# Sentiments and rationalizability

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# SENTIMENTS AND RATIONALIZABILITY

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Abstract. Sentiments are characteristics of players' beliefs. I propose two notions of sentiments, confidence and optimism, and I study their role in shaping the set of rationalizable strategy profiles in (incomplete information) games with complementarities. Confidence is related to a player's perceived precision of information; optimism is the sentiment that the outcome of the game will be "favorable." I prove two main results on how sentiments and payoffs interact to determine the *size* and *location* of the set of rationalizable profiles. The first result provides an explicit upper bound on the size of the set of rationalizable strategy profiles, relating complementarities and confidence; the second gives an explicit lower bound on the change of location, relating complementarities and optimism. I apply these results to four areas. In models of currency crisis (Morris and Shin [16]), the results suggest that the most confident investors may drive financial markets. In models of empirical industrial organization (Aradillas-Lopez [2]), the paper provides a classification of the parameter values for which the model is identified. In non-Bayesian updating (Epstein [7]), the results clarify the strategic implications of certain biases. Finally, the results generalize and clarify the uniqueness result of global games (Carlsson and van Damme [4] and Frankel et al. [10]).

## 1. Introduction

In all social or economic interactions, whether they take place in financial markets, in elections or joint-ventures, the beliefs of the actors contribute to shape the set of possible outcomes. Bank runs are often caused by shifts of agents' beliefs which are unrelated to the real economy (Diamond and Dybvig [6]). In financial markets, the trading volume of investors with brokerage accounts is excessive compared to rational wisdom (Odean [21], [22]). And more than fifty thousand corporations are established every month, despite the combination of having much to lose and the seemingly low chance of success (Cooper, Woo, and Dunkelberg [5]). For the game-theorist, the richness of outcomes appears in the set of rationalizable strategy profiles. If there are many such profiles, far apart from one another, then very different outcomes may ensue. Similarly, the set of rationalizable strategy profiles may be narrow, but if its location within the set of all profiles is unknown, or if it changes in time or across populations, then very different outcomes may also ensue.

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I propose two notions of sentiments, confidence and optimism, which are characteristics of players' beliefs, and I study their role in rationalizability for games with complementarities. The paper contains two main results. The first result provides an explicit upper bound on the distance between any two rationalizable profiles, relating complementarities and confidence. The more confident the players are, the smaller that distance is. Naive intuition suggests that a "confident player" is, by definition, not influenced by others' actions; instead such a player makes his own choices regardless of the actions of others, and this intuitively favors equilibrium uniqueness and tight sets of rationalizable strategies. While this naive explanation is misleading, a confident player, as defined in the paper, will act *as if* he were not affected much by others' strategies. The second result provides a lower bound on the amount by which the rationalizable profiles increase after optimism increases. As a player becomes more optimistic, he is willing to choose actions that he were previously not willing to take. This causes a shift in the set of rationalizable strategy profiles. Optimism thus contributes to locating the set of rationalizable strategy profiles.

The main advantage of this approach is that sentiments are primitive objects which do not specify the origin of the beliefs. Belief formation is completely general. Players need not share a common prior; they can have heterogenous beliefs. This seems natural in financial markets where traders can have priors with different means (Varian [26]). Moreover, players need not even be Bayesian; they can have systematic biases, such as a prior or an overreaction bias (Epstein [7]).

I first describe the model in order to define sentiments. The model is a family of games with incomplete information. Players have types, which are either payoff-relevant, or informative about the state of nature which is payoff-relevant. The state of nature represents the physical reality, such as the weakness of a currency or the fundamentals of the economy. Players can take one of finitely many ordered actions. They only care about an aggregate, i.e. a summary statistic, of what their opponents are doing. This aggregate could be the average action of the opponents, the proportion of opponents playing some action, or other statistics. Players do not know the types of their opponents, and they may not know the state of nature. Based on their information, they formulate beliefs about the state and about their opponents' types. With their beliefs, they can assess the distribution of the aggregate. The games under consideration are games with complementarities. Each player wants to play larger actions when others do so as well, and/or when the state of nature or their type increases. Finally, these games admit dominance regions, which are "tail regions" of the state space for which the extremal actions are strictly dominant.

I now define confidence and optimism. Confidence is the sentiment that one has faith in one's information or abilities. In games of incomplete information, it is natural to interpret confidence as a precision-related concept. The notion that I adopt is related to the perceived precision of one's information (see Section 2). Since each player has two types of beliefs, about the state and about others' types, confidence will have two dimensions. The first dimension, called *state confidence*, measures the perceived precision of types in regards to the state. When a player's type increases, by how much does he think the state will increase on average? The answer is state confidence and it measures how good the player thinks his type is at reproducing changes in the state. A confident player thinks that his type "picks up" the right magnitude of shifts, so state confidence should be relatively large. The second dimension, called *aggregate confidence*, measures the perceived precision of a player's type in regards to others' types. The player considers a hypothetical situation with counterfactual information: His opponents decrease their strategies but simultaneously his type increases. By how much does the player think the aggregate will decrease on average? The answer is aggregate confidence and it gives the counterfactual information which receives more weight; as such it is an indicator of confidence (in one's type). In the counterfactuals, the first piece of information is bad news, because the aggregate should decrease on average. The second piece of information is good news, because others should receive larger types as well, hence play larger actions. A player who displays full aggregate confidence follows his information and answers zero to the question. In general, the smaller the answer, the larger aggregate confidence.

Under incomplete information, optimism measures how favorable a player expects the outcome to be. By convention, an outcome is said to be more favorable if it is larger.[1] From a player's perspective, the outcome becomes more favorable when the aggregate and the state are larger, because of the complementarities. Optimism also has two dimensions. For state beliefs, a player becomes more optimistic if, with the same informational type, he now believes larger states are more likely. The change in optimism is measured by the amplitude of his belief shift. For many distributions, this is equivalent to asking the player how much he expects the state to change on average.[2] The other dimension of optimism is the change in the expected aggregate forecasted by a player with the same informational type.

As mentioned before, the paper contains two main results. In games with strategic complementarities, there exist a largest and a smallest rationalizable strategy profile (Milgrom and Roberts [15]), and the distance between them gives the size of the set of rationalizable strategy profiles.

The first result provides an upper bound on the size of the set of rationalizable profiles; this upper bound relates complementarities and confidence. The lessons for comparative statics are enlightening. The set of rationalizable strategy profiles tends to shrink as players become more confident. This fact is strong because it holds across belief structures. In particular, it implies that if players are fully confident, then there is a unique rationalizable profile. Further, rationalizable profiles tend to get closer when payoffs become more sensitive to the state. Finally, rationalizable profiles tend to get further apart when strategic complementarities become stronger.

The explanation behind the result is intuitive. A confident player follows his information, hence his course of actions, even when his opponents modify their strategies, because he does not expect the aggregate action to change much if his type compensates for it. This favors uniqueness. However, the beliefs of a poorly confident player are easily swayed by others' strategies. This gives bite to the complementarities, and favors multiplicity. The payoff incentives are also quite clear. On the one hand, as players become more sensitive to the state, they become more sensitive to

[1] Recall that actions are ordered, so outcomes/profiles can be ranked accordingly. That larger outcomes are "better" or more favorable is not an objective statement. Sometimes the difference between optimism and pessimism is a bit vague, depending on the interpretation.

[2] This is true for all location-scale families.

variation in their private information. A variation in type induces a more or less drastic change in actions, regardless of what others are doing. This favors uniqueness. On the other hand, strategic complementarities create interdependence, and favor multiplicity.

The second result provides an explicit lower bound on the amount by which the extremal rationalizable profiles must vary after optimism increases; this lower bound relates complementarities and optimism. As optimism increases, it follows from Milgrom and Roberts [15] that the set of rationalizable profiles moves up. This paper is concerned with the amplitude of this change in location, because this will say, for instance, how much more likely a currency attack or a bank run should be. When a player becomes more optimistic, he is willing to play larger actions at lower types. But "how low" is he willing to go? To answer, the information of the optimistic player is worsened up to the point where he is indifferent again. The comparative statics lessons are insightful. Everything else equal, if players become more optimistic and/or less confident, then the minimal amount by which the extremal rationalizable profiles must rise increases. This is because good news becomes better and better news. Interestingly, confidence also comes into play. If a player is poorly confident, hence believes his information is not precise, then as he becomes more optimistic, it takes very low types to convince him that his optimism was unfounded. So, the change in location is larger for lower confidence levels.

The results are particularly interesting in four areas: Financial markets, empirical industrial organization, non-Bayesian updating, and global games.

This paper suggests that financial markets may be driven by the most confident investors in the sense that small changes in their strategies lead to larger changes in the strategies of the poorly confident investors. The results cover the case of asymmetric confidence levels across players, in which case multiplicity may reappear as groups of investors become less confident. Why? Poorly confident investors have beliefs which are relatively insensitive to their type, yet their beliefs must change across the multiple rationalizable profiles, for otherwise they would not change their investment strategies. Therefore, the strategies of others must have swayed their beliefs. I illustrate these ideas in Section 6.1 in the currency crisis model of Morris and Shin [16].

In econometric models of empirical industrial organization, the results can provide a classification of the parameter values for which the model is identified. A recent literature in empirical industrial organization aims to estimate models with incomplete information. The econometrician is assumed to know the family of joint-distributions of the noise variables. But there is no common prior or Bayesian assumption. The econometrician assumes, however, that the data he observes come from a unique equilibrium. In Section 6.2, I illustrate how my results can be used for identification purposes in a simplified version of Aradillas-Lopez [2]'s model.

This paper also sheds some light on the strategic implications of non-Bayesian updating. Epstein [7] reports several updating biases and provides an axiomatic model. While it is beyond the scope of this paper to offer a comprehensive treatment, I consider two biases, the prior bias and the overreaction bias, and I study their impact on rationalizability in games with complementarities

(Section 6.3). The prior bias lowers confidence, while the overreaction bias magnifies it. Therefore, the former favors multiplicity, and the latter uniqueness.

This paper generalizes and clarifies the traditional global games uniqueness results (Carlsson and van Damme [4], Frankel, Morris and pauzner [10], Morris and Shin [18]).[3] The generalization comes from the fact that players need not share a common prior. Players in global games become fully confident and this implies that there is a unique rationalizable profile (Section 4.2.2).[4] The explanation offered in Mathevet [14] is explicit here: The global game information structure dampens the complementarities to the point where uniqueness is obtained. The paper is also part of a recent effort, started by Morris and Shin [19] and Izmalkov and Yildiz [12], to understand rationalizability beyond the common prior assumption. Morris and Shin [19] revisits the belief foundations of global games in a general class of binary-action games with virtually no modeling assumptions. They provide different sets of conditions on beliefs which ensure uniqueness, or even characterize rationalizability. Izmalkov and Yildiz [12] introduces sentiments into the study of global games. They define notions of optimism, and they analyze partnership games and games of currency crisis. In two-player games, their notion of optimism is the probability with which a player believes that his opponent receives a higher type than his. They are able to fully characterize the unique equilibrium in terms of optimism. In Section 4.2.1, I explain the relationship between my results and Izmalkov and Yildiz [12] in the context of partnership games.

The remainder of the paper is organized as follows. There are four major sections. The first one illustrates the main concepts. The following section presents the model and the assumptions. The third section formally defines confidence and optimism and it contains all the results. The last section applies the results to currency crises, empirical industrial organization, and non-Bayesian updating.

## 2. The Intuition Behind Confidence and Optimism

A simple global game example is helpful to derive natural notions of confidence and optimism. The general definitions will be substantial extensions of these ideas. The example will also bring out "stylized" facts which will prove to be general comparative statics results.

Consider the following investment game (Morris and Shin [18]). Two players are deciding whether to invest (1) or not (0). Each player receives a net profit that depends on the action profile and on the fundamental of the economy $\theta \in \mathbb{R}$.

| | 1 | 0 |
|---|---|---|
| 1 | $\theta, \theta$ | $\theta - 1, 0$ |
| 0 | $0, \theta - 1$ | $0, 0$ |

Players share a common prior about $\theta$ which is a normal distribution with mean $y$ and standard

[3] It is important to note that the traditional global game results hold for compact action spaces and non-aggregative games.

[4] It is not trivial to show that players become fully confident, because this convergence has to be uniform in type and strategies.

deviation $\tau$. Each player $i$ receives a linear type $t_i = \theta + \epsilon_i$ about the state, where $\epsilon_i$ is normally distributed with mean 0 and standard deviation $\nu$. Assume that each $\epsilon_i$ is independently distributed from $\epsilon_j$ $(j \neq i)$ and $\theta$.

Upon receiving his type $t_i$, player $i$ formulates beliefs about the state and about the other player's type. Following Morris and Shin [18], $i$'s posterior beliefs about $\theta$ is a normal distribution with mean $\mu = (\nu^2 y + \tau^2 t_i)/(\nu^2 + \tau^2)$ and standard deviation $\sqrt{\nu^2\tau^2/(\nu^2 + \tau^2)}$. His posterior beliefs about $j$'s type are also normal, with mean $\mu$ and standard deviation $\sqrt{(2\nu^2\tau^2 + \nu^4)/(\nu^2 + \tau^2)}$.

In this context, it is natural to say that a player is confident if $\nu$ is small, because his type becomes a perfect predictor of the state and of his opponent's type. Indeed, if $\nu$ is small then $\mu \approx t_i$ and both standard deviations are nearly zero. It is also natural to say that a player becomes more optimistic if $y$ increases, because both the prior and the posterior beliefs increase in $y$ with respect to first-order stochastic dominance. Better fundamentals and larger types are expected.

Morris and Shin [18] offer a detailed account of equilibrium multiplicity. They derive an equation whose zeroes correspond to symmetric equilibria, as shown in Figure 1. Each intersection with the $x$-axis is a symmetric equilibrium cutoff type. For example, if the intersection occurs at 1/2, then it means that the strategies that consist in playing 0 below type 1/2 and 1 above form a symmetric equilibrium. Each panel is associated with its own level $\nu$. The right panel has a smaller $\nu$ than the left one. In each panel, the lower curve correspond to prior mean $y$ and the upper curve corresponds to a more optimistic situation $y' > y$.

Three facts appear in Figure 1. First, it seems that the set of rationalizable strategy profiles shrink as $\nu \to 0$, that is, as players become more confident. This is part of the well-known global games result, where uniqueness is obtained for small $\nu$ (Carlsson and van Damme [4], Frankel, Morris and pauzner [10], Morris and Shin [18]). Second, as players become more optimistic, they invest earlier. This also appears in the picture since the extremal intersections occur earlier for the upper curve. Third, as players become more confident ($\nu \to 0$), the positive effect of optimism on investment becomes weaker; when $\nu$ is very small, increasing $y$ seems to have no effect.

The global game analysis implicitly assumes that there is common knowledge of the linear signaling functions. So there is a sense in which $\nu$ could be some *objective* noise in everyone's type. In general, however, it could be that a player only knows his own signaling function, believes it has noise $\nu_i$, and believes others have different noises $\nu_j$. In this case, it seems appropriate to talk about the player's *perceived* precision. Even more generally, players may not have linear signaling functions, they may not share a common prior, and they may not even be Bayesian statisticians.

What are the relevant definitions of confidence and optimism in the general case? Are the effects of confidence and optimism preserved in general?

It is worth mentioning that the general definition of confidence is not a trivial extension of the above. The reason is that the above argument confuses reliability and precision. It is well-known that there is a unique equilibrium when the prior is uniform (Frankel, Morris, and Pauzner [10]), regardless of the noise level $\nu$. Thus confidence ought to capture this aspect as well.

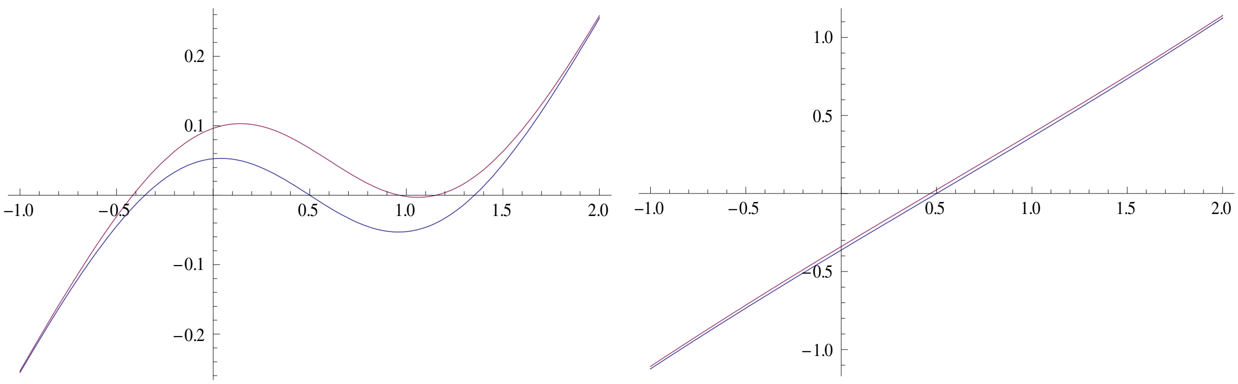


FIGURE 1. (L) Multiplicity and Large Upward-Shift, and (R) Uniqueness and Small Updward-Shift.

Among the many ways of formalizing confidence, the relevant notion will measure the perceived content of *informational change*. Healy and Moore [11] reports several definitions from the psychology and finance literature. Some are related to the estimation, or ranking, of one's own performance compared to others' (Erev, Wallsten, and Budescu [8]), and other definitions are related to the perceived precision of one's information. Odean ([21], [22]) studies trading in retail brokerage accounts and makes a strong case for the latter. He reports studies of the calibration of subjective probabilities where people tend to overestimate the precision of their knowledge (Alpert and Raiffa [1], Fischhoff, Slovic and Lichtenstein [9]). Investment bankers (von Holstein and Carl-Axel [28]), engineers (Kidd [13]), entrepreneurs (Cooper, Woo, and Dunkelberg [5]), lawyers (Wagenaar and Keren [29]), negotiators (Neale and Bazerman [20]), and managers (Russo and Schoemaker [23]) have all been observed to exhibit overconfidence in their judgements. In most existing works, precision is understood as reliability. Usually, a confident player is one who believes his information has a smaller variance than what it actually does. In Odean ([21], [22]) the concept is comparative, because it also includes the player's perception of his opponents' variances. The present definition is related to these concepts. For example, if a player thinks the variance of his type is lower than what it actually is, then he will display state overconfidence.

## 3. The Model

The model is described by the following game with incomplete information. The set and the number of players is $N < \infty$.[5] The set of types of player $i$ is $T_i = \mathbb{R}$ with generic element $t_i$. Denote $T_{-i} = \mathbb{R}^{n-1}$. Player $i$'s action set is a finite and linearly ordered set $A_i = \{a_{i,1}, \ldots, a_{i,M_i}\}$, where

[5] The paper and its results can be extended to $N = \infty$, as in Section 6.1.

the actions are indexed in increasing order. Let $A_{-i} = \prod_{i \neq i} A_j$. The payoff function of player $i$, denoted $u_i$, will be defined later.

A strategy for player $i$ is a measurable function from $T_i$ into $A_i$. In the class of games to be studied, only strategies which are monotone in a player's type will be relevant. Given the finite number of actions, those strategies are step functions, which are fully characterized by their cutoffs. Thus, any (monotone) strategy is representable by a vector (of cutoffs) in $\mathbb{R}^{M_i - 1}$. Without loss of generality, let $S_i \subset \mathbb{R}^{M_i-1}$ denote the compact set of (monotone) strategies for player $i$. Let $S = \prod_i S_i$ denote the set of strategy profiles and let $S_{-i} = \prod_{j \neq i} S_j$ denote the profiles of strategies for players other than $i$.

3.1. **The Payoffs: Aggregation and Complementarities.** There is a state of nature represented by a variable $\theta \in \mathbb{R}$. In a currency crisis model, $\theta$ represents the weakness of the currency. In other models, this variable represents the strength of the fundamentals of the economy. Each player $i$ only cares about an aggregate $\Gamma_i$ of his opponents' actions. This aggregate is an increasing and non-constant function, which maps action profiles and states (or types) from $A_{-i} \times \mathbb{R}$ onto a linearly ordered set $\mathcal{G}_i$. For example, a player could care about the sum of her opponents' actions, or about the proportion of her opponents playing more than some threshold which depends on the state.[6] Each player $i$'s utility is assumed to depend only on the state, $u_i(a_i, \Gamma_i(a_{-i}, \theta), \theta)$, or only on the player's own type, $u_i(a_i, \Gamma_i(a_{-i}), t_i)$. That is, the payoff structure includes *common values* and *private values*, but no mixtures of the two.

The utility functions are subject to the following assumptions.

3.1.1. *The Assumptions.* Let $X$ and $T$ be lattices (see Topkis). A function $f : X \times T \to \mathbb{R}$ has (strictly) increasing differences in $(x, t)$ if for all $x' > x$ and $t' > t$, $f(x', t') - f(x, t')\,(>) \geq f(x', t) - f(x, t)$. To avoid redundancy, the assumptions are given for the common values, but they identically translate to private values by replacing $\theta$ with the player's type. Player $i$'s utility function has increasing differences in $(a_i, a_{-i})$ for each $\theta$, and strictly increasing differences in $(a_i, \theta)$ for each $a_{-i}$.[7] For each profile $a$, $i$'s utility function is bounded on compact sets of states $\theta$. Finally, there are dominance regions: There exist states $\overline{\theta}$ and $\underline{\theta}$ such that for states above $\overline{\theta}$, it is a strictly dominant strategy for each player to play his largest action, and for states below $\underline{\theta}$, it is a strictly dominant strategy for each player to play his smallest action.

The first condition introduces strategic complementarities, by which a player wants to increase his action when others do so as well. The second requirement introduces state monotonicity, by which a player wants to increase his action when the state is larger. The third is a technical condition, and the last one imposes dominance regions.

[6] In these cases, write $\Gamma_i(a_{-i}, \theta) = \sum_{j \neq i} a_j$ and $\Gamma_i(a_{-i}, \theta) = (\sum_{j \neq i} 1_{a_j \geq a^*(\theta)})/(N-1)$.

[7] There are applications where these assumptions can be weakened. For example, if there exists a value $\gamma'$ of $\Gamma_i$ which always occurs with strictly positive probability (in equilibrium), then it is sufficient for $u_i(a_i, \gamma', \theta)$ to have strictly increasing differences in $(a_i, \theta)$ (see Section 6.1).

The applied literature is replete with games that satisfy these conditions. In currency crisis models (Morris and Shin [16]), speculators have to decide whether or not to attack a currency. Each speculator only cares about the proportion of people who attack. If there are enough speculators who attack, the currency is devaluated. The dominance regions correspond to regions where the currency is so weak, or so strong, that a particular action is strictly dominant. The bank run model of Morris and Shin [17] satisfies these assumptions, as well as the investment game of Carlsson and van Damme [4] and the model of merger waves of Toxvaerd [24]. There are voting situations and search models which fit into the framework. In a search model, agents only care about the sum of the effort of their potential partners. The more people search, the more an agent wants to search. If the probability to find a partner is non-zero, even if a single agent searches, as long as he puts maximal effort, and if the search cost increases very slowly for large types, then there will be dominance regions. Finally, a recent literature in empirical industrial organization estimates models that satisfy these assumptions (Aradillas-Lopez [2]. See Section 6.2).

3.2. **Beliefs and Aggregate Distribution.** A player formulates type-dependent beliefs about the state of nature and his opponents' types. Decompose these beliefs into two parts. First, player $i$'s type-dependent beliefs about the state of nature, called state beliefs, are represented by a distribution function $F_i(\cdot|t_i)$ with density function $f_i(\theta|t_i)$. Second, player $i$ formulates beliefs, which for each type and state of nature, assigns a probability measure on $T_{-i}$. These beliefs are represented by $\mu_i : T_i \times \mathbb{R} \to \mathcal{M}_{-i}$ where $\mathcal{M}_{-i}$ is the set of all probability measures on $T_{-i}$.

Under private values, there is no state of nature, and de facto no state beliefs. But this case is technically equivalent to a common values case where types are fully informative about the state. That is for each $t_i$, assume $F_i(\cdot|t_i)$ is derived from the Dirac measure where the singleton set $\{t_i\}$ receives measure 1.[8]

3.2.1. *Aggregate Distribution.* Conditionally on his type $t_i$ and the state $\theta$, player $i$ constructs the distribution of the aggregate as a function of others' strategies $s_{-i}$. To do so, the player uses his beliefs $\mu_i$. Consider the set $L(t'_{-i}) = \{t_{-i} \in T_{-i} : t_j \le t'_j \text{ for all } j \neq i\}$ which is the set of type vectors lower than $t_{-i}$. Take $\ell \in \mathbb{N}^{n-1}$ and denote by $a_{-i,\ell}$ the vector of actions where each $j \neq i$ plays action $a_{j,\ell_j}$. Define $A_{-i}(\gamma, \theta) = \{\ell \in \mathbb{N}^{n-1} : \Gamma_i(a_{-i,\ell}, \theta) = \gamma\}$ to be the set of combinations of actions which yield aggregate value $\gamma$ at state $\theta$. Recall that $s_{-i} = (s_j)$ where each $s_j = (s_{j,h})$ is a vector of cutoff types; $j$ plays action $a_h$ if and only if his type is in $[s_{j,h}, s_{j,h+1}]$. I use the shorthand $\tau_i = (\theta, s_{-i}, t_i)$. The aggregate distribution is described by the following probability mass function

$$g_i(\gamma|\tau_i) = \mu_i(t_i, \theta) \left[ \bigcup_{\ell \in A_{-i}(\gamma,\theta)} \left\{ L((s_{j,\ell_j+1})_j) \bigcap \overline{L((s_{j,\ell_j})_j)} \right\} \right].$$

Denote by $G_i(\cdot|\tau_i)$ the cumulative distribution function obtained from $g_i$. $G_i$ gives the conditional probability that $\Gamma_i$ is *strictly* less than $\gamma$.

[8] The Dirac measure gives measure 1 to every set that contains $t_i$, and 0 to others. Then, $\int_{\mathbb{R}} u(\theta) f_i(\theta|t_i) d\theta = u(t_i)$ for every function $u$.

3.2.2. *The Assumptions.* Players believe that larger states are more likely when their type increases. That is, if $t_i' > t_i$, then $F_i(.|t_i') >_{st} F_i(.|t_i)$.[9] Players also believe that larger opponents' types are more likely when their type and the state increase. Formally, if $(t_i', \theta') \geq (t_i, \theta)$, then $\mu_i(t_i', \theta') \geq_{st} \mu_i(t_i, \theta)$.[10] Further, the likelihood of states which are excessively far from a player's type is null. There exists $D_i$ such that $f_i(\theta|t_i) = 0$ whenever $|t_i - \theta| > D_i$.[11] Finally, for each $\gamma$ and $s_{-i}$, $g_i(\gamma|\cdot, s_{-i}, \cdot)$ and $f_i$ are continuous and bounded.

Beyond these assumptions, beliefs formation is free. Players may not share the same prior beliefs, hence they can have heterogeneous beliefs. Players need not even be Bayesian, and they can have updating biases.

## 4. Confidence and Rationalizability

This section defines a notion of confidence and investigates its role in determining the size of the set of rationalizable strategy profiles. As noted before, there are many ways of formalizing confidence and several definitions are possible. The definition adopted here is related to a player's perceived precision of his information, which he may misconstrue.

The basic ingredients of the definition are the average state and the average aggregate. Let $F_i^k(\theta|t_i) = F_i(\theta - k|t_i)$ denote $i$'s state beliefs $F_i$ after a rightward shift by an amount $k \geq 0$. Based on his assessment of the aggregate distribution, each player $i$ predicts that the average aggregate is the expectation of $G_i(\cdot|\tau_i)$, denoted $\Gamma_i^e[G_i(\tau_i)]$.

Since a player produces beliefs about the state and about others' types, confidence has two dimensions. State confidence is defined first. Throughout, $v$ will be any positive number.

**Definition 1.** *Player $i$'s state confidence is represented by function $k_F^i$ where $k_F^i(v)$ is the supremum of all $k$ such that $F_i(\cdot|t_i + v) \geq_{st} F_i^k(\cdot|t_i)$ for all $t_i$.*

Confidence of state beliefs is the minimal shift in player $i$'s state beliefs after an increase in type. If the state beliefs belong to a location-scale family,[12] such as the normal or logistic distribution, then state confidence answers the question: When a player's type increases by $v$, by how much does he think the state will increase on average? This is asking the player how good he thinks his type is at *reproducing changes* of the state. A confident player will think that his type "picks up" the right magnitude of shifts, so $k_F^i$ should be relatively large. In the case of normal state beliefs with mean $\alpha t_i$, as in Section 2, $k_F^i(v) = \alpha v$.

[9] $>_{st}$ stands for the (strict) first-order stochastic dominance ordering. $>_{st}$ means that for every strictly increasing function $u$ on $\mathbb{R}$, $\int_{\mathbb{R}} u(\theta) dF_i(\cdot|t_i') > \int_{\mathbb{R}} u(\theta) dF_i(\cdot|t_i)$.

[10] This is the multidimensional stochastic-dominance ordering.

[11] Instead, one could assume that the likelihood of states which are excessively far from a player's type is arbitrarily small. Then, for the results to hold, assume $u_i$ is bounded and continuous.

[12] Let $f(\theta)$ be any pdf. Then for $k \in \mathbb{R}$ and any $\sigma > 0$, the family of pdfs $(1/\sigma) f((\theta - k)/\sigma)$ indexed by $(k, \sigma)$ is called the location-scale family with standard pdf $f$. Many distributions such as the normal distribution form location-scale families.

The other notion of confidence, defined below, involves a player's beliefs about others' types. Suppose the player considers the counterfactual that his opponents decrease their strategies while his type increases. The first piece of information is bad news, because the aggregate should decrease, while the second piece of information is good news, because the state and others' types are larger and this should lead to larger actions. The counterfactual information which receives more weight is the indicator of confidence.

Define $c(v) = (\boldsymbol{v}, v, k_F^i(v))$ where $\boldsymbol{v}$ is a vector with identical entries $v$. The vector $\tau_i + c(v)$, equal to $(s_{-i} + \boldsymbol{v}, t_i + v, \theta + k_F^i(v))$, represents the counterfactual information: The opponents decrease their strategies while the type (hence the state) is higher. Think of strategies as bins, delimited by cutoffs. When a player's type falls into a bin, he plays the corresponding action. As players other than $i$ raise their cutoffs from $s_{-i}$ to $s_{-i} + \boldsymbol{v}$, it translates all the bins upwards by $v$, thereby delaying the play of larger actions. Simultaneously, $i$'s type rises by $v$. By how much does player $i$ think the aggregate will decrease on average? Asking this question is an indirect way of asking player $i$ how correlated to others' types he thinks his type is. In other words, it indicates how good the player thinks his type is at *reproducing changes* in others' types. If the player believes his type is very good at it, then his opponents' types should have increased by $v$ (more or less), since his type has increased by $v$. If so, his opponents' types should fall into the same bins as before despite the translation. Therefore, the aggregate should not decrease too much.

A 'really' confident, or overconfident, player would believe that $G_i(\tau_i + c(v)) \geq_{st} G_i(\tau_i)$. Despite the contradictory information, that player believes that larger aggregates are at least as likely as before. As a result, if asked by how much the average aggregate should decrease, he would answer zero. The smaller the answer, the higher the confidence. All of this is captured in the next definition.

**Definition 2.** *Player $i$'s aggregate confidence is represented by function $k_G^i$, defined such that $k_G^i(v) \geq \Gamma_i^e[G_i(\tau_i) \vee G_i(\tau_i + c(v))] - \Gamma_i^e[G_i(\tau_i + c(v))]$ for all $\tau_i$.*[13]

In the absence of state of nature under private values, aggregate confidence is the only confidence indicator. This case is technically equivalent to the common value case with full state confidence.

A player becomes more confident if $k_F^i$ increases and $k_G^i$ decreases, both uniformly. The relationship between state and aggregate confidence is one-sided. If a player gains state confidence, then his aggregate confidence rises. The explanation is quite simple. If a player believes that the state is higher on average than he first thought, then he will believe higher aggregates are more likely.

4.1. **The Main Theorem.** The main result features function $\varepsilon$ which synthesizes the forces that determine the size of the set of rationalizable profiles,

$$\varepsilon(F, G, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow M_*(k_F^i(v), t_i) - k_G^i(v) C^*(t_i) > 0, \forall\, t_i, i\}. \tag{4.1}$$

[13]Recall that $\vee$ stands for the least upper-bound (w.r.t. $\geq_{st}$ here) between two elements of a set. Moreover, as the definition indicates, if a player is confident with level $k_G^i$ then he trivially is confident with level $k_G'^i > k_G^i$.

Functions $M_*$ and $C^*$ are defined in the appendix as (A.7) and (A.8). Each of them measures one kind of complementarities. The former is the expected *minimal* amount of complementarities between own action and state. The latter is the expected *maximal* amount of strategic complementarities.

**Theorem 1.** *In the game of incomplete information, the distance between any two profiles of rationalizable strategies is less than* $\varepsilon(F, G, \boldsymbol{u})$*.*[14]

The proof is relegated to the appendix, but I provide an intuitive treatment. Consider a three-player game where each player only cares about the sum of his opponents' actions, such as in a search model. For simplicity, suppose that each player only has two actions and that the extremal equilibria $\overline{s}$ and $\underline{s}$ are symmetric. Say that each player is very confident, and consider player 1. Going from the largest to the smallest equilibrium, 2 and 3's strategies decrease (i.e. they search less) and 1's cutoff type increases from $\overline{s}_1$ to $\underline{s}_1$, which represents the fact that 1 delays the play of the larger action at the smallest equilibrium. At type $\underline{s}_1$, player 1 expects roughly the same total action, on average, from $\underline{s}_{-1}$ as what he expected from $\overline{s}_{-1}$ when his type was $\overline{s}_1$. This is because he is confident. However, his type is now strictly higher, so player 1 cannot be indifferent between the two actions at $\underline{s}_1$. Hence a high level of confidence cannot support two equilibria.

Note that full confidence is not necessary for uniqueness; full aggregate confidence alone implies uniqueness. The above argument shows that a confident player does not expect his opponents' actions to change much across profiles. It appears in $\varepsilon(\cdot)$ that a confident player acts *as if* he were not affected much by the complementarities, and as such he tends to play actions regardless of others' strategies. This favors uniqueness. On the other hand, the beliefs of a poorly confident player are easily swayed by others' strategies. This gives bite to the complementarities, and favors multiplicity.

There are two main comparative statics lessons to learn from the theorem. The first one is that state sensitivity tends to shrink the set of rationalizable strategy profiles, whereas strategic complementarities tend to enlarge it. Function $\varepsilon$ is, indeed, decreasing in $M_*$ while it is increasing in $C^*$. The explanation is intuitive. State sensitivity disconnects a player from the others, by inciting him to base his action on his type, while strategic complementarities connect players together. Interestingly, the strategic complementarities, which are known to favor multiplicity, may also enlarge the equilibrium set when they get stronger.

The second lesson is that confidence tends to shrink the set of rationalizable profiles. This fact, which is the object of the next corollary, is strong because it holds across belief structures.

**Corollary 1.** *If players become more confident, that is* $k^i_F \geq k^i_{F'}$ *and* $k^i_G \leq k^i_{G'}$ *for all* $i \in N$*, then* $\varepsilon(F, G, \boldsymbol{u}) \leq \varepsilon(F', G', \boldsymbol{u})$*.*

The mechanism by which confidence affects the size appears clearly in $\varepsilon(\cdot)$. As mentioned above, when players become more confident, it directly lowers the impact of strategic complementarities.

[14] The distance between two profiles $s = (s_{i,\ell})$ and $s' = (s'_{i,\ell})$ is given by the maximal distance between any two cutoffs $\max_i \max_{\ell=1,\ldots,M_i-1} |s'_{i,\ell} - s_{i,\ell}|$.

At the same time, confidence strengthens the monotonic relationship with states; and thus players want to play larger actions. As a consequence, uniqueness is obtained in the case where $k^i_G = 0$ (and $k^i_F(v) = v$). I call this case *full confidence.*

**Corollary 2.** *If players are fully confident, then there is a unique equilibrium.*[15]

4.2. **Applications.**

4.2.1. *The Investment Game.* In the game of Section 2, it is easy to compute $M^*(k^i_F(v), t_i) = k^i_F(v)$ and $C^*(t_i) = 1$. The theorem implies that all rationalizable strategy profiles are contained in a set of diameter

$$\varepsilon(F, G, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow k^i_F(v) > k^i_G(v), \forall\, t_i, i\}. \tag{4.2}$$

Here aggregate confidence is always between 0 and 1, so let $k^*_i = \sup_{v \geq 0} k^i_G(v)$. Thus $\varepsilon(F, G, \boldsymbol{u}) \leq \inf\{v : k^i_F(v) > k^*_i, \forall\, t_i, i\}$. If $F_i$ is a normal distribution with mean $\frac{3}{4}t_i$, and $k^*_i = \frac{3}{16}$, then $\varepsilon(F, G, \boldsymbol{u}) \leq \frac{1}{4}$.

4.2.2. *Global Games.* Global games (Carlsson and van Damme [4], Frankel, Morris and Pauzner [10], Morris and Shin [18]) give a nice illustration of these concepts. The main global game result is a uniqueness result as $\nu \to 0$ (see Section 2). As $\nu \to 0$, the signal becomes perfectly reliable, and so we approach full confidence. Formally, $\lim_{\sigma \to 0} k^i_F(v) = v$ and $\lim_{\sigma \to 0} k^i_G(v) = 0$;[16] this implies uniqueness by Corollary 2. The other uniqueness result is concerned with a uniform prior and holds for any $\nu > 0$. In this case, players are also fully confident according to the present definitions, despite the unreliability of the signals. The signal may indeed be unreliable, yet it is considered good at reproducing changes, because of the absence of any prior information.

4.2.3. *"Non Global Games".* The result can also be used in non global-games scenarios (see also Sections 6.2 and 6.3). Consider the arms race model of Baliga and Sjostrom [3]. Two countries decide whether to invest in a weapons program (0) or not (1). Let $d > 0$ represent the disutility of having a less advanced weapons system than the other country. A country that builds the new weapons system while the other does not receives a gain of $\mu > 0$. A country that acquires new weapons has to bear a psychological or monetary cost $t_i \in (\mu, d)$. The game is summarized by the following payoff matrix:

| | 1 | 0 |
|---|---|---|
| 1 | $0, 0$ | $-d, \mu - t_2$ |
| 0 | $\mu - t_1, -d$ | $-t_1, -t_2$ |

Unlike global games, types $t_1$ and $t_2$ are independently drawn from the same distribution $F$ with pdf $f$ and support $[0, d]$. As Baliga and Sjostrom argue, costs of acquiring weapons may depend on political considerations that are specific to a certain country or a certain leader, hence the independence assumption. Note that this game is under private values. The theorem implies that

[15]If players are fully confident, then $k^i_G(\cdot) = 0$. Then $\varepsilon(F, G, \boldsymbol{u}) = 0$ because $M_*(k^i_F(v), t_i) > 0$ by assumption.

[16]It is not trivial to show this because convergence has to be uniform in type and strategies.

if $1/(d-\mu) > f(t)$ for all $t$, then there is a unique equilibrium. This condition is sufficient for uniqueness but not necessary. Baliga and Sjostrom obtain a weaker sufficient condition.

## 5. Optimism and Rationalizability

This section investigates the role of optimism and confidence in determining how the rationalizable profiles change position within the set of all profiles.

5.1. **Optimism.** Optimism is the sentiment that the outcome of a situation will be "favorable," where favorable means "large outcomes" in this context. The main reason for studying optimism is for comparative purposes across groups of players or across time periods. The starting point is to define what it means to become more optimistic. In games with incomplete information, becoming more optimistic will be interpreted as having a better outlook on the aggregate value and the state with the same (informational) type. But this is not enough to determine the movements of the set of rationalizable strategy profiles; a player may become more optimistic but his newly-found optimism may be fragile in the sense that the slightest decrease in type could bring him back to his old "beliefs." In the latter case, this only causes minor changes in the equilibria. This explains why the robustness of optimism has to be addressed.

Let $G_i$ and $G_i'$ be the aggregate distributions derived from beliefs $\mu_i$ and $\mu_i'$.

**Definition 3.** *Player $i$ becomes more optimistic from $(F_i, \mu_i)$ to $(F_i', \mu_i')$, if $F_i'(\cdot|t_i) \geq_{st} F_i(\cdot|t_i)$ for all $t_i$, and $G_i'(\cdot|\tau_i) \geq_{st} G_i(\cdot|\tau_i)$ for all $\tau_i$.*

A player becomes more optimistic if, with the same (informational) type as before, he now believes that larger states and larger aggregates are more likely. From Milgrom and Roberts [15], an increase in optimism leads the largest and the smallest rationalizable strategy profiles to increase. By how much? The question is important because the answer will say how much more likely a currency attack or a bank run should be, for example. The answer relies on quantifying the variations in optimism. Before doing so, I return to the notion of confidence.

Interestingly, state confidence will play a role in positioning rationalizable profiles as well. It appears under a different form than previously defined. This is the next definition.

**Definition 4.** *Player $i$'s upper state confidence is the function $K_F^i$, where $K_F^i(v)$ is the infimum of all $k$ such that $F_i(\theta + k|t_i + v) \geq F_i(\theta|t_i)$ for all $\theta$ and all $t_i$.*

Upper state confidence is the amount by which a stochastically dominant distribution should be shifted down to become dominated. It is always greater than state confidence as defined by Definition 1 and thus coined lower state confidence. Figure 2 depicts the difference between them.

When the beliefs' shape changes dramatically after a change in type, the two concepts can give different values. It should be clear from the picture that both notions are equivalent when changes in types lead to uniform translation of the distribution. This is the case for well-behaved beliefs such as the location-scale families. They are also equivalent in private values.

As usual, since players have two types of beliefs, there are two notions of change in optimism. State optimism is dealt with first.

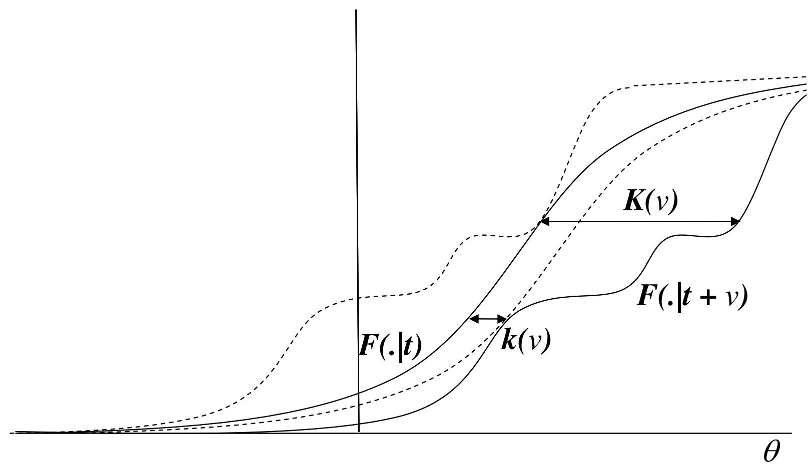


FIGURE 2. Upper and Lower State Confidence

**Definition 5.** *Player $i$ becomes more optimistic from $F_i$ to $F_i'$ by an amount $\omega_1^i$, defined as the supremum of all $\omega$ such that $F_i'(\cdot|t_i) \geq_{st} F_i^{\omega}(\cdot|t_i)$ for all $t_i$.*

If a player becomes more optimistic about the state, then his state beliefs shift up, and $\omega_1$ measures the amplitude of the shift. For all location-scale families, this is simply the amount by which a player expects the state to increase on average. The other notion of optimism involves the beliefs about other's types. The definition is similar in spirit. The magnitude of an increase in optimism is the amount by which a player thinks the average aggregate will increase.

Even though a player may become a lot more optimistic, this could be very fragile, in the sense that slight decreases in type may temper it. In other words, a player may believe that the average aggregate will increase a lot, but minor bad news could destroy his new beliefs. In this situation, the rationalizable profiles should not change much. To account for this, the definition has to test for the robustness or persistence of optimism.

Define $o(v) = (K_F^i(v) - \omega_1^i, \mathbf{0}, v)$ where $\mathbf{0}$ is a vector with identical entries 0. The vector $\tau_i - o(v)$, equal to $(\theta - K_F^i(v) + \omega_1^i, s_{-i}, t_i - v)$, represents an optimistic perspective on the state, deteriorated by a bad news. The perspective on the state is more optimistic than under $\tau_i$, because the state is larger by $\omega_1^i$, which is good news; however, the type decreases by $v$ (which in turn decreases the state by the confidence level). Recall that $G_i'$ is more "optimistic" than $G_i$ in the stochastic-dominance sense. To measure robustness, the expected values of $G_i'(\tau_i - o(v))$ and $G_i(\tau_i)$ are to be compared for every $v$. This corresponds to the following scenario. Take a player who has become more optimistic, worsen his information, and ask him by how much he thinks the expected aggregate has changed. When $v$ is small, the optimistic player with $G_i'(\tau_i - o(v))$ is still more optimistic than before, by an amount measured by the difference in expected values. When $v$ is large, the player who once was more optimistic is now more pessimistic, because $G_i'(\tau_i - o(v))$ is dominated by $G_i(\tau_i)$. In this situation, I assume that the player uses a worst-case rule. What is the most pessimistic forecast that he can make? The supremum of the two distributions serves as the most optimistic scenario. The worst prediction the player can make is to believe that the average aggregate will decrease by the difference from the most optimistic case.

Define distribution

$$\chi(\tau_i, v) = \begin{cases} G_i(\tau_i), & \text{if } G'_i(\tau_i - o(v)) \geq_{st} G_i(\tau_i) \\ G_i(\tau_i) \vee G'_i(\tau_i - o(v)), & \text{otherwise.} \end{cases}$$

This distribution will express the above worst-case rule in the next definition.

**Definition 6.** *The robustness of player i's optimism change from $\mu_i$ to $\mu'_i$ is given by function $\omega_2^i$, where $\omega_2^i(v)$ is defined such that $\omega_2^i(v) \leq \Gamma_i^e[G'_i(\tau_i - o(v))] - \Gamma_i^e[\chi_i(\tau_i, v)]$ for all $\tau_i$.*

5.2. **Main Theorem.** The main result provides a lower bound on the distance covered by the extremal equilibria after a change in optimism. For all $B = (F_i, G_i)_i$ and $B' = (F'_i, G'_i)_i$, define

$$\delta(B, B', \boldsymbol{u}) = \sup \left\{ v : M_*(\omega_1^i - K_F^i(v), t_i) + \min\{\omega_2^i(v) C_*(t_i), \omega_2^i(v) C^*(t_i)\} \geq 0, \, \forall \, t_i, i \right\} \tag{5.1}$$

where $C_*$ is defined analogously to $C^*$ (with min instead of max). The main result summarizes the forces that contribute to position the set of rationalizable profiles.

**Theorem 2.** *In the game of incomplete information, if each player $i \in N$ becomes more optimistic from $(F_i, \mu_i)$ to $(F'_i, \mu'_i)$, then the extremal rationalizable profiles both increase by at least $\delta(B, B', \boldsymbol{u})$.*

The comparative statics lessons are twofold. First, the more optimistic players become, the more the rationalizable strategy profiles tend to increase. This result holds across belief structures. That is, if optimism increases more from one belief structure to another than for another pair of belief structures, then the rationalizable strategy profiles tend to cover more distance for the first pair.

Interestingly, confidence is involved in positioning the rationalizable profiles as well. Its role is intuitive. If a player has little confidence in his type, then as he becomes more optimistic, it takes a lot to convince him that his newly-found optimism was unfounded, which leads to larger shifts in the rationalizable strategies. All of this is summarized in the next corollary.

**Corollary 3.** *Everything else equal, if players become more optimistic and less confident, then the minimal amount by which the extremal rationalizable profiles must rise increases.*

The second lesson is concerned with the effects of payoffs. Players' sensitivity to the state, $M_*$, determines how they react to the changes in states that they foresee. As this sensitivity increases, the effect of optimism, part of which is to expect larger states, is enhanced. Players then want to play larger actions. The role of strategic complementarities is ambiguous. On the one hand, when a player becomes more optimistic, he foresees larger aggregate values, and the strength of the complementarities determines his reaction to it. On the other hand, recall that location is tied to the robustness of optimism. When a player's information is worsened, so he becomes pessimistic, the effect of strong complementarities is reversed. Bad news become worse news.

5.2.1. *The Investment Game.* Izmalkov and Yildiz [12] define a notion of optimism in the investment game of Section 2. Players have a uniform prior about $\theta$ and linear types of the form $t_i = \theta + \nu\epsilon_i$ where $\epsilon_i \in [-1,1]$. Each player $i$ is allowed to have his own subjective beliefs about $(\epsilon_1, \epsilon_2)$ given by $\Pr_i$. The uniform distribution of $\theta$ implies that players exhibit full state confidence; so $k_F^i(v) = v$. Izmalkov and Yildiz consider fixed state beliefs, so $\omega_1 = 0$. The linear payoffs give $C^*(t_i) = C_*(t_i) = 1$ and $M_*(-K_F^i(v), t_i) = -k_F^i(v) = -v$. As a result, Theorem 2 says

$$\delta(B, B', \boldsymbol{u}) = \sup\left\{v : -v + \omega_2(v) \geq 0, \forall\, t_i, i\right\}. \tag{5.2}$$

The homogeneity in the setup allows to go further. Since the game is symmetric, the relevant aggregate distribution is $\Pr_i(t_j > t_i|\theta, t_i)$, which is $\Pr_i(\epsilon_j > \epsilon_i)$. Denote the latter by $q$. $q$ is the definition of optimism in Izmalkhov and Yildiz [12]. It is an intuitive indicator on second-order beliefs, which gives the probability that a player believes that his opponent receives a higher type than his. Since $\omega_2(v) = \Delta q$, the variation in $q$, (5.2) is consistent with Izmalkov and Yildiz [12], according to which the unique profile varies by $\Delta q$.

## 6. Applications

6.1. **Currency Crises.** Investors' sentiments play an important role in financial markets. To study this role, I introduce confidence into the currency crisis model of Morris and Shin [16].[17]

The previous results will suggest that the most confident investors may drive financial markets. The model will also suggest that less optimism tends to enlarge the set of outcomes. If investors become less optimistic, then they believe that an attack is less likely, so they are inclined not to attack. However, if they maintain a sufficiently high level of confidence, there may be bandwagon effects when private information becomes favorable.

Consider a continuum of players $[0,1]$. Each player takes one of two actions, 0 and 1, where action 0 is to keep the unit of an asset, and action 1 is to sell it short. Keeping the asset (action 0) yields a net payoff of zero. The consequence of selling (action 1) depends on the proportion of agents who also choose to sell the asset short and on the state of nature $\theta$. The proportion of agents taking action 1 determines the regime of the economy, which can either be the status quo or the new regime. The threshold at which the economy changes regime is given by $r(\theta)$. Let $\int_{[0,1]} a_i di$ be the proportion of players choosing 1. If this proportion exceeds $r(\theta)$, then the economy enters the new regime. A player cannot change the regime on his own, so he only cares about whether or not the others' actions change the regime. Formally, $\Gamma_i(a_{-i}, \theta) = \mathbf{1}_{\{\int_{[0,1]} a_i di > r(\theta)\}}$. Player $i$'s payoffs are given by the following matrix:

| | $\Gamma_i = 1$ | $\Gamma_i = 0$ |
|---|---|---|
| 1 | $u(\theta) - c$ | $-c$ |
| 0 | 0 | 0 |

[17] Izmalkhov and Yildiz have already provided an insightful study of optimism in this model.

Partition the set $[0, 1]$ of players into finitely many subsets, each containing either a single player, or a continuum of identical players.[18] I abuse notation and denote (a representative player from) each group by $i$. The following conditions are imposed: $u(\cdot) > c$ and $u$ is strictly increasing; and there exist $\overline{\theta}$ and $\underline{\theta}$ such that $r(\theta) = 0$ when $\theta > \overline{\theta}$ and $r(\theta) = 0$ when $\theta < \underline{\theta}$. Under these conditions, the payoff assumptions are satisfied, except state monotonicity which only holds weakly. This implies a minor change in (A.5) (Appendix) whose first part becomes

$$M(k_F^i, \tau_i, v) = E_{\theta|t_i}[(u(\theta + k_F^i(v)) - u(\theta))g_i(1|\tau_i + c(v))].^{19} \tag{6.1}$$

In turn, this implies a change in function $\varepsilon$, which becomes

$$\varepsilon(F, G, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow M(k_F^i, \tau_i, v) > k_G^i(v)C^*(t_i), \forall \tau_i, i\}. \tag{6.2}$$

Theorem 1 holds such that the distance between rationalizable profiles is bounded by (6.2). This has several implications, described in the next propositions and a remark.

**Proposition 1.** *If each investor $i$ becomes more optimistic and more confident, then $\varepsilon(F, G, \boldsymbol{u}) \leq \varepsilon(F', G', \boldsymbol{u})$.*

The presence of the aggregate distribution in (6.1) obfuscates the role of confidence. Confidence alone may no longer "control" the upper bound on the size of the rationalizable set, because the probability of an attack, as perceived by each player, is a crucial ingredient. If investors believe an attack is less likely, and de facto are less optimistic, then they may decrease their strategies. This may cause the smallest rationalizable profile to decrease. This need not be, however, if the level of confidence is high enough, for investors may not believe that lower strategies will have serious consequences on actions (when information is appropriately favorable). Overall, the effect is unclear. What remains clear is that full confidence, while demanding, still implies uniqueness.

**Proposition 2.** *If investors are fully confident, then there is a unique equilibrium.*

**Remark**. Theorem 1 suggests that *the most confident investors can drive financial markets*. Although it is beyond the scope of this paper to explore this claim in detail, I give some intuition. Suppose that there are two groups of investors, 1 and 2. If both groups are very confident, then there is a unique equilibrium. Instead, assume group 2 is a group of poorly confident investors. Depending on the size of both groups, the above results say that multiplicity may reappear. When it is the case, denote the extremal rationalizable profiles by $\overline{s} = (\overline{s}_1, \overline{s}_2)$ and $\underline{s} = (\underline{s}_1, \underline{s}_2)$. One way of measuring the impact of others' strategies on a group's beliefs is via $\overline{s}_i - \underline{s}_i$. This is the amount by which the equilibrium cutoff type has to increase to offset the change of beliefs caused by the change of others' strategies $\overline{s}_j - \underline{s}_j$. This is an indicator of who influences who, because it says which group has to increase its strategy more in response to a (smaller) change in the other group's strategy. In this case, confidence implies $\overline{s}_2 - \underline{s}_2 \geq \overline{s}_1 - \underline{s}_1$. That is, the poorly confident investors

[18] Identical players share the same state and aggregate beliefs. Frankel et al. [10] use this technique to extend their results to continuum of players.

[19] At any rationalizable profile, $g_i(1|\tau_i) > 0$, because any rationalizable strategy must play 1 for large types.

change their strategies more in response to a smaller change in the opposing strategies (compared to the confident ones). As such, they can be seen as following the most confident investors.[20]

6.2. **Empirical Industrial Organization.** A recent literature in industrial organization aims to estimate models with incomplete information. In these models, the econometrician cannot assume the existence of a common prior, nor can he assume specific signaling structures. Moreover, data do not always support Bayesian updating. To estimate the model, the econometrician assumes, however, that the data he observes come from a unique equilibrium. My results help appreciate the nature of this assumption, and they provide a classification of the parameter values for which the model is identified.

I present a simplified version of Aradillas-Lopez [2]'s incomplete information model. Two firms play a simultaneous-move game. Think, for example, of a technology adoption game for complementary products. Each firm has to decide whether to provide a technology. Firm 1 would prefer to provide technology 1 if firm 2 provides technology 2 where $t_i$ represents the market prospects from $i$'s perspective. The exogenous determinants of firm $i$'s profit are represented by $X_i \in \mathbb{R}^{n_i}$. The payoffs, known to both firms, are given by the following matrix:

| | $Y_2 = 1$ | $Y_2 = 0$ |
|---|---|---|
| $Y_1 = 1$ | $X_1'\beta_1 + t_1 + \alpha_1, X_2'\beta_2 + t_2 + \alpha_2$ | $X_1'\beta_1 + t_1, 0$ |
| $Y_1 = 0$ | $0, X_2'\beta_2 + t_2$ | $0, 0$ |

The game is assumed to have strategic complementarities: $\alpha_1, \alpha_2 \geq 0$.[21] At the beginning of the game, each firm $i$ observes its type $t_i$ and $X = (X_1, X_2)$. The firms know the joint distribution $H(t_1, t_2)$, and using $H$ they formulate beliefs about the other firm's type. The econometrician is assumed to know $H$ but he does not know the coefficients $\alpha_i$ and $\beta_i$, $i = 1, 2$. Given a sample $\{(Y_{1,t}, X_{1,t}, Y_{2,t}, X_{2,t})\}$, the econometrician tries to estimate these coefficients. The main purpose of the exercise is to predict the effect of an exogenous change in $X_1$ or $X_2$ on the likelihood of investment.

To estimate the coefficients, the econometrician assumes that the sample comes from a unique equilibrium (of the incomplete information game) at the true parameter values. This assumption guarantees that the likelihood function is well-defined, so it is important to understand which implications it has on $H$ and the coefficients.

This is a private information setup, hence state confidence is $k^i_F(v) = v$. It is easy to compute $M_*(v, t_i) = v$ and $C^*_i(\theta) = \alpha_i$. Note $G_i(1|s_j, t_i) = H(s_{j1}|t_i)$ where $H(t_j|t_i)$ is the conditional distribution computed from $H$.

[20] This is a relative claim. Since there are multiple equilibria, the most confident investors also follow the least confident ones, but not as much in comparison to the poorly confident ones.

[21] Aradillas-Lopez [2] is interested in a larger class of games than games with complementarities, such as entry games for example. So he does not assume $\alpha_i \geq 0$.

Although Theorem 1 does not characterize equilibrium uniqueness, its estimate of the size is quite accurate for games with linear payoffs. It gives a simple relationship between $G$ and the values of coefficients for which uniqueness holds and the model is identified

$$\varepsilon(F, G, \boldsymbol{u}) = \left\{ \underline{v} : v > \underline{v} \Rightarrow \frac{1}{\alpha_i} > \frac{k_G^i(v)}{v} \ \forall i \right\}.$$

First, the coefficients $\beta_i$'s do not play any role in uniqueness. Only the $\alpha_i$'s are relevant. Second, in semi-parametric estimations, $H$ is assumed to belong to a family of distributions where certain coefficients are unknown. For example, $H$ could be a normal distribution with means $\mu_1$ and $\mu_2$, and variance-covariance matrix $V_{12}$. Consider the set of all $(\mu_1, \mu_2, V_{12}, \alpha_1, \alpha_2)$ to be the parameter space. Theorem 1 gives the econometrician a description of the regions of the parameter space for which the model is identified, before he actually estimates the model. A priori, if all values of $\alpha_i$'s were possible, then uniqueness would nearly require full aggregate confidence. That is, $k_G^i(v) = 0$ for $i = 1, 2$. This is a strong assumption, because it requires $t_1$ and $t_2$ to be perfectly correlated under $H$.

6.3. **The Effect of Updating Biases.** This section studies the strategic implications of updating biases on the outcomes of a game. While the Bayesian paradigm is standard, it is conceivable that real-life agents may depart from it in more or less systematic ways. Epstein [7] provides an axiomatic model of non-Bayesian updating where he reports different types of biases. I analyze the strategic implications of the prior and overreaction bias.

Players have a prior (cumulative) distribution $P_i$. In common values, $P_i : \Theta \to [0, 1]$ represents the prior beliefs about the state of nature. In private values, $P_i : T_{-i} \to [0, 1]$ represents the prior beliefs about the types of $i$'s opponents. Upon receiving type $t_i$, a player using Bayesian updating would have posterior beliefs $BU_i(\cdot|t_i)$. To satisfy the assumptions, suppose that Bayesian beliefs are increasing in type with respect to first-order stochastic dominance. Let $Q_i(\cdot|t_i)$ be the posterior beliefs that $i$ actually holds upon receiving $t_i$.

6.3.1. *Prior Bias and Underreaction.* A player who has a prior bias gives "too little" weight to observation and "too much" weight to his prior knowledge. In the spirit of Epstein [7], this can be modeled as

$$Q_i(\cdot|t_i) = \alpha P_i(\cdot) + (1 - \alpha) BU_i(\cdot|t_i), \tag{6.3}$$

where $\alpha \in [0, 1]$ measures the magnitude of the bias. Since $P_i$ gives no weight to the data/type, it is clear that $Q_i$ displays less confidence than $BU_i(\cdot|t_i)$, because the stochastic dominance shift is reduced by the presence of the prior.

In light of previous results, such a prior bias tends to favor multiplicity in general. Note that $Q_i$ and $BU_i$ are not ranked regarding optimism. The shape of beliefs (6.3) can change dramatically as the type varies, in a similar fashion as Figure 2. In common values, this leads to wonder whether the current definition of state confidence underestimates the role of the type. The stochastic dominance shift can be small, while the average state increases widely. To capture this effect, state confidence could be defined as the change in the average state. This new definition would change slightly the expression of $\varepsilon$, but not necessarily its accuracy.

6.3.2. *Overreaction.* A player who is subject to overreaction gives "too much" weight to observation, leading him to overestimate the importance of his type. Let $\mu$ be the expectation under $P_i$. This bias can be modeled as

$$Q_i(\cdot|t_i) = BU_i(\cdot|t_i + \alpha(t_i - \mu)), \tag{6.4}$$

where $\alpha \in [0, 1]$ measures the magnitude of the bias. A biased player believes at $t_i$ what a Bayesian player would believe at $t_i + \alpha(t_i - \mu)$. In other words, after receiving $t_i > (<)\mu$, the player interprets his type as a better (worse) news than what it actually is. Because $t_i + \alpha(t_i - \mu) = (1 + \alpha)t_i - \alpha\mu$, overreaction leads to larger confidence level than Bayesian updating. Therefore, it promotes tighter rationalizable sets.

## 7. Conclusion

In this paper, I have introduced confidence and developed optimism, two notions of sentiments that capture essential features of the beliefs that are involved in shaping the set of rationalizable strategy profiles. The main advantage of the approach is twofold. First, it does not specify the origin of the beliefs, and thus it subsumes the case of heterogenous priors, general signaling technologies, and even non-Bayesian updating. Second, it synthesizes these sentiments and the properties of the payoffs within explicit expressions that can give insightful comparative statics. The paper also includes a number of applications; one of them suggests that the most confident investors may be more influential than the least confident investors. Thoroughly studying this claim is an interesting avenue for future research.

## Appendix A. Proofs

The entire argument of the first result is as follows:

(1) The games of incomplete information under consideration are games with strategic complementarities (GSC). This implies the existence of a largest and a smallest equilibrium, as in Milgrom-Roberts [15] and Vives [27].
(2) Furthermore, the payoffs display some monotonicity between actions and states, and the beliefs display monotonicity in type. As a direct implication of Van Zandt and Vives [25], (a) best-responses to monotone (in-type) strategies are monotone and (b) the extremal equilibria are in monotone strategies.
(3) I prove that the best-reply mapping, restricted to monotone strategies, is a contraction for all pairs of profiles that are distant enough. Since the extremal equilibria are in monotone strategies, they can be no further apart than this distance.
(4) Since extremal equilibria bound the set of profiles in rationalizable strategies in GSC, this gives a distance between any pair of rationalizable profiles.

In view of (2), I will only consider monotone (in-type) strategies. Any such strategy can be represented as a finite sequence of cutoff points, because there is a finite number of actions. I call those cutoff points *real cutoffs* as opposed to the *fictitious cutoffs* defined next. The relationship between the two families of cutoffs is given in Section A.2. Player $i$'s strategy $s_i = (s_{i,\ell})_{\ell=1}^{M_i-1}$ where each $s_{i,\ell}$ is the threshold type below which $i$ plays $a_\ell$, and above which he plays $a_{\ell+1}$.

According to the next definition, the fictitious cutoff between two actions is the (unique) type at which a player is indifferent between them. A fictitious cutoff between two actions may not always exist.

**Definition 7.** *For $i \in N$, the fictitious cutoff point between $a_n$ and $a_m$, denoted $c_{n,m}$ is defined, if it exists, as the (only) type $t_i$ such that $Eu_i(a_n, s_{-i}, t_i) - Eu_i(a_m, s_{-i}, t_i) = 0$.*

Recall $\Delta u_i(\gamma, \theta) = u_i(a_n, \gamma, \theta) - u_i(a_m, \gamma, \theta)$ be the difference in utility from playing $a_n$ over $a_m$ at state $\theta$ when others play actions generating $\gamma$. Define

$$Eu_i(a_i, s_{-i}, t_i) = \int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} u_i(a_i, \gamma, \theta) g_i(\gamma|\theta, s_{-i}, t_i) f_i(\theta|t_i) d\theta.$$

A.1. **Proposition 3.**

**Proposition 3.** *If $v > \varepsilon(F, G, \boldsymbol{u})$, then for all pairs of actions $(a_n, a_m)$, all types $t_i$, strategies $s_{-i}$, and $i \in N$ such that*

$$Eu_i(a_n, s_{-i}, t_i) - Eu_i(a_m, s_{-i}, t_i) \geq 0 \tag{A.1}$$

*the following inequality holds*

$$Eu_i(a_n, s_{-i} + \boldsymbol{v}, t_i + v) - Eu_i(a_m, s_{-i} + \boldsymbol{v}, t_i + v) > 0 \tag{A.2}$$

*Proof.* Suppose (A.1) is satisfied. From the definition of state confidence $F_i(\cdot|t_i + v) \geq_{st} F^{k_F^i(v)}(\cdot|t_i)$; thus if the following inequality holds, then it implies (A.2)

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma|\theta, s_{-i} + \boldsymbol{v}, t_i + v) f_i(\theta - k_F^i(v)|t_i) d\theta > 0, \tag{A.3}$$

because $\sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma|\theta, s_{-i}, t_i + v)$ is increasing in $\theta$. After a change of variables, (A.3) becomes

$$E_{\theta|t_i} \left[ \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta + k_F^i(v)) g_i(\gamma|\tau_i + c(v)) \right] > 0. \tag{A.4}$$

If the following inequality holds, then it implies (A.4) (because (A.1) holds)

$$E_{\theta|t_i} \left[ \sum_{\gamma \geq \underline{\gamma}} (\Delta u_i(\gamma, \theta + k_F^i(v)) - \Delta u_i(\gamma, \theta)) g_i(\gamma|\tau_i + c(v)) \right] + E_{\theta|t_i} \left[ \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta)(g_i(\gamma|\tau_i + c(v)) - g_i(\gamma|\tau_i)) \right] > 0. \tag{A.5}$$

The first member of (A.5) is strictly positive, because $\Delta u_i$ is strictly increasing in $\theta$. Although the second member is not always positive, there is a lower bound on how negative it can be. For any $\gamma \in \mathcal{G}_i$, define $\sigma(\gamma) = \min\{\gamma' \in \mathcal{G}_i : \gamma' > \gamma\}$ to be the successor of $\gamma$. By convention, let $G_i(\sigma(\overline{\gamma})|\cdot) = 1$. Note that

$$\sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta)(g_i(\gamma|\tau_i + c(v)) - g_i(\gamma|\tau_i)) =$$
$$\sum_{\gamma \geq \underline{\gamma}} (G_i(\sigma(\gamma)|\tau_i + c(v)) - G_i(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(\sigma(\gamma), \theta)). \quad \text{(A.6)}$$

Define

$$C^*(\theta) = \max \left\{ \frac{\Delta u_i(\sigma(\gamma), \theta) - \Delta u_i(\gamma, \theta)}{\sigma(\gamma) - \gamma} : \gamma \in \mathcal{G}_i \right\} \quad \text{(A.7)}$$

to be the largest amount of complementarities in $i$'s payoffs. Let $g_i^*$ be the probability mass function of distribution $G_i(\tau_i) \vee G_i(\tau_i + c(v))$. Since $\Delta u_i$ is increasing in $\gamma$, it follows from the definition of confidence that

$$\begin{aligned}
& \textstyle\sum_{\gamma \geq \underline{\gamma}} (G_i(\sigma(\gamma)|\tau_i + c(v)) - G_i(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(\sigma(\gamma), \theta)) \\
\geq \quad & \textstyle\sum_{\gamma \geq \underline{\gamma}} (G_i(\sigma(\gamma)|\tau_i + c(v)) - G_i^*(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(\sigma(\gamma), \theta)) \\
\geq \quad & \textstyle\sum_{\gamma \geq \underline{\gamma}} (G_i(\sigma(\gamma)|\tau_i + c(v)) - G_i^*(\sigma(\gamma)|\tau_i))(\gamma - \sigma(\gamma))C^*(\theta) \\
= \quad & C^*(\theta) \textstyle\sum_{\gamma} \gamma(g_i(\gamma|\tau_i + c(v)) - g_i^*(\gamma|\tau_i)) \\
\geq \quad & -C^*(\theta) k_G^i(v)
\end{aligned}$$

For $x \in \mathbb{R}$, define

$$M_*(\theta, x) = \min_{(\gamma, n, m)} \Delta u_i(a_m, a_n, \gamma, \theta + x) - \Delta u_i(a_m, a_n, \gamma, \theta) \quad \text{(A.8)}$$

to be the smallest amount of state monotonicity in $i$'s payoff. Therefore, if the following inequality holds, then it implies (A.5)

$$E_{\theta|t_i}[M_*(\theta, k_F^i(v))] + k_G^i(v) E_{\theta|t_i}[C^*(\theta)] > 0 \quad \text{(A.9)}$$

By definition of $\varepsilon(F, G, \boldsymbol{u})$, if $v > \varepsilon(F, G, \boldsymbol{u})$, then (A.9) holds for all pairs of actions $a_n$ and $a_m$, types $t_i$, strategies $s_{-i}$, and $i \in N$. This implies that (A.5), hence (A.4) and (A.2) are satisfied for all these parameters. □

A.2. **Real vs. Fictitious Cutoffs and Proposition 5.** The real cutoffs were defined as the threshold types that separate an action from its successor. They are sufficient to represent any increasing strategy. How to recover the real cutoffs from the fictitious cutoffs? The following example illustrates the problem.

**Example 1.** Consider a game satisfying all the assumptions. Let there be two players. let $A_1 = A_2 = \{0, 1, 2\}$. There will be three fictitious cutoffs, $c_{1,0}$, $c_{2,0}$ and $c_{2,1}$, but only two are needed to represent a player's best-response. Which ones? For instance, suppose strategy $(0.2, 0.8)$ is a best-response for $i$ to some strategy $s_j$ of player $j$. It consists in playing 0 for types below 0.2, 2 for types above 0.8, and 1 in between. In this case, the first real cutoff, $s_{i,0}$, that separates 0 and 1 is $0.2 = c_{1,0}$. The second real cutoff, $s_{i,1}$, that separates 1 and 2 is $0.8 = c_{2,1}$. Now, consider the following best-response $(0.4, 0.4)$ to $s'_j$. In this case, the player never plays 1 except possibly on a set of measure zero (when receiving exactly type 0.4). The first real cutoff, $s'_{i,0}$, that separates 0 and 1 is $0.4 = c'_{2,0}$, but the second real cutoff, $s'_{i,1}$, is also $c'_{2,0}$, because 1 is not played. So the real cutoffs can change which fictitious cutoff they take value of.

This leads to the following definition where the real cutoffs are defined inductively from the fictitious ones.[22]

**Definition 8.** *Given $s_{-i}$, the largest real cutoff, $s_{i,M_i-1}$, is the fictitious cutoff $c_{M_i,\alpha}$ for which there exists $\epsilon > 0$ such that $E\Delta u_i(a_{i,M_i}, a_i, s_{-i}, t_i) > 0$ for all actions $a_i \neq a_{i,M_i}$ whenever $t_i > c_{M_i,\alpha}$, and $E\Delta u_i(a_{i,\alpha}, a_i, s_{-i}, t_i) > 0$ for all actions $a_i \neq a_{i,\alpha}$ whenever $t_i \in (c_{M_i,\alpha} - \epsilon, c_{M_i,\alpha})$. Assuming $s_{i,\ell} = c_{n,m}$ (with $n > m$), the real cutoff $s_{i,\ell-1} = c_{n,m}$ if $\ell > m$. Otherwise, if $\ell = m$, then $s_{i,\ell} = c_{m,\beta}$ for which there exists $\epsilon > 0$ such that $E\Delta u_i(a_{i,m}, a_i, s_{-i}, t_i) > 0$ for all actions $a_i \neq a_{i,m}$ whenever $t_i > c_{m,\beta}$, and $E\Delta u_i(a_{i,\beta}, a_i, s_{-i}, t_i) > 0$ for all actions $a_i \neq a_{i,\beta}$ whenever $t_i \in (c_{m_i,\beta} - \epsilon, c_{m_i,\alpha} - \epsilon)$.*

The definition is actually straightforward. The dominance regions imply that $a_{i,M_i}$ will be played. So, the largest real cutoff is the fictitious cutoff between $a_{i,M_i}$ and the action $a_{i,\alpha}$ played right before. All actions in between are not played, and so they receive the same real cutoff. Then we proceed in a downward fashion to find the action which was played right before $a_{i,\alpha}$, and so on.

The next proposition shows that if an action is strictly dominated by another action for all types against some opposing profile, then it must be strictly dominated by that same action for all types and against all opposing profiles. As a result, the same set of fictitious cutoffs will exist across opposing strategy profiles.

**Proposition 4.** *Let $\varepsilon(F, G, \boldsymbol{u}) < \infty$. For any actions $a_i, a'_i \in A_i$, if there exists $s'_{-i} \in \mathbb{R}$ such that $Eu_i(a'_i, s'_{-i}, t_i) > Eu_i(a_i, s'_{-i}, t_i)$ for all $t_i \in \mathbb{R}$, then $Eu_i(a'_i, s_{-i}, t_i) > Eu_i(a_i, s_{-i}, t_i)$ for all $s_{-i} \in \mathbb{R}$ and $t_i \in \mathbb{R}$.*

*Proof.* Let $\varepsilon(F, G, \boldsymbol{u}) < \infty$. Suppose first $a'_i > a_i$. If there is $s'_{-i}$ such that $Eu_i(a'_i, s'_{-i}, t_i) > Eu_i(a_i, s'_{-i}, t_i)$ for all $t_i$, then Proposition 3 implies that for all $v > \varepsilon(F, G, \boldsymbol{u})$,

$$Eu_i(a'_i, s'_{-i} + \boldsymbol{v}, t_i + v) - Eu_i(a_i, s'_{-i} + \boldsymbol{v}, t_i + v) > 0, \tag{A.10}$$

for all $t_i$. Take any $s_{-i}$ and choose $v > \varepsilon(F, G, \boldsymbol{u})$ for which $s'_{-i} + \boldsymbol{v} \geq s_{-i}$ (so $s_{-i}$ is a larger strategy). Since larger strategies lead to larger aggregates, the strategic complementarities imply

[22] Existence of the fictitious cutoffs poses no problem in the definition, for if a real cutoff takes on the value of a fictitious cutoff, that fictitious cutoff must exist.

(by (A.10))

$$Eu_i(a'_i, s_{-i}, t_i + v) - Eu_i(a_i, s_{-i}, t_i + v) > 0$$

for all $t_i$. This is equivalent to $Eu_i(a'_i, s_{-i}, t_i) - Eu_i(a_i, s_{-i}, t_i) > 0$ for all $t_i$; since $s_{-i}$ was arbitrary, it proves the claim. Suppose now that $a'_i < a_i$. If there is $s'_{-i}$ such that $Eu_i(a'_i, s'_{-i}, t_i) > Eu_i(a_i, s'_{-i}, t_i)$ for all $t_i$, then Proposition 3 implies that for all $v > \varepsilon(F, G, \boldsymbol{u})$,

$$Eu_i(a'_i, s'_{-i} - \boldsymbol{v}, t_i - v) - Eu_i(a_i, s'_{-i} - \boldsymbol{v}, t_i - v) > 0, \tag{A.11}$$

for all $t_i$. Take any $s_{-i}$ and choose $v > \varepsilon(F, G, \boldsymbol{u})$ for which $s_{-i} \geq s'_{-i} - \boldsymbol{v}$ (so $s_{-i}$ is a smaller strategy). It follows from (A.11) and the strategic complementarities that

$$Eu_i(a'_i, s_{-i}, t_i - v) - Eu_i(a_i, s_{-i}, t_i - v) > 0$$

for all $t_i$, which is equivalent to $Eu_i(a'_i, s_{-i}, t_i) - Eu_i(a_i, s_{-i}, t_i) > 0$. □

The next proposition is an important piece of the main theorem. If all of $i$'s fictitious cutoffs contract in response to a variation of $s_{-i}$, then so do all of $i$'s real cutoffs. That is, $i$'s best-reponse contracts as well.

**Proposition 5.** *Suppose $\varepsilon(F, G, \boldsymbol{u}) < \infty$. If, for some $v > 0$, $|c'_{n,m} - c_{n,m}| < v$ for all $n$ and $m$ such that both (fictitious) cutoffs exist, then $|s_{i,\ell} - s'_{i,\ell}| < v$ for all $\ell = 1, \ldots, M_i - 1$.*

*Proof.* I prove the result by induction. Suppose that, for some $v > 0$, $|c'_{n,m} - c_{n,m}| < v$ for all $n$ and $m$ for which both $c'_{n,m}$ and $c_{n,m}$ exist. First, I show that it is true for the largest real cutoff $s_{i,M_i}$. Then it extends to all other real cutoffs.

The largest action $a_{i,M_i}$ is always played for large enough types. So the largest real cutoff always takes on the value of the fictitious cutoff between $a_{i,M_i}$ and some other action. Suppose that $s_{i,M_i-1} = c_{M_i,w}$ and $s'_{i,M_i-1} = c'_{M_i,z}$ where $a_{i,w}$ and $a_{i,z}$ are some actions. Proposition 4 implies that $c_{M_i,z}$ must exist. To see why, suppose $c_{M_i,z}$ did not exist. Since $a_{M_i}$ must be played, it would mean that $a_{M_i}$ strictly dominates $a_z$ for all $t_i$ against $s_{-i}$; Proposition 4 would then imply that $a_{M_i}$ strictly dominates $a_z$ for all $t_i$ and all opposing strategies, $s'_{-i}$ in particular, making the existence of $c'_{M_i,z}$ impossible. So, $s'_{i,M_i-1} - s_{i,M_i-1} = c'_{M_i,z} - c_{M_i,w} = c'_{M_i,z} - c_{M_i,z} + c_{M_i,z} - c_{M_i,w}$. Note that $c_{M_i,z} - c_{M_i,w} \leq 0$ by definition 8. Indeed, $s_{i,M_i-1} = c_{M_i,w}$ implies that $a_{i,M_i}$ is played right after $a_{i,w}$, and so it must be that $a_{i,M_i}$ was preferred to $a_{i,z}$ for some lower types. That is, $c_{M_i,z} \leq c_{M_i,w}$. Since $c'_{M_i,z} - c_{M_i,z} < v$, then $s'_{i,M_i-1} - s_{i,M_i-1} < v$. The proof is similar for $s_{i,M_i-1} - s'_{i,M_i-1}$, hence $|s'_{i,M_i-1} - s_{i,M_i-1}| < v$.

For the other real cutoffs, the situation is more difficult, because the action may or may not be played. There will be several cases, depending on whether the action is played. By induction hypothesis, suppose that $|s'_{i,\ell+1} - s_{i,\ell+1}| < v$. The objective is to show that it implies $|s'_{i,\ell} - s_{i,\ell}| < v$.

**Case 1**: Action $a_{i,\ell}$ is played both under $s_i$ and $s'_i$. This case is similar to the case of the largest real cutoff, and the proof is identical.

**Case 2**: Action $a_{i,\ell}$ is played neither under $s_i$ nor $s'_i$. Then, by definition, $s_{i,\ell} = s_{i,\ell+1}$ and $s'_{i,\ell} = s'_{i,\ell+1}$. By induction hypothesis, $|s'_{i,\ell} - s_{i,\ell}| < v$.

**Case 3**: Action $a_{i,\ell}$ is not played under $s_i$, but it is played under $s'_i$. Then, $s_{i,\ell} = c_{w,z}$ for some actions $a_{i,w}$ and $a_{i,z}$ such that $z < \ell < w$, and $s'_{i,\ell} = c'_{\ell,x}$ for some $a_{i,x}$. Write $s'_{i,\ell} - s_{i,\ell} = c'_{\ell,x} - c_{w,z}$.

First, I establish that both $c_{w,\ell}$ and $c'_{w,\ell}$ exist. Action $a_{i,w}$ is played (under $s_i$) against $s_{-i}$ but it cannot strictly dominate $a_{i,\ell}$ for all types $t_i$, because if it did, then Proposition 4 would imply that it is also the case (under $s'_i$) against $s'_{-i}$ (thus $a_{i,\ell}$ could not be played under $s'_i$, yet it is). Therefore, $c_{w,\ell}$ must exist. This implies that for all $t_i \geq c_{w,\ell}$,

$$Eu_i(a_{i,w}, s_{-i}, t_i) > Eu_i(a_{i,\ell}, s_{-i}, t_i). \tag{A.12}$$

Let $\boldsymbol{h} = (h, \ldots, h)$ where $h > \varepsilon(F, G, \boldsymbol{u})$ is large enough such that $s_{-i} + \boldsymbol{h} \geq s'_{-i}$. It follows from Proposition 3 and (A.12) that for all $t_i \geq c_{w,\ell}$,

$$Eu_i(a_{i,w}, s_{-i} + \boldsymbol{h}, t_i + h) > Eu_i(a_{i,\ell}, s_{-i} + \boldsymbol{h}, t_i + h)$$

and thus by strategic complementarities,

$$Eu_i(a_{i,w}, s'_{-i}, t_i + h) > Eu_i(a_{i,\ell}, s'_{-i}, t_i + h),$$

for all $t_i \geq c_{w,\ell}$. We know $a_{i,\ell}$ is played (under $s'_i$) against $s'_{-i}$, so the last inequality implies that $c'_{w,\ell}$ exists.

Second, I prove that real cutoff contracts. The following inequality must hold, $c'_{w,\ell} \geq c'_{\ell,x}$, because $a_{i,\ell}$ is played under $s'_i$ in an open set of types above $c'_{\ell,x}$ (so it is only for types larger than $c'_{\ell,x}$ that $a_{i,w}$ can be preferred to $a_{i,\ell}$). Similarly, $c_{w,\ell} \leq c_{w,z}$, because $a_{i,w}$ is played under $s_i$ in an open set of types above $c_{w,z}$, hence $a_{i,w}$ started to be preferred to $a_{i,\ell}$ for smaller types. As a result,

$$s'_{i,\ell} - s_{i,\ell} = c'_{\ell,x} - c_{w,z} \leq c'_{w,\ell} - c_{w,\ell},$$

so $s'_{i,\ell} - s_{i,\ell} < v$. By a similar reasoning, $s_{i,\ell} - s'_{i,\ell} \leq c'_{\ell,z} - c_{\ell,z}$, and so $s_{i,\ell} - s'_{i,\ell} < v$. Putting everything together, $|s'_{i,\ell} - s_{i,\ell}| < v$.

**Case 4**: Action $a_\ell$ is played under $s_i$ but it is not played under $s'_i$. The argument is similar to case 3. $\square$

A.3. **Proof of Theorem 1.** The theorem relies on the concept of a $q$-contraction, so I define it first.

**Definition 9.** *Let $(X, d)$ be a metric space. If $\xi : X \to X$ satisfies the condition $d(\xi(x), \xi(y)) < d(x, y)$ for all $x, y \in X$ such that $d(x, y) > q$, then $\xi$ is called a $q$-contraction.*

A traditional contraction mapping "shrinks" the images of all points. A $q$-contraction only "shrinks" those of points that are sufficiently far apart (further apart than $q$).

*Proof.* Player $i$'s expected utility of playing $a_i$ when his type is $t_i$ and the other players play $s_{-i}$ is:

$$Eu_i(a_i, s_{-i}, t_i) = \int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} u_i(a_i, \gamma, \theta) g_i(\gamma | \theta, s_{-i}, t_i) f_i(\theta | t_i) d\theta. \tag{A.13}$$

Now pick $n, m \in \{1, \ldots, M_i\}$ such that $n > m$. If it exists, the fictitious cutoff between $a_{i,n}$ and $a_{i,m}$ is defined as the type $t_i$ such that

$$Eu_i(a_{i,m}, s_{-i}, t_i) = Eu_i(a_{i,n}, s_{-i}, t_i),$$

that is,

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma | \theta, s_{-i}, c_{n,m}) f_i(\theta | c_{n,m}) d\theta = 0. \tag{A.14}$$

By state monotonicity, we know that $\Delta u_i$ is strictly increasing in $\theta$, and increasing in $\gamma$. Since $F_i$ is strictly increasing in $t_i$ w.r.t. first-order stochastic dominance, and since $G_i$ is increasing in $(\theta, t_i)$ w.r.t. to first-order stochastic dominance, there can be only one type $t_i$ that satisfies (A.14). As a result, the best-replies (which are cutoff strategies) are almost everywhere functions, and not correspondences. Consider two profiles of strategies for players $-i$, $s_{-i} = (s_{j,\ell})$ and $s'_{-i} = (s'_{j,\ell})$. Denote $v_{j,\ell} = |s'_{j,\ell} - s_{j,\ell}|$ for $\ell = 1, \ldots, M_j - 1$. Let $v = \max_{j \neq i} \max_{\ell \in \{1,\ldots,M_j-1\}} v_{j,\ell}$. At $s_{-i}$, the cutoff between $a_{i,n}$ and $a_{i,m}$ is $c_{n,m}$ and satisfies (A.14). At $s'_{-i}$, the cutoff between $a_{i,n}$ and $a_{i,m}$ is $c'_{n,m}$. By way of contradiction, assume $c'_{n,m} = c_{n,m} + v$ so that

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma | \theta, s'_{-i}, c_{n,m} + v) f_i(\theta | c_{n,m} + v) d\theta = 0. \tag{A.15}$$

If $v > \varepsilon(F, G, \boldsymbol{u})$, Proposition 3 says that (A.14) and (A.15) cannot hold simultaneously. In words, $c'_{n,m} = c_{n,m} + v$ cannot be the fictitious cutoff at $s'_{-i}$ if $c_{n,m}$ is the cutoff at $s_{-i}$. Clearly, this is also true for $c'_{n,m} \geq c_{n,m} + v$. Therefore, $c'_{n,m} - c_{n,m} < v$. The same argument applies to show that if $c'_{n,m}$ is the cutoff, then it cannot be that $c_{n,m} \geq c'_{n,m} + v$ is the cutoff at $s'_{-i}$. The conclusion is that if $v > \varepsilon(F, G, \boldsymbol{u})$, then $|c'_{n,m} - c_{n,m}| < v$ for all $n, m$ such that both cutoffs exist, and all $i \in N$. Proposition 5 implies that each $i$'s best-reply is an $\varepsilon(F, G, \boldsymbol{u})$-contraction. From Milgrom and Roberts [15], it follows that there exist two extremal equilibria, $\overline{s}$ and $\underline{s}$, which correspond to the extremal profiles of rationalizable strategies. Let $d$ be the sup-norm metric. Since $br_i$ is an $\varepsilon(F, G, \boldsymbol{u})$-contraction, if $d(\overline{s}, \underline{s}) > \varepsilon(F, G, \boldsymbol{u})$, then we have

$$\begin{array}{rcl} d(\overline{s}, \underline{s}) & = & d(br(\overline{s}), br(\underline{s})) \\ & = & \max_{i \in N} d(br_i(\overline{s}_{-i}), br_i(\underline{s}_{-i})) \\ & \leq & \max_{i \in N} d(br_i(\underline{s}_{-i} - d(\overline{s}, \underline{s})), br_i(\underline{s}_{-i})) \\ & < & d(\overline{s}, \underline{s}), \end{array}$$

where the first inequality holds because best-replies are increasing (in games of strategic complements).[23] This string of inequalities leads to a contradiction, and thus $d(\overline{s}, \underline{s}) \leq \varepsilon(F, G, \boldsymbol{u})$.

□

[23]Notice $\underline{s}_{-i} - d(\overline{s}, \underline{s})$ is a larger strategy than $\overline{s}_{-i}$.

A.4. **Theorem 2.** I first establish a proposition which will be used in the proof.

**Proposition 6.** *Let* $\{c_{n,m}\}$ *be the set of fictitious cutoffs under* $B$*, and let* $\{c_{n,m}\}$ *be the set of fictitious cutoffs under* $B'$*, where* $B'$ *is more optimistic than* $B$*. If, for some* $v > 0$*,* $c_{n,m} - c'_{n,m} \geq v$ *for all* $n$ *and* $m$ *such that both fictitious cutoffs exist, then* $s_{i,\ell} - s'_{i,\ell} \geq v$ *for all* $\ell = 1, \ldots, M_i - 1$*.*

*Proof.* The result is proved by induction. Suppose that, for some $v > 0$, $c_{n,m} - c'_{n,m} \geq v$ for all $n$ and $m$ such that both fictitious cutoffs exist. First, I show it is true for the largest real cutoff $s_{i,M_i}$. Then it extends to all other real cutoffs.

Suppose that $s_{i,M_i-1} = c_{M_i,w}$ and $s'_{i,M_i-1} = c'_{M_i,z}$ where $a_{i,w}$ and $a_{i,z}$ are some actions. Because $c'_{M_i,z}$ exists, $a_{i,z}$ is preferred to $a_{i,M_i}$ for all $t_i \leq c'_{M_i,z}$ (under $B'$). $B$ is less optimistic than $B'$, so $a_{i,z}$ is also preferred to $a_{i,M_i}$ for all $t_i \leq c'_{M_i,z}$ under $B$. At some point, this relationship is reversed, because $a_{i,M_i}$ is always played, hence $c_{M_i,z}$ must exist. Write $s_{i,M_i-1} - s'_{i,M_i-1} = c_{M_i,w} - c'_{M_i,z} = c_{M_i,w} - c_{M_i,z} + c_{M_i,z} - c'_{M_i,z}$. By definition 8, $c_{M_i,w} - c_{M_i,z} \geq 0$, because $s_{i,M_i-1} = c_{M_i,w}$ implies that $a_{i,M_i}$ is played right after $a_{i,w}$, and so it must be that $a_{i,M_i}$ was preferred to $a_{i,z}$ for some lower types (the argument is similar to Proposition 5). That is, $c_{M_i,z} \leq c_{M_i,w}$. Since $c_{M_i,z} - c'_{M_i,z} \geq v$, it must be that $s_{i,M_i-1} - s'_{i,M_i-1} \geq v$.

By induction hypothesis, suppose that $s_{i,\ell+1} - s'_{i,\ell+1} \geq v$. The objective is to show that it implies $s_{i,\ell} - s'_{i,\ell} \geq v$. Consider four cases.

**Case 1**: Action $a_{i,\ell}$ is played both under $s_i$ and $s'_i$. This case is similar the case of the largest real cutoff, and the proof is identical.

**Case 2**: Action $a_{i,\ell}$ is played neither under $s_i$ nor $s'_i$. Then, $s_{i,\ell} = s_{i,\ell+1}$ and $s'_{i,\ell} = s'_{i,\ell+1}$. By induction hypothesis, $s_{i,\ell} - s'_{i,\ell} \geq v$.

**Case 3**: Action $a_{i,\ell}$ is not played under $s_i$, but it is played under $s'_i$. Then, $s_{i,\ell} = c_{w,z}$ for some actions $a_{i,w}$ and $a_{i,z}$ such that $z < \ell < w$, and $s'_{i,\ell} = c'_{\ell,x}$ for some $a_{i,x}$. For types $t_i \geq c_{w,z}$, $a_{i,w}$ is preferred to $a_{i,\ell}$ under $B$, and so is it under $B'$, because $B'$ is more optimistic than $B$. Since $a_{i,\ell}$ is played under $B'$, there are also types at which $a_{i,\ell}$ is preferred to $a_{i,w}$; so $c'_{w,\ell}$ must exist. To show that $c_{w,\ell}$ exists, recall that for types $t_i \geq c_{w,z}$, $a_{i,w}$ is preferred to $a_{i,\ell}$ under $B$. For types $t_i \leq c'_{\ell,x}$, $a_{i,\ell}$ is preferred to $a_{i,w}$ under $B'$, for otherwise $a_{i,w}$ would be preferred to $a_{i,\ell}$ for all types above $c'_{\ell,x}$. Thus, because $B$ is less optimistic than $B'$, $a_{i,\ell}$ must also be preferred to $a_{i,w}$ for types $t_i \leq c'_{\ell,x}$. Under $B$, there are types such that the preference between $a_{i,\ell}$ and $a_{i,w}$ goes both ways, so $c_{w,\ell}$ exists. Write $s_{i,\ell} - s'_{i,\ell} = c_{w,z} - c'_{\ell,x}$. Note that $c'_{w,\ell} \geq c'_{\ell,x}$, because $\ell$ is played under $s'_i$ in an open set of types above $c'_{\ell,x}$ (so it is only for larger types that $a_{i,w}$ will be preferred). Further, note that $c_{w,\ell} \leq c_{w,z}$, because $a_{i,w}$ is played under $s_i$ in an open set of types above $c_{w,z}$, hence $a_{i,w}$ is preferred to $a_{i,\ell}$ for smaller types. As a result, $s_{i,\ell} - s'_{i,\ell} \geq c_{w,\ell} - c'_{w,\ell}$, and so $s_{i,\ell} - s'_{i,\ell} \geq v$.

**Case 4**: Action $a_\ell$ is played under $s_i$ but it is not played under $s'_i$. The argument is close to case 3. Since $a_{i,\ell}$ is played under $s_i$, it must dominate $a_{i,z} < a_{i,\ell}$ for some types; because beliefs $B'$ are more optimistic than $B$, $a_{i,\ell}$ must also dominate $a_{i,z}$ for those types. But $a_{i,z}$ is played under $s'_i$, and as a result, $c'_{\ell,z}$ exists. Likewise, $a_{i,z}$ dominates $a_{i,\ell}$ under $B'$ for some types, because it is played. For those types, it must also be the case under $B$, because $B$ is less optimistic than $B'$. So $c_{\ell,z}$ exists. □

*Proof.* In supermodular games, recall that the largest (smallest) equilibrium coincide with the largest (smallest) profile of rationalizable strategies. Consider the largest (smallest) equilibrium, denoted by $\overline{s}$ ($\underline{s}$), under beliefs $(F_i, G_i)$, $i = 1, \ldots, n$. At strategy profile $\overline{s}$, $i$'s (fictitious) cutoff between $a_n$ and $a_m$ satisfies

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i(\gamma|\theta, \overline{s}_{-i}, c_{n,m}) f_i(\theta|c_{n,m}) d\theta = 0. \tag{A.16}$$

Since beliefs $(F_i', G_i')$ are more optimistic than $(F_i, G_i)$,

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i'(\gamma|\theta, \overline{s}_{-i}, c_{n,m}) f_i'(\theta|c_{n,m}) d\theta \geq 0, \tag{A.17}$$

because $\Delta u_i$ is increasing in $\theta$ and $\gamma$. This implies that the (fictitious) cutoff between $a_n$ and $a_m$ must be smaller under $(F_i', G_i')$ than $(F_i, G_i)$. The proof will say how much smaller that fictitious cutoff has to be under $(F_i', G_i')$. For $s_{-i}$ and $t_i$, take any $v \geq 0$ such that if

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i(\gamma|\theta, s_{-i}, t_i) f_i(\theta|t_i) d\theta = 0 \tag{A.18}$$

holds, then

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i'(\gamma|\theta, \overline{s}_{-i}, t_i - v) f_i'(\theta|t_i - v) d\theta > 0. \tag{A.19}$$

If $v$ satisfies (A.19), then $t_i - v$ cannot be the (fictitious) cutoff under $(F_i', G_i')$ (because $t_i - v$ is too high). Look for a larger $v$ (that is, a lower $t_i - v$). It follows from the definition of optimism and upper-confidence that

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i'(\gamma|\theta, \overline{s}_{-i}, t_i - v) f_i(\theta - \omega_1 + K^i(v)|t_i) d\theta > 0 \tag{A.20}$$

implies (A.19). After a change of variables, (A.20) is equivalent to

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta + \omega_1 - K^i(v)) g_i'(\gamma|\theta + \omega_1 - K^i(v), \overline{s}_{-i}, t_i - v) f_i(\theta|t_i) d\theta > 0. \tag{A.21}$$

If (A.18) holds, then (A.21) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}} \sum_{\gamma} (\Delta u_i(\gamma, \theta + \omega_1 - K^i(v)) - \Delta u_i(\gamma, \theta)) g_i'(\gamma|\theta + \omega_1 - K^i(v), \overline{s}_{-i}, t_i - v) f_i(\theta|t_i) d\theta \\ &\quad + \int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) f_i(\theta|t_i) (g_i'(\gamma|\theta + \omega_1 - K^i(v), \overline{s}_{-i}, t_i - v) - g_i(\gamma|\theta, \overline{s}_{-i}, t_i)) d\theta > 0. \end{aligned} \tag{A.22}$$

Consider each member of (A.22) successively and find a lower for this expression. Take the first member. By definition of $M_*$,

$$\begin{aligned} &\int_{\mathbb{R}} \sum_{\gamma} (\Delta u_i(\gamma, \theta + \omega_1 - K^i(v)) - \Delta u_i(\gamma, \theta)) g_i'(\gamma|\theta + \omega_1 - K^i(v), \overline{s}_{-i}, t_i - v) f_i(\theta|t_i) d\theta \geq \\ &\qquad \int_{\mathbb{R}} M_*(\omega_1 - K^i(v), t_i) f_i(\theta|t_i) d\theta \end{aligned} \tag{A.23}$$

Take the second member of (A.22). Note that

$$\sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma,\theta)(g_i'(\gamma|\theta+\omega_1-K^i(v),\overline{s}_{-i},t_i-v)-g_i(\gamma|\theta,\overline{s}_{-i},t_i)) =$$

$$\sum_{\gamma \geq \underline{\gamma}} (G_i'(\sigma(\gamma)|\theta+\omega_1-K^i(v),\overline{s}_{-i},t_i-v)-G_i(\sigma(\gamma)|\theta,s_{-i},t_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(\sigma(\gamma),\theta)) \quad \text{(A.24)}$$

For notational ease, $G_i(\tau_i)\vee G_i(\tau_i-o(v))$ is denoted $G_i^*(\tau_i)$, and $G_i(\tau_i)\wedge G_i(\tau_i-o(v))$ is denoted $G_{*,i}(\tau_i)$. The same notation applies to the probability mass functions. Like (A.7), define

$$C_*(\theta)=\min\left\{\frac{\Delta u_i(\sigma(\gamma),\theta)-\Delta u_i(\gamma,\theta)}{\sigma(\gamma)-\gamma} : \gamma\in\mathcal{G}_i\right\} \quad \text{(A.25)}$$

to be the minimum amount of complementarities at state $\theta$. Suppose first that $G_i(\tau_i-o(v))\geq_{st} G_i(\tau_i)$ for all $\tau_i$. So $w_2(v)\leq \Gamma_i^e[G_i'(\tau_i-o(v))]-\Gamma_i^e[G_i(\tau_i)]$ for all $\tau_i$. Since $\Delta u_i$ is increasing in $\gamma$, optimism implies that for all $\tau_i$ (so it is particularly true for the extremal rationalizable strategies $\overline{s}_{-i}$ and $\underline{s}_{-i}$),

$$\begin{aligned}
& \textstyle\sum_{\gamma\geq\underline{\gamma}}(G_i'(\sigma(\gamma)|\tau_i-o(v))-G_i(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(\sigma(\gamma),\theta))\\
= & \textstyle\sum_{\gamma\geq\underline{\gamma}}(G_i'(\sigma(\gamma)|\tau_i-o(v))-G_{*,i}(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(\sigma(\gamma),\theta))\\
\geq & \textstyle\sum_{\gamma\geq\underline{\gamma}}(G_i'(\sigma(\gamma)|\tau_i-o(v))-G_{*,i}(\sigma(\gamma)|\tau_i))(\gamma-\sigma(\gamma))C_*(\theta)\\
= & C_*(\theta)\textstyle\sum_{\gamma}\gamma(g_i'(\gamma|\tau_i-o(v))-g_{*,i}(\gamma|\tau_i))\\
\geq & C_*(\theta)w_2^i(v)
\end{aligned}$$

Suppose now that $G_i(\tau_i-o(v))\not\geq_{st} G_i(\tau_i)$ for some $\tau_i$. So $w_2(v)\leq \Gamma_i^e[G_i'(\tau_i-o(v))]-\Gamma_i^e[G_i'(\tau_i-o(v))\vee G_i(\tau_i)]$ for all $\tau_i$. Then for all $\tau_i$,

$$\begin{aligned}
& \textstyle\sum_{\gamma\geq\underline{\gamma}}(G_i'(\sigma(\gamma)|\tau_i-o(v))-G_i(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(\sigma(\gamma),\theta))\\
\geq & \textstyle\sum_{\gamma\geq\underline{\gamma}}(G_i'(\sigma(\gamma)|\tau_i-o(v))-G_i^*(\sigma(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(\sigma(\gamma),\theta))\\
\geq & \textstyle\sum_{\gamma\geq\underline{\gamma}}(G_i'(\sigma(\gamma)|\tau_i-o(v))-G_i^*(\sigma(\gamma)|\tau_i))(\gamma-\sigma(\gamma))C^*(\theta)\\
= & C^*(\theta)\textstyle\sum_{\gamma}\gamma(g_i'(\gamma|\tau_i-o(v))-g_i^*(\gamma|\tau_i))\\
\geq & C^*(\theta)w_2^i(v)
\end{aligned}$$

Putting this together with (A.23), we have that if (A.18) holds, then

$$\int_{\mathbb{R}} M_*(\theta, \omega_1 - K^i(v)) f_i(\theta|t_i) d\theta - \min\left\{\int_{\mathbb{R}} \omega_2(v) C_*(\theta) f_i(\theta|t_i) d\theta, \int_{\mathbb{R}} \omega_2(v) C^*(\theta) f_i(\theta|t_i) d\theta\right\} > 0 \tag{A.26}$$

implies (A.19). Let $M_*(\omega_1 - K^i(v), t_i) = \int_{\mathbb{R}} M_*(\theta, \omega_1 - K^i(v)) f_i(\theta|t_i) d\theta$, $C^*(t_i) = \int_{\mathbb{R}} C^*(\theta) f_i(\theta|t_i) d\theta$ and $C_*(t_i) = \int_{\mathbb{R}} C_*(\theta) f_i(\theta|t_i) d\theta$. Hence, $\delta(B, B', \boldsymbol{u})$ gives the infimum value of $v$ such that (A.26) is satisfied for all pair of actions, strategies of players $-i$, and player $i$. This means that $c_{n,m} - c'_{n,m} \geq d(B, B', \boldsymbol{u})$ for all $n$ and $m$. Proposition 6 completes the proof. $\square$

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