{I}_{\{Y_i<t\}} - F(x)| > \delta S_n\right) = 0. \tag{A.9}$$

Therefore, the Lindeberg condition is satisfied:

$$\lim_{n\to\infty}\frac{1}{S_n^2}\sum_{i=1}^n \int_{\{|Y_i-\mu_i|>\delta S_n\}} (Y_i-\mu_i)^2 dP = 0. \tag{A.10}$$

This ensures that the Central Limit Theorem holds for $Y_1, Y_2, \ldots$ and

$$\frac{\sum_{i=1}^n w_i\mathbb{I}_{\{X_i<t\}} - \sum_{i=1}^n w_iF(t)}{\sqrt{F(t)[1-F(t)]\sum_{i=1}^n w_i^2}} \xrightarrow{d} N(0,1). \tag{A.11}$$

The latter is equivalent to

$$\frac{\sum_{i=1}^n w_i}{\sqrt{\sum_{i=1}^n w_i^2}}[F_n(t) - F(t)] \xrightarrow{d} N(0, F(t)[1-F(t)]). \tag{A.12}$$

Recall that the Kolmogorov-Smirnov distribution KS is a distribution of $\sup_{0\le y\le 1}|B(y)|$, where $B(y)$ is a Brownian bridge, i.e. $B(y) \sim N(0, y(1-y))$, see e.g. Lehmann and Romano (2005, p. 585). Therefore, putting $y = F(t)$ and taking the supremum, we obtain that

$$\frac{\sum_{i=1}^n w_i}{\sqrt{\sum_{i=1}^n w_i^2}} \sup_{t\in\mathbb{R}} |F_n(t) - F(t)| \xrightarrow{d} KS. \tag{A.13}$$

□

Proof of Lemma 3.3. Let $\mathbb{X}_t = (X_1, X_2, ..., X_t)$. Observe that

$$X_{t,MR} = \mathbb{I}_{R_t=MR} X_t + \mathbb{I}_{R_t\neq MR}[\alpha + (1-\beta)X_{t-1,MR} + \sigma\epsilon_t], \tag{A.14}$$

where ϵ_t has the standard Gaussian distribution. Thus,

$$\begin{aligned} E(X_{t,MR}|\mathbb{X}_t) &= P(R_t = MR|\mathbb{X}_t)X_t + \\ &\quad + P(R_t \neq MR|\mathbb{X}_t)\left[\alpha + (1-\beta)E(X_{t-1,MR}|\mathbb{X}_t, R_t \neq MR) + \sigma E(\epsilon_t|\mathbb{X}_t, R_t \neq MR)\right] = \\ &= P(R_t = MR|\mathbb{X}_t)X_t + P(R_t \neq MR|\mathbb{X}_t)\left[\alpha + (1-\beta)E(X_{t-1,MR}|\mathbb{X}_{t-1}) + \sigma E(\epsilon_t)\right] = \\ &= P(R_t = MR|\mathbb{X}_t)X_t + P(R_t \neq MR|X_1, ..., X_t)\left[\alpha + (1-\beta)E(X_{t-1,MR}|\mathbb{X}_{t-1})\right]. \end{aligned}$$

Analogously,

$$\begin{aligned} E(X_{t,MR}^2|\mathbb{X}_t) &= P(R_t = MR|\mathbb{X}_t)X_t^2 + P(R_t \neq MR|\mathbb{X}_t)E\left[(\alpha + (1-\beta)X_{t-1,MR} + \sigma\epsilon_t)^2|\mathbb{X}_t, R_t \neq MR\right] = \\ &= P(R_t = MR|\mathbb{X}_t)X_t^2 + P(R_t \neq MR|\mathbb{X}_t)\Big\{2\sigma E\left[(\alpha + (1-\beta)X_{t-1,MR})\epsilon_t|\mathbb{X}_t, R_t \neq MR\right] + \\ &\quad + E\left[\alpha^2 + 2\alpha(1-\beta)X_{t-1,MR} + (1-\beta)^2 X_{t-1,MR}^2|\mathbb{X}_t, R_t \neq MR\right] + \sigma^2 E[\epsilon_t^2|\mathbb{X}_t, R_t \neq MR]\Big\}. \end{aligned}$$

From the law of iterated expectation and basic properties of conditional expected values:

$$E\left[(\alpha + (1-\beta)X_{t-1,MR})\epsilon_t|\mathbb{X}_{t-1}\right] = 0, \tag{A.15}$$

yielding

$$\begin{aligned} E(X_{t,MR}^2|\mathbb{X}_t) &= P(R_t = MR|\mathbb{X}_t)X_t^2 + \\ &\quad + P(R_t \neq MR|\mathbb{X}_t)\left[\alpha^2 + 2\alpha(1-\beta)E(X_{t-1,MR}|\mathbb{X}_{t-1}) + (1-\beta)^2 E(X_{t-1,MR}^2|\mathbb{X}_{t-1}) + \sigma^2\right]. \end{aligned}$$

Finally, substituting variables $(X_1, X_2, ..., X_t)$ with their observed values $(x_1, x_2, ...x_t)$ completes the proof. □

References

Bally, V., Talay, D., (1996) The law of the Euler scheme for stochastic differential equations: II. Convergence rate of the density. Monte Carlo Methods and Applications 2, 93-128.

Billingsley, P. (1986) Probability and Measure (2nd ed.) Wiley, New York.

Brockwell, P.J., Davis, R.A. (1996) Introduction to Time Series and Forecasting (2nd ed.) Springer-Verlag, New York.

Carrasco, M., Hu, L., Ploberger W. (2004) Optimal test for Markov switching. Working paper. University of Rochester.

Cetin, M., Comert, G. (2006) Short-term traffic flow prediction with regime switching models. Transportation Research Record: Journal of the Transportation Research Board 1965, 23-31.

Cho, J.S., White, H. (2007) Testing for regime switching. Econometrica 75(6), 1671-1720.

Eydeland, A., Wolyniec, K. (2003) Energy and Power Risk Management. Wiley, Hoboken, NJ.

Garcia, R. (1998) Asymptotic null distribution of the likelihood ratio test in Markov switching models. International Economic Review 39, 763-788.

Haeusler, E., Mason, D.M., Newton, M.A. (1991) Weighted bootstrapping of the means. CWI Quaterly, 213-228.

Hamilton, J. (1990) Analysis of time series subject to changes in regime. Journal of Econometrics 45, 39-70.

Hamilton, J. (1996) Specification testing in Markov-switching time series models. Journal of Econometrics 70, 127-157.

Huang, M.L., Brill, P.H. (2004) A distribution estimation method based on level crossings. Journal of Statistical Planning and Inference 124, 45-62.

Huisman, R., de Jong, C. (2003). Option pricing for power prices with spikes. Energy Power Risk Management 7.11, 12-16.

Janczura, J., Weron, R. (2010) An empirical comparison of alternate regime-switching models for electricity spot prices. Energy Economics, *forthcoming*. Working paper version available at MPRA: http://mpra.ub.uni-muenchen.de/20661/.

Janczura, J., Weron, R. (2010) Efficient estimation of Markov regime-switching models. Working paper.

Juang, B.-H., Rabiner, L.,R. (1985) Mixture autoregressive hidden Markov models for speech signals. IEEE Transactions on Acoustics, Speech, and Signal Processing, ASSP-30(6), 1404-1413.

Kim, C.-J. (1994) Dynamic linear models with Markov-switching. Journal of Econometrics 60, 1-22.

Lehmann, E.L., Romano, J.P. (2005) Testing Statistical Hypotheses (3rd ed.) Springer, New York.

Luo, Q., Mao, X. (2007) Stochastic population dynamics under regime switching. Journal of Mathematical Analysis and Applications, 334(1), 69-84.

Maiboroda, R.E. (1996) Estimates for distributions of components of mixtures with varying concentrations. Ukrainian Mathematical Journal 48(4), 618-622.

Maiboroda, R.E. (2000) A test for the homogeneity of mixtures with varying concentrations. Ukrainian Mathematical Journal 52(8), 1256-1263.

Sen, R., Hsieh, F. (2009) A note on testing regime switching assumption based on recurrence times. Statistics and Probability Letters 79, 2443-2450.

Trück, S., Weron, R., Wolff, R. (2007) Outlier treatment and robust approaches for modeling electricity spot prices. Proceedings of the 56th Session of the ISI. Available at MPRA: http://mpra.ub.uni-muenchen.de/4711/.

Vasas, K., Eleka, P., Márkusa, L. (2007) A two-state regime switching autoregressive model with an application to river flow analysis. Journal of Statistical Planning and Inference, 137(10), 3113-3126.

Vasicek, O. (1977) An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177-188.

Weron, R. (2006) Modeling and Forecasting Electricity Loads and Prices: A Statistical Approach. Wiley, Chichester.

Weron, R. (2009) Heavy-tails and regime-switching in electricity prices. Mathematical Methods of Operations Research 69(3), 457-473.

Withers, C.S., Nadarajah, S. (2010) The distribution and quantiles of functionals of weighted empirical distributions when observations have different distributions. arXiv:1002.4338v1.