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ABSTRACT

This paper investigates Goodwin's Lotka-Volterra model in disaggregative form. It is found that the dynamic behavior of the model depends on the material input coefficients matrix.

Key words: Goodwin's Lotka-Volterra model, principal coordinates, disaggregated models, multiple degree of freedom system

JEL Classification: C02, E32

1. Introduction

It is well known that, due to the method of 'principal coordinates', the Goodwin's Lotka-Volterra model (G-L-V hereafter) can be written in disaggregative form.[1] This form can be easily recognizable as a multiple degree of freedom vibration system.[2] Goodwin (1986, p. 208) mentions that the dynamic behavior of the whole system depends not on the possible complex eigenvalues of material input coefficients matrix, but on the sectoral interrelationships of the system. In this paper, we shall deal with this model and show that the behavior of the model depends on the eigenvalues of material input coefficients matrix.

The remainder of the paper is structured as follows. Section 2 presents the model. Section 3 evaluates the finds of our investigation. Section 4 concludes.

2. The Model

The well-known G-L-V model may be described, in disaggregative form, by the following relations (Goodwin, 1984, 1986, 1989; Goodwin and Punzo, 1987, pp. 106-112)

$$Y_j \equiv (1-\lambda_j) X_j, \lambda_j < 1 \ , \ j = 1,2,...,n \tag{1}$$

$$L_j / X_j = B_j, \hat{B}_j = -a_j \tag{2}$$

$$\hat{N}_j = n_j \tag{3}$$

$$\hat{W}_j = \rho_j (L_j / N_j) - \gamma_j, \gamma_j < \rho_j \tag{4}$$

$$\dot{X}_j = \mu_j [k_j \dot{X}_j - (X_j - \lambda_j X_j - W_j B_j X_j)], k_j \mu_j > 1 \tag{5}$$

where $a_j, n_j, \rho_j, \gamma_j, \mu_j$ are positive constants. As usual, a 'dot' ('hat') above a variable denotes time derivative (logarithmic derivative), *i.e.,* $\dot{y} = dy/dt$ ($\hat{y} = \dot{y}/y$). Furthermore, X_j, Y_j, L_j, B_j, N_j, k_j, W_j and μ_j denote the gross output, net output, employment, direct labour coefficient, labour force, desired capital-output ratio, real wage rate, and the speed of adjustment of output to excess demand of the *jth* 'eigensector', respectively.[3]

Finally, λ_j denotes the j eigenvalue of the diagonalizable $n \times n$ matrix of material input coefficients, $\mathbf{A}$.[4] It is also assumed that the system is viable, *i.e.*, the Perron-Frobenius (P-F hereafter) eigenvalue, $\lambda_{\mathbf{A}}$, is less than 1 (for more details, see Kurz and Salvadori, 1995, chs 3-4).

Relation (1) captures the assumption that the capital lasts for one period of production. Relations (2) and (3) capture the assumption of steady ('disembodied') technical progress and steady growth of the labour force, respectively. Relation (4) captures the assumption that the real wage rate rise in the neighborhood of full employment. Finally, relation (5) captures the adjustment of output to excess demand. We also assume that all wages and not profits are consumed, and the capital-output ratio and all prices are both constant.

From the definition of workers' share of the *jth* 'eigensector', $U_j \equiv W_j L_j / Y_j$, we obtain

$$U_j = \theta_j / (1 - \lambda_j) \tag{6}$$

where $\theta_j \equiv W_j B_j = U_j (1 - \lambda_j)$ is the unit labour cost of the *jth* eigensector. Furthermore, combining (5) and (6) and rearranging, we get

$$\hat{X}_j = [\mu_j / (k_j \mu_j - 1)](1 - \lambda_j)(1 - U_j) \tag{7}$$

Logarithmic differentiation of $U_j \equiv W_j L_j / Y_j$ yields

$$\hat{U}_j = \hat{W}_j + \hat{L}_j - \hat{Y}_j \tag{8}$$

But from relations (1) and (2) we also have $\hat{L}_j = \hat{Y}_j - a_j$. So, substituting the above relation and relation (4) in (8), we get

$$\hat{U}_j = \rho_j (L_j / N_j) - a_j - \gamma_j \tag{9}$$

From the definition of employment rate of the *jth* eigensector, $V_j \equiv L_j / N_j$, relation (9) can be written as

$$\hat{U}_j = \rho_j V_j - (a_j + \gamma_j) \tag{10}$$

Logarithmic differentiation of $V_j \equiv L_j / N_j$ yields

$$\hat{V}_j = \hat{L}_j - \hat{N}_j \tag{11}$$

Furthermore, from (1) we obtain $\hat{X}_j = \hat{Y}_j$. Therefore, from $\hat{L}_j = \hat{Y}_j - a_j$, (3), (7) and (11) we get

$$\hat{V}_j = [\mu_j / (k_j \mu_j - 1)](1 - \lambda_j)(1 - U_j) - (a_j + n_j) \tag{12}$$

Consequently, the model reduces to the $2n$ dimensional dynamical system of (10) and (12). The singular points (equilibrium) of the system is given by

$$\dot{U}_j = \dot{V}_j = 0 \text{ or } \left(U_j^*, V_j^*\right) = \left\{(0,0), \left(\alpha_j / b_j, c_j / d_j\right)\right\} \tag{13}$$

where $\alpha_j \equiv [\mu_j / (k_j \mu_j - 1)](1 - \lambda_j) - (a_j + n_j)$, $b_j \equiv [\mu_j / (k_j \mu_j - 1)](1 - \lambda_j)$, $c_j \equiv a_j + \gamma_j$, $d_j \equiv \rho_j$.

Linearazing the system around the non-zero point we get

$$\dot{U}_j = [d_j (\alpha_j / b_j)] V_j' \tag{14}$$

$$\dot{V}_j = -[b_j (c_j / d_j)] U_j' \tag{15}$$

where $V_j' \equiv V_j - V_j^*$ and $U_j' \equiv U_j - U_j^*$.

Relations (14) and (15) can be written in a matrix form as

$$\dot{\mathbf{U}} = <\mathbf{\Phi}> \mathbf{V}' \tag{16}$$

$$\dot{\mathbf{V}} = - <\mathbf{\Omega}> \mathbf{U}' \tag{17}$$

where $\dot{\mathbf{U}} \equiv [\dot{U}_j]$, $\mathbf{U}' \equiv [U_j']$, $\dot{\mathbf{V}} \equiv [\dot{V}_j]$, $\mathbf{V}' \equiv [V_j']$ and $<\mathbf{\Phi}>$, $<\mathbf{\Omega}>$ denote the diagonal matrices formed from the elements $[d_j(\alpha_j / b_j)]$ and $[b_j(c_j / d_j)]$, respectively.

Moreover, from (16) we obtain the following system of second-order differential equations

$$\ddot{\mathbf{U}} = <\mathbf{\Phi}> \dot{\mathbf{V}} \tag{18}$$

Substituting (17) in (18) and taking into consideration $<\mathbf{M}> \equiv <\mathbf{\Phi}><\mathbf{\Omega}>$, it follows that relation (18) becomes

$$\ddot{\mathbf{U}} = - <\mathbf{M}> \mathbf{U}' \tag{19}$$

which represent a set of n uncoupled single degree of freedom equations of motion.

It is important to note that 'these eigensectors do not exist–they are mere accounting devices: no decisions are taken by such fictitious units' (Goodwin and Punzo, 1987, p. 60). Regarding this, Goodwin observes that we can always go back to the 'actual quantities'. Through a 'coordinate transformation', these n independent single degree of freedom systems (see, relation (19)) are transformed back to a multiple degree of freedom system ('actual quantities') and *vice versa*. Thus, pre-multiplying relation (19) by $\mathbf{Q}$ and taking into account that the vector $\mathbf{u} \equiv \mathbf{QU}$ ($\mathbf{v} \equiv \mathbf{QV}$) denotes the workers' shares (the employment rates) of the actual system, we obtain[5]

$$\ddot{\mathbf{u}} = -\mathbf{M}\mathbf{u}' \tag{20}$$

where $\ddot{\mathbf{u}} \equiv [\ddot{u}_j] \equiv \mathbf{Q}\ddot{\mathbf{U}}$, $\mathbf{u}' \equiv [u'_j] \equiv \mathbf{Q}\mathbf{U}'$, $\mathbf{M} \equiv \mathbf{Q} < \mathbf{M} > \mathbf{Q}^{-1}$, and $\mathbf{Q} \equiv [q_j]$ ($\mathbf{Q}^{-1} \equiv [q'_j]$) is the $n \times n$ matrix which is formed from the right-hand side (left-hand side) eigenvectors of $\mathbf{A}$, *i.e.*, it denotes the 'modal matrix' of $\mathbf{A}$.[6]

Similarly, pre-multiplying relation (17) by $\mathbf{Q}$ we obtain

$$\dot{\mathbf{v}} = -\mathbf{\Omega}\mathbf{u}' \tag{21}$$

where $\dot{\mathbf{v}} \equiv [\dot{v}_j] \equiv \mathbf{Q}\dot{\mathbf{V}}$ and $\mathbf{\Omega} \equiv \mathbf{Q} < \mathbf{\Omega} > \mathbf{Q}^{-1}$.

Hence, from (20) we obtain the solutions u_j, in terms of t. Finally, given $\mathbf{u}'$ we obtain the solution of (21).

3. Discussion

The system (20) is easily recognizable as a 'free vibration of undamped n degree of freedom system' where the identity matrix, $\mathbf{I}$, and the matrix $\mathbf{M}$ are, respectively, the 'mass' and 'stiffness' matrices (see, for example, Shabana, 1996). Moreover, the examination of the system (10), (12) shows that there are two possibilities regarding the eigenvalues of matrix $\mathbf{A}$. These possibilities are:

1. All eigenvalues are real
2. Some eigenvalues are complex

We shall deal with these cases in turn:

Case 1: All eigenvalues are real

If all eigenvalues are real, then the result is a $2n$ dimensional system, with n Lotka-Volterra (L-V hereafter) oscillating pairs, *i.e.*, all pairs U, V will oscillate with different periods and phases. Moreover, taking into consideration that the actual system can be described as a 'free vibration of undamped n degree of freedom system', the motion of each actual sector ('general motions') can be expressed as a linear combination of those simpler motions ('eigenoscillations'), each of which has a definite frequency.[7] If the ratios of the frequencies are rational numbers, these motions are periodic. On the other hand, if one or more pairs of frequencies form an irrational ratio, these motions will be erratic, never repeating. It should be remarked

that these motions depends 'solely' on the 'form of the system', *i.e.*, on the sectoral interrelationships of the system.

Case 2: Some eigenvalues are complex

In this case, some of the pairs of L-V equations involve complex parameters which present serious interpretatory problems. It is worth recalling that the equation of motion of each actual sector can be expressed as a linear combination of the motions of the n independent single degree of freedom systems. Although, in contrast with the previous case, the actual system cannot be studied by means of standard analytical methods, *i.e.*, there is no known theory available for a system consist of n coupled systems of L-V differential equations some of which involve complex parameters.[8] Instead, we shall apply simulation methods to investigate the dynamical behavior of the system. A set of numerical simulations gives oscillations into monotonic explosions.[9] It should be mentioned that the present analysis is based on the assumption of a same labour market (or $a_j, n_j, \rho_j, \gamma_j$) and μ_j, k_j in every 'eigensector'. If particular $a_j, n_j, \gamma_j, \mu_j, k_j$ is assumed then the elements of matrix $\mathbf{M}$ will be complex, *i.e.*, the actual system is recognizable as a 'hysteretically damped n degree of freedom system' (see Maia and Silva, 1997, pp. 62-4).[10] Thus, the system (20) may be written as

$$\ddot{\mathbf{u}} + \mathbf{N}\mathbf{u}' + i\mathbf{D}\mathbf{u}' = \mathbf{0} \qquad (20a)$$

where $\mathbf{N}$, $\mathbf{D}$ are $n \times n$ 'stiffness', 'hysteric damping' matrices, respectively, $i(\equiv \sqrt{-1})$ is the imaginary unit, and $\mathbf{M} \equiv \mathbf{N} + i\mathbf{D}$. It must be noted that (i) there is no an economic interpretation of the system (20a); and (ii) 'this type of damping has been defined only for the particular case of forced harmonic vibration and presents some difficulties to rigorous free vibration […] analysis' (*ibid.*, p. 62).

4. Concluding Remark

The examination of the G-L-V model in disaggregative form showed that there are two possibilities regarding the eigenvalues of material input coefficients matrix. It is worth recalling that if all the eigenvalues are real then the result is a $2n$ dimensional system with n L-V oscillating pairs, where the motion of each actual sector is a linear combination of these pairs. On the other hand, if some eigenvalues are complex then

we should assume the same labour market, desired capital-output ratio, and speed of adjustment of output to excess demand in all eigensectors, in order to avoid serious interpretatory problems. Furthermore, by introducing this assumption, it is found that the actual system cannot be studied by means of standard analytical methods. Instead, we applied simulation methods, which gave oscillations into monotonic explosions. These no satisfactory solutions are likely to have derived from limitations of the present model, such as the absence of the assumption that capacity is not fully used and, therefore, the lack of an investment demand function *à la* Bhaduri and Marglin (1990), which is positively related to the profits share and to capacity utilization (for this line of research, see Mainwaring, 1991; Barbosa-Filho and Taylor, 2006; Mariolis, 2006, pp. 202-214). Future work should investigate the possibility to overcome this drawback, by introducing into the original model this assumption.

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Footnotes

[1] The method of 'principal coordinates' is originally proposed from R. M. Goodwin at 1976 in *Use of Normalised General Co-ordinates in Linear Value and Distribution Theory* (1976, ch 7 1983). With regard to this issue, see, also, Aruka, 1991; Boggio, 1991; Steenge, 1995.

[2] It is worth mentioning that the degrees of freedom of a system are the number of independent coordinates necessary to completely describe the motion of that system.

[3] We consider, in accordance with Goodwin, n independent labour markets, each with its particular, given growth rates of productivity and labour force.

[4] Matrices (and vectors) are denoted by boldface letters.

[5] Note that $\mathbf{M}$ and $<\mathbf{M}>$ are similar matrices.

[6] In other words, we follow the reciprocal of the method, which leads to a set of n independent systems.

[7] It should be noted that it is possible u_j (v_j) to exceed unity. As is known, the original G-L-V model (Goodwin, 1967) suffers from the same basic defect (see, for example, Desai *et al.*, 2006, pp.5). Recent works (Weber, 2005, pp. 17-26; Desai *et al.*, 2006) show that the original model can be reformulated to ensure that these variables cannot exceed unity.

[8] Taking into account that some of the eigenvalues of $\mathbf{A}$ are complex, it is then not too difficult to show that some of the eigenvalues of matrix $\mathbf{M}$ are complex too. See Appendix A.

[9] In the Appendix B, we present a numerical example illustrating the points made above.

[10] See Appendix A.

APPENDIX A

Let us consider a 3×3 matrix $\mathbf{A}$ with one real eigenvalue (the P-F eigenvalue), $\lambda_1(<1)$ and a pair of complex conjugate eigenvalues λ_2, λ_3. From (20) we get

$$\mathbf{M} \equiv \mathbf{Q} < \mathbf{M} > \mathbf{Q}^{-1} \tag{A.1}$$

where

$$< \mathbf{M} >= \begin{pmatrix} \alpha_1 c_1 & 0 & 0 \\ 0 & a_2 c_2 & 0 \\ 0 & 0 & a_3 c_3 \end{pmatrix}$$

and $\alpha_j \equiv [\mu_j /(k_j \mu_j - 1)](1-\lambda_j) - (a_j + n_j)$, $c_j \equiv a_j + \gamma_j$. Since λ_2, λ_3 are complex, then $\alpha_2 c_2$ and $\alpha_3 c_3$ are complex numbers and, therefore, the matrix $\mathbf{M}$ has complex eigenvalues. Hence, if matrix $\mathbf{A}$ has complex eigenvalues, then matrix $\mathbf{M}$ has also complex eigenvalues.

Moreover, relation (A.1) after rearrangement gives:

$$\mathbf{M}=\begin{pmatrix} c_1\alpha_1 q_{11}q'_{11}+c_2\alpha_2 q_{12}q'_{21}+c_3\alpha_3 q_{13}q'_{31} & c_1\alpha_1 q_{11}q'_{12}+c_2\alpha_2 q_{12}q'_{22}+c_3\alpha_3 q_{13}q'_{32} & c_1\alpha_1 q_{11}q'_{13}+c_2\alpha_2 q_{12}q'_{23}+c_3\alpha_3 q_{13}q'_{33} \\ c_1\alpha_1 q_{21}q'_{11}+c_2\alpha_2 q_{22}q'_{21}+c_3\alpha_3 q_{23}q'_{31} & c_1\alpha_1 q_{21}q'_{12}+c_2\alpha_2 q_{22}q'_{22}+c_3\alpha_3 q_{23}q'_{32} & c_1\alpha_1 q_{21}q'_{13}+c_2\alpha_2 q_{22}q'_{23}+c_3\alpha_3 q_{23}q'_{33} \\ c_1\alpha_1 q_{31}q'_{11}+c_2\alpha_2 q_{32}q'_{21}+c_3\alpha_3 q_{33}q'_{31} & c_1\alpha_1 q_{31}q'_{12}+c_2\alpha_2 q_{32}q'_{22}+c_3\alpha_3 q_{33}q'_{32} & c_1\alpha_1 q_{31}q'_{13}+c_2\alpha_2 q_{32}q'_{23}+c_3\alpha_3 q_{33}q'_{33} \end{pmatrix}$$

(A.1a)

Since λ_1 is the P-F eigenvalue, the first column (row) of $\mathbf{Q}$ (of $\mathbf{Q}^{-1}$), is real and positive. On the other hand, the corresponding eigenvectors to λ_2, λ_3 will ordinary involve negative and complex numbers. Therefore, the element m_{11} can be expressed as:

$$\alpha_1 c_1 q_{11}q'_{11}+(\pm\varepsilon\pm\eta i)(\pm\sigma\pm\beta i)(\pm\zeta\pm\delta i)+(\pm\varepsilon\mp\eta i)(\pm\sigma\mp\beta i)(\pm\zeta\mp\delta i)$$

where $\sigma,\beta,\zeta,\delta\geq 0$ and $\varepsilon,\eta,\alpha_1 c_1 q_{11}q'_{11}>0$.

The above relation is a sum of real numbers, and, therefore the element κ_{11} is real. By contrast, if $a_2\neq a_3$ or $n_2\neq n_3$ or $k_2\neq k_3$ or $\gamma_2\neq\gamma_3$ or $\mu_2\neq\mu_3$ then $\alpha_2 c_2\neq\alpha_3 c_3$ and then m_{11} is complex (and the same holds true for any m_{ij} of a *nxn* matrix). Hence, if each 'eigensector' has its particular $a_j, n_j, \gamma_j, \mu_j, k_j$, then the elements of matrix $\mathbf{M}$ will be complex.

APPENDIX B

In what follows we present two numerical examples illustrating the points made above.

Example 1

Consider the following system:

$$\mathbf{A}=\begin{pmatrix} 0.1 & 0.015 & 0.001 \\ 0.16 & 0.35 & 0.06 \\ 0.14 & 0.14 & 0.275 \end{pmatrix}$$

with

$$\mathbf{Q}\cong\begin{pmatrix} 0.035 & -0.040 & -0.841 \\ 0.700 & -0.364 & 0.452 \\ 0.713 & 0.930 & 0.297 \end{pmatrix},$$

$$\mathbf{Q}^{-1} \cong \begin{pmatrix} 0.670 & 0.976 & 0.411 \\ -0.145 & -0.773 & 0.766 \\ -1.154 & 0.077 & -0.019 \end{pmatrix},$$

$$\lambda_1 \cong 0.419,\ \lambda_2 \cong 0.214,\ \lambda_3 \cong 0.0916$$

Also, we assume $\gamma_j = \gamma$, $\rho_j = \rho$, $a_j = a$, $n_j = n$, $[\mu_j/(k_j\mu_j - 1)] = [\mu/(k\mu - 1)]$ where $j = 1,2,3$. Thus for $\gamma = 0.03,\ \rho = 0.04,\ a = 0.005,\ n = 0.06$ and $[\mu/(k\mu - 1)] = 1$ we get

$$\mathbf{M} \cong \begin{pmatrix} -0.029 & 0.0005 & 0.00004 \\ 0.006 & -0.020 & 0.002 \\ 0.005 & 0.005 & -0.023 \end{pmatrix}$$

and

$$\lambda^{\mathrm{M}}{}_1 \cong \text{-}0.030,\ \lambda^{\mathrm{M}}{}_2 \cong \text{-}0.025, \text{ and } \lambda^{\mathrm{M}}{}_3 \cong \text{-}0.018$$

where $\lambda^{\mathrm{M}}{}_j$ the j eigenvalue of $\mathbf{M}$.

Moreover, we set $u_1'(0) = u_2'(0) = u_3'(0) = 0.1$, $\dot{u}_1'(0) = \dot{u}_2'(0) = \dot{u}_3'(0) = 0.1$. Therefore, using Mathematica, we get

$$u_1' = 0.007\cos(0.134t) + 0.0006\cos(0.159t) + 0.092\cos(0.172t) + 0.054\sin(0.134t) + 0.004\sin(0.159t) + 0.536\sin(0.172t)$$

$$u_2' = 0.144\cos(0.134t) + 0.006\cos(0.159t) - 0.050\cos(0.172t) + 1.071\sin(0.134t) + 0.035\sin(0.159t) - 0.289\sin(0.172t)$$

$$u_3' = 0.147\cos(0.134t) - 0.014\cos(0.159t) - 0.033\cos(0.172t) + 1.092\sin(0.134t) - 0.089\sin(0.159t) - 0.190\sin(0.172t)$$

If we set $C_1 = C_2 = C_3 = 0$, where $C_{1,2,3}$ the constants of integration of $\dot{\mathbf{v}} = -\mathbf{\Omega}\mathbf{u}'$, then from the above solution we get

$$v_1' = 0.203\cos(0.134t) + 0.016\cos(0.159t) + 2.482\cos(0.172t) - 0.027\sin(0.134t) - 0.003\sin(0.159t) - 0.426\sin(0.172t)$$

$$v_2' = 4.052\cos(0.134t) + 0.152\cos(0.159t) - 1.333\cos(0.172t) - 0.544\sin(0.134t) - 0.024\sin(0.159t) + 0.229\sin(0.172t)$$

$$v_3' = 4.131\cos(0.134t) - 0.387\cos(0.159t) - 0.876\cos(0.172t) - 0.555\sin(0.134t) + 0.0615\sin(0.159t) + 0.150\sin(0.172t)$$

Finally, Figures 1 and 2 represent the path of u', v', respectively

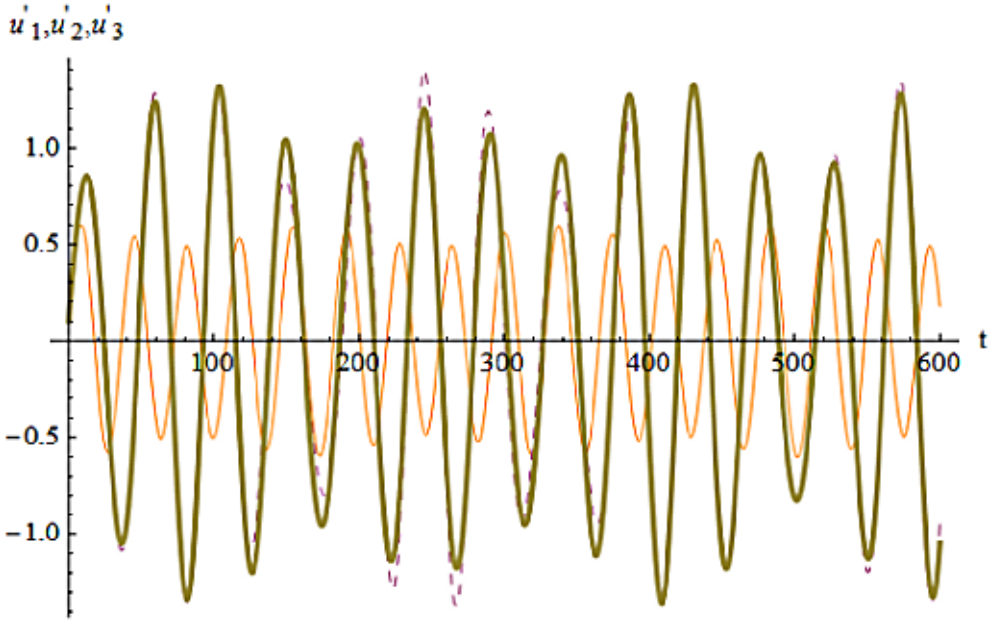

Figure 1. The path of $u'_{1,2,3}$

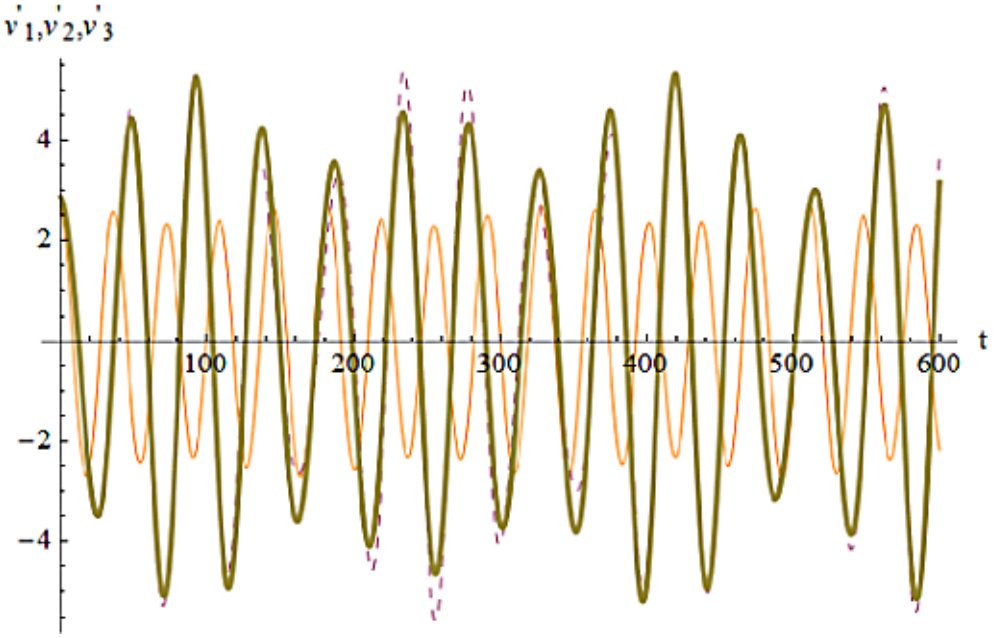

Figure 2. The path of $v'_{1,2,3}$

Example 2

Consider the following system:

$$\mathbf{A} = \begin{pmatrix} 0.4 & 0.9 & 0.02 \\ 0.08 & 0.1 & 0.2 \\ 0.4 & 0.21 & 0.009 \end{pmatrix}$$

with

$$\mathbf{Q} \cong \begin{pmatrix} -0.802 & 0.128 - 0.621i & 0.128 + 0.621i \\ -0.274 & 0.117 + 0.385i & 0.117 - 0.385i \\ -0.531 & -0.660 & -0.660 \end{pmatrix},$$

$$\mathbf{Q}^{-1} \cong \begin{pmatrix} -0.667 & -1.075 & -0.321 \\ 0.268 + 0.319i & 0.432 - 0.784i & -0.628 - 0.077i \\ 0.268 - 0.319i & 0.432 + 0.784i & -0.628 + 0.077i \end{pmatrix},$$

$$\lambda_1 \cong 0.721,\ \lambda_2 \cong \text{-}0.106\text{+}0.253\,i\,,\ \lambda_3 \cong \text{-}0.106\text{-}0.253\,i$$

Furthermore, we assume that $\gamma = 0.03,\ \rho = 0.04,\ a = 0.005,\ n = 0.06$ and $[\mu/(k\mu - 1)] = 1.$

Thus it is obtained that

$$\mathbf{M} \cong \begin{pmatrix} -0.019 & 0.032 & 0.0007 \\ 0.003 & -0.029 & 0.007 \\ 0.014 & 0.007 & -0.032 \end{pmatrix}$$

and

$$\lambda^{\mathrm{M}}{}_1 \cong \text{-}0.036\text{-}0.009\,i\,,\ \lambda^{\mathrm{M}}{}_2 \cong \text{-}0.036\text{+}0.009\,i\,,\ \lambda^{\mathrm{M}}{}_3 \cong \text{-}0.007$$

Moreover, we set $u_1'(0) = u_2'(0) = u_3'(0) = 0.1$ and $\dot{u}_1'(0) = \dot{u}_2'(0) = \dot{u}_3'(0) = 0.1.$ Therefore, using Mathematica, we get

$$u_1' = e^{-0.023t}[0.165e^{0.023t}\cos(0.087t) + 0.046\cos(0.192t) - 0.112e^{0.046t}\cos(0.192t) + 1.912e^{0.023t}\sin(0.087t) - 0.172\sin(0.192t) - 0.149e^{0.046t}\sin(0.192t)]$$

$$u_2' = e^{-0.023t}[0.057e^{0.023t}\cos(0.087t) + 0.027\cos(0.192t) + 0.017e^{0.046t}\cos(0.192t) + 0.654e^{0.023t}\sin(0.087t) + 0.110\sin(0.192t) + 0.117e^{0.046t}\sin(0.192t)]$$

$$u_3' = e^{-0.023t}[0.1106e^{0.023t}\cos(0.087t) - 0.185\cos(0.192t) + 0.176e^{0.046t}\cos(0.192t) + 1.267e^{0.023t}\sin(0.087t) - 0.011\sin(0.192t) - 0.082e^{0.046t}\sin(0.192t)$$

If we set $C_1 = C_2 = C_3 = 0$, then from the above solution we get

$$v_1' = -0.466\sin(0.087t) + 11.233[0.480\cos(0.087t) - 0.080e^{-0.023t}\cos(0.192t) - 0.073e^{0.023t}\cos(0.192t) - 0.013e^{-0.023t}\sin(0.192t) + 0.043e^{0.023t}\sin(0.192t)]$$

$$v_2' = -0.160\sin(0.087t) + 11.233[0.164\cos(0.087t) + 0.049e^{-0.023t}\cos(0.192t) + 0.054e^{0.023t}\cos(0.192t) - 0.017e^{-0.023t}\sin(0.192t) - 0.002e^{0.023t}\sin(0.192t)]$$

$$v_3' = -0.309\sin(0.087t) + 11.233[0.318\cos(0.087t) + 0.004e^{-0.023t}\cos(0.192t) - 0.029e^{0.023t}\cos(0.192t) + 0.085e^{-0.023t}\sin(0.192t) - 0.084e^{0.023t}\sin(0.192t)]$$

Finally, Figures 3 and 4 represent the path of u', v', respectively

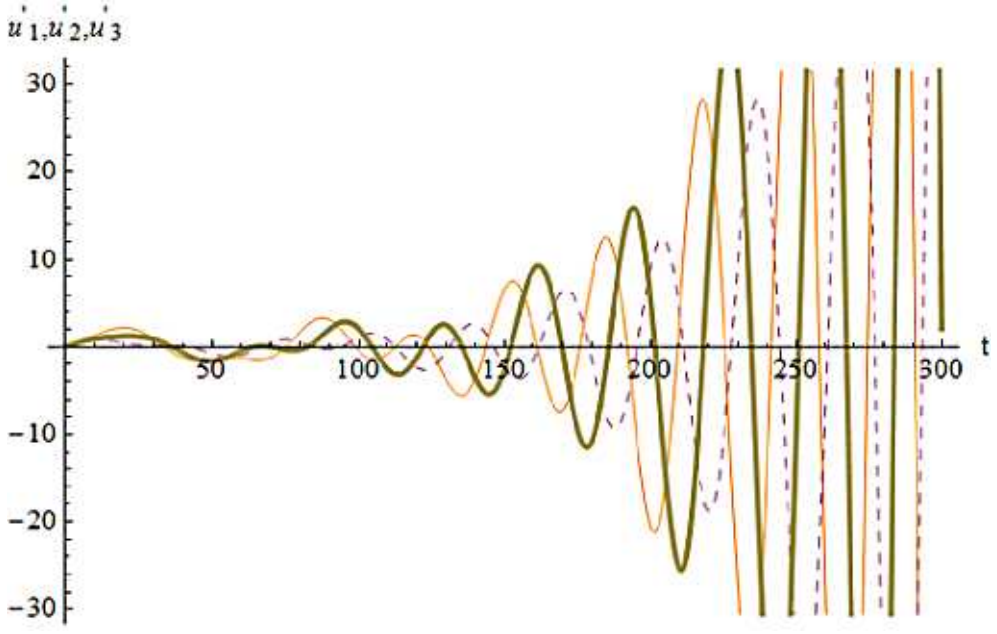

Figure 3. The path of $u_{1,2,3}'$

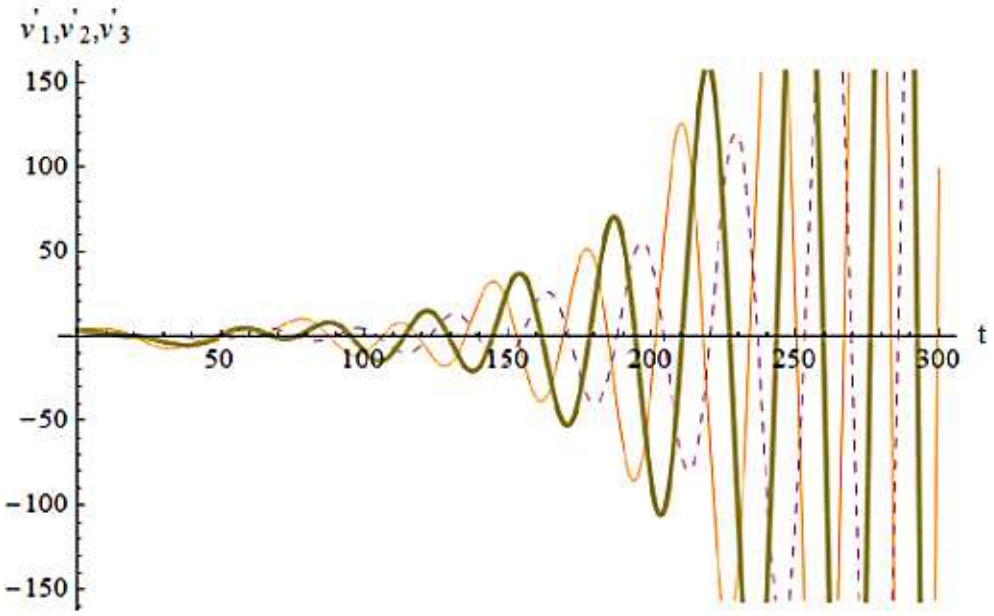

Figure 4. The path of $v'_{1,2,3}$

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