Origins of scaling in FX markets

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Abstract: Typical data sets employed by economists and financial analysts do not exceed a few hundred or thousand observations per series. However, in the last decade data sets containing tick-by-tick observations have become available. The studies of these data have turned up new and interesting facts about the pricing of assets.

In this article we show that foreign exchange (FX) rate returns satisfy scaling with an exponent significantly different from that of a random walk. But what is more important, we also show that the conditionally exponential decay (CED) model can be used to solve a long standing problem in the analysis of intra-daily data, i.e. it can be used to identify the mathematical structure of the distributions of FX returns corresponding to the empirical scaling laws.

Keywords: FX market, scaling law, volatility, CED model, high frequency data

1 Introduction

It is well known starting from Mandelbrot [24, 25] and Fama [11, 12] that market returns are not normally distributed. In search for satisfactory descriptive models of economic data, large numbers of distributions have been tried and even entire classes of distributional types have been constructed. These include mixtures of normal distributions [20, 34], Student t [4], hyperbolic [10, 21] and normal inverse Gaussian (NIG) laws [3], alternative stable distributions [28, 32] and subordinated processes [7, 17] to name a few. In any particular case it is always possible to find a distribution that fits the data well, provided one works within a suitably broad and flexible class of candidates. For example, in Fig. 1 some alternatives to the normal distribution are fitted to DJIA index daily returns. Statistical inference implies that, among the four tested distributions, the NIG law gives the best fit.

However, it is not enough to fit given data through the choice of a "good distribution". It is much more important to explain returns data behavior with a statistical model that predicts the data's main characteristics. This is the main problem we address in this article.

The Efficient Market Hypothesis (EMH), including Arbitrage Pricing Theory (APT) and the Capital Asset Pricing Model (CAPM), was very successful in making the mathematical environment easier, but unfortunately is not justified by the real data. Instead, there is a need to seek for a market hypothesis that fits the observed data better and takes into account why markets exist to begin with. In the EMH place, the Fractal and the Heterogeneous Market Hypotheses (FMH, HMH) have been proposed recently [29, 31]. Based on current developments

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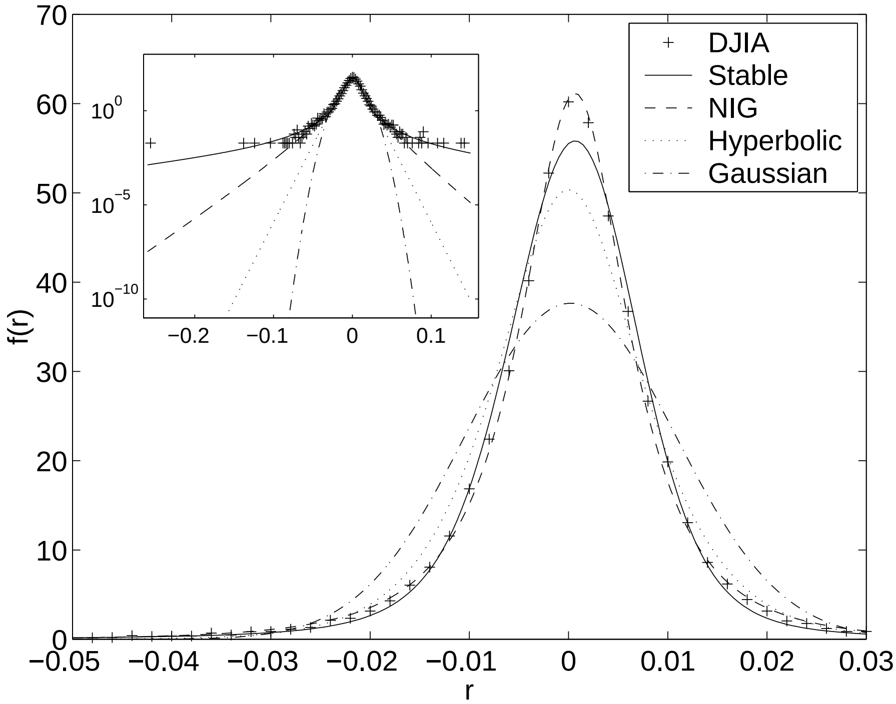

Figure 1: Histogram of the empirical density (kernel estimator) of the Dow Jones Industrial Average (DJIA) daily returns for the period January 1901 – December 1999 and four estimating densities: stable, normal inverse Gaussian (NIG), hyperbolic and Gaussian. The inset illustrates the same plot on a semilogarithmic scale.

of chaos theory and using the fractal objects whose disparate parts are self-similar, these hypotheses provide a new framework for a more precise modeling of turbulence, discontinuity, and nonperiodicity that truly characterize today's financial markets.

2 Fractal and Heterogeneous Market Hypotheses

The success of the fractal approach gave rise to the hypothesis that the market itself is fractal, with heterogeneous trading behavior. The Fractal Market Hypothesis (FMH) emphasizes the impact of information and investment horizons on the behavior of investors. In traditional finance theory, the investor is generic. Basically, an investor is anyone who wants to buy, sell, or hold a security because of the available information, which is also treated as a generic item. The investor is considered a rational price-taker, i.e. someone who always wants to maximize return and knows how to value current information. This generic approach, where information and investors are general cases, implies that all types of information impact all investors equally. That is where it fails.

The following assumptions were proposed for the FMH [31]:

- the market is made up of many individuals with a large number of different investment horizons;
- information has different impact on different investment horizons;
- the stability of the market is largely a matter of liquidity (balancing of supply and demand) – liquidity is present when the market is composed of many investors with many different investment horizons;

- prices reflect a combination of short-term technical trading and long-term fundamental valuation;
- if a security has no tie to the economic cycle, then there will be no long term trend – trading, liquidity, and short-term information will dominate.

The purpose of the FMH is to give a model of investor behavior and market price movements that fits our observations. When markets are considered stable, the EMH and CAPM seem to work fine. However, during panics and stampedes, these models break down, like singularities in physics. This is not unexpected, because the EMH, APT, and the CAPM are equilibrium models. They cannot handle the transition to turbulence. Unlike the EMH, the FMH says that information is valued accordingly to the investment horizon of the investor. The key fact is that under the FMH the market is stable when it has no characteristic time scale or investment horizon. Instability occurs when the market looses its fractal structure and assumes a fairly uniform investment horizon.

A statistical study of financial data from the fractal point of view is based on the analysis of time intervals Δt of different sizes. A reasonable question to ask is: what is the relation between volatility and the size of time intervals? The answer to this question is the scaling law reported in Müller et al. [30]

$$v(t) = c(\Delta t)^D, \tag{1}$$

which relates the volatility $v(t)$ with Δt. Here c is an empirical constant and D is an empirical drift exponent. Traditionally volatility is defined as the l^2 norm (standard deviation of logarithmic price changes), however, the Olsen & Associates group used the l^1 norm (mean of absolute logarithmic price changes) as it is better suited for the analysis of high-frequency data.

In spite of its elementary nature, a scaling law study is immediately able to reject the Gaussian hypothesis and reveal an important property of financial time series. For the Gaussian case the above formula is true with a drift exponent of 0.5, while the empirical values of drift exponents D are significantly different from that value. These and other recently found properties of empirical time series have led the researchers at Olsen & Associates to the Heterogeneous Market Hypothesis (HMH) as opposed to the assumption of a homogeneous market where all participants interpret news and react to news in the same way. The HMH is characterized by the following interpretations of the empirical findings:

- different actors in the heterogeneous market have different time horizons and dealing frequencies;
- the market is heterogeneous with a fractal structure of the participants' time horizons as it consists of short-term, medium-term and long-term components;
- different actors are likely to settle for different prices and decide to execute their transactions in different market situations, in other words, they create volatility;
- the market is also heterogeneous in the geographic location of the participants.

The market participants of the HMH also differ in the other aspects beyond the time horizons and the geographic locations: they can have different degrees of risk aversion, institutional constraints, and transaction costs [16, 29].

Guillaume et al. [16] present stylized facts concerning the spot intra-daily FX market. They uncover a new wealth of structures that demonstrates the complexity of this market. Further they group empirical regularities of the FX market data under three major topics: the distribution of price changes, the process of price formation, and the heterogeneous structure of the market. Independently Gallucio et al. [14] addressed the same problem and found that FX rates have a far more complex nature than random walk, since hidden correlations are present in the data. In both papers the following problem is posed:

Problem: *How to characterize the mathematical structure of the distributions of price changes corresponding to the empirical scaling law for volatility?*

In what follows we show that the CED model provides a natural solution to this problem. It clarifies the ideas of the Fractal and Heterogeneous Market Hypotheses and provides a rigorous mathematical framework for further analysis of financial complex processes. Moreover, the obtained distributional form of returns has important implications for theoretical and empirical analyses in economics and finance, since portfolio and option pricing theories are typically based on distributional assumptions [35, 36].

3 The CED model

The market is made up of participants, from tick traders to long-term investors. Each has a different investment horizon that can be ordered in time. When all investors with different horizons are trading simultaneously the market is stable. The stability of the market relies, however, not only on a random diversification of the investment horizons of the participants but also on the fact that the different horizons value the importance of the information flow differently. Hence, both the information flow and the investment horizons should have their own contribution to the observed global market features. In general, the locally random markets have a global statistical structure that is non-random.

We assume that the model is a discrete time economy with a finite number of trading dates from time 0 to time T and its uncertainty has a global impact on the market index returns on the interval $[0, T]$. As a proxy of the capital market index we can use a stock index, for example DJIA or S&P500, and as a proxy of the FX market index for a particular currency – the prevailing exchange rate, i.e. the local currency rate against one of the global currencies (e.g. USD, EUR). In the family of all world investors let us identify those N who are acting on a given market described by a chosen index. Call them $I_{1N}, I_{2N}, ..., I_{NN}$. Let R_{iN} be the positive (or the absolute value of negative) part of the ith investor's return. Each ith investor is related to a cluster of agents acting simultaneously on common complement markets. The influence of this cluster of agents is of the type of short-range (intra-cluster) interactions and is reflected by a random risk-aversion factor A_i. The long-range type of interactions are imposed on the ith investor by the inter-cluster relationship manifested by the random risk factors B^i_j for all $j \neq i$. They reflect how fast the information flows to the ith investor, see [33, 38].

Condition CED1: For the ith investor the following property holds:

$$\begin{aligned} \phi_{iN}(r|a,b) &= \mathbf{P}\Big(R_{iN} \geq r \Big| A_i = a, b_N^{-1} \max(B^i_1, \ldots, B^i_{i-1}, B^i_{i+1}, \ldots, B^i_N) = b\Big) \\ &= \exp(-[a \min(r,b)]^c), \end{aligned} \tag{2}$$

where r, a and b are non-negative constants, b_N is a suitable, positive normalizing constant and $c \geq 1$.

The dependence in the CED model measured by the conditional return excess decays similarly as in some conditionally heteroscedastic models, i.e. in an exponential way, see (2), but reflects both short- as well as long-range effects. This idea concerns systems in which the behavior of each individual entity strongly depends on its short- and long-range random interactions.

Condition CED2: Investors have different investment horizons ("short-range interaction") affected by different information sets ("long-range interaction"). The investment horizon of the investor is reflected by the random variable A_i, while $\{B_j^i, j = 1, 2, \ldots, N, j \neq i\}$ reflect the information flow to this investor.

The probability that the return R_{iN} will be not less than r is conditioned by the value a taken by the random variable A_i and by the value b taken by the maximum of the set of random variables $\{B_j^i, j = 1, 2, \ldots, N, j \neq i\}$. Therefore (2) can be rewritten as follows:

$$\phi_{iN}(r|a,b) = \begin{cases} 1 & \text{for} \quad r = 0 \\ \exp(-(ar)^c) & \text{for} \quad r < b \\ \exp(-(ab)^c) & \text{for} \quad r \geq b, \end{cases}$$

i.e. the conditional return excess $\phi_{iN}(r|a,b)$ decays exponentially with a decay rate a and exponent c as r tends to the value b. Then it takes a constant value $\ll 1$. The basic statistical assumption is that

Condition CED3: The random variables $A_1, A_2, \ldots$ and $B_1^i, B_2^i, \ldots$ form independent and convergent (with respect to addition and maximum, respectively [19]) sequences of non-negative, independent, identically distributed (i.i.d.) random variables. The variables $R_{1N}, \ldots, R_{NN}$ are also non-negative, i.i.d. for each N.

Observe that the dependence on external conditions is expressed by the relationship (2) of each R_{iN} with A_i and $\max(B_1^i, \ldots, B_{i-1}^i, B_{i+1}^i, \ldots, B_N^i)$. Condition CED3 can be partially justified by the following argument. Institutional trading is a major factor in the determination of asset prices. If professional investment managers have similar beliefs, then the i.i.d. distributions assumption may hold as a first approximation. Professional managers are likely to have similar beliefs because they have access to similar information sources. This uniformity of information over time would tend to generate similar beliefs [33].

The basic result which allows us to see the structure of distributions describing market returns is the following: if the global behavior of the market is given by

$$\phi(r) = \mathbf{P}\Big(\lim_{N\to\infty} r_N \min(R_{1N}, \ldots, R_{NN}) \geq r\Big), \tag{3}$$

where r_N is a suitable, positive normalizing constant, than under conditions CED1–CED3 the function $\phi(r)$ fulfills the global return equation:

$$\frac{d\phi}{dr}(r) = -\alpha\lambda(\lambda r)^{\alpha-1}\left(1 - \exp\left(-\frac{(\lambda r)^{-\alpha}}{k}\right)\right)\phi(r), \tag{4}$$

where the parameters $\lambda > 0, k > 0$ and $\alpha > 0$ are determined by the limiting procedure in (3).

A more general type of the above equation has been recently studied in the context of complex stochastic systems [19, 37]. It turns out that the solution of (4) has the following integral form

$$\phi(r) = \exp\left[-\frac{1}{k}\int_0^{k(\lambda r)^\alpha}\left(1 - e^{-1/s}\right)ds\right].$$

The function $\phi(r)$ monotonically decreases from $\phi(0) = 1$ to $\phi(\infty) = 0$. Moreover, the probability density of the global return $f(r) = -\frac{d\phi(r)}{dr}$ is given by the formula

$$f(r) = \alpha\lambda(\lambda r)^{\alpha-1}\left[1 - \exp\left(-\frac{(\lambda r)^\alpha}{k}\right)\right]\exp\left[-\frac{1}{k}\int_0^{k(\lambda r)^\alpha}\left(1 - e^{-1/s}\right)ds\right], \tag{5}$$

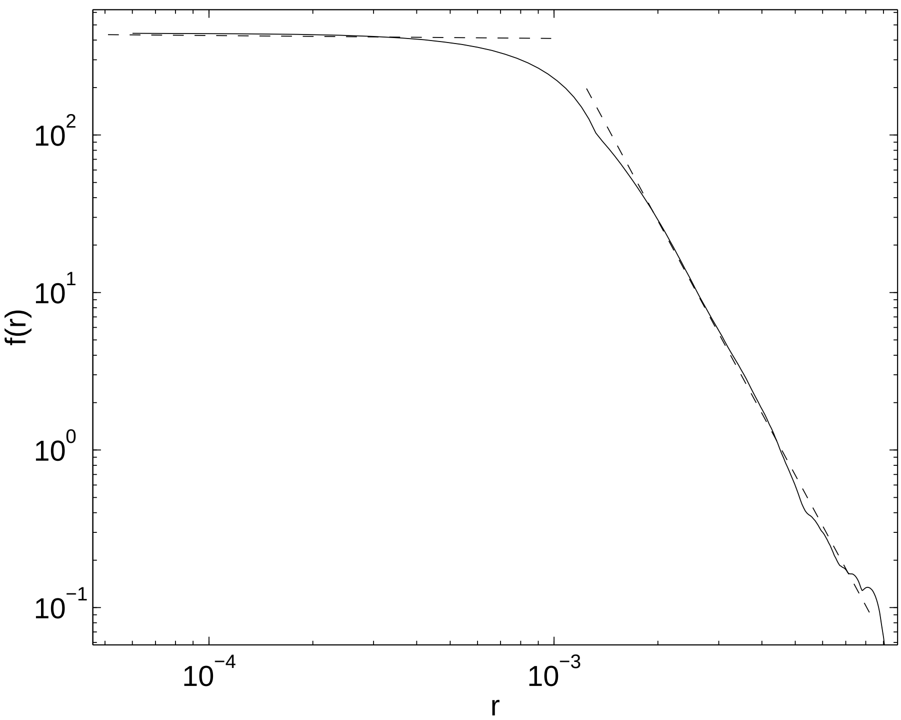

Figure 2: Double logarithmic plot of the empirical density (kernel estimator) of the USD/CHF exchange rate 30 minute negative returns. Dashed lines represent the two power laws. For the absolute value of negative returns the direct approach yielded the following values: $\alpha^- = 1.04$, $\lambda^- = 925.4$ and $k^- = 0.250$.

and exhibits the two power-laws property, see Fig. 2

$$f(r) \propto \begin{cases} (\lambda r)^{\alpha - 1} & \text{for} \quad \lambda r \ll 1, \\ (\lambda r)^{-\frac{\alpha}{k} - 1} & \text{for} \quad \lambda r \gg 1. \end{cases} \tag{6}$$

Hence, the global return distribution is characterized by the following three parameters: α, λ and k, where α is the shape and λ is the scale parameter. In general, parameter α slows down – in comparison with an individual investor – the return rate $\alpha\lambda(\lambda r)^{\alpha-1}$ of the global market return distribution and parameter k decides how fast the information flow is spread out in the market. If $k \to 0$ then the long-range interaction is neglected and (4) takes the form of the Weibull probability density function.

4 Estimation

Now, that we have a model which can be used to identify the mathematical structure of the distributions of returns corresponding to the empirical scaling laws, the basic question to ask is: how can we fit the CED model to financial data? In what follows we present three different estimation methods and use the best one in a detailed analysis of high frequency data.

But before we start we want to emphasize the fact that the CED probability density is defined only on the positive half line. As a consequence, in all three presented methods, we have to carry out the same analysis for positive returns (CED$^+$) and then for absolute value of negative returns (CED$^-$). This results in obtaining two sets of estimators: $\{\widehat{\alpha}^+, \widehat{\lambda}^+, \widehat{k}^+\}$ and $\{\widehat{\alpha}^-, \widehat{\lambda}^-, \widehat{k}^-\}$. To make notation simpler in the next two sections we describe how to obtain generic estimators $\{\widehat{\alpha}, \widehat{\lambda}, \widehat{k}\}$ without specifying if we are using positive or absolute value of negative returns.

One might criticize the CED model by saying that it is possible to approximate pretty well any empirical distribution with a six parameter law. However, as we will show later, we can reduce the number of parameters to a three element set: $\{\widehat{\lambda}, \widehat{k}^{+}, \widehat{k}^{-}\}$, because in most cases $\widehat{\alpha}^{+} = \widehat{\alpha}^{-} = 1$ and $\widehat{\lambda}^{+} = \widehat{\lambda}^{-} = \widehat{\lambda}$ both for positive and for negative returns.

4.1 Direct approach

Probably the most natural method of estimating α, λ and k makes use of the two power-laws property (6) of the CED density. In this method we have to calculate the kernel density estimator $\widehat{f}(r)$ of the empirical returns and then plot it on a double logarithmic paper, see Fig. 2. If the empirical density of returns is unimodal then – to reduce estimation errors – we first have to center the density around its mode. Otherwise it would be impossible to "glue" the positive and the negative part of the CED density.

If the empirical data follows the CED law than the plot should be linear both for small and for large λr. Namely, if for $0 < \lambda r \ll 1$ the best linear fit of $\widehat{f}(r)$ is given by

$$\log\left(\widehat{f}(r)\right) = a + b\log(r),$$

then comparing it with the logarithm of (6) we obtain estimators of α and λ

$$\widehat{\alpha} = b + 1, \qquad \widehat{\lambda} = \left(\frac{\exp(a)}{b+1}\right)^{1/(b+1)}. \tag{7}$$

Similarly, if for $\lambda r \gg 1$ the best linear fit is given by

$$\log\left(\widehat{f}(r)\right) = c + d\log(r),$$

then comparing it with the logarithm of (6) we obtain

$$\widehat{k} = -\frac{\widehat{\alpha}}{d+1}. \tag{8}$$

Unfortunately some serious problems arise when we apply this method [27]. Namely, we need a precise, yet as smooth as possible, estimate of the empirical density. This can be done – only up to a certain degree – by manipulating the window and the kernel of the density estimator. For a given sample $R = (R_1, R_2, \ldots, R_n)$, e.g. absolute value of 30 minute negative returns of the USD/CHF exchange rate as in Fig. 2, and for any real r the kernel density estimator [9, 18] provides an approximate value of the density in the form

$$\widehat{f}_n(r) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{b_n}K\left(\frac{r-R_i}{b_n}\right).$$

The kernel $K(u)$ is a continuous, nonnegative and symmetric function satisfying

$$\int_{-\infty}^{\infty} K(u)du = 1,$$

whereas the window $\{b_n\}_{n\in\mathbf{N}}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} nb_n = \infty$. In our estimation procedures we used the Bertlett kernel

$$K(u) = \frac{3}{4}(1-u^2)\mathbf{1}_{[-1,1]}(u),$$

with $b_n = 2.3\sigma n^{-1/5}$, where σ is the standard deviation of the sample, as it seemed to give the best results.

Another weakness of the presented method is caused by the fact that a small number of observations in the tails of the distribution (i.e. for $0 < \lambda r \ll 1$ and for $\lambda r \gg 1$) may introduce a large bias to the estimators. Ironically, to estimate α, λ and k we only use values from the tails of the distribution! Thus we lose most of the information carried by the sample. Moreover, there is still a problem of selecting data points for the linear fit on the log-log plot. We were unable to automate this procedure and the estimators depended on subjective choice of data points.

4.2 Minimization in L^p norm and maximum likelihood method

To overcome weak points of the direct approach we turned to more classical estimation methods: minimization of distance in the L^p norm and the maximum likelihood method. In our case the former one reduces to finding a minimum of a function of three variables

$$\int_0^\infty \left|\widehat{f}(r) - f(r;\alpha,\lambda,k)\right|^p dr. \tag{9}$$

Unfortunately, it also uses an estimate of the empirical density $\widehat{f}(r)$ and thus is subject to unnecessary estimation errors.

On the other hand the maximum likelihood method is free of this flaw. In short, it is a recipe for producing an estimator $\widehat{\theta}$ of the vector of parameters $\theta = (\alpha, \lambda, k)$, called the maximum likelihood estimator (MLE). The MLE is defined as an estimator such that it is the value of the argument θ which maximizes the likelihood function [6]

$$L_\theta(R_1, \ldots, R_n) = \prod_{k=1}^{n} f(R_k),$$

where $R = (R_1, \ldots, R_n)$ is the sample and f is the probability density, as a function of θ.

Notice that $L_\theta(R)$ obtains a maximum exactly for the same values of θ as the so called log-likelihood function

$$\log L_\theta(R_1, \ldots, R_n) = \sum_{k=1}^{n} \log f(R_k).$$

As it happens in our case, this observation usually leads to much easier maximization algorithms. Namely, from formula (5) we can calculate the log-likelihood function of the CED model

$$\begin{aligned}\log L_\theta(R_1, \ldots, R_n) \quad = \quad & \sum_{k=1}^{n} \left\{ \log \alpha\lambda^\alpha + (\alpha - 1)\log(R_k) + \log\left[1 - \exp\left(-\frac{(\lambda R_k)^{-\alpha}}{k}\right)\right]\right\} \\ & - \sum_{k=1}^{n} \left\{ \frac{1}{k} \int_0^{k(\lambda R_k)^\alpha} \left(1 - e^{-1/s}\right) ds \right\}.\end{aligned}$$

Then using the Nelder-Mead simplex minimization procedure (MATLAB implementation, see also [22]) applied to the function

$$-\log L_\theta(R_1, \ldots, R_n)$$

we obtain estimates of α, λ and k. Note that this method uses information carried by all returns and not only those in the tails of the distribution. MLEs are in general – for almost all data sets and sampling intervals (Δt's) – much better than those obtained using the direct approach or by minimizing the distance in the L^p norm.

Table 1: Empirical and CED drift exponents for the analyzed FX rates.

FX rate	$\frac{N_0}{N}$	D	D_0	$\frac{D_0-D}{D}$	c_λ	D_{CED}	$\frac{D_{CED}-D}{D}$
USD/DEM	20%	0.46	0.51	10.9%	0.48	0.46	-0.4%
GBP/USD	22%	0.42	0.48	13.2%	0.42	0.42	-0.4%
USD/JPY	22%	0.43	0.49	12.8%	0.42	0.43	-0.2%
USD/CHF	23%	0.46	0.52	12.5%	0.48	0.46	-1.2%
AUD/USD	25%	0.42	0.48	15.6%	0.42	0.42	-0.3%
DEM/JPY	26%	0.44	0.51	15.4%	0.45	0.44	-0.8%
GBP/DEM	27%	0.48	0.55	13.4%	0.45	0.47	-3.2%
USD/FRF	27%	0.42	0.49	16.3%	0.43	0.42	-1.4%
CAD/USD	28%	0.40	0.47	16.7%	0.41	0.39	-2.8%
DEM/FRF	29%	0.32	0.40	23.8%	0.35	0.31	-4.3%
DEM/ITL	29%	0.43	0.50	15.9%	0.45	0.43	-0.6%
DEM/FIM	48%	0.34	0.48	40.6%	0.35	0.34	-1.1%

5 Empirical analysis

The empirical studies were conducted on a data set released by Olsen & Associates for the Second International Conference on High Frequency Data in Finance, Zürich, April 1-3, 1998. The data set comprised exchange rates of all major currencies, metal prices (gold, silver, platinum) and two most popular stock indices (DJIA, S&P500) for the period January 1, 1996 – December 31, 1996. The data came in files where time (GMT for FX rates and metals, EST for indices) and FX rates, metal prices or index values, respectively, were reported sequentially in 30 minute intervals. Thus the number of data points was 17520 for each asset. It is worth noting, that these data sets are quotations of foreign currencies and metals available from international vendors like Reuters, Knight-Ridder and Telerate, and do not correspond to real prices in the global FX or metals market. The actual deals are usually made over the telephone and the transaction prices may differ from the offers or even no transactions may take place at the offered prices. On the other hand, stock index values are neither transaction prices nor nominal quotes but sums of selected sets of stock prices. Among the three types of assets, the FX rates were the most actively traded. For this reason we begin our empirical analysis with these data sets.

5.1 FX market

One of the main features of the FX spot market is the fact that it is a 24 hours global market, which is mostly inactive during weekends and national holidays. The first observation of the week arrives at 22:30 Greenwich Mean Time (GMT) on Sunday with the opening of the Asian markets and the last observation comes from the West Coast of the USA at about 22:30 GMT on Friday [16], see Fig. 3. In contrast to traditional high frequency analysis [30], we exclude inactive periods from the calculations, because they introduce a large bias to the estimators and make comparison of scaling laws for different instruments much more difficult. In the second column of Table 1 we give the percent of zero returns in the data, which corresponds to the inactivity of a given market. The inactivity ranges from 20% for the most actively traded in 1996 exchange rate – USD/DEM, to 48% for the DEM/FIM exchange rate.

Analysis of high frequency (intra-daily) data relies on definitions of the variables under study – in our case – the price and the volatility. Although these definitions are probably well known, we give them for the sake of completeness. The (logarithmic) price at time t_i is defined as

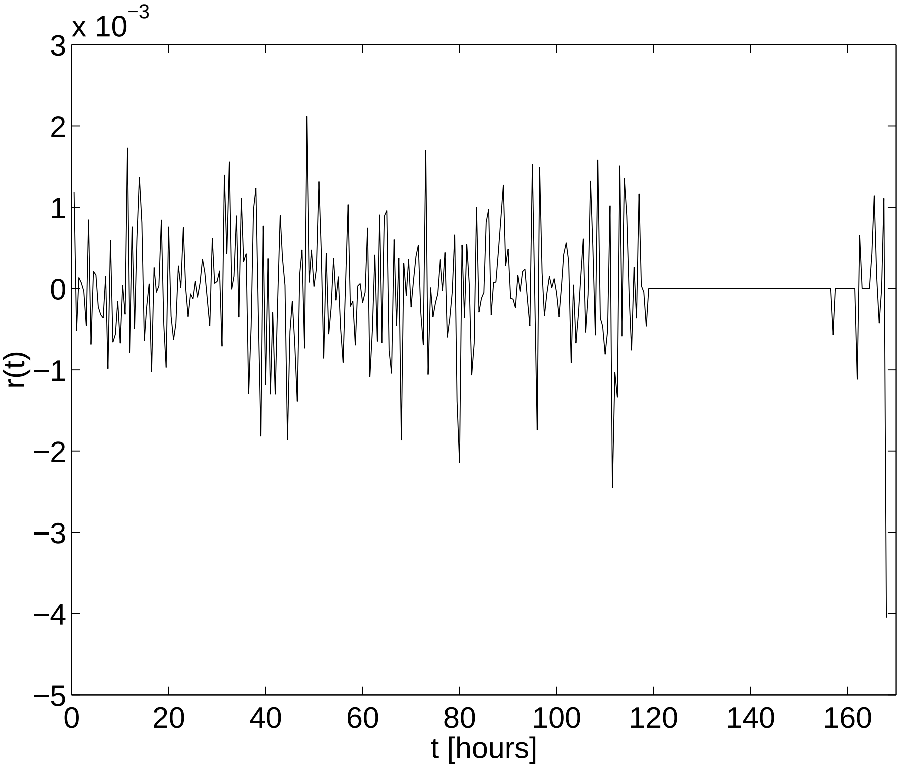

Figure 3: Half hour returns of the USD/CHF exchange rate during the second week of 1996, i.e. January 8th till January 14th.

[16, 38]

$$x(t_i) \equiv x(t_i, \Delta t) \equiv \frac{1}{2}\Big(\log p_{bid}(t_i) + \log p_{ask}(t_i)\Big),$$

where $\{t_i\}$ is the sequence of the regular spaced in time data, Δt is the time interval ($\Delta t = 30$ min., $\Delta t = 1$ hour, etc.) and $p_{bid}(t_i)$ ($p_{ask}(t_i)$) is the arithmetic average of the bid (ask) quotes just prior to and just after time t_i. The definition takes the average of the bid and ask price rather than either the bid or the ask series as a better approximation of the transaction price. The volatility $v(t_i)$ at time t_i is defined as the average of unsigned changes of the logarithmic price (return)

$$v(t_i) \equiv v(t_i, \Delta t, T) \equiv \frac{1}{n}\sum_{k=1}^{n}\Big|x(t_{i-k}) - x(t_{i-k} - \Delta t)\Big|, \tag{10}$$

where T is the sample period on which the volatility is computed (e.g. one day, one year) and n is a positive integer with $T = n\Delta t$.

In the third and fourth column of Table 1 we give the empirical drift exponent (see eq. (1)) for data without inactive periods (D) and for data with inactive periods (i.e. with zero returns included, D_0), whereas the difference (in percent) between these two exponents is given in the fifth column. We can clearly see that for all FX rates exponent D is significantly smaller than 0.5. Moreover, the difference between D and D_0 increases with the percent of zero returns (Tab. 1, column 2), thus showing that the inclusion of inactive periods causes overestimation of the drift exponent.

Intensive studies of all twelve FX rates and all 48 sampling intervals ($\Delta t = 30, 60, ..., 1440$ minutes) lead us to several interesting conclusions [27]. Firstly, we observed that we could reduce the initial six parameter set to three parameters $\{\widehat{\lambda}, \widehat{k}^{+}, \widehat{k}^{-}\}$. It turned out that for all analyzed FX rates the shape parameter α was almost equal to one both for positive and for absolute value

Table 2: Mean values (over all Δt's) of the shape parameters α^+ and α^- and drift exponents of the scale parameters λ^+ and λ^- for the analyzed FX rates.

FX rate	$\langle\alpha^+\rangle$	$\langle\alpha^-\rangle$	c_{λ^+}	c_{λ^-}
USD/DEM	1.04	1.00	0.47	0.48
GBP/USD	1.00	1.01	0.44	0.40
USD/JPY	1.02	1.00	0.45	0.40
USD/CHF	1.01	0.99	0.49	0.46
AUD/USD	1.04	1.06	0.44	0.40
DEM/JPY	1.07	1.03	0.46	0.43
GBP/DEM	1.01	0.99	0.50	0.42
USD/FRF	1.05	1.04	0.42	0.43
CAD/USD	0.98	1.02	0.41	0.41
DEM/FRF	0.99	0.97	0.34	0.35
DEM/ITL	1.00	1.01	0.44	0.47
DEM/FIM	0.96	0.99	0.34	0.37

of negative returns and the scale parameter λ was almost the same for positive and for absolute value of negative returns, see Table 2 and Fig. 4.

Secondly, we found that λ was related to the time interval Δt by a power law

$$\lambda(\Delta t) = \Lambda(\Delta t)^{-c_\lambda}. \tag{11}$$

This phenomenon is illustrated in Fig. 5 for the USD/CHF rate, separately for positive (λ^+) and for absolute value of negative returns (λ^-). The straight dashed and dotted lines represent the best linear fits (linear regression) for all positive and negative returns, respectively. The slopes are given by $-c_{\lambda^+} = -0.46 \pm 0.03$ and $-c_{\lambda^+} = -0.49 \pm 0.03$. The values of these exponents are almost equal and their mean value c_λ closely approximates the empirical drift exponent D, see Table 1.

As a consequence of the two above observations the CED density has finite and non-zero limits at point zero. Moreover, the left-hand limit is equal to the right-hand limit and we can make the CED density continuous on the whole real line by setting $f(0) = \lambda$.

The third parameter k, which decides how fast the information flow is spread out in the market, is qualitatively different for positive and for absolute value of negative returns. This difference is responsible for the asymmetry of the density of returns. Since k defines the tails of the distribution its estimator is very fragile. This causes a large dispersion of estimates and prevents us from identifying k as a function of Δt, see Fig. 6.

We can see from Table 1 that the drift exponent c_λ closely approximates the empirical drift exponent D. Yet we can do much better [27]. Recall that if a random variable R has density $f(r)$ then its mean value is given by

$$\langle R \rangle = \int_A r f(r) dr,$$

where A is the support of $f(r)$. Unfortunately in our case it is not clear how to integrate $\int_0^\infty r f(r) dr$, because the CED probability density function itself is a quite complicated function. However, the mean value of R can also be obtained [13] by taking the limit

$$\langle R \rangle = - \left. \frac{d}{dt} \right|_{t=0} \varphi(t),$$

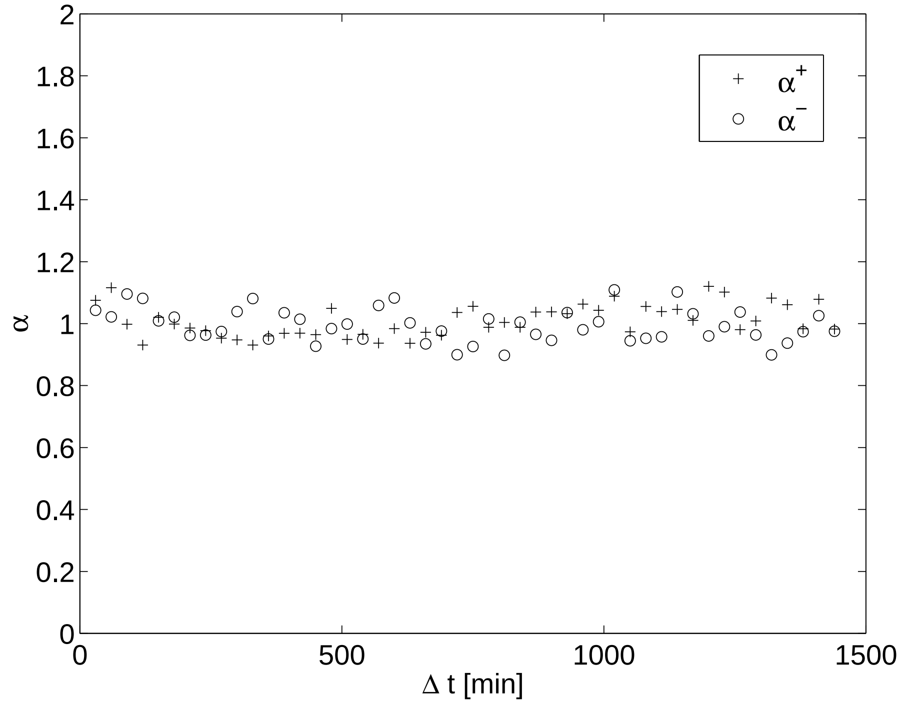

Figure 4: Shape parameter α for the USD/CHF exchange rate and for all Δt's.

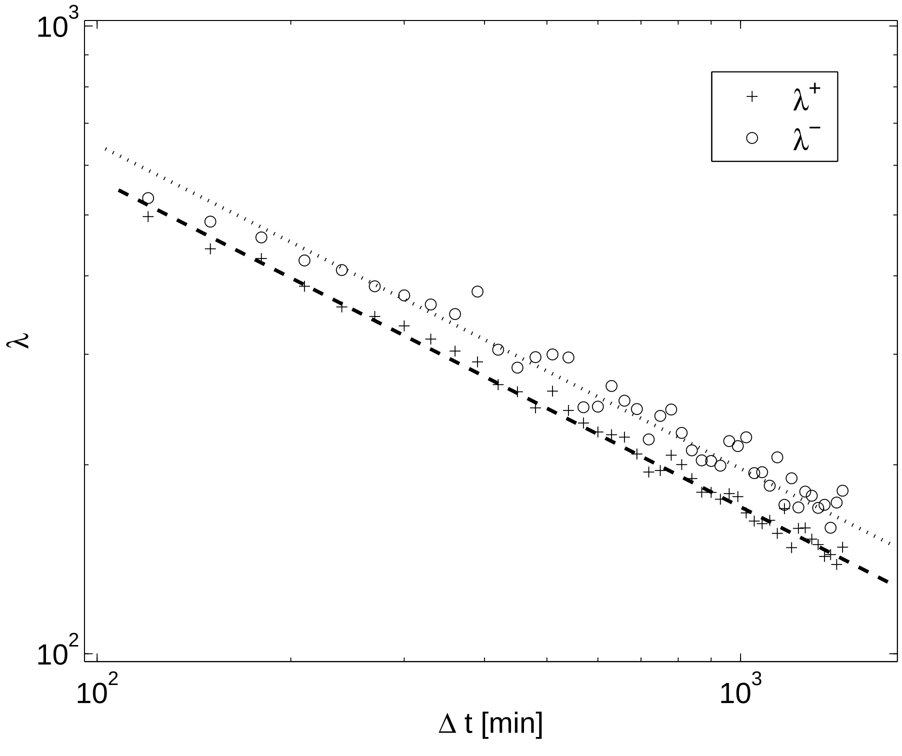

Figure 5: Scale parameter λ for the USD/CHF exchange rate and for all Δt's.

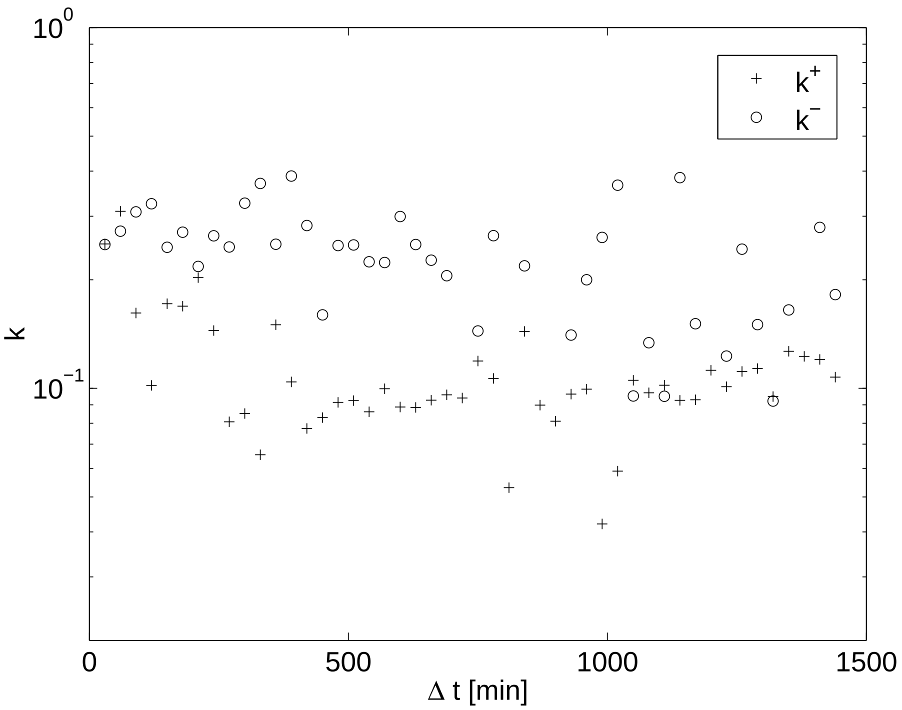

Figure 6: Parameter k for the USD/CHF exchange rate and for all Δt's.

where $\varphi(t)$ is the moment generating function $\varphi(t) = E\{e^{-tR}\}$. This leads us to the following

$$\langle R \rangle = -\left.\frac{d}{dt}\right|_{t=0} \int_0^\infty f(r) e^{-tr} dr = \frac{1}{\lambda} \int_0^\infty \exp\left\{ -\frac{1}{k} \int_0^{kr^\alpha} \left[1 - \exp(-s^{-1}) \right] ds \right\} dr. \tag{12}$$

Introducing the function

$$\Psi(\alpha, k) = \int_0^\infty \exp\left\{ -\frac{1}{k} \int_0^{kr^\alpha} \left[1 - \exp(-s^{-1}) \right] ds \right\} dr,$$

we can rewrite (12) as $\langle R \rangle = \frac{1}{\lambda} \Psi(\alpha, k)$ and the mean of the absolute value of return is given by

$$\langle |R| \rangle = \frac{\langle R^+ \rangle + \langle -R^- \rangle}{2}.$$

Analytic analysis of $\Psi(\alpha, k)$ is quite difficult. However, from our earlier empirical studies we know that $\alpha = 1$. This simplifies things and in Fig. 7 we can see Ψ as a function of $1/k$. Clearly for small k (large $1/k$) $\Psi(k)$ is very close to one. This justifies our earlier use of c_λ as an approximation of the empirical drift exponent D. But if we include $\Psi(k)$ in our calculations then the approximation is even better, see Table 1 where the CED estimate D_{CED} and the difference in percent between D_{CED} and D is given in the last two columns. This is also illustrated in Fig. 8, where almost a perfect match is obtained for the USD/CHF exchange rate.

5.2 Metals and indices

Like foreign currency, metals data sets contain bid and ask quotes which were recorded by international financial data vendors. The actual deals are typically made over the telephone and the transaction prices usually differ from the offers. Although global, the metals market is less

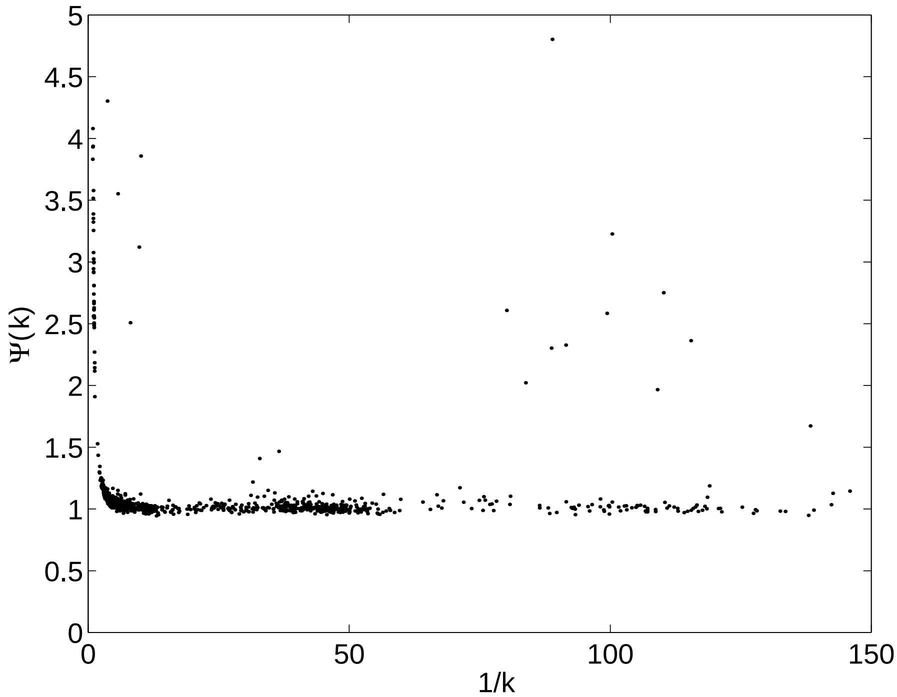

Figure 7: Function $\Psi(\alpha, k)$ for all twelve exchange rates and for all Δt's.

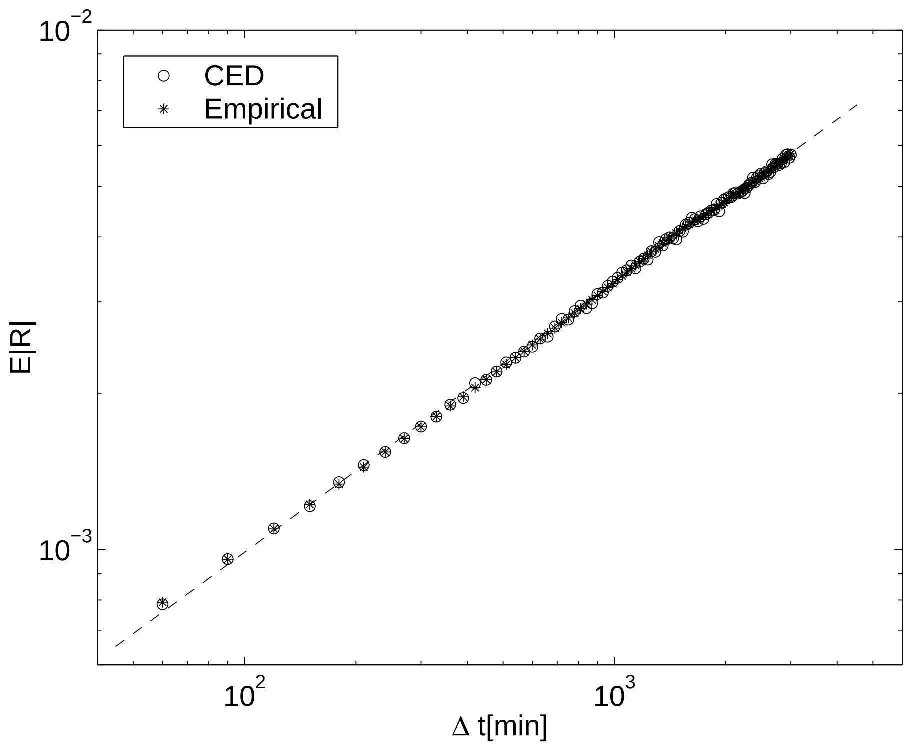

Figure 8: Scaling law for volatility. A comparison of the CED and empirical volatilities for the USD/CHF exchange rate and for all Δt's. The dashed line represents the best linear fit to the CED volatilities.

Table 3: Empirical and CED drift exponents, mean values of the shape parameters α^+ and α^- and drift exponents of the scale parameters λ^+ and λ^- for the analyzed metals.

Metal	$\frac{N_0}{N}$	D	D_0	$\frac{D_0-D}{D}$	D_{CED}	$\frac{D_{CED}-D}{D}$	$\langle\alpha^+\rangle$	$\langle\alpha^-\rangle$	c_{λ^+}	c_{λ^-}
Gold	41%	0.40	0.46	15.1%	0.50	25.7%	1.27	1.28	0.49	0.56
Silver	51%	0.30	0.40	32.2%	0.34	13.7%	1.28	1.27	0.44	0.30
Platinum	71%	0.38	0.43	13.4%	0.41	7.7%	1.36	1.38	0.42	0.47

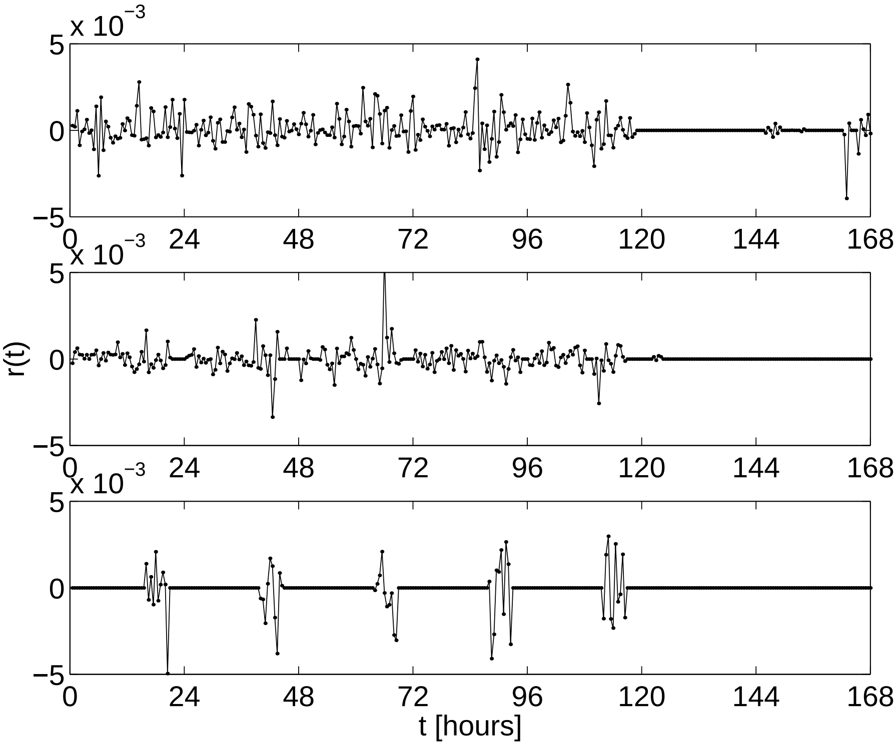

Figure 9: Half hour returns of the USD/CHF exchange rate (*top panel*), gold (*middle panel*) and the S&P500 index (*bottom panel*) during the ninth week of 1996, i.e. February 26th till March 3rd. Clearly visible are the working hours of the New York Stock Exchange.

active than most foreign exchange markets. Comparison of the second columns in Tables 1 and 3 places the most actively traded metal – gold – somewhere between DEM/ITL and DEM/FIM – the least actively traded of the twelve analyzed exchange rates. The inactivity of other metals markets is even higher, with platinum being traded for only 29% of the time! This property of the global metals market is also clearly visible in Fig. 9 where a sample week of the USD/CHF exchange rate, gold and the S&P500 index 30-minute returns is presented.

Intensive studies of all three metals prices and all 48 sampling intervals ($\Delta t = 30, 60, ..., 1440$ minutes) allowed us to observe that the shape parameter α was almost the same for positive and for absolute value of negative returns. It averaged 1.27 for gold and silver and 1.37 for platinum, see Table 3. The larger, than for FX rates, value of α could originate either from estimation errors (smaller samples due to lower activity of the market) or different characteristics of the global metals market.

The scale parameter λ was also very close for both sets of returns, however, the differences were higher than for most FX rates. Moreover, like for FX rates, we found that λ was related

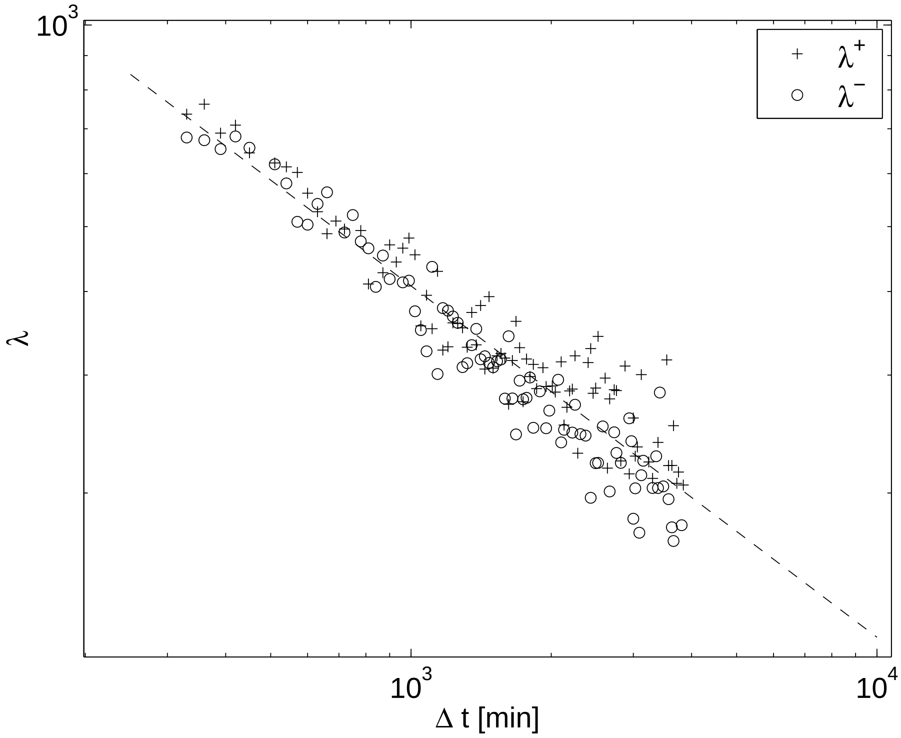

Figure 10: Scale parameter λ for gold and for all Δt's.

to the time interval Δt by a power law (11), see Fig. 10, and that k, which decides how fast the information flow is spread out in the market, was qualitatively different for positive and for absolute value of negative returns resulting in asymmetry of the density of returns.

Compared to the metals market, the inactivity of the US stock market is higher – 80-85%, averaging to about 5-6.5 trading hours per working day. Differences in the empirical drift exponents (1) between data with and without inactive periods ranged from 100% for S&P500 to 250% for DJIA. The CED model did not fit to the data well and percentage errors were two orders of magnitude higher than for FX rates. One possible explanation for this behavior is the deficiency of data points – we were analyzing less than 3500 index values – which could cause large estimation errors. Another possible explanation stems from the fact that the US stock market, although the largest and most influential in the world, is not a global one. The prices of US stocks reflect the current state of the US economy and hardly depend on the situation in Europe or Asia.

6 Conclusions

High-frequency data means a very large amount of data. The number of observations in one single day of a liquid market is equivalent to the number of daily data within 30 years. High-frequency data open the way for studying financial markets at very different time scales, ranging from minutes to years or even decades [1, 8]. As it turns out, some empirical properties are similar at different scales, leading to fractal or scale-invariant behaviors.

But why look for fractals, scaling and power laws to begin with? The reason for this is that complex phenomena often give rise to power laws which are universal in the sense that they are independent – to a large extent – of the microscopic details. These power laws emerge from collective action and transcend individual specificities [5]. The knowledge of such universal laws would be very helpful, not only in the understanding of how the markets work, but also in constructing efficient and profitable trading strategies.

Scaling properties have been the objects of study since the early work of Mandelbrot [24] on cotton prices. In 1990, the research group of Olsen & Associates working with high-frequency data published empirical studies of volatility scaling in FX markets [30]. Later, these and other scaling properties were found for different financial and economic time series (see eg. [2, 15, 23, 26]).

In this article we have shown that the CED model can provide theoretical background for the observed stylized facts, in particular the scaling law for volatility and a power law decay of the distribution of returns. It clarifies the ideas of the Fractal and Heterogeneous Market Hypotheses and provides a rigorous mathematical framework for further analysis of financial complex processes.

References

[1] T.G. Andersen, Some reflections on analysis of high-frequency data, J. Business & Economic Statistics 18 (2000) 146-153.

[2] H. Aoyama, W. Souma, Y. Nagahara, M.P. Okazaki, H. Takayasu, M. Takayasu, Pareto's law for income of individuals and debt of bankrupt companies, Fractals 8 (2000) 293-300.

[3] O.E. Barndorff-Nielsen, Processes of the normal inverse Gaussian type, Finance & Stochastics 2 (1998) 41-68.

[4] R.C. Blattberg, N.J. Gonedes, A comparison of the stable and Student distributions as statistical models for stock prices, J. Business 47 (1974) 244-280.

[5] J.-P. Bouchaud, Power laws in economics and finance: some ideas from physics, Quantitative Finance 1 (2001) 105-112.

[6] G. Casella, R.L. Berger, Statistical Inference, 2nd ed., Duxbury, Pacific Grove, 2001.

[7] P.K. Clark, A subordinated stochastic process model with finite variance for speculative prices, Econometrica 41 (1973) 135-156.

[8] M.M. Dacorogna, R. Gencay, U.A. Müller, R.B. Olsen, O.V. Pictet, An Introduction to High-Frequency Finance, Academic Press, San Diego, 2001.

[9] L. Devroye, A Course on Density Estimation, Birkhäuser, Boston, 1987.

[10] E. Eberlein, U. Keller, Hyperbolic distributions in finance, Bernoulli 1 (1995) 281-299.

[11] E. Fama, Mandelbrot and the stable Paretian hypothesis, J. Business 35 (1963) 420-429.

[12] E. Fama, The behavior of stock market prices, J. Business 38 (1965) 34-105.

[13] W. Feller, An Introduction to Probability Theory and Its Applications, 2nd edn. Wiley, New York, 1971.

[14] S. Galluccio, G. Caldarelli, M. Marsili, Y.-C. Zhang, Scaling in currency exchange, Physica A 245 (1997) 423-436.

[15] I. Gleria, R. Matsushita, S. Da Silva, Scaling power laws in the Sao Paulo Stock Exchange, Economics Bulletin (2002) in print.

[16] D.M. Guillaume, M.M. Dacorogna, R.R. Dave, U.A. Müller, R.B. Olsen, O.V. Pictet, From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets, Finance & Stochastics 1 (1997) 95-129.

[17] S.R. Hurst, E. Platen, S.T. Rachev, Subordinated market index models: A comparison, Financial Engineering & Japanese Markets 4 (1997) 97-124.

[18] A. Janicki, A. Weron, Simulation and Chaotic Behavior of α-Stable Stochastic Processes, Marcel Dekker, New York, 1994.

[19] A. Jurlewicz, A. Weron, K. Weron, Asymptotic behavior of stochastic systems with conditionally exponential decay property, Applicationes Mathematicae 23 (1996) 379-394.

[20] S.J. Kon, Models of stock returns - A comparison, J. Finance 39 (1984) 147-165.

[21] U. Küchler, K. Neumann, M. Sørensen, A. Streller, Stock returns and hyperbolic distributions, Mathematical & Computer Modelling 29 (1999) 1-15.

[22] J.C. Lagarias, J. A. Reeds, M. H. Wright, P. E. Wright, Convergence properties of the Nelder-Mead simplex method in low dimensions, SIAM Journal of Optimization 9 (1998) 112-147.

[23] T. Lux, M. Ausloos, Market fluctuations I: Scaling, multi-scaling and their possible origins, in A. Bunde, H.-J. Schellnhuber eds., Theories of disasters, Springer, Berlin, 2001.

[24] B.B. Mandelbrot, The variation of certain speculative prices, J. Business 36 (1963) 394-419.

[25] B.B. Mandelbrot, Fractals and Scaling in Finance: Discontinuity, Concentration, Risk, Springer-Verlag, New York, 1997.

[26] R.N. Mantegna, H.E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge, 1999.

[27] S. Mercik, R. Weron, Scaling in currency exchange: A Conditionally Exponential Decay approach, Physica A 267 (1999) 239-250.

[28] S. Mittnik, S.T. Rachev, Modeling asset returns with alternative stable distributions, Econometric Rev. 12 (1993) 261-329.

[29] U.A. Müller, M.M. Dacorogna, R.D. Dave, O.V. Pictet, R.B. Olsen, J.R. Ward, Unpublished manuscript, Olsen & Associates, Zürich, 1993.

[30] U.A. Müller, M.M. Dacorogna, R.B. Olsen, O.V. Pictet, M. Schwarz, C. Morgenegg, Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis, J. Banking & Finance 14 (1990) 1189-1208.

[31] E.E. Peters, Fractal Market Analysis: Applying Chaos Theory to Investment and Economics, Wiley & Sons, New York, 1994.

[32] S.T. Rachev, S. Mittnik, Stable Paretian Models in Finance, Wiley, 2000.

[33] S.T. Rachev, A. Weron, K. Weron, Conditionally exponential dependence model for asset returns, Appl. Math. Lett. 10 (1997) 5-9.

[34] A.L. Tucker, L. Bond, The probability distribution of foreign exchange price changes: Tests of candidate processes, Rev. Economics & Statistics 70 (1988) 638-647.

[35] A. Weron, S. Mercik, R. Weron, Origins of the scaling behaviour in the dynamics of financial data, Physica A 264 (1999) 562-569.

[36] A. Weron, R. Weron, Financial Engineering: Derivatives Pricing, Computer Simulations, Market Statistics, (in Polish) WNT, Warsaw, 1998.

[37] K. Weron, A. Jurlewicz, Two forms of self-similarity as a fundamental feature of the power-law dielectric relaxation, J. Physics A: Mathematical & General 26 (1993) 395-410.

[38] R. Weron, K. Weron, A. Weron, A conditionally exponential decay approach to scaling in finance, Physica A 264 (1999) 551-561.