Dynamic Contracting with Persistent Shocks

Zhang, Yuzhe

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Yuzhe Zhang[†]

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Abstract:

In this paper, we develop continuous-time methods for solving dynamic principal-agent problems in which the agent's privately observed productivity shocks are persistent over time. We characterize the optimal contract as the solution to a system of ordinary differential equations and show that, under this contract, the agent's utility converges to its lower bound—immiserization occurs. Unlike under risk-neutrality, the wedge between the marginal rate of transformation and a low-productivity agent's marginal rate of substitution between consumption and leisure will not vanish permanently at her first high-productivity report; also, the wedge increases with the duration of a low-productivity report. We apply the methods to numerically solve the Mirrleesian dynamic taxation model, and find that the wedge is significantly larger than that in the independently and identically distributed (i.i.d.) shock case.

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[†]Department of Economics, Henry B. Tippie College of Business, University of Iowa, Iowa City, IA, 52242, U.S.A.; yuzhe-zhang@uiowa.edu.

1. Introduction

A common assumption in the dynamic mechanism design literature is that the agent's privately observed shocks are independently and identically distributed (i.i.d.). As pointed out by Fernandes and Phelan (2000), this assumption is merely for the sake of tractability. It implies that, at the beginning of a given date, an agent's forward-looking utility of following a given strategy when facing a given contract is independent of past histories.[1]

However, in many economic environments with hidden information, the agent's shocks are highly persistent. For example, in the design of optimal health insurance, a customer's health condition today is strongly correlated with her previous condition. And in unemployment insurance where an unemployed worker's searching effort is hidden, it is reasonable to conjecture that the worker's chance of finding a new job depends not only on her current effort but also on her searching effort in the past.

In this paper, we develop continuous-time methods for solving dynamic contracting problems and apply them to an optimal taxation model in which the agent's privately observed productivity shocks are persistent over time. Productivity process is modeled as a finite-state Markov chain with transitions arriving as a Poisson process. A key technique that we develop is that, in continuous time, the incentive constraints are transformed into a system of differential equations and inequalities. This system of equations connects the principal's choice variables and the evolution of the promised utilities, thus allows us to rewrite the contracting problem as a stochastic control problem. We then study the stochastic control problem and obtain a sharp characterization of the optimal contract.

We find that the cost of delivering a utility vector is increasing in the promised utility but decreasing in the transitional utility. We also find that for each level of promised utility to the low-productivity agent, there is an efficient (cost-minimizing) level of transitional utility. However, when the agent reports low productivity, the principal has to move the transitional utility strictly below the efficient level, since the high-productivity type has an incentive to misreport and reduce her effort. This feature makes the persistent shock contract open to renegotiation, because moving back to the efficiency level makes the agent indifferent and the principal strictly better off. This is different from the i.i.d. shock case, where the contract is always renegotiation proof.

There are many features that persistent shock models share with i.i.d. shock models. The agent's promised utility moves up with a high report and moves down with a low report. The consumption and output levels for a high report are both higher than those for a low report when compared at similar levels of promised utilities. The immiserization result continues to hold in persistent shock models, because the martingale property (associated with the inverse Euler equation) remains valid.[2]

[1] This notion of *common knowledge about preferences* was first discussed in Fudenberg, Holmstrom, and Milgrom (1990).

[2] Farhi and Werning (2007) show that immiserization is avoided when the principal is more patient than the agent.

Based on these findings, we conclude that, *qualitatively*, the persistent shock models are similar to i.i.d. shocks. However, *quantitatively*, persistence is still an important issue that should not be ignored. Through a numerical example, we find that the distortions in the persistent shock model are much larger than in the i.i.d. shock model. Thus, using i.i.d. shocks as an approximation to the true productivity shocks would seriously underestimate the role of the tax system in a Mirrleesian model.

1.1 Related Literature

Our formulation of the problem is based largely on ideas in Fernandes and Phelan (2000). They developed a recursive formulation for contracting problems in which private types are serially correlated. In these situations, different types of agents derive different continuation utilities from the same continuation contract due to type-specific priors. When the agent chooses between truth-telling and lying, she compares the continuation utility as a truth-teller and the continuation utility as a liar. Thus, the principal finds it necessary to enforce a vector of utilities for all the potentially different types. They showed that this vector of continuation utilities is the state variable in their recursive formulation. Our work provides an analytical characterization of the optimal contract, which is different from Fernandes and Phelan (2000), who solved the optimal contract by numerical iteration following Abreu, Pearce, and Staccheti (1990).

This paper is motivated by the literature on continuous-time contracting with hidden actions (Holmstrom and Milgrom (1987), Schattler and Sung (1993), Cvitanic, Wan, and Zhang (2007), Williams (2006), Westerfield (2006), Sannikov (2007a,b)). The literature shows that setting principal-agent models in continuous time could allow for more explicit characterization of the solution. The novelty in this paper involves our modeling of the random process. Since the traditional continuous-time methods with diffusion process cannot be readily adopted to study hidden information models with persistent shocks (see Section 7 for an explanation), we model the agent's type as a finite-state Markov chain, which introduces techniques that are different from, but complementary to, the above literature.

There are a few recent papers studying dynamic contracts with private and persistent shocks. Kapicka (2007) studied an optimal taxation model with a continuum of productivity shocks in a discrete-time model. By assuming the validity of the first-order approach, he reduced the infinite-dimensional state variable to a vector of two numbers: promised utility and its derivative. This simplification allows him to numerically compute a taste-shock model, which sheds light on the properties of the tax system in dynamic taxation models. Williams (2008) also studied persistent shocks with a continuum of shocks, but in continuous time. He considered a related first-order approach, but by using the continuous-time techniques he had developed in Williams (2006), he provided sufficient conditions to check whether the solution from the first-order approach is fully incentive compatible. Furthermore, he found that the inverse Euler equation does not hold in his environment and the immiserization property disappears.[3]

Finally, our paper is closely related to the literature on dynamic social insurance and taxation (Albanesi (2006), Albanesi and Sleet (2006), Atkeson and Lucas (1992), Golosov, Kocherlakota, and Tsyvinski (2003), Golosov and Tsyvinski (2006, 2007), Kocherlakota (2005), Kapicka (2006)). Our model is a simplified version (with neither aggregate resource constraint nor capital accumulation) of Golosov, Kocherlakota, and Tsyvinski (2003) and Kocherlakota (2005). However, our research focuses more on developing a methodology and understanding the analytical properties of the optimal contract. Battaglini and Coate (2003) studied a persistent shock dynamic taxation model with risk-neutral agents. The distortion in their model eventually vanishes in two senses. First, it vanishes permanently with any high-productivity report. Second, the distortion decreases to zero for an agent who always reports low productivity. We can apply the continuous-time methods to study their model and confirm their findings. However, when we study risk-averse utilities, we find that the distortion increases with the low-productivity report and never vanishes. These differences suggest that risk aversion is critical for the patterns of the distortion.

The remainder of the paper is organized as follows. Section 2 lays out the economic environment and sets up the social planner's contracting problem. In Section 3, we derive the continuous-time evolution of the state variable as differential equations and inequalities. The resulting differential equations are put to use in Section 4 to characterize the set of implementable utility pairs and in Section 5 to study the long-run dynamics of the optimal contract. In Section 6, through a numerical example, we show that the models with persistent shocks imply significantly larger wedges than the models with i.i.d. shocks. The last section concludes. All proofs are collected in the Appendices.

2. A Dynamic Contracting Problem

2.1 The Environment and the Shock Process

Time is continuous. Consider a risk-neutral principal and a risk-averse agent who engage in long-term contracting at time 0. Both the principal and the agent are able to commit. The preferences of the agent are

$$E\left[\int_0^\infty e^{-rt}\left[u(c_t) - v(y_t)/\theta_t\right]dt\right], \tag{1}$$

where c_t and y_t are the agent's consumption and output at time t, r is her discount rate, θ_t is her private taste shock, and E is an expectations operator. The principal has the same discount rate r and

[3] The first-order approach is able to reduce the dimension of the state variable to a small number, but its validity is still unestablished. Williams (2008) provided sufficient conditions for the first-order approach, but these conditions may either be overly stringent or fail to hold.

minimizes

$$E\left[\int_0^\infty e^{-rt}\left[c_t - y_t\right]dt\right], \tag{2}$$

which is the expected discounted cost of the consumption-output plan. We assume that $u : [0, \bar{c}] \to [0, \bar{u}]$ is twice continuously differentiable, increasing, and strictly concave ($u' > 0$ and $u'' < 0$). The disutility function $v : [0, \bar{y}] \to [0, \bar{v}]$ is twice continuously differentiable, increasing, and strictly convex ($v' > 0$ and $v'' > 0$). The agent's privately observed taste shock θ_t can be re-interpreted as a productivity shock if $v(y) = y^\gamma, \gamma > 1$. Define $\phi = \theta^{1/\gamma}$ as the agent's productivity. She is able to transform one unit of labor into ϕ units of output. Her disutility depends on the amount of labor $l = y/\phi$ she spends to produce y, thus the disutility is $v(l) = v(y)/\theta$.

The shock process $(\theta_t)_{t\geq 0}$ is a time-homogeneous, continuous-time Markov chain with a finite state space $\Theta = \{\theta_1, \theta_2, ..., \theta_N\}$, where $0 < \theta_1 < \theta_2 < ... < \theta_N$, and a generator matrix $Q = (q_{ij})_{1\leq i,j\leq N}$. Let $\mathscr{N} = \{1, 2, ..., N\}$ be the set of indices. A probability space $(\Omega, \mathscr{F}, P)$ is described as follows. Let sample space Ω be

$$\left\{\omega : [0,\infty) \to \mathscr{N} \,\middle|\, \omega \text{ is right continuous, has a finite number of jumps in any interval } [0,t], t \geq 0\right\}.$$

Each $\omega \in \Omega$ describes a complete sample path of random indices. For $t \geq 0$, the random variable $\iota_t(\omega) = \omega(t)$ describes the state at t. To keep track of information, endow Ω with a filtration, i.e., a nondecreasing family $\{\mathscr{F}_t\}_{t\geq 0}$ of σ-fields, where $\mathscr{F}_t = \sigma((\iota_s)_{0\leq s\leq t})$ and $\mathscr{F} = \sigma\left(\cup_{t\geq 0}\mathscr{F}_t\right)$. $\mathscr{F}_t$ denotes the information from 0 up to t, including the number of jumps up to t, the timing, and the destinations of these jumps. The generator matrix Q satisfies the following conditions:

(i) $-q_{ii} > 0$ for all i; (For convenience, we use q_i to denote $-q_{ii}$)

(ii) $q_{ij} > 0$ for all $i \neq j$;

(iii) $\sum_j q_{ij} = 0$ for all i.

Each entry q_{ij} $(i \neq j)$ is the rate of moving from state i to state j, and $q_i = \sum_{j\neq i} q_{ij} > 0$ is the rate of leaving state i. For all $t, h \geq 0$, conditional on $\iota_t = i$,

$$\Pr(\iota_{t+h} = j | \iota_t = i) = \delta_{ij} + q_{ij}h + o(h),$$

where δ_{ij} is 1 if $i = j$, and 0 otherwise. An equivalent way to describe the continuous-time Markov chain is that, conditional on $\iota_t = i$, the holding time S (which records the duration that the chain stays in i before a transition) is an exponential random variable of parameter q_i, and once a transition occurs, it jumps to state j $(j \neq i)$ with probability q_{ij}/q_i. We can always endow the measurable space $(\Omega, \mathscr{F})$ with a probability measure P such that under P, $(\iota_t)_{t\geq 0}$ is a Markov chain with the generator matrix Q and the properties mentioned above (see, for example, Norris (1997, Chapter 2)).

2.2 The Contracting Problem

The agent knows her initial type and privately observes $(\iota_t)_{t\geq 0}$ afterwards, while the principal cannot observe the realizations and holds a belief that the agent's initial type is i with probability p_i, $1 \leq i \leq N$. At time 0, the principal offers a contract, which the agent may either accept or reject. If the agent rejects, then she gets the outside option $\bar{U}_i$ if her initial type is i. Otherwise, she sequentially reports the newly observed shocks to the principal, and the principal implements the contract based on reported histories.

In each period t, based on realized shocks $(\iota_s)_{0\leq s\leq t}$, the agent makes a report σ_t to the principal. Collectively, with sample path ω realized, the agent's reported history is $\sigma = (\sigma_t)_{t\geq 0}$. Therefore, we define the agent's reporting strategy to be a measurable function $\sigma : (\Omega, \mathscr{F}) \to (\Omega, \mathscr{F})$. Moreover, at time t, since an agent is unable to distinguish between two sample paths ω^1, ω^2, where $\omega^1(s) = \omega^2(s), \forall s \in [0, t]$, the reported paths have to satisfy $\sigma(\omega^1)(s) = \sigma(\omega^2)(s), \forall s \in [0, t]$. This means that σ should also be measurable from $(\Omega, \mathscr{F}_t)$ to $(\Omega, \mathscr{F}_t)$ for any $t \geq 0$. We could write $\sigma_t(\omega) = \sigma_t(\omega^{[0,t]})$, where $\omega^{[0,t]}$ is the restriction of ω on $[0, t]$. Notice that this definition of a reporting strategy implicitly imposes restrictions on the agent's reports; when the reports at different t are pieced together, the reported history $\sigma(\omega)$ has to be right continuous and admits finite jumps in finite time. This restriction is innocuous. Since the true sample paths have these properties, if the agent cheats and, for example, her reports fail to be right continuous, then cheating will be identified by the principal. Intuitively, right continuity prohibits the agent from immediately reverting to truth-telling after misreporting at t, and the finite-jump condition prohibits her from switching between truth-telling and cheating too often. A reporting strategy σ is *truth-telling* if $\sigma(\omega) = \omega$, for all $\omega \in \Omega$. We use σ^* to denote the truth-telling strategy.

The principal offers a contract $\mathscr{C} = (c_t, y_t)_{t\geq 0}$ at time 0, where the measurable functions $c_t : (\Omega, \mathscr{F}_t) \to [0, \bar{c}]$ and $y_t : (\Omega, \mathscr{F}_t) \to [0, \bar{y}]$ specify the agent's consumption and output at t, respectively. We use Ω to denote both the set of true realizations and the set of reports. As a collection of subsets that contain reports, the algebra $\mathscr{F}_t$ describes the principal's information structure; while as a collection of subsets that contain the true shocks, it describes the agent's information structure. When the reporting strategy is σ, the principal's information in the space of true shocks, $\sigma^{-1}(\mathscr{F}_t) = \{A \subseteq \Omega : \sigma(A) \in \mathscr{F}_t\}$, is weakly coarser than that of the agent. Here for the principal, Ω is the set of reports and the measurability condition requires that c_t and y_t be based only upon the reports available from 0 up to (and including) t. We assume that a contract is progressively measurable; in other words, the mapping $(s, \omega) \to (c_s(\omega), y_s(\omega)) : ([0, t] \times \Omega, \mathscr{B}([0, t]) \otimes \mathscr{F}_t) \to (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$ is measurable for all $t \geq 0$.[4] Note that the information structure here is similar to that in the discrete-time literature. Within a period

[4]This technical condition requires the joint measurability of the contract as a function of (s, ω) in the product space, which, together with the measurability of σ, guarantees that the following promised utilities are well-defined.

(or instant), the timing is that an agent first makes her report then receives her compensation. In both cases, consumption and output under the contract are allowed to jump instantaneously with the arrival of a new report, rather than being required to be predictable functions of past reports.

It is useful to study the agent's promised utility vector based on different histories, which turns out to be the state variable in a recursive formulation. Denote the set of possible histories before the realization of ι_t by

$$\mathscr{N}^{t-} = \left\{ \iota^{t-} : [0,t) \to \mathscr{N} \middle| \; \iota^{t-} \text{ is right continuous, has a finite number of jumps} \right\}.$$

An agent's future discounted utility when she has a history of reports $h^{t-} \in \mathscr{N}^{t-}$, her realization of ι_t is i and she follows a strategy σ is

$$\begin{aligned} & w_i(h^{t-};\sigma,\mathscr{C}) \\ = \; & E_t\left[\int_t^\infty e^{-r(s-t)}\left[u(c_s(h^{t-},(\sigma(\omega))^{[t,\infty)})) - v(y_s(h^{t-},(\sigma(\omega))^{[t,\infty)}))/\theta_{\omega(s)}\right] ds \middle| \omega(t) = i\right], \end{aligned}$$

where $(\sigma(\omega))^{[t,\infty)}$ is the restriction of the report $\sigma(\omega)$ on $[t,\infty)$, and $(h^{t-},(\sigma(\omega))^{[t,\infty)})$ denotes a sample path (or report) where h^{t-} is followed by $(\sigma(\omega))^{[t,\infty)}$. In particular, if $\sigma = \sigma^*$,

$$\begin{aligned} & w_i(h^{t-};\sigma^*,\mathscr{C}) \\ = \; & E_t\left[\int_t^\infty e^{-r(s-t)}\left[u(c_s(h^{t-},\omega^{[t,\infty)})) - v(y_s(h^{t-},\omega^{[t,\infty)}))/\theta_{\omega(s)}\right] ds \middle| \omega(t) = i\right] \\ = \; & E_t\left[\int_t^\infty e^{-r(s-t)}\left[u(c_s(h^{t-},\omega^{[t,s]})) - v(y_s(h^{t-},\omega^{[t,s]}))/\theta_{\omega(s)}\right] ds \middle| \omega(t) = i\right]. \end{aligned} \tag{3}$$

It will be crucial to distinguish between *persistent* promised utility and *transitional* promised utilities. The report at an instant t may either be the same as previous reports or indicate a transition. Accordingly, there are two types of promised utilities: the utility associated with no transition and the utilities associated with all possible transitions. More precisely, let $i_t^* = \lim_{s\uparrow t} h^{t-}(s)$ be the report of type immediately before t. Then $w_{i_t^*}(h^{t-};\sigma^*,\mathscr{C})$ is called the *persistent* promised utility and $w_i(h^{t-};\sigma^*,\mathscr{C})$, $(i \neq i_t^*)$, are called the *transitional* promised utilities. When the agent's current report is the same as the previous reports, she receives the persistent promised utility; otherwise, if there is a transition to state i, she receives $w_i(h^{t-};\sigma^*,\mathscr{C})$. Detailed discussion of the promised utilities is provided in the next section.

A contract $\mathscr{C}$ is said to be *incentive compatible* (I.C.) if for any t, any $h^{t-} \in \mathscr{N}^{t-}$, and any strategy σ,

$$w_i(h^{t-};\sigma^*,\mathscr{C}) \;\geq\; w_i(h^{t-};\sigma,\mathscr{C}), \text{ for all } i. \tag{4}$$

In this environment with commitment, it follows from the revelation principle that we could restrict attention to I.C. contracts. The principal's problem can then be written as

$$
\begin{aligned}
(5) \qquad & \min_{\mathscr{C}} \quad \sum_{i=1}^{N} p_i E\left[\int_0^\infty e^{-rt}\left[c_t(\omega)-y_t(\omega)\right]dt \middle| \omega(0)=i\right] \\
\text{subject to} \quad & \mathscr{C} \text{ is I.C.}, \\
& w_i(\emptyset;\sigma^*,\mathscr{C}) \geq \bar{U}_i, \text{ for all } i,
\end{aligned}
$$

where $\emptyset$ denotes an empty history and $\bar{U}_i$ is the type i agent's outside option.

3. Incentive Constraints and the Evolution of $(w_i(h^{t-}))_{1\leq i\leq N}$

In this section, we show that $(w_i(h^{t-};\sigma^*,\mathscr{C}))_{1\leq i\leq N}$ is the state variable for a dynamic programming problem, and that the incentive constraints can be simplified as differential equations (and inequalities) that describe the evolution of the state variable.

In the following discussion, the discounted utility $w_i(h^{t-};\sigma^*,\mathscr{C})$ will be simplified to $w_i(h^{t-})$ when σ^* and $\mathscr{C}$ are well understood. $i^{[t,s)}$ denotes a sample path (or report) of type i from t to s (not including s).

Fix an I.C. contract $\mathscr{C}$. First, consider the continuity property of $w_i(h^{t-})$ as a function of t. Recall that i_t^* denotes the report immediately before t. When limits are taken from the left, $w_{i_t^*}$ is the persistent promised utility and other w_i ($i \neq i_t^*$) are transitional promised utilities. Things become complicated when limits are taken from the right, since there are N possible paths of reports. Starting from t, the agent might report $i = i_t^*$, which the principal interprets as no transition, or she might report $i \neq i_t^*$, which the principal interprets as a transition to i. Following history $(h^{t-},(i_t^*)^{[t,s)})$, $w_{i_t^*}$ is still the persistent promised utility. However, following $(h^{t-},i^{[t,s)})$ ($i \neq i_t^*$), w_i would replace $w_{i_t^*}$ to be the persistent promised utility.

The following lemma shows that the persistent promised utility is continuous, while the transitional promised utilities have both left and right limits but allow for downward jumps.

Lemma 1 *If $\mathscr{C}$ is I.C., then*

(i) For right-hand limits,

$$
\begin{aligned}
(6) \qquad w_i(h^{t-}) &= \lim_{s\downarrow t} w_i(h^{t-},i^{[t,s)}), \text{ for all } i, \\
(7) \qquad w_j(h^{t-}) &\geq \lim_{s\downarrow t} w_j(h^{t-},i^{[t,s)}), \text{ for all } j \neq i.
\end{aligned}
$$

(ii) For left-hand limits, let h^{s-} denote the restriction of h^{t-} on $[0,s)$ $(s<t)$,

$$
\begin{aligned}
(8) \qquad w_{i_t^*}(h^{t-}) &= \lim_{s\uparrow t} w_{i_t^*}(h^{s-}), \\
(9) \qquad w_j(h^{t-}) &\leq \lim_{s\uparrow t} w_j(h^{s-}), \text{ for all } j \neq i_t^*.
\end{aligned}
$$

It is useful to understand the meaning of equation (7) with various values of i and j. If $j \neq i_t^*$ (she has a transition to j), then she could not gain by delaying the transition report and reporting i_t^* for a short time (followed by truth-telling), or by reporting a transition to a different state i ($i \neq j, i \neq i_t^*$) for a short time (followed by truth-telling). If $j = i_t^*$ (she does not have a transition), then she could not gain by reporting a transition to i ($i \neq i_t^*$) for a short time (followed by truth-telling).[5] Similarly, equation (9) simply means that an agent with a transition to j before t would not delay the transition report until t.

Next we will obtain a sharper characterization of the evolution of utility w_i along any reported history. Fix a report h^{t-}. It follows from the definition of $\mathscr{N}^{t-}$ that there exists a finite collection of jump times $t_0 = 0 < t_1 < t_2 < ... < t_n < t_{n+1} = t$ and a finite history of past types $(i_0, i_1, ..., i_n)$, such that $h^{t-}(s) = i_k$, if $s \in [t_k, t_{k+1})$, $0 \leq k \leq n$. In the interval $[t_k, t_{k+1})$, w_{i_k} is the persistent promised utility and, in addition to being continuous, its evolution is described by a differential equation. w_i ($i \neq i_k$) is a transitional promised utility in $[t_k, t_{k+1})$. Although it could have a countable number of downward jumps, its evolution is described by a differential inequality (an upper bound on dw_i/dt is found). These differential equations (and inequalities) are both necessary and sufficient conditions for a contract to be I.C. We turn next to this key characterization result.

THEOREM 1 *Let $\mathscr{C}$ be a contract, and $(w_i(h^{t-}))_{1\leq i\leq N}$, $(w_i(h^{t-}) : \mathscr{N}^{t-} \to \mathbb{R}$, for all $t \geq 0)$, be an arbitrary stochastic process.*

(i) (necessity) If $\mathscr{C}$ is I.C., and $(w_i(h^{t-}))_{1\leq i\leq N}$ are the promised utilities defined in (3), then for any history h^{t-} with the form $(t_0, t_1, ..., t_n, t_{n+1}; i_0, i_1, ..., i_n)$, $w_i(h^{s-})$ is differentiable for all i and almost every (a.e.) $s \in [0, t)$. If $i = i_k$, then for a.e. $s \in [t_k, t_{k+1})$,

$$(10) \qquad \frac{dw_i(h^{s-})}{ds} = (r + q_i)w_i(h^{s-}) - \sum_{j\neq i} q_{ij}w_j(h^{s-}) - (u(c(h^s)) - v(y(h^s))/\theta_i).$$

If $i \neq i_k$, then for a.e. $s \in [t_k, t_{k+1})$,

$$(11) \qquad \frac{dw_i(h^{s-})}{ds} \leq (r + q_i)w_i(h^{s-}) - \sum_{j\neq i} q_{ij}w_j(h^{s-}) - (u(c(h^s)) - v(y(h^s))/\theta_i).$$

(ii) (sufficiency) Assume $(w_i(h^{t-}))_{1\leq i\leq N}$ is a bounded process and for any history h^{t-} with the form $(t_0, t_1, ..., t_n, t_{n+1}; i_0, i_1, ..., i_n)$, $w_i(h^{s-})$ is differentiable for all i and a.e. $s \in [0, t)$. If $(w_i(h^{t-}))_{1\leq i\leq N}$ and $\mathscr{C}$ satisfy (10) and (11), then $(w_i(h^{t-}))_{1\leq i\leq N}$ satisfies (3), and contract $\mathscr{C}$ is I.C.

[5] Notice that, strictly speaking, a type j agent at time t may not be the same person as a type j agent at s, ($s > t$), but when explaining the intuition, we implicitly assume that the agent has no transition between t and s, since the probability of having a transition is small when s is close to t.

It is helpful to understand the meanings of (10) and (11), because all of our remaining results are derived from this system of differential equations and inequalities. Equation (10) is a *promise-keeping* condition. We can rewrite the right side of equation (10) as

$$\left[rw_i(h^{s-}) - (u(c(h^s)) - v(y(h^s))/\theta_i)\right] + \left[\sum_{j\neq i} q_{ij}(w_j(h^{s-}) - w_j(h^{s-}))\right],$$

where the first term is the natural (instantaneous) rate of change of promised utility $w_i(h^{s-})$ when there is no uncertainty. For $j \neq i$, each term $q_{ij}(w_i(h^{s-}) - w_j(h^{s-}))$ captures the additional rate of change of $w_i(h^{s-})$ due to the transition to state j at arrival rate q_{ij}. The promise-keeping condition in discrete time is

$$(12) \qquad w_i(s) \;=\; (u(c(s)) - v(y(s))/\theta_i)dt + e^{-rdt}\left[\sum_{j\neq i}(q_{ij}dt)w_j(s+dt) + e^{-q_i dt}w_i(s+dt)\right],$$

where s and $s+dt$ denote two periods in discrete time and $q_{ij}dt$ is the transitional probability in short time dt. Equation (10) can be informally derived by taking limit $dt \to 0$ in (12). (The formal proof can be found in APPENDIX A.) Inequality (11) is an *incentive compatibility* condition, the intuition for which is similar to that of (10). If (11) holds as equality, then type i obtains $w_i(h^{s-})$ by reporting i_k, thus she is indifferent between truth-telling and reporting i_k; otherwise, if $w_i(h^{s-})/ds$ is less than the right side of (11), then reporting i_k makes her strictly worse off.

With the conditions on the derivatives of $(w_i(h^{t-}))_{1\leq i\leq N}$, the principal's problem is transformed into a dynamic stochastic control problem. With truth-telling, $(w_i(\iota^{t-}))_{1\leq i\leq N}$ and ι_t are endogenous and exogenous state variables, respectively, and (c_t, y_t) are control variables. Given the current report i and before the next transition, the system evolves according to a differential inclusion (in the following discussion, $w_i(h^{t-})$, $c(h^t)$, and $y(h^t)$ will be simplified to $w_i(t)$ (or w_i), c_t, and y_t when h^{t-} and h^t are well understood):

$$\begin{aligned}\frac{dw_i}{dt} &= (q_i + r)w_i - \sum_{j\neq i} q_{ij}w_j - u(c_t) + v(y_t)/\theta_i,\\ \frac{dw_j}{dt} &\in \left(-\infty, (q_j + r)w_j - \sum_{k\neq j} q_{jk}w_k - u(c_t) + v(y_t)/\theta_j\right], j \neq i.\end{aligned}$$

Introducing $(N-1)$ slack control variables μ_j, $\mu_j \geq 0$, $j \neq i$, the system is

$$\begin{aligned}\frac{dw_i}{dt} &= (q_i + r)w_i - \sum_{j\neq i} q_{ij}w_j - u(c_t) + v(y_t)/\theta_i,\\ \frac{dw_j}{dt} &= (q_j + r)w_j - \sum_{k\neq j} q_{jk}w_k - u(c_t) + v(y_t)/\theta_j - \mu_j, j \neq i.\end{aligned}$$

When a downward jump of w_j happens, we interpret it as $\mu_j = \infty$. Given initial states i and $(w_j)_{1\leq j\leq N}$, if $(c_t(\omega^{[0,t]}), y_t(\omega^{[0,t]}), \mu_i(\omega^{[0,t]}))_{t\geq 0, 1\leq i\leq N}$ is the optimal policy for the stochastic control problem, then the cost $V_i((w_j)_{1\le j\le N})$ is

$$V_i((w_j)_{1\le j\le N}) = E\left[\int_0^\infty e^{-rt}\left[c_t(\omega^{[0,t]}) - y_t(\omega^{[0,t]})\right]dt\middle|\omega(0)=i\right].$$

$(V_i)_{1\le i\le N}$ can be directly used to solve the principal's problem in (5). If the prior belief is degenerate, ($p_i = 1, p_j = 0, \forall j \neq i$, i.e., initial type is known to the principal), then the principal can pick an initial state $(w_j)_{1\le j\le N}$ to start the optimal control problem; except the participation constraint $w_i \ge \bar{U}_i$, the other states $w_j (j \neq i)$ are transitional utilities, and are free to be chosen by the principal. When the prior belief is not degenerate, the principal can choose a type-dependent state variable to start the optimal control problem: when the initial report is i, the principal picks an initial state $(w^i_j)_{1\le j\le N}$. To prevent a type i agent from misreporting j and immediately reporting a transition to i, and obtaining the transitional promised utility w^j_i, the incentive constraints $w^i_i \ge w^j_i$ must be imposed. To summarize the above discussion, the principal's problem in (5) is equivalent to

$$\begin{aligned} \min_{(w^i_j)_{1\le i,j\le N}} \quad & \sum_{i=1}^N p_i V_i((w^i_j)_{1\le j\le N}) \\ \text{subject to} \quad & w^i_i \ge \bar{U}_i, \\ & w^i_i \ge w^j_i, \text{ for all } i, j. \end{aligned}$$

Remark 1 The conditions in (10) and (11) are still necessary when the instantaneous utility function is unbounded. However, for sufficiency, some form of transversality condition is needed. One sufficient condition is that, for any reporting strategy σ,

$$\lim_{t\to\infty} e^{-rt} E\left[w_{\iota_t}((\sigma(\omega))^{[0,t)})\right] \quad = \quad 0.$$

4. The Set of Implementable Utilities

To simplify the exposition, in the remainder of the paper we will consider only the case in which $N = 2$. This section studies the set of implementable utilities, defined as,

$$W = \left\{(w_1(\emptyset;\sigma^*,\mathscr{C}), w_2(\emptyset;\sigma^*,\mathscr{C})) \in \mathbb{R}^2 : \mathscr{C} \text{ is I.C. }\right\},$$

which is the domain of the value functions. In the next section, we examine the properties of the value functions and the long-run dynamics implied by them.[6]

[6] Both of these sections use the differential equations we developed earlier heavily; however, there is not much dependence between them, and either one could be read first. In the next section, we focus mainly on unbounded utility and disutility functions, and W with unbounded utilities is much easier to obtain than with bounded utilities (in the next section, W is the whole set $\mathbb{R}^2$), as explained in the following Remark 2. The reader interested in the dynamics of the optimal contract could skip most of this section, and proceed to Remark 2 and the next section.

The common approach in the literature is to compute this set by iteration. Following Abreu, Pearce, and Staccheti (1990), begin with an initial guess that contains W, then iterate until the sequence of sets converges to W, which is the largest fixed point of the operator. However, using continuous-time methods, we will show that this set can be obtained directly. In fact the boundary of W can be characterized by differential equations. The remainder of this section will be devoted to this characterization.

We first study some simple contracts. If the contract always specifies maximal consumption $\bar{c}$ and minimal output 0, regardless of reports (i.e., $c_t(h^t) = \bar{c}, y_t(h^t) = 0, \forall h^t \in \mathscr{N}^t$), then the contract can implement the pair $(\bar{u}/r, \bar{u}/r)$, which is the upper-right corner of W. If consumption 0 and output $\bar{y}$ are always specified, then the lower-left corner is implemented. We denote it by $-(x_1\bar{v}, x_2\bar{v})$, where $x_1 = ((q_2+r)/\theta_1 + q_1/\theta_2)/(r(q_1+q_2+r)), x_2 = (q_2/\theta_1 + (q_1+r)/\theta_2)/(r(q_1+q_2+r))$. Similarly, the "consumption 0, output 0" contract implements the utility pair $(0,0)$, while the "consumption $\bar{c}$, output $\bar{y}$" contract implements $(-x_1\bar{v} + \bar{u}/r, -x_2\bar{v} + \bar{u}/r)$.

Next consider four families of contracts. The first two families are indexed by $c^* \in (0, \bar{c})$. A contract $\mathscr{C}^{1c^*}$ in the first family is $(c_t^{1c^*}(h^t), y_t^{1c^*}(h^t)) = (c^*, 0)$, for all $h^t \in \mathscr{N}^t$, which implements the utility pair

$$(u(c^*)/r, u(c^*)/r), c^* \in (0, \bar{c}). \tag{13}$$

A contract $\mathscr{C}^{2c^*}$ in the second family is $(c_t^{2c^*}(h^t), y_t^{2c^*}(h^t)) = (c^*, \bar{y})$, for all $h^t \in \mathscr{N}^t$, which implements the utility pair

$$(-x_1\bar{v} + u(c^*)/r, -x_2\bar{v} + u(c^*)/r), c^* \in (0, \bar{c}). \tag{14}$$

The third and fourth families of contracts are indexed by $t^* \in (0, \infty)$. A contract $\mathscr{C}^{3t^*}$ in the third family is

$$(c_t^{3t^*}(h^t), y_t^{3t^*}(h^t)) = \begin{cases} (0,0), t \le t^* \\ (0, \bar{y}), t > t^*, \end{cases}$$

while in the fourth family, a contract $\mathscr{C}^{4t^*}$ is

$$(c_t^{4t^*}(h^t), y_t^{4t^*}(h^t)) = \begin{cases} (\bar{c}, \bar{y}), t \le t^* \\ (\bar{c}, 0), t > t^*. \end{cases}$$

The utility pair $(w_1^{3t^*}, w_2^{3t^*})$ implemented by $\mathscr{C}^{3t^*}$ can be solved in the following way. Under contract $\mathscr{C}^{3t^*}$, and when $t \le t^*$, the promised utility evolves according to the differential equation system

$$\frac{dw_1}{dt} = (q_1 + r)w_1 - q_1 w_2$$
$$\frac{dw_2}{dt} = (q_2 + r)w_2 - q_2 w_1,$$

and (w_1, w_2) will hit $(-x_1\bar{v}, -x_2\bar{v})$ at time t^*. Therefore, solving the differential equations together with the boundary condition yields

$$w_1^{3t^*} = -\bar{v}\left[\frac{q_1(1/\theta_1 - 1/\theta_2)}{(q_1+q_2)(q_1+q_2+r)}e^{-(q_1+q_2+r)t^*} + \frac{q_2/\theta_1 + q_1/\theta_2}{(q_1+q_2)r}e^{-rt^*}\right], \tag{15}$$

$$w_2^{3t^*} = -\bar{v}\left[\frac{q_2(1/\theta_2 - 1/\theta_1)}{(q_1+q_2)(q_1+q_2+r)}e^{-(q_1+q_2+r)t^*} + \frac{q_2/\theta_1 + q_1/\theta_2}{(q_1+q_2)r}e^{-rt^*}\right]. \tag{16}$$

Similarly,

$$w_1^{4t^*} = \bar{v}\left[\frac{q_1(1/\theta_1 - 1/\theta_2)}{(q_1+q_2)(q_1+q_2+r)}e^{-(q_1+q_2+r)t^*} + \frac{q_2/\theta_1 + q_1/\theta_2}{(q_1+q_2)r}e^{-rt^*}\right] - x_1\bar{v} + \bar{u}/r, \tag{17}$$

$$w_2^{4t^*} = \bar{v}\left[\frac{q_2(1/\theta_2 - 1/\theta_1)}{(q_1+q_2)(q_1+q_2+r)}e^{-(q_1+q_2+r)t^*} + \frac{q_2/\theta_1 + q_1/\theta_2}{(q_1+q_2)r}e^{-rt^*}\right] - x_2\bar{v} + \bar{u}/r. \tag{18}$$

It turns out that the utility pairs delivered by $(\mathscr{C}^{1c^*}, \mathscr{C}^{2c^*}, \mathscr{C}^{3t^*}, \mathscr{C}^{4t^*})$ form the boundary of W (see Figure 1).

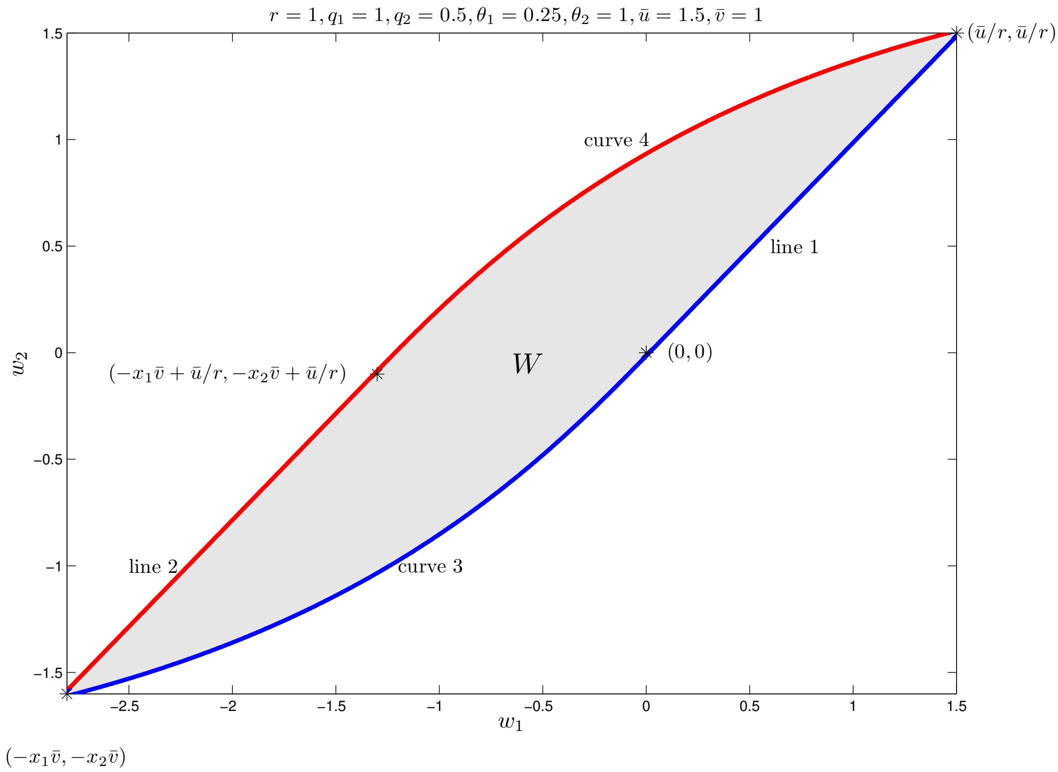

Figure 1: The set of implementable utility pairs.

Theorem 2 *The boundaries of W consist of the four points $(\bar{u}/r, \bar{u}/r)$, $(-x_1\bar{v}, -x_2\bar{v})$, $(0,0)$, and $(-x_1\bar{v} + \bar{u}/r, -x_2\bar{v} + \bar{u}/r)$, and the four pieces of curves that connect these points. The lower boundary is specified in equations (13), (15), and (16), while the upper boundary is specified in equations (14), (17), and (18).*

It is intuitive that curves in (14), (17), and (18) specify the upper boundary. Since for a fixed w_1, in order to increase w_2, the principal could increase consumption and output in a way that makes the low-productivity agent indifferent but the high-productivity agent better off, it is not surprising to see that, at curves on the upper boundary, output is maintained at the maximum level $\bar{y}$. Similarly, output is at the minimum level 0 on the lower boundary.

Remark 2 When utility is unbounded, it is typically easier to determine W. For example, when utility and disutility can take any real number, consider the no-information contract, where constant c^* and y^* are specified regardless of reports. This is trivially I.C. and delivers utility

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1/r & -x_1 \\ 1/r & -x_2 \end{pmatrix} \begin{pmatrix} u(c^*) \\ v(y^*) \end{pmatrix},$$

$$\text{where} \quad x_1 = \frac{(q_2+r)/\theta_1 + q_1/\theta_2}{r(q_1+q_2+r)}, x_2 = \frac{q_2/\theta_1 + (q_1+r)/\theta_2}{r(q_1+q_2+r)}.$$

Since the matrix has full rank, any $(w_1, w_2) \in \mathbb{R}^2$ can be implemented by choosing the appropriate $u(c^*)$ and $v(y^*)$. Therefore, $W = \mathbb{R}^2$ in this case.

5. Dynamics of the Optimal Contract

In previous sections, we simplified the incentive constraints and used them to characterize the set of implementable utility pairs. In this section, we address the question of the dynamic behavior of the state variables under the optimal contract. To simplify the analysis, we focus on a special case where the shock space consists of two elements, $\Theta = \{\theta_1, \theta_2\}, \theta_1 < \theta_2$, and the agent has logarithmic utility and disutility functions, $u(c) = \log(c), c > 0$, and $v(y) = -\log(-y), y < 0$. We assume that the value functions V_1 and V_2 in this case are *twice* differentiable and an optimal contract always exists for any utility pair (w_1, w_2). We use $V_{i,1}, V_{i,2}, V_{i,11}, V_{i,12}, V_{i,22}$ to denote $\frac{\partial V_i}{\partial w_1}, \frac{\partial V_i}{\partial w_2}, \frac{\partial^2 V_i}{\partial w_1 \partial w_1}, \frac{\partial^2 V_i}{\partial w_1 \partial w_2}, \frac{\partial^2 V_i}{\partial w_2 \partial w_2}$, respectively, for $i = 1, 2$.

It is helpful to first preview this lengthy section. We find two parallel efficiency lines, which separate the state space $\mathbb{R}^2$ into three regions. We then show that the bottom region is absorbing. With a high-productivity report, the state variable moves upward and along the efficiency line, while with a low-productivity report, the state variable moves downward and enters the interior of the region. The dynamics of the system are described by an ordinary differential equation (ODE) system, and by studying the system, we derive many sample path properties, which are summarized in Theorem 3.

Consider first the homogeneity of the value functions. It follows from $\log(\exp(\lambda)c) = \lambda + \log(c)$ that a contract $\mathscr{C} = (c_t, y_t)_{t \geq 0}$ implements (w_1, w_2), if and only if $(\exp(\lambda)c_t, \exp(\lambda)y_t)_{t \geq 0}$ implements

$$(w_1 + (1/r + x_1)\lambda, w_2 + (1/r + x_2)\lambda), \text{ for all } \lambda \in \mathbb{R},$$

where $x_1 = ((q_2+r)/\theta_1 + q_1/\theta_2)/(r(q_1+q_2+r)), x_2 = (q_2/\theta_1 + (q_1+r)/\theta_2)/(r(q_1+q_2+r))$. Therefore, $(c_t^*, y_t^*)_{t\geq 0}$ is the optimal contract to implement (w_1, w_2) if and only if $(\exp(\lambda)c_t^*, \exp(\lambda)y_t^*)_{t\geq 0}$ is the optimal contract at $(w_1 + (1/r + x_1)\lambda, w_2 + (1/r + x_2)\lambda)$. Moreover, the speed vectors of the time paths starting from these two initial conditions are identical, i.e., for $(w_1', w_2') = (w_1 + (1/r + x_1)\lambda, w_2 + (1/r + x_2)\lambda)$,

$$\frac{dw_1(i^{[0,t)})}{dt} = \frac{dw_1'(i^{[0,t)})}{dt}, \quad \frac{dw_2(i^{[0,t)})}{dt} = \frac{dw_2'(i^{[0,t)})}{dt}, \quad i = 1, 2. \tag{19}$$

The next lemma states this homogeneity and uses it to establish other elementary properties of V_1 and V_2.

LEMMA 2 *The value functions V_1, V_2 have the following properties:*

(i) (Homogeneity) For any $\lambda \in \mathbb{R}$,

$$V_i(w_1 + (1/r + x_1)\lambda, w_2 + (1/r + x_2)\lambda) = \exp(\lambda)V_i(w_1, w_2), i = 1, 2; \tag{20}$$

(ii) V_1 and V_2 are weakly convex;

(iii) (Monotonicity) $V_{1,2} \leq 0, V_{1,1} > 0, V_{2,1} \leq 0, V_{2,2} > 0$.

Part (iii) of the above lemma states that value functions are monotonic, increasing in the persistent promised utility but decreasing in the transitional utility. The transitional utility is used as a threat: it is what a cheater can hope to get if she pretends to be the reported type but immediately reports a transition to her true type afterwards. For this reason, the transitional utility is also called the *threat* utility. The monotonicity in the persistent promised utility is straightforward because the principal needs to give more consumption to (and require less output from) the agent if he promises more expected utility to her. The intuition for $V_{1,2} \leq 0$ is as follows. The principal can instantaneously and freely lower the transitional utility and keep a tighter threat at zero cost; however, once the transitional utility is moved to a lower level, it is not I.C. to jump back immediately. Thus the cost function cannot be increasing in the transitional utility. Next we will show that it is strictly decreasing when the transitional utility is sufficiently low, i.e., for a w_1, $V_{1,2} < 0$ if w_2 is sufficiently low.

LEMMA 3 *For all $w_1 \in \mathbb{R}$, $\{w_2 \in \mathbb{R} : V_{1,2}(w_1, w_2) < 0\} \neq \emptyset$, $\{w_2 \in \mathbb{R} : V_{1,2}(w_1, w_2) = 0\} \neq \emptyset$. And for all $w_2 \in \mathbb{R}$, $\{w_1 \in \mathbb{R} : V_{2,1}(w_1, w_2) = 0\} \neq \emptyset$, $\{w_1 \in \mathbb{R} : V_{2,1}(w_1, w_2) < 0\} \neq \emptyset$.*

Since $V_{1,2} \leq 0$, V_1, as a function of w_2, could be strictly decreasing forever or be flat forever. The above lemma rules out these two possibilities and, together with the convexity of V_1, it implies that there is an intermediate level of w_2, below which the value function V_1 is strictly decreasing and above which it is flat. Formally, we can define two curves, f_1 and f_2 :

$$f_1(w_1) = \min\{w_2 \in \mathbb{R} : V_{1,2}(w_1, w_2) = 0\},$$
$$f_2(w_2) = \min\{w_1 \in \mathbb{R} : V_{2,1}(w_1, w_2) = 0\}.$$

Since $V_{1,2}(w_1, w_2) = 0$ if and only if $V_{1,2}(w_1 + (1/r + x_1)\lambda, w_2 + (1/r + x_2)\lambda) = 0$, it follows that f_1 and f_2 are parallel straight lines with slope $(1/r + x_2)/(1/r + x_1)$. We call f_1 and f_2 the *efficiency* lines, because for each level of promised utility, they indicate the optimal level of transitional utility to minimize the cost. For example, if initially the principal holds the belief $p_1 = 1$ and he wants to deliver utility w_1 to the agent, then he needs to choose a transitional promised utility as one of the initial state variables. The optimal level to choose is $w_2 = f_1(w_1)$.

These lines are also critical for the study of the dynamics. Starting from above f_1, the state variable will jump downward to the efficiency line f_1 with a report 1, and starting from below f_2, the state variable will jump leftward to f_2 with a report 2. We further show that, although the transitional utility might make downward jumps contingent on the report of a transition, it never jumps when the report remains unchanged. When it jumps with a report of transition, it always jumps onto the efficiency lines and stays below them until another transition occurs. The fact that the time path is smooth when there is no transition is intuitive; since the utility function is strictly concave, making abrupt changes in transitional utilities without arrival of new information increases the cost for the principal. The properties of the time path of the state variable are summarized in the following lemma.

Lemma 4 *For a reported history h^{t-}, if $w_2(h^{t-}) > f_1(w_1(h^{t-}))$,*

$$\lim_{s \downarrow t} w_2(h^{t-}, 1^{[t,s)}) = f_1(w_1(h^{t-})).$$

Furthermore, $w_2(h^{t-}, 1^{[t,s)})$ is a continuous function of $s \in (t, \infty)$, so that both w_1 and w_2 are continuous following the path of $(h^{t-}, 1^{[t,s)})$. Similarly, if $w_1(h^{t-}) > f_2(w_2(h^{t-}))$,

$$\lim_{s \downarrow t} w_1(h^{t-}, 2^{[t,s)}) = f_2(w_2(h^{t-})).$$

The definition of efficiency lines and the immediate jumps make it clear that

$$\begin{aligned} V_1(w_1, w_2) &= V_1(w_1, \min(w_2, f_1(w_1))), \\ V_2(w_1, w_2) &= V_2(\min(w_1, f_2(w_2)), w_2). \end{aligned}$$

In the region below the efficiency lines, no further jumps occur in the state variable. Since the evolution of the state variable is controlled by differential equations in this region, one can lay out the *Hamilton–Jacobi–Bellman (HJB)* equations that the value functions satisfy.[7] For any (w_1, w_2) with $w_2 \leq f_1(w_1)$, V_1 satisfies

$$\begin{aligned} (q_1 + r)V_1(w_1, w_2) &= \min_c\{c - (V_{1,1} + V_{1,2})u(c)\} + \min_y\{-y + (V_{1,1}/\theta_1 + V_{1,2}/\theta_2)v(y)\} \\ &\quad + q_1 V_2(w_1, w_2) + V_{1,1}((q_1 + r)w_1 - q_1 w_2) \\ &\quad + \min_{\mu_2 \geq 0}\{V_{1,2}((q_2 + r)w_2 - q_2 w_1 - \mu_2)\}. \end{aligned} \tag{21}$$

[7]See, for example, Fleming and Soner (2006, equation (7.13), p.134), or see Wälde (2006) for a general reading on continuous-time methods in economics.

Similarly, for (w_1, w_2) with $w_1 \leq f_2(w_2)$,

$$\begin{aligned}(q_2+r)V_2(w_1,w_2) &= \min_c\{c-(V_{2,1}+V_{2,2})u(c)\}+\min_y\{-y+(V_{2,1}/\theta_1+V_{2,2}/\theta_2)v(y)\}\\ &\quad + q_2V_1(w_1,w_2)+\min_{\mu_1\geq 0}\{V_{2,1}((q_1+r)w_1-q_1w_2-\mu_1)\}\\ &\quad + V_{2,2}((q_2+r)w_2-q_2w_1).\end{aligned}$$

In addition, notice that μ_2 (μ_1) can be non-zero only if $V_{1,2}=0$ ($V_{2,1}=0$). Therefore, if $w_2 < f_1(w_1)$, we can rewrite the HJB equation as

$$\begin{aligned}(q_1+r)V_1(w_1,w_2) &= \min_c\{c-(V_{1,1}+V_{1,2})u(c)\}+\min_y\{-y+(V_{1,1}/\theta_1+V_{1,2}/\theta_2)v(y)\}\\ &\quad + q_1V_2(w_1,w_2)+V_{1,1}((q_1+r)w_1-q_1w_2)\\ &\quad + V_{1,2}((q_2+r)w_2-q_2w_1).\end{aligned}$$

Totally differentiating (21) with respect to w_1 and applying the envelop theorem yield

$$\begin{aligned}(q_1+r)V_{1,1} &= -(V_{1,11}+V_{1,12})u(c)+(V_{1,11}/\theta_1+V_{1,12}/\theta_2)v(y)\\ &\quad +q_1V_{2,1}+V_{1,1}(q_1+r)+V_{1,11}((q_1+r)w_1-q_1w_2)\\ &\quad -V_{1,2}q_2+\min_{\mu_2\geq 0}\{V_{1,12}((q_2+r)w_2-q_2w_1-\mu_2),\end{aligned}$$

which is simplified as

$$\begin{aligned}0 &= V_{1,11}((q_1+r)w_1-q_1w_2-u(c)+v(y)/\theta_1)\\ &\quad + V_{1,12}((q_2+r)w_2-q_2w_1-u(c)+v(y)/\theta_2-\mu_2)+q_1V_{2,1}-q_2V_{1,2}.\end{aligned}$$

Using $dw_1/dt = ((q_1+r)w_1-q_1w_2-u(c)+v(y)/\theta_1)$, and $dw_2/dt = ((q_2+r)w_2-q_2w_1-u(c)+v(y)/\theta_2-\mu_2)$, we get

$$\frac{dV_{1,1}}{dt} = V_{1,11}\frac{dw_1}{dt}+V_{1,12}\frac{dw_2}{dt} = q_2V_{1,2}-q_1V_{2,1}.$$

Similarly, totally differentiating (21) with respect to w_2 yields

$$\frac{dV_{1,2}}{dt} = q_1V_{1,1}-q_1V_{2,2}+(q_1-q_2)V_{1,2}.$$

The above equations together with $dw_1/dt = ((q_1+r)w_1-q_1w_2-u(c)+v(y)/\theta_1)$, $dw_2/dt = ((q_2+r)w_2-q_2w_1-u(c)+v(y)/\theta_2-\mu_2)$ constitute an ODE system to describe the dynamics with report 1:

$$\frac{dw_1}{dt} = (q_1+r)w_1-q_1w_2-u(c)+v(y)/\theta_1 \tag{22}$$

$$\frac{dw_2}{dt} = (q_2+r)w_2-q_2w_1-u(c)+v(y)/\theta_2-\mu_2 \tag{23}$$

$$\frac{dV_{1,1}}{dt} = q_2V_{1,2}-q_1V_{2,1} \tag{24}$$

$$\frac{dV_{1,2}}{dt} = q_1V_{1,1}-q_1V_{2,2}+(q_1-q_2)V_{1,2}, \tag{25}$$

$$\text{where} \quad c = (V_{1,1}+V_{1,2}), y = -(V_{1,1}/\theta_1+V_{1,2}/\theta_2). \tag{26}$$

Notice that (26) comes from the minimization problems $\min_c\{c - (V_{1,1} + V_{1,2})u(c)\}$ and $\min_y\{-y + (V_{1,1}/\theta_1 + V_{1,2}/\theta_2)v(y)\}$ and the assumption of logarithmic utility and disutility functions. Similarly, the ODE system with report 2 is

$$\frac{dw_1}{dt} = (q_1 + r)w_1 - q_1 w_2 - u(c) + v(y)/\theta_1 - \mu_1 \tag{27}$$

$$\frac{dw_2}{dt} = (q_2 + r)w_2 - q_2 w_1 - u(c) + v(y)/\theta_2 \tag{28}$$

$$\frac{dV_{2,1}}{dt} = q_2 V_{2,2} - q_2 V_{1,1} + (q_2 - q_1)V_{2,1} \tag{29}$$

$$\frac{dV_{2,2}}{dt} = q_1 V_{2,1} - q_2 V_{1,2}, \tag{30}$$

$$\text{where} \quad c = (V_{2,1} + V_{2,2}), y = -(V_{2,1}/\theta_1 + V_{2,2}/\theta_2). \tag{31}$$

Below f_1 (or above f_2), the slack control variable μ_2 (or μ_1) would be 0; i.e., $V_{1,2} < 0$ implies $\mu_2 = 0$, and $V_{2,1} < 0$ implies $\mu_1 = 0$.

By studying the evolution of partial derivatives of the value functions (we may call them the shadow prices of promised utilities), these two ODE systems provide plenty of information about the dynamics of the state variable. In the next step, we use them to show that the two efficiency lines do not coincide. In fact, line f_1 is above f_2, and they split the state space into three regions.

LEMMA 5 *Line f_1 is strictly above f_2, i.e., $f_1(w_1) > (f_2)^{-1}(w_1)$.*

The following lemma characterizes how the state variable evolves with the two reports.

LEMMA 6 *With report 2, the time path starting from f_2 will remain on f_2 and move toward (∞, ∞) (see Figure 3). With report 1, the time path starting from f_1 will move below f_1 (see Figure 2). If the Markov chain is symmetric ($q_1 = q_2$), then the region below f_2 is absorbing. More precisely, starting from f_2 and with report 1, (w_1, w_2) enters the interior of the region.*

Intuitively, as long as the agent claims that her productivity is low, the contract specifies a low level of output, but in order to prevent a high-productivity agent from lying, the contract necessarily lowers the utility of a potential liar. On one hand, this keeps incentive compatibility; on the other hand, maintaining a low threat moves w_2 below the efficient level $f_1(w_1)$.[8]

In the absorbing region below f_2, the dynamics could be summarized as a *clockwise triangle.* With a high report, the state variable moves up along the efficiency line f_2. It starts moving below the line

[8] The dynamics of the time path with a low report also imply that the optimal contract with commitment is no longer renegotiation-proof in the environment with persistent shocks. To see this, suppose the contract starts from the line f_1, and the agent experiences a period of low shocks, then the time path moves below f_1, which is $w_2 < f_1(w_1)$ and $V_1(w_1, w_2) > V_1(w_1, f_1(w_1))$. Should the principal have the chance to renegotiate with the agent, he would be willing to move the state from (w_1, w_2) to $(w_1, f_1(w_1))$; doing this makes the agent indifferent and lowers the principal's cost. However it violates the ex ante incentive constraints that prevent the high-productivity agent from lying.

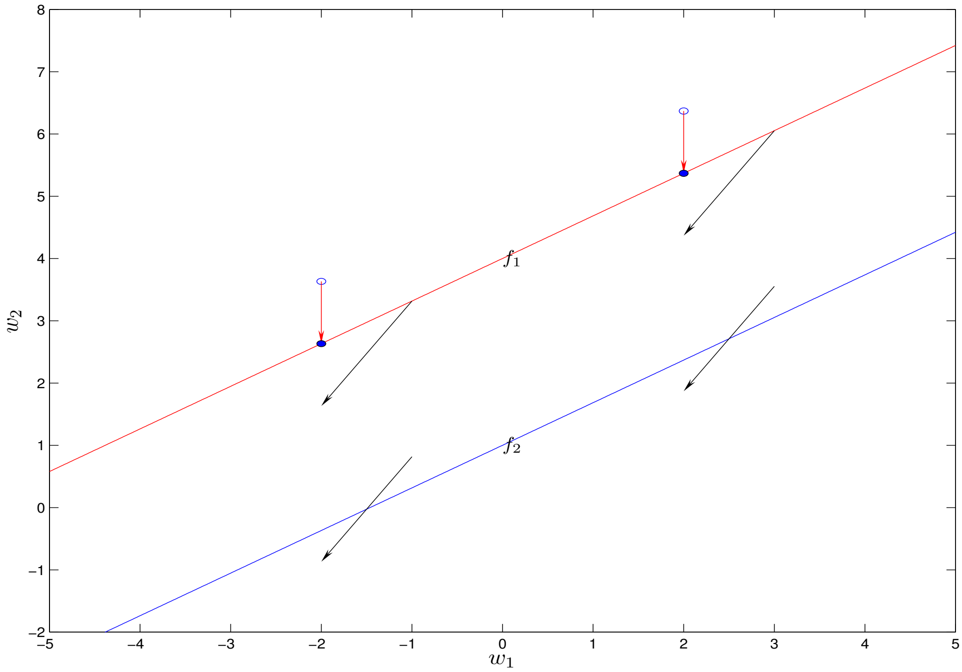

Figure 2: The dynamics with report 1.

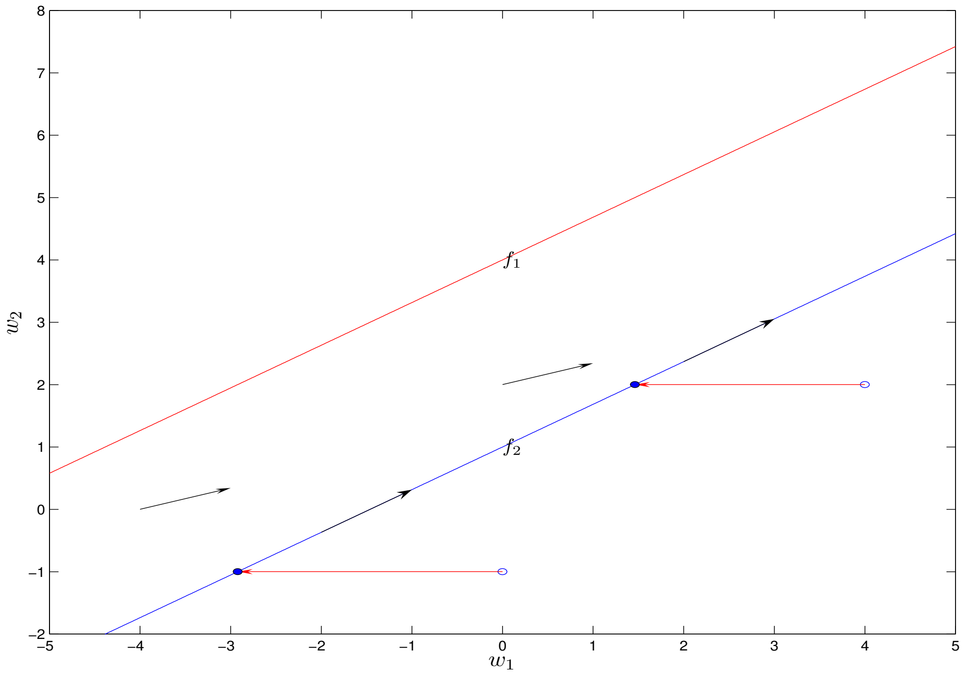

Figure 3: The dynamics with report 2.

when a low shock arrives. It will keep moving down toward $(-\infty, -\infty)$, until another high shock arrives, which moves the state back to the efficiency line f_2 by an immediate jump to the left. Then the state variable moves up again and starts the next triangle. The next main theorem focuses on the optimal policies along the time paths in the absorbing region.

THEOREM 3 *If the Markov chain is symmetric,*[9] *i.e.,* $q_1 = q_2 = q$*, then in the region below* f_2*, the following properties hold.*

(i) The dynamics with a report 1 *is described by an ODE system.*

$$\frac{dw_1}{dt} = (q+r)w_1 - qw_2 - u(c) + v(y)/\theta_1 \tag{32}$$

$$\frac{dw_2}{dt} = (q+r)w_2 - qw_1 - u(c) + v(y)/\theta_2 \tag{33}$$

$$\frac{dV_{1,1}}{dt} = qV_{1,2} \tag{34}$$

$$\frac{dV_{1,2}}{dt} = qV_{1,1} - qV_{2,2}(f_2(w_2), w_2), \tag{35}$$

$$\text{where} \quad c = (V_{1,1} + V_{1,2}), y = -(V_{1,1}/\theta_1 + V_{1,2}/\theta_2). \tag{36}$$

(ii) $V_1(w_1, w_2) \geq V_2(w_1, w_2)$.

(iii) (Monotonicity of the policy functions and promised utilities) Following a report 1*, both the persistent and the transitional promised utilities fall, consumption falls, and output increases. The opposite happens when following a report* 2.

(iv) At a transition from 2 *to* 1*, consumption jumps downward and output jumps downward. The opposite happens at a transition from* 1 *to* 2.

(v) With a report 1*, the agent's consumption-leisure decision is distorted (regardless of her previous reports), and the distortion increases with the duration of report* 1*. With a report* 2*, there is no consumption-leisure distortion.*

(vi) (Immiserization) The inverse Euler equation holds, i.e., $1/u'(c(\iota^t)) = c(\iota^t)$ *is a martingale. Immiserization still holds; consumption converges to its lower bound, and output converges to its upper bound almost surely (a.s.).*

The implications that we derive are similar to the i.i.d. case. The low-productivity agent receives subsidy, and her future utility moves downward; while the high-productivity agent pays tax and she is promised to be treated better in the future. The principal distorts the consumption-leisure decision of a low-productivity agent to obtain better incentives, since in the system, a high-productivity agent has

[9] If the chain is asymmetric, then we cannot prove that the bottom region is absorbing. Little is known about the dynamics of the contract in this case.

an incentive to misreport. The optimal system makes the high-productivity agent indifferent between truth-telling and cheating, because in equation (33), μ_2 is 0.

Most of these findings are consistent with those in Williams (2008). In his private and persistent income model, a positive innovation in the reported endowment leads to an increase in the promised utility and vice versa. However, because the inverse Euler equation is no longer valid in Williams (2008), the immiserization does not hold and consumption has a positive drift and increasing variability.

Remark 3 It is interesting to compare the results of this paper to those of Battaglini and Coate (2003). Our continuous-time method can be used to study the risk-neutral utility function as well. With $u(c) = c$, the value function satisfies a different type of homogeneity, i.e., $V_i(w_1 + \lambda, w_2 + \lambda) = V_i(w_1, w_2) + \lambda$, which implies that $V_{i,1} + V_{i,2} = 1$, for $i = 1, 2$. A key feature in their model is that for certain pairs of persistent and transitional promised utilities, the full information contract is implementable. This means that there are two 45-degree lines similar to f_1 and f_2 (we can call them g_1 and g_2, and g_1 is above g_2) that split the state space into three regions. In the region between the two lines, the full information contract is implementable. However, in the region below g_2, for a level w_1 promised to the low-productivity type, the transitional promised utility is forced to be below the efficiency level ($V_{1,2} < 0$) to prevent the high-productivity agent from misreporting. Battaglini and Coate (2003) studied the optimal contract starting from this region. They showed that once the low-productivity agent has a transition, the contract becomes efficient by jumping leftward to line g_2. Even with report 1, the state variable will eventually approach the efficiency line, implying the consumption-leisure distortion disappears in the long run. These implications can be easily derived with continuous-time methods. In the bottom region, $V_{1,1} > 1$, $V_{1,2} < 0$. Equation (24) implies that $V_{1,1}$ is decreasing. Since homogeneity implies that $V_{1,1} \geq 1$ in this case, it has to be that $\lim_{t\to\infty} V_{1,1}(t) = 1$, which implies that the time path approaches g_2. These patterns are in sharp contrast with our model with risk-averse utility functions. In Battaglini and Coate (2003), the efficiency line g_2 is absorbing, while in Lemma 6, we show that with report 1, the state leaves the efficiency line f_2 and moves farther below it. This difference generates different implications for distortions. While they showed that the distortion is eliminated permanently after the agent's first report of type 2 and decreases even when the agent always reports 1, in our model, the distortion always exists with report 1 and increases with the duration of the report. Battaglini and Coate (2003) showed that the distortion vanishes is robust to the introduction of small amounts of risk aversion; however, we have shown that this conclusion is reversed when the risk aversion is large enough (in our model, we choose the log function as both utility and disutility functions, implying that the utilities in consumption and leisure have similar risk aversions). Risk aversion seems to be critical for the pattern of distortions, but exploring the correlation of these two is beyond the scope of this paper and left for future research.

The monotonicity and immiserization results of the optimal contract are similar to an i.i.d. shock model. This suggests that, qualitatively, incentive constraints work similarly in these two models. The difference lies in the quantitative effects of persistence. To study these effects concretely, we turn to a numerical example.

6. A Numerical Example

In this section, we numerically solve the model with hidden productivity shocks. First we choose the parameters so that they match observed empirical facts. Then we artificially decrease the persistence of the productivity process (by increasing the value of q) to approach the i.i.d. shocks, and keep all the other elements of the model fixed. We shall make comparisons between the implications of the persistent shock model and the i.i.d. shock model.

We assume that the agent's preferences are

$$E\left[\int_0^\infty e^{-rt}\left(\frac{c_t^{1-\sigma}}{1-\sigma} - \kappa\frac{y_t^{1+\gamma}}{1+\gamma}/\theta_t\right)dt\right]. \tag{37}$$

We set r to 0.0408 to match an annual discount factor of 0.96. We follow Albanesi and Sleet (2006) in setting σ and κ to be 1.461 and 1.1840 respectively, and follow Chari, Kehoe, and McGrattan (1998) in setting γ to be 2. This implies that the elasticity of the labor supply is 0.5. We choose parameter values for θ_1, θ_2, and q to match the unconditional mean, unconditional variance, and the covariance (between period t and $t+1$) of the skill process described in Golosov and Tsyvinski (2007). This implies values of 0.2652, 7.4094, and 0.0249 for θ_1, θ_2, and q, respectively. The productivity process is highly persistent, which is the driving force of the pattern of wedges shown below. Notice that by i.i.d. shock model, we specifically mean a discrete-time model with independent shocks in which one period corresponds to one unit of time in continuous time (i.e., the discount factor β is e^{-r}). The i.i.d. shock model will match our continuous-time model when $q = 0.5$, because then the average holding times (of a productivity state) will be equal in the two models.

6.1 The Wedges

We first define three wedges discussed in Albanesi and Sleet (2006). For a given reported history h^{t-}, if the agent makes a report of type i at time t, then denote consumption by $c_t(h^{t-}, i)$ and output by $y_t(h^{t-}, i)$.

(i) The *insurance wedge* $\frac{u'(c_t(h^{t-},1))}{u'(c_t(h^{t-},2))} - 1$ measures the consumption smoothing implied by the optimal contract.

(ii) The *consumption-leisure wedge* $\frac{u'(c_t(h^{t-},i))}{v'(y_t(h^{t-},i))/\theta_i} - 1$, $i = 1, 2$, measures the ratio of marginal utility to marginal disutility for the type i agent at time t.

Table 1: The Wedges in the Taxation Model

	persistent	i.i.d.
Insurance wedge	0.35	0.19×10^{-1}
Consumption-Leisure wedge	$(0.55\ \ , -0.32 \times 10^{-3})$	$(0.9 \times 10^{-2}, 0)$
Intertemporal wedge	$(0.65 \times 10^{-2}, 0.87 \times 10^{-2})$	$(0.25 \times 10^{-3}, 0.46 \times 10^{-3})$

(iii) The *intertemporal wedge* $\frac{E_t[u'(c_{t+1})|\iota_t=i]}{u'(c_t(h^{t-},i))} - 1$, $i = 1, 2$, measures the ratio of marginal utility at $t+1$ to marginal utility at t for the type i agent. In the above, c_{t+1} denotes the uncertain consumption at $t+1$, which depends on the realization of types at $t+1$.

The wedges defined above measure the degree of insurance from different dimensions. It is easy to see that in the full-information allocation, all the wedges should be 0. The larger the wedge, the worse the insurance is, and the larger the distortion is in the allocation.

6.2 Numerical Results

In order to study the wedge patterns in the model, we pick an endogenous state variable $(w_1, w_2) = (-60.9024, -59.5591)$, and then report values of the three wedges at this point. This particular choice of the state is not essential for the pattern reported in Table 1.[10] We can see from the table that the persistent shock model implies a much larger insurance wedge and consumption-leisure wedge (with report 1). It is also helpful to draw these wedges as functions of q. The insurance wedge and consumption-leisure wedge both decrease rapidly with the decrease in persistence (see Figure 4).[11] These findings are broadly consistent with those in Williams (2008). In a hidden income model, he found that the agent's exposure to risk depends positively on the persistence of the information. With less persistence (bigger q), the exposure is smaller, and the consumption is better smoothed (i.e., smaller wedges).

A distinctive feature of the results from the i.i.d. shock model is that all the wedges are close to zero. This implies that the allocation with the i.i.d. shocks is close to the full information (the first-best) allocation. The intuition for the results is as follows. Given that the discount factor e^{-r} is close to 1, the agent cares about her utility as the long-run average. If the shocks are i.i.d. and thus transitory,

[10] The state is chosen to match the utility level that the agent can achieve in autarky in the i.i.d. case. Choosing other levels of promised utilities would not change the results reported in Table 1 significantly. Alternatively, we could calculate averages of the wedges in the long run. Because of the immiserization result, the system does not imply a steady-state distribution of the state variable, so we need to impose a lower bound on the state variable to obtain a steady state. Using the averages would not change the pattern of the wedges either. See Zhang (2006).

[11] Since the consumption-leisure wedge with report 2 and the intertemporal wedge remain small for all levels of q, we do not draw them in the figure.

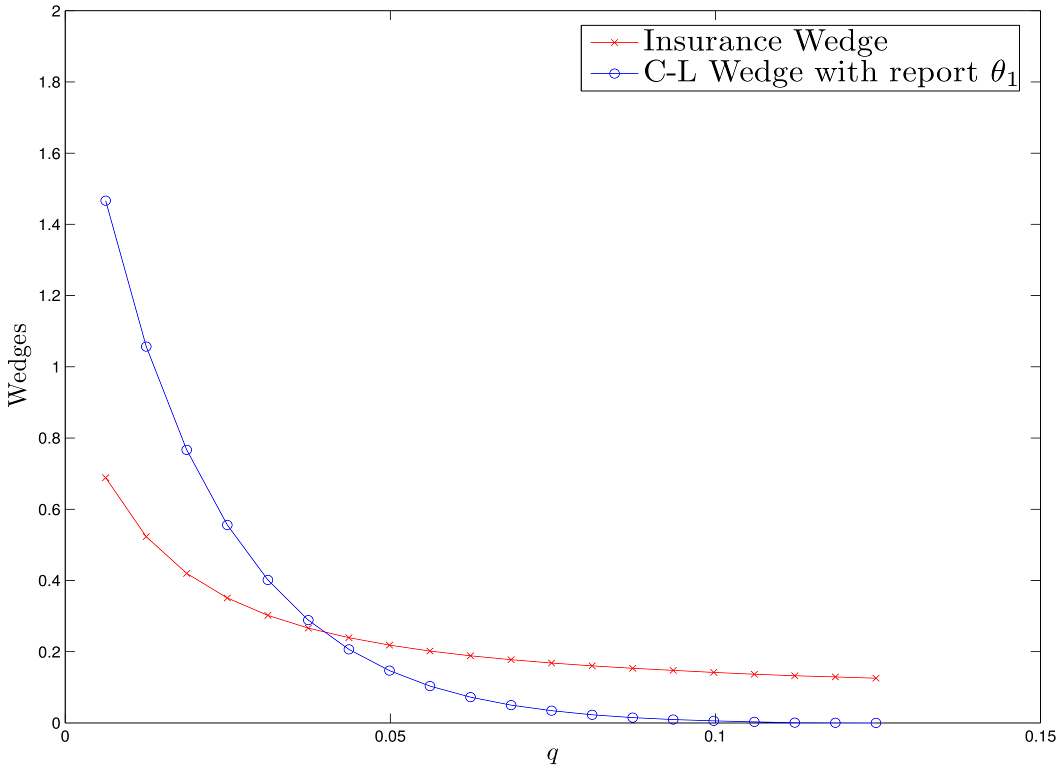

Figure 4: The wedges as functions of q.

the effect of any productivity shock at t is small and will be smoothed into many periods in the future. If the agent has a bad shock, the principal will still provide the consumption level close to that of the high-productivity agent but will lower the discounted utility from $t + 1$ on. In the long run, by the law of large numbers, the effects of high- and low-productivity shocks cancel out, and the agent does not experience large deviations from the first-best allocation. The intertemporal taxation and subsidy play an essential role in the optimal contract to smooth consumption.

The patterns of wedges with persistent shocks are significantly different from the i.i.d. shock model. We see that the insurance wedge is more than ten times bigger than the wedge in the i.i.d. case, implying that the consumption smoothing is far from being perfect. The consumption-leisure wedge is also quantitatively large, meaning that the low-productivity agent is distorted in her consumption-leisure decision. Despite this, the persistent shock model does not imply a large intertemporal wedge. To understand these patterns, it would be helpful to consider the permanent-shock model, which is the opposite extreme of the i.i.d. shocks. Suppose that the agent initially has a permanent productivity shock that is only privately observed. Then the optimal allocation requires a type-specific but constant stream of consumption and output for each type. It is well known that the optimal allocation implies consumption-leisure distortion only for the low-productivity agent. The intertemporal wedge is 0 simply because the consumption process is deterministic. Our results show that the pattern of wedges with persistent shocks is similar to a permanent shock model. Quantitatively, this is driven by the low value of q. The productivity process is so persistent that it is almost permanent.[12]

7. Concluding Remarks

This paper studies a continuous-time version of the dynamic taxation model with persistent shocks. Merely putting the problem in continuous time rather than in discrete time would not generate new economic implications; however, many implications that are difficult to obtain in a discrete-time model can be derived in its continuous-time analogue. The differential equations in Section 3 and the phase-diagram analysis in Section 5 provide a lot of information about the properties of the optimal contract. The advantages of the continuous-time method come from the fact that the phase-diagram analysis is traditionally carried out using differential equations, thus there are more mathematical tools available.

Our results are derived under several restrictive assumptions. First, since we use the recursive formulation in Fernandes and Phelan (2000), the dimension of the state vector is equal to the number of states in the agent's private information process. If the process is a diffusion process (with a continuum of possible states), then our state variable will be infinite dimensional, thus making it extremely difficult to study the dynamics of the contract. In this paper, we limit our attention to the case of two shocks, where the phase-diagram analysis is still tractable. Second, we use logarithmic utility and disutility functions. In APPENDIX C, we extend our results to functional forms including $\frac{c^{1-\sigma}}{1-\sigma}$ and $-\exp(-\sigma c)$, but some form of homogeneity is indispensable (note that the qualitative analysis in the i.i.d. case also assumes some form of homogeneity; for example, see Atkeson and Lucas (1992)). With the homogeneity property, the efficiency curves f_1 and f_2 are straight lines (see Figures 2, 3), thus we can easily show that f_1 is above f_2 and they split the state space into three regions. This greatly simplifies the analysis. Without the homogeneity property, f_1 and f_2 could be curves and (in principle) could intersect multiple times and separate the state space into many small regions, making the phase-diagram analysis intractable. Third, when we show that the bottom region is absorbing, an additional assumption of symmetry is used. This assumption helps us to show that, when the system starts on f_2 with report 1, $dw_1 < 0$ and the slope of the path ($\frac{dw_2}{dw_1}$) is bigger than that of f_2, thus the region below f_2 absorbs this path. It is still unclear whether the bottom region is absorbing when the Markov chain is asymmetric. We view our paper as the first step toward understanding the implication of persistent shocks and leave this question and further generalizations for future research.

[12] Similar effects of persistent shocks can be observed in other dynamic models as well. For example, in incomplete market model with idiosyncratic income shocks, a persistent shock changes future expected income more than a temporary shock does, thus has a greater effect on a consumer's consumption-saving behavior.

APPENDIX A: Proofs of Main Results

Proof of Lemma 1: We prove only the right-hand limits, since the proofs for the left-hand limits are similar. Let $-B$ be a lower bound on the instantaneous disutility; for example, we could define $B = \bar{v}/\theta_1$. We first prove a preliminary result,

$$(38) \qquad w_i(h^{t-}) \quad \geq \quad w_i(h^{t-}, j^{[t,s)})e^{-r(s-t)}e^{-q_i(s-t)} - \frac{B}{r}\left(1 - e^{-r(s-t)}e^{-q_i(s-t)}\right), 1 \leq i \leq N.$$

Consider a type i agent who reports j from time t to s and tells the truth from s onward if her type is still i. Her strategies in other contingencies need not be specified. The payoff from this strategy is at least

$$\int_t^s e^{-r(x-t)}(-B)dx + e^{-r(s-t)}\left[e^{-q_i(s-t)}w_i(h^{t-}, j^{[t,s)}) + (1 - e^{-q_i(s-t)})\frac{-B}{r}\right],$$

since with probability $e^{-q_i(s-t)}$, her type remains i, and with probability $(1 - e^{-q_i(s-t)})$, she obtains at least the lower bound. The above is the right side of (38), and since the contact is I.C., utility from truth-telling is higher.

Conditional on $\iota_t = i$, the holding time S_1 (it first leaves state i at time $t + S_1$) is an exponential random variable of parameter q_i. Let $A = \{\omega \in \Omega : \omega(t) = i, t + S_1 \geq s\}$, $A^C = \{\omega \in \Omega : \omega(t) = i, t + S_1 < s\}$,

$$\begin{aligned}
w_i(h^{t-}) &= E_t\left[\int_t^\infty e^{-r(x-t)}\left[u(c_x(h^{t-}, \omega^{[t,x]})) - v(y_x(h^{t-}, \omega^{[t,x]}))/\theta_{\omega(x)}\right]dx\middle|\omega(t) = i\right] \\
&= E_t\left[\int_t^\infty e^{-r(x-t)}\left[u(c_x(h^{t-}, \omega^{[t,x]})) - v(y_x(h^{t-}, \omega^{[t,x]}))/\theta_{\omega(x)}\right]dx\middle|A\right].\Pr(A|i) \\
&\quad + E_t\left[\int_t^\infty e^{-r(x-t)}\left[u(c_x(h^{t-}, \omega^{[t,x]})) - v(y_x(h^{t-}, \omega^{[t,x]}))/\theta_{\omega(x)}\right]dx\middle|A^C\right].\Pr(A^C|i) \\
&= \left[\int_t^s e^{-r(x-t)}\left[u(c_x(h^{t-}, i^{[t,x]})) - v(y_x(h^{t-}, i^{[t,x]}))/\theta_i\right]dx + e^{-r(s-t)}w_i(h^{t-}, i^{[t,s)})\right].\Pr(A|i) \\
&\quad + E_t\left[\int_t^\infty e^{-r(x-t)}\left[u(c_x(h^{t-}, \omega^{[t,x]})) - v(y_x(h^{t-}, \omega^{[t,x]}))/\theta_{\omega(x)}\right]dx\middle|A^C\right].\Pr(A^C|i).
\end{aligned}$$

Since $\Pr(A|i) = \exp(q_i(s-t)), \Pr(A^C|i) = 1-\exp(q_i(s-t))$, letting $s \downarrow t$, we see that $\lim_{s\downarrow t} w_i(h^{t-}, i^{[t,s)})$ exists and equals $w_i(h^{t-})$. The inequality (7) follows directly from the inequality (38). The only issue left is the existence of $\lim_{s\downarrow t} w_j(h^{t-}, i^{[t,s)})$, when $j \neq i$. By contradiction, suppose $\limsup_{s\downarrow t} w_j(h^{t-}, i^{[t,s)}) > \liminf_{s\downarrow t} w_j(h^{t-}, i^{[t,s)})$. Then there is a $\epsilon > 0$, such that for all $\delta > 0$, we can find $t < s_1 < s_2 < t + \delta$, such that $w_j(h^{t-}, i^{[t,s_1)}) < w_j(h^{t-}, i^{[t,s_2)}) - \epsilon$. But this would be a contradiction to inequality (38) when s_2 is close to s_1. *Q.E.D.*

Proof of Theorem 1:

(i) (necessity) The proof is divided into three steps: in step (a), we show that $w_i(h^{s-})$ is differentiable a.e.; in step (b), we derive equation (39) as a continuous-time analog of equation (12); in step (c), we take limits in equation (39) to finish the proof.

(a) We first show that $w_i(h^{s-})$ has, at most, countable discontinuous points, is of bounded variation, and thus is differentiable a.e. Define

$$\begin{aligned} V^+(x) &= \sup\left\{\sum_{k=1}^{n}(w_i(h^{t_k-}) - w_i(h^{t_{k-1}-}))^+ : P = \{t_0, ..., t_n\} \text{ is a partition of } [0,x)\right\}, \\ V^-(x) &= \sup\left\{\sum_{k=1}^{n}(w_i(h^{t_k-}) - w_i(h^{t_{k-1}-}))^- : P = \{t_0, ..., t_n\} \text{ is a partition of } [0,x)\right\}. \end{aligned}$$

We show that $V^+(x) < \infty$. Recall from equation (38),

$$w_i(h^{t_{k-1}-}) \geq w_i(h^{t_k-})e^{-(r+q_i)(t_k-t_{k-1})} - \frac{B}{r}(1 - e^{-(r+q_i)(t_k-t_{k-1})}).$$

Hence

$$\begin{aligned} (w_i(h^{t_k-}) - w_i(h^{t_{k-1}-}))^+ &\leq \left(\left(w_i(h^{t_k-}) + \frac{B}{r}\right)\left(1 - e^{-(r+q_i)(t_k-t_{k-1})}\right)\right)^+ \\ &\leq \frac{2B}{r}(r + q_i)(t_k - t_{k-1}). \end{aligned}$$

Therefore, $V^+(x) \leq \frac{2B}{r}(r + q_i)x$. Since $V^+(x) - V^-(x) = w_i(h^{x-}) - w_i(h^{0-})$, $V^-(x)$ is also finite. It is easy to verify that both V^+ and V^- are monotonic functions, and V^+ is continuous. Although V^- could be discontinuous, Theorem 29.7 in Aliprantis and Burkinshaw (1990) asserts that a monotonic function has, at most, countable discontinuities. Since the difference of two monotonic functions is of bounded variation, $w_i(h^{s-})$ has bounded variation and, by Theorem 29.11 in Aliprantis and Burkinshaw (1990), w_i is differentiable on path h^{t-} a.e.

(b) Pick $s \in [t_k, t_{k+1})$, such that w_j is differentiable for all $1 \leq j \leq N$ at s. Conditional on $\omega(s) = i = i_k$, the holding time S_1 (it first leaves state i at time $s + S_1$) is an exponential random variable of parameter q_i. For any $a > s$,

$$\begin{aligned} & w_i(h^{s-}) \\ = \; & E_s\left[\int_s^\infty e^{-r(x-s)}\left[u(c(h^{s-}, \omega^{[s,x]})) - v(y(h^{s-}, \omega^{[s,x]}))/\theta_{\omega(x)}\right] dx\middle|\omega(s) = i\right] \\ = \; & E_s\left[\int_s^{(s+S_1)\wedge a} e^{-r(x-s)}\left[u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i\right] dx\middle|\omega(s) = i\right] \\ & + E_s\left[\int_{(s+S_1)\wedge a}^\infty e^{-r(x-s)}\left[u(c(h^{s-}, \omega^{[s,x]})) - v(y(h^{s-}, \omega^{[s,x]}))/\theta_{\omega(x)}\right] dx\middle|\omega(s) = i\right]. \end{aligned}$$

By Fubini's theorem, the first term on the right is

$$\begin{aligned} & E_s\left[\int_s^{(s+S_1)\wedge a} e^{-r(x-s)}\left[u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i\right] dx\middle|\omega(s) = i\right] \\ = \; & E_s\left[\int_s^a \chi_{\{S_1 \geq (x-s)\}} e^{-r(x-s)}\left[u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i\right] dx\middle|\omega(s) = i\right] \end{aligned}$$

$$
\begin{aligned}
&= \int_s^a e^{-r(x-s)} \left[u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i\right] E_s \left[\chi_{\{S_1 \geq (x-s)\}} \Big| \omega(s) = i\right] dx \\
&= \int_s^a e^{-(r+q_i)(x-s)} (u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i) dx,
\end{aligned}
$$

where χ in the second line is the indicator function. Let $A = \{\omega \in \Omega : \omega(s) = i, s + S_1 \geq a\}$, $A^C = \{\omega \in \Omega : \omega(s) = i, s + S_1 < a\}$, then the second term is

$$
\begin{aligned}
& E_s \left[\int_{(s+S_1)\wedge a}^{\infty} e^{-r(x-s)} \left[u(c(h^{s-}, \omega^{[s,x]})) - v(y(h^{s-}, \omega^{[s,x]}))/\theta_{\omega(x)}\right] dx \Big| \omega(s) = i\right] \\
= \; & E_s \left[\int_a^{\infty} e^{-r(x-s)} \left[u(c(h^{s-}, \omega^{[s,x]})) - v(y(h^{s-}, \omega^{[s,x]}))/\theta_{\omega(x)}\right] dx \Big| A\right] . \Pr(A|i) \\
& + E_s \left[\chi_{A^C} \int_{s+S_1}^{\infty} e^{-r(x-s)} \left[u(c(h^{s-}, \omega^{[s,x]})) - v(y(h^{s-}, \omega^{[s,x]}))/\theta_{\omega(x)}\right] dx \Big| \omega(s) = i\right] \\
= \; & e^{-r(a-s)} w_i(h^{s-}, i^{[s,a)}) e^{-q_i(a-s)} + E_s \left[\chi_{A^C} e^{-rS_1} \sum_{j \neq i} \frac{q_{ij}}{q_i} w_j(h^{s-}, i^{[s,s+S_1)}) \Big| \omega(s) = i\right] \\
= \; & e^{-(r+q_i)(a-s)} w_i(h^{s-}, i^{[s,a)}) + \int_s^a e^{-(r+q_i)(x-s)} \sum_{j \neq i} q_{ij} w_j(h^{s-}, i^{[s,x)}) dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
w_i(h^{s-}) \; = \; & \int_s^a e^{-(r+q_i)(x-s)} (u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i) dx \\
& + e^{-(r+q_i)(a-s)} w_i(h^{s-}, i^{[s,a)}) + \int_s^a e^{-(r+q_i)(x-s)} \sum_{j \neq i} q_{ij} w_j(h^{s-}, i^{[s,x)}) dx. \qquad (39)
\end{aligned}
$$

(c) By assumption, $\lim_{a \to s} \frac{w_i(h^{s-}, i^{[s,a)}) - w_i(h^{s-})}{a-s}$ exists, and $\lim_{a \to s} w_j(h^{s-}, i^{[s,a)}) = w_j(h^{s-})$, for all j. Furthermore, since $c(h^{s-}, i^{[s,x]})$ and $y(h^{s-}, i^{[s,x]})$ are measurable functions of x, Theorem 29.4 in Aliprantis and Burkinshaw (1990) states

$$
\lim_{a \to s} \frac{1}{a-s} \int_s^a e^{-(r+q_i)(x-s)} (u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i) dx = u(c(h^s)) - v(y(h^s))/\theta_i,
$$

for a.e. s. We now have

$$
\begin{aligned}
& \lim_{a \to s} \frac{w_i(h^{s-}, i^{[s,a)}) - w_i(h^{s-})}{a-s} \\
= \; & \lim_{a \to s} \frac{1 - e^{-(r+q_i)(a-s)}}{a-s} w_i(h^{s-}, i^{[s,a)}) - \lim_{a \to s} \frac{1}{a-s} \int_s^a e^{-(r+q_i)(x-s)} \sum_{j \neq i} q_{ij} w_j(h^{s-}, i^{[s,x)}) dx \\
& - \lim_{a \to s} \frac{1}{a-s} \int_s^a e^{-(r+q_i)(x-s)} (u(c(h^{s-}, i^{[s,x]})) - v(y(h^{s-}, i^{[s,x]}))/\theta_i) dx \\
= \; & (r + q_i) w_i(h^{s-}) - \sum_{j \neq i} q_{ij} w_j(h^{s-}) - (u(c(h^s)) - v(y(h^s))/\theta_i), \text{ for a.e. } s.
\end{aligned}
$$

If $i \neq i_k$, the proof for inequality (11) is similar to the above. Since a type i agent could misreport

i_k up to $(s+S_1)\wedge a$, and starts truth-telling after $(s+S_1)\wedge a$, I.C. implies

$$\begin{aligned} w_i(h^{s-}) \quad &\geq \quad \int_s^a e^{-r(x-s)}e^{-q_i(x-s)}(u(c(h^{s-},{i_k}^{[s,x]}))-v(y(h^{s-},{i_k}^{[s,x]}))/\theta_i)dx \\ &\quad + e^{-(r+q_i)(a-s)}w_i(h^{s-},{i_k}^{[s,a)})+\int_s^a e^{-(r+q_i)(x-s)}\sum_{j\neq i}q_{ij}w_j(h^{s-},{i_k}^{[s,x)})dx. \end{aligned}$$

Taking limit yields

$$\lim_{a\to s}\frac{w_i(h^{s-},{i_k}^{[s,a)})-w_i(h^{s-})}{a-s}\leq (r+q_i)w_i(h^{s-})-\sum_{j\neq i}q_{ij}w_j(h^{s-})-(u(c(h^s))-v(y(h^s))/\theta_i).$$

(ii) (sufficiency) By using Dynkin's formula (see, for example, Fleming and Soner (2006, Appendix B)), we first represent the state variable $w_i(h^{t-})$ as the sum of the discounted value of $(u(c)-v(y)/\theta)$ and the discounted value of slack control variable μ. Then we show that for truth-telling, the discounted value of μ is 0, thus truth-telling achieves $w_i(h^{t-})$.

The Dynkin formula states that for a stochastic process $w(t)$ (possibly multidimensional) driven by

$$\frac{dw(t)}{dt} \quad = \quad f(t,w(t),\iota_t),$$

and for a real-valued smooth function $\Phi(t,w,\iota_t)$, the following equality holds

$$E_t\left[\Phi(s,w(s),\iota_s)\right] \quad = \quad \Phi(t,w(t),\iota_t)+E_t\left[\int_t^s A\Phi(x,w(x),\iota_x)dx\right], s>t,$$

where

$$A\Phi(x,w(x),\iota_x) \quad = \quad \frac{\partial\Phi}{\partial x}+f(x,w(x),\iota_x)\frac{\partial\Phi}{\partial w}+\sum_{j\neq\iota_x}q_{\iota_x j}[\Phi(x,w(x),j)-\Phi(x,w(x),\iota_x)].$$

Given the current state $(w_i(h^{t-}))_{1\leq i\leq N}$ and a strategy σ, according to equations (10) and (11), the process $w(\sigma(h^{t-},\omega^{[t,s)}))$ evolves as

$$\begin{aligned} \frac{dw_i(\sigma(h^{t-},\omega^{[t,s)}))}{ds} \quad &= \quad (r+q_i)w_i(\sigma(h^{t-},\omega^{[t,s)}))-\sum_{j\neq i}q_{ij}w_j(\sigma(h^{t-},\omega^{[t,s)})) \\ &\quad -(u(c(\sigma(h^{t-},\omega^{[t,s]})))-v(y(\sigma(h^{t-},\omega^{[t,s]})))/\theta_i)-\mu_i(\sigma(h^{t-},\omega^{[t,s]})), \end{aligned}$$

where $\mu_i(\sigma(h^{t-},\omega^{[t,s]}))=0$, if $\sigma_s(h^{t-},\omega^{[t,s]})=i$. Define $\Phi(s,w,\iota_s)=e^{-r(s-t)}w_{\iota_s}$. By Dynkin's formula,

$$E_t\left[\Phi(s,w(s),\iota_s)\right] \quad = \quad \Phi(t,w(t),\iota_t)+E_t\left[\int_t^s A\Phi(x,w(x),\iota_x)dx\right].$$

Taking limit $s\to\infty$ and using the fact that w is bounded, we have

$$0=w_{\iota_t}(h^{t-}) \quad + \quad E_t\left[\int_t^\infty A\Phi(x,w(\sigma(h^{t-},\omega^{[t,x)})),\iota_x)dx\right],$$

where

$$\begin{aligned}
A\Phi &= -re^{-r(x-t)}w_{\iota_x}(\sigma(h^{t-},\omega^{[t,x)})) + e^{-r(x-t)}\left(\frac{dw_{\iota_x}(\sigma(h^{t-},\omega^{[t,x)}))}{dx}\right) \\
&\quad + \sum_{j\neq\iota_x} e^{-r(x-t)}q_{\iota_x j}[w_j(\sigma(h^{t-},\omega^{[t,x)})) - w_{\iota_x}(\sigma(h^{t-},\omega^{[t,x)}))] \\
&= -e^{-r(x-t)}(u(c(\sigma(h^{t-},\omega^{[t,x]}))) - v(y(\sigma(h^{t-},\omega^{[t,x]})))/\theta_{\iota_x}) - e^{-r(x-t)}\mu_{\iota_x}(\sigma(h^{t-},\omega^{[t,x]})).
\end{aligned}$$

Thus,

$$\begin{aligned}
w_{\iota_t}(h^{t-}) &= E_t\left[\int_t^\infty e^{-r(x-t)}(u(c(\sigma(h^{t-},\omega^{[t,x]}))) - v(y(\sigma(h^{t-},\omega^{[t,x]})))/\theta_{\iota_x})dx\right] \\
&\quad + E_t\left[\int_t^\infty e^{-r(x-t)}\mu_{\iota_x}(\sigma(h^{t-},\omega^{[t,x]}))dx\right].
\end{aligned}$$

Since μ is nonnegative, the payoff from any strategy σ is weakly below $w_{\iota_t}(h^{t-})$. If $\sigma = \sigma^*$, then the second term is 0, and truth-telling achieves $w_{\iota_t}(h^{t-})$.

Q.E.D.

Proof of Theorem 2: We verify two things. First, any point between the two boundaries can be implemented by some contract. Second, any point either above the upper boundary or below the lower boundary cannot be implemented by any contract.

From the definition of boundary curves, any point on the boundary can be implemented. To implement a point (w_1, w_2) between the two boundaries, the principal may start with the policy $(c_t, y_t) = (0, 0)$, and let the promised utilities evolve according to the following differential equations,

$$\begin{aligned}
\frac{dw_1}{dt} &= (q_1 + r)w_1 - q_1 w_2 \\
\frac{dw_2}{dt} &= (q_2 + r)w_2 - q_2 w_1,
\end{aligned}$$

until time s^*, the time when the path hits some point (w_1^*, w_2^*) on the boundary. Then starting from s^*, the principal implements (w_1^*, w_2^*) using the contracts that define the boundary curves.

Second, we will show that any point below the lower boundary cannot be implemented by any contract. The proof for points above the upper boundary is analogous. Let function $g : [-x_1\bar{v}, \bar{u}/r] \to [-x_2\bar{v}, \bar{u}/r]$ be the lower boundary of W. Pick a point (w_1, w_2) with $w_2 < g(w_1)$ and a contract $\mathscr{C}$. We will prove that the continuation utility will eventually be impossible to implement under the history of reporting type 1 for a long time. To see this, let us calculate the distance between the lower boundary and the continuation utility $(w_1(1^{[0,t)}), w_2(1^{[0,t)}))$ under contract $\mathscr{C}$. Recall

$$\begin{aligned}
\frac{dw_1}{dt} &= (q_1 + r)w_1 - q_1 w_2 - u(c_t) + v(y_t)/\theta_1, \\
\frac{dw_2}{dt} &= (q_2 + r)w_2 - q_2 w_1 - u(c_t) + v(y_t)/\theta_2 - \mu_2.
\end{aligned}$$

The distance between w_2 and $g(w_1)$ satisfies

$$\begin{aligned}\frac{d(g(w_1)-w_2)}{dt} &\geq \frac{dg(w_1)}{dw_1}((q_1+r)w_1-q_1w_2-u(c_t)+v(y_t)/\theta_1)\\ &\quad -((q_2+r)w_2-q_2w_1-u(c_t)+v(y_t)/\theta_2)\\ &= \frac{dg(w_1)}{dw_1}((q_1+r)w_1-q_1w_2)-((q_2+r)w_2-q_2w_1)\\ &\quad +\left(\frac{dg(w_1)}{dw_1}/\theta_1-1/\theta_2\right)v(y_t)+\left(1-\frac{dg(w_1)}{dw_1}\right)u(c_t).\end{aligned}$$

Since $\frac{\theta_1}{\theta_2} \leq \frac{dg(w_1)}{dw_1} \leq 1$ and $w_2 \leq g(w_1)$,

$$\begin{aligned}\frac{d(g(w_1)-w_2)}{dt} &\geq \frac{dg(w_1)}{dw_1}((q_1+r)w_1-q_1g(w_1))-((q_2+r)w_2-q_2w_1)\\ &= ((q_2+r)g(w_1)-q_2w_1)-((q_2+r)w_2-q_2w_1)\\ &= (q_2+r)(g(w_1)-w_2).\end{aligned}$$

Therefore, the distance is increasing exponentially and, in finite time, w_2 will be less than $-x_2\bar{v}$. This is a contradiction because the worst scenario for the θ_2 type agent is "consumption 0 and maximal output $\bar{y}$," which provides utility $-x_2\bar{v}$. *Q.E.D.*

Proof of Lemma 2: We give a proof only for V_1, since the proof for V_2 is the same.

(i) It is because $(c_t^*, y_t^*)_{t\geq 0}$ is the optimal contract to implement (w_1, w_2) if and only if $(\exp(\lambda)c_t^*, \exp(\lambda)y_t^*)_{t\geq 0}$ is the optimal contract for $(w_1 + (1/r + x_1)\lambda, w_2 + (1/r + x_2)\lambda)$.

(ii) This follows from the fact that the contracting problem has a convex objective function and a linear constraint set when the control variables are u and v. For $\lambda \in [0, 1]$, pick two implementable pairs $(w_1, w_2), (w_1', w_2')$. Suppose $\mathscr{C}, \mathscr{C}'$ implement (w_1, w_2) and (w_1', w_2'), respectively; then the convex combination of $\mathscr{C}$ and $\mathscr{C}'$ will implement $\lambda(w_1, w_2) + (1-\lambda)(w_1', w_2')$. The new contract $(\lambda u(c(h^t)) + (1-\lambda)u(c'(h^t)), \lambda v(y(h^t)) + (1-\lambda)v(y'(h^t)))$ will still be I.C., because the differential equation conditions in Theorem 1 hold after the convex combination.

(iii) We show $V_{1,2} \leq 0$ first. Pick (w_1, w_2) and (w_1', w_2') with $w_1 = w_1', w_2 < w_2'$. Notice that initially the type is 1, so the evolution of (w_1', w_2') is controlled by

$$\begin{aligned}\frac{dw_1'}{dt} &= (q_1+r)w_1'-q_1w_2'-u(c_t)+v(y_t)/\theta_1,\\ \frac{dw_2'}{dt} &= (q_2+r)w_2'-q_2w_1'-u(c_t)+v(y_t)/\theta_2-\mu_2.\end{aligned}$$

By picking $\mu_2(0) = \infty$, the system could jump to (w_1, w_2) immediately, and then follow the consumption-output plan starting from (w_1, w_2). Thus $V_1(w_1', w_2') \leq V_1(w_1, w_2)$. To see that $V_{1,1} > 0$, differentiating equation (20) at $\lambda = 0$, we have

$$V_1(w_1, w_2) = V_{1,1}(w_1, w_2)(1/r + x_1) + V_{1,2}(w_1, w_2)(1/r + x_2). \tag{40}$$

It follows from $V_1 > 0, V_{1,2} \leq 0$ that $V_{1,1} > 0$.

Q.E.D.

PROOF OF LEMMA 3: We give a proof only for V_1, since the proof for V_2 is the same. We will need several preliminary results. First, let $\bar{V}_i(w_i)$ be the first-best cost function if the shock is public information and the agent starts with initial state i and promised utility w_i, $i = 1, 2$. It is obvious that for all w_1, w_2,

$$V_1(w_1, w_2) \geq \bar{V}_1(w_1), V_2(w_1, w_2) \geq \bar{V}_2(w_2).$$

Second, using the same proof as that in proving equation (39), we can show, for $t > 0$,

$$\begin{aligned} V_1(w_1, w_2) &= \int_0^t e^{-(r+q_1)s}(c(1^{[0,s]}) - y(1^{[0,s]}))ds + e^{-(r+q_1)t}V_1(w_1(1^{[0,t)}), w_2(1^{[0,t)})) \\ &\quad + \int_0^t e^{-(r+q_1)s} q_1 V_2(w_1(1^{[0,s)}), w_2(1^{[0,s)}))ds, \end{aligned} \tag{41}$$

where $c(1^{[0,s]}), y(1^{[0,s]})$ are the optimal control variables and $w_1(1^{[0,s)}), w_2(1^{[0,s)})$ are the time paths of the state variables under the optimal control. Both the control variables and the time paths implicitly depend the initial state (w_1, w_2).

Third, if $a > 0, b > 0$ are two numbers, and $B > 0$ is an upper bound for $\int_0^t e^{-as}c(s)ds$, then

$$\int_0^t e^{-bs}u(c(s))ds \leq \int_0^t e^{-bs}c(s)ds \leq e^{(a+b)t}B. \tag{42}$$

Similarly, if $B > 0$ is an upper bound for $\int_0^t e^{-as}(-y(s))ds$, then

$$\int_0^t e^{-bs}v(c(s))ds \geq \int_0^t e^{-bs}y(s)ds \geq -e^{(a+b)t}B. \tag{43}$$

(i) To show $\{w_2 \in \mathbb{R} : V_{1,2}(w_1, w_2) < 0\} \neq \emptyset$, it suffices to show that $\lim_{w_2 \to -\infty} V_1(w_1, w_2) = \infty$, since V_1 is a (weakly) convex and decreasing function of w_2. By contradiction, assume that there is $B > 0$, such that $\lim_{w_2 \to -\infty} V_1(w_1, w_2) \leq B$. Starting at (w_1, w_2), the evolution with report 1 is

$$\begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{pmatrix} = \begin{pmatrix} q_1 + r & -q_1 \\ -q_2 & q_2 + r \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} -u(c_t) + v(y_t)/\theta_1 \\ -u(c_t) + v(y_t)/\theta_2 - \mu_2 \end{pmatrix}.$$

Since $\begin{pmatrix} q_1 + r & -q_1 \\ -q_2 & q_2 + r \end{pmatrix} = \begin{pmatrix} -\frac{q_2}{q_1+q_2} & \frac{q_2}{q_1+q_2} \\ \frac{q_2}{q_1+q_2} & \frac{q_1}{q_1+q_2} \end{pmatrix}^{-1} \begin{pmatrix} q_1 + q_2 + r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} -\frac{q_2}{q_1+q_2} & \frac{q_2}{q_1+q_2} \\ \frac{q_2}{q_1+q_2} & \frac{q_1}{q_1+q_2} \end{pmatrix}$,
we can solve the equation and obtain

$$\begin{aligned} w_1(1^{[0,t)}) &= \frac{q_1}{q_1+q_2} w_1 e^{(q_1+q_2+r)t} + \frac{q_2}{q_1+q_2} w_1 e^{rt} - \frac{q_1}{q_1+q_2} w_2 (e^{(q_1+q_2+r)t} - e^{rt}) \\ &\quad + \frac{q_1}{q_1+q_2} \int_0^t e^{(q_1+q_2+r)(t-s)} (1/\theta_1 - 1/\theta_2) v(y(1^{[0,s]}))ds \\ &\quad + \int_0^t e^{r(t-s)} \left(-u(c(1^{[0,s]})) + \left(\frac{q_2/\theta_1}{q_1+q_2} + \frac{q_1/\theta_2}{q_1+q_2} \right) v(y(1^{[0,s]})) \right) ds \\ &\quad + \frac{q_1}{q_1+q_2} \int_0^t (e^{(q_1+q_2+r)(t-s)} - e^{r(t-s)}) \mu_2(1^{[0,s]})ds. \end{aligned} \tag{44}$$

Fix t and w_1, and let $w_2 \to -\infty$. Equation (41) and $\sup_{w_2} V_1(w_1, w_2) \le B$ imply that $\int_0^t e^{-(r+q_1)s}(c(1^{[0,s]}) - y(1^{[0,s]}))ds$ is uniformly (when w_2 varies) bounded by B. Using (42),(43), we see that the fourth and fifth terms on the right of (44) are bounded from below. Since the first and second terms are fixed, and the last term is nonnegative, taking limit in (44) yields

$$\lim_{w_2 \to -\infty} w_1(1^{[0,t)}) \quad = \quad \infty.$$

Using (41), we see that, as $w_2 \to -\infty$,

$$V_1(w_1, w_2) \quad \ge \quad e^{-(r+q_1)t} V_1(w_1(1^{[0,t)}), w_2(1^{[0,t)})) \ge e^{-(r+q_1)t} \bar{V}_1(w_1(1^{[0,t)})) \to \infty.$$

(ii) To show that $\{w_2 \in \mathbb{R} : V_{1,2}(w_1, w_2) = 0\} \neq \emptyset$, by contradiction, suppose $V_{1,2}(w_1, w_2) < 0$ for all w_2. Then with report 1, the slack variable $\mu_2 = 0$, the system is

$$\begin{aligned} \frac{dw_1}{dt} &= (q_1 + r)w_1 - q_1 w_2 - u(c_t) + v(y_t)/\theta_1, \\ \frac{dw_2}{dt} &= (q_2 + r)w_2 - q_2 w_1 - u(c_t) + v(y_t)/\theta_2. \end{aligned}$$

Fix t and w_1. Pick w_2^* and B such that $e^{-(r+q_1)t}\bar{V}_1(B) \ge V_1(w_1, w_2^*)$. If $w_2 \ge w_2^*$, then $V_1(w_1, w_2) \le V_1(w_1, w_2^*)$ and equation (41) imply that

$$\sup_{w_2 \in [w_2^*, \infty)} \sup_{s \in [0,t]} w_1(1^{[0,s)}) \quad \le \quad B. \tag{45}$$

We can solve the differential equation and obtain, for $s \in [0, t]$,

$$\begin{aligned} q_2 w_1(1^{[0,s)}) + q_1 w_2(1^{[0,s)}) &= (q_2 w_1 + q_1 w_2)e^{rs} - \int_0^s e^{r(s-x)}(q_1 + q_2)u(c(1^{[0,x]}))dx \\ &\quad + \int_0^s e^{r(s-x)}(q_2/\theta_1 + q_1/\theta_2)v(y(1^{[0,x]}))dx. \end{aligned}$$

Since $\int_0^s e^{-(r+q_1)x}(c(1^{[0,x]}) - y(1^{[0,x]}))dx \le V_1(w_1, w_2) \le V_1(w_1, w_2^*)$, (42) and (43) imply

$$\begin{aligned} -\int_0^s e^{r(s-x)}(q_1 + q_2)u(c(1^{[0,x]}))dx &\ge -e^{rt}(q_1 + q_2)e^{(2r+q_1)t}V_1(w_1, w_2^*), \\ \int_0^s e^{r(s-x)}(q_2/\theta_1 + q_1/\theta_2)v(y(1^{[0,x]}))dx &\ge -e^{rt}(q_2/\theta_1 + q_1/\theta_2)e^{(2r+q_1)t}V_1(w_1, w_2^*). \end{aligned}$$

The above two inequalities and (45) yield

$$\begin{aligned} q_1 w_2(1^{[0,s)}) &\ge (q_2 w_1 + q_1 w_2)e^{rs} - e^{rt}(q_1 + q_2)e^{(2r+q_1)t}V_1(w_1, w_2^*) \\ &\quad - e^{rt}(q_2/\theta_1 + q_1/\theta_2)e^{(2r+q_1)t}V_1(w_1, w_2^*) - q_2 B, \end{aligned}$$

thus $\inf_{s \in [0,t]} w_2(1^{[0,s)}) \to \infty$ as $w_2 \to \infty$. This and (41) imply, as $w_2 \to \infty$,

$$\begin{aligned} V_1(w_1, w_2) &\ge \int_0^t e^{-(r+q_1)s} q_1 V_2(w_1(1^{[0,s)}), w_2(1^{[0,s)}))ds \\ &\ge \int_0^t e^{-(r+q_1)s} q_1 \bar{V}_2(w_2(1^{[0,s)}))ds \to \infty, \end{aligned}$$

which contradicts our assumption that $V_{1,2}(w_1, w_2) < 0$ for all w_2.

Q.E.D.

Proof of Lemma 4: (41) implies that

$$V_1(w_1(h^{t-}), w_2(h^{t-})) = \lim_{s\downarrow t} V_1(w_1(h^{t-}, 1^{[t,s)}), w_2(h^{t-}, 1^{[t,s)})) = V_1(w_1(h^{t-}), \lim_{s\downarrow t} w_2(h^{t-}, 1^{[t,s)})). \tag{46}$$

When $w_2(h^{t-}) > f_1(w_1(h^{t-}))$,

$$V_1(w_1(h^{t-}), f_1(w_1(h^{t-}))) = V_1(w_1(h^{t-}), w_2(h^{t-})) = V_1(w_1(h^{t-}), \lim_{s\downarrow t} w_2(h^{t-}, 1^{[t,s)})).$$

Thus $\lim_{s\downarrow t} w_2(h^{t-}, 1^{[t,s)}) \geq f_1(w_1(h^{t-}))$. If $\lim_{s\downarrow t} w_2(h^{t-}, 1^{[t,s)}) > f_1(w_1(h^{t-}))$, then the contracts starting from $(w_1(h^{t-}), w_2(h^{t-}))$ and $(w_1(h^{t-}), f_1(w_1(h^{t-})))$ cannot be equal to each other almost surely (a.s.). Convex combination can be used to lower the cost at $(w_1(h^{t-}), w_2(h^{t-})/2 + f_1(w_1(h^{t-}))/2)$. This contradicts the fact that $V_1(w_1, w_2) = V_1(w_1, f_1(w_1))$, for all $w_2 \geq f_1(w_1)$.

Next we show that for all $s > t$, $w_2(h^{t-}, 1^{[t,s)}) \leq f_1(w_1(h^{t-}, 1^{[t,s)}))$. This implies that, once the state variable jumps onto the efficiency line, it stays on or below it forever, unless a new transition occurs, which may require the state variable to jump to the other efficiency line. By contradiction, suppose for some $s > t$, $w_2(h^{t-}, 1^{[t,s)}) > f_1(w_1(h^{t-}, 1^{[t,s)}))$. Since $\lim_{s'\uparrow s} w_2(h^{t-}, 1^{[t,s')}) \geq w_2(h^{t-}, 1^{[t,s)})$, and w_2 is continuous a.e., we can find an $s' < s$, such that

$$w_2(h^{t-}, 1^{[t,s')}) > f_1(h^{t-}, 1^{[t,s')}) \text{ and } w_2 \text{ is continuous at } s'.$$

At s', the time path is continuous and above the efficiency line f_1, thus does not immediately jump onto it, which is a contradiction to what we have shown in the first step.

Last, we show that $w_2(h^{t-}, 1^{[t,s)}) = \lim_{s'\downarrow s} w_2(h^{t-}, 1^{[t,s')})$. Now the state is below the efficiency line and, by definition, $V_{1,2} < 0$ in this region. $w_2(h^{t-}, 1^{[t,s)}) > \lim_{s'\downarrow s} w_2(h^{t-}, 1^{[t,s')})$ contradicts equation (46). *Q.E.D.*

Proof of Lemma 5: By contradiction, first suppose that f_2 is above f_1, i.e., $f_1(w_1) < (f_2)^{-1}(w_1)$. Lemma 7 in APPENDIX B implies that $V_{1,1}(w_1, w_2) - V_{2,2}(w_1, w_2)$ is strictly decreasing in w_2, which implies

$$V_{1,1}(w_1, f_1(w_1)) - V_{2,2}(w_1, f_1(w_1)) > V_{1,1}(w_1, (f_2)^{-1}(w_1)) - V_{2,2}(w_1, (f_2)^{-1}(w_1)),$$

which contradicts Lemma 8 in APPENDIX B.

Next, suppose that f_1 and f_2 coincide, i.e., $f_1(w_1) = (f_2)^{-1}(w_1)$. At $(w_1, f_1(w_1))$, the definition of the efficiency lines and Lemma 8 imply that $V_{1,2} = V_{2,1} = 0$, $V_{1,12} = V_{1,22} = V_{2,11} = V_{2,12} = 0$, $V_{1,1} = V_{2,2} > 0$. Using equation (24),(25) with report 1, we find

$$V_{1,11}\frac{dw_1}{dt} = V_{1,11}\frac{dw_1}{dt} + V_{1,12}\frac{dw_2}{dt} = \frac{dV_{1,1}}{dt} = q_2 V_{1,2} - q_1 V_{2,1} = 0$$
$$V_{1,22}\frac{dw_2}{dt} = V_{1,21}\frac{dw_1}{dt} + V_{1,22}\frac{dw_2}{dt} = \frac{dV_{1,2}}{dt} = q_1 V_{1,1} - q_1 V_{2,2} + (q_1 - q_2) V_{1,2} = 0,$$

therefore $\frac{dw_1(1^{[0,t)})}{dt}|_{t=0} = \frac{dw_2(1^{[0,t)})}{dt}|_{t=0} = 0$. Similarly, with report 2, the state variable does not move either. However, we know that $V_{1,1} + V_{1,2} = V_{2,1} + V_{2,2}$ and $V_{1,1}/\theta_1 + V_{1,2}/\theta_2 > V_{2,1}/\theta_1 + V_{2,2}/\theta_2$ on f_1. Using (26),(31), we see that $c_1 = c_2, -y_1 > -y_2$, which gives a contradiction,

$$\begin{aligned}
0 = \frac{dw_2(1^{[0,t)})}{dt}|_{t=0} &= (q_2 + r)w_2 - q_2 w_1 - u(c_1) + v(y_1)/\theta_2 - \mu_2 \\
&\leq (q_2 + r)w_2 - q_2 w_1 - u(c_1) + v(y_1)/\theta_2 \\
&< (q_2 + r)w_2 - q_2 w_1 - u(c_2) + v(y_2)/\theta_2 \\
&= \frac{dw_2(2^{[0,t)})}{dt}|_{t=0} = 0.
\end{aligned}$$

Q.E.D.

Proof of Lemma 6: We first show that starting from $(f_2(w_2), w_2)$, with report 2, $w_1(2^{[0,t)}) = f_2(w_2(2^{[0,t)}))$, for all $t \geq 0$. Lemma 10 in APPENDIX B states that $V_{1,1} = V_{2,2}$ on f_2. Lemma 7 in APPENDIX B implies that $V_{1,1} < V_{2,2}$ if $w_1 < f_2(w_2)$. If the state moves to the left of f_2, solving equation (29) yields

$$\begin{aligned}
&V_{2,1}(w_1(2^{[0,t)}), w_2(2^{[0,t)})) - e^{(q_2-q_1)t} V_{2,1}(f_2(w_2), w_2) \\
= &\int_0^t e^{(q_2-q_1)(t-s)} q_2 (V_{2,2}(w_1(2^{[0,s)}), w_2(2^{[0,s)})) - V_{1,1}(w_1(2^{[0,s)}), w_2(2^{[0,s)}))) ds > 0,
\end{aligned}$$

which contradicts $V_{2,1} \leq 0$ from part (iii) in Lemma 2. Therefore, the state variable remains on the efficiency line f_2 with report 2. Furthermore, equation (30) and the fact that $V_{2,1} = 0, V_{1,2} < 0$ on f_2, imply that

$$V_{2,22}\frac{dw_2}{dt} = V_{2,21}\frac{dw_1}{dt} + V_{2,22}\frac{dw_2}{dt} = \frac{dV_{2,2}}{dt} = q_1 V_{2,1} - q_2 V_{1,2} > 0,$$

therefore $dw_2/dt > 0$ and the state moves up along f_2.

Lemma 10 in APPENDIX B states that $V_{1,1} < V_{2,2}, V_{1,2} = 0$ on f_1. With report 1 and starting from the line f_1, equation (25) implies that

$$\frac{dV_{1,2}}{dt} = q_1 V_{1,1} - q_1 V_{2,2} + (q_1 - q_2)V_{1,2} < 0,$$

which means that the time path leaves f_1 and moves below f_1.

To understand the pattern when the report is θ_1 and the state starts from f_2, assume that $q_1 = q_2 = q > 0$. Equation (25) is

$$V_{1,21}\frac{dw_1}{dt} + V_{1,22}\frac{dw_2}{dt} = \frac{dV_{1,2}}{dt} = qV_{1,1} - qV_{2,2} = 0.$$

Therefore, starting from f_2, $\frac{dw_2}{dw_1} = -V_{1,21}/V_{1,22}$. Substituting into equation (24), we find

$$0 > qV_{1,2} = \frac{dV_{1,1}}{dt} = V_{1,11}\frac{dw_1}{dt} + V_{1,12}\frac{dw_2}{dt} = V_{1,11}\frac{dw_1}{dt}\left(1 - \frac{V_{1,12}V_{1,21}}{V_{1,11}V_{1,22}}\right).$$

It follows from the convexity of V_1 ($V_{1,12}V_{1,21} \leq V_{1,11}V_{1,22}$) that $dw_1/dt < 0$. Equation (48) and $V_{1,2} < 0$ on f_2 imply that $-V_{1,21}/V_{1,22} > (1/r + x_2)/(1/r + x_1)$, thus dw_2/dw_1 is larger than the slope of f_2 ($(1/r + x_2)/(1/r + x_1)$). So the time path of (w_1, w_2) will move southwest, and be below f_2, and thus will enter the interior of the region. *Q.E.D.*

PROOF OF THEOREM 3:

(i) This follows from the system described in equations (22), (23), (24), (25), (26). Notice that in the region below f_2, $V_{2,1}(w_1, w_2) = 0$ and $V_{2,2}(w_1, w_2) = V_{2,2}(f_2(w_2), w_2)$. That $\mu_2 = 0$ follows from $V_{1,2} < 0$.

(ii) This follows from LEMMA 9 and LEMMA 10 in APPENDIX B.

(iii) We provide a proof only for a report 1. Since $V_{1,2} < 0$ and $V_{1,1} > V_{2,2}$, equations (34),(35) imply that $V_{1,1}$ decreases and $V_{1,2}$ increases. Now we show that $dw_2/dt < 0$ for all t. Suppose this is not the case, since initially $dw_2/dt < 0$, there is some t^*, $\left.\frac{dw_2(t)}{dt}\right|_{t=t^*} = 0$. Equation (34) and $V_{1,11} > 0$ imply that $\left.\frac{dw_1(t)}{dt}\right|_{t=t^*} < 0$. If f denotes the straight line that passes $(w_1(t^*), w_2(t^*))$ and is below but parallel to f_2, then this implies that the time path moves leftward and crosses f from below. Since the time path starts from f_2, it has to cross f from above at least once before. The time path crosses f from different directions twice, contradicting equation (19). A similar argument shows that $dw_1/dt < 0$ for all t.

Next, we will show that $V_{1,1}(t) + V_{1,2}(t)$ is strictly decreasing in t. Adding (34) and (35) yields

$$\frac{d(V_{1,1} + V_{1,2})}{dt} = qV_{1,2} + qV_{1,1} - qV_{2,2}(f_2(w_2), w_2). \tag{47}$$

Thus it suffices to show that $V_{1,1}(t) + V_{1,2}(t) < V_{2,2}(f_2(w_2(t)), w_2(t))$, for all $t \geq 0$. By contradiction, suppose at some time t^*, $V_{1,1}(t^*) + V_{1,2}(t^*) \geq V_{2,2}(f_2(w_2(t^*)), w_2(t^*))$. For $t^{**} > t^*$, solving (47) yields

$$\begin{aligned}
(V_{1,1} + V_{1,2})(t^{**}) &= e^{q(t^{**}-t^*)}(V_{1,1} + V_{1,2})(t^*) - \int_{t^*}^{t^{**}} e^{q(t^{**}-s)}V_{2,2}(f_2(w_2(s)), w_2(s))ds \\
&> e^{q(t^{**}-t^*)}(V_{1,1} + V_{1,2})(t^*) - \int_{t^*}^{t^{**}} e^{q(t^{**}-s)}V_{2,2}(f_2(w_2(t^*)), w_2(t^*))ds \\
&= e^{q(t^{**}-t^*)}((V_{1,1} + V_{1,2})(t^*) - V_{2,2}(f_2(w_2(t^*)), w_2(t^*))) \\
&\quad + V_{2,2}(f_2(w_2(t^*)), w_2(t^*)) \\
&> V_{2,2}(f_2(w_2(t^{**})), w_2(t^{**})),
\end{aligned}$$

where the inequalities follow from that $V_{2,2}(f_2(w_2), w_2)$ is strictly increasing in w_2 and $\frac{dw_2(t)}{dt} < 0$.

Now for $t > t^{**}$,

$$\begin{aligned}(V_{1,1}+V_{1,2})(t) &= e^{q(t-t^{**})}(V_{1,1}+V_{1,2})(t^{**}) - \int_{t^{**}}^{t} e^{q(t-s)}V_{2,2}(f_2(w_2(s)),w_2(s))ds \\ &> e^{q(t-t^{**})}(V_{1,1}+V_{1,2})(t^{**}) - \int_{t^{**}}^{t} e^{q(t-s)}V_{2,2}(f_2(w_2(t^{**})),w_2(t^{**}))ds \\ &= e^{q(t-t^{**})}((V_{1,1}+V_{1,2})(t^{**}) - V_{2,2}(f_2(w_2(t^{**})),w_2(t^{**}))) \\ &\quad +V_{2,2}(f_2(w_2(t^{**})),w_2(t^{**})).\end{aligned}$$

Since $(V_{1,1}+V_{1,2})(t^{**}) > V_{2,2}(f_2(w_2(t^{**})),w_2(t^{**}))$, it is easily seen that $\lim_{t\to\infty}(V_{1,1}+V_{1,2})(t) = \infty$. This contradicts the fact that $\frac{dV_{1,1}}{dt} < 0$ (from equation (34)) and $V_{1,1}+V_{1,2} < V_{1,1}$.

We further know that $V_{1,1}(t)/\theta_1 + V_{1,2}(t)/\theta_2 = V_{1,1}(t)(1/\theta_1 - 1/\theta_2) + (V_{1,1}(t)+V_{1,2}(t))/\theta_2$ is strictly decreasing in t. These and (36) imply that consumption falls and output increases with duration of the report 1. The properties with report 2 are obvious.

(iv) Part (iii) and equation (40) imply that

$$V_{1,1}(1/r+x_1) + V_{1,2}(1/r+x_2) \geq V_{2,1}(1/r+x_1) + V_{2,2}(1/r+x_2) = V_{2,2}(1/r+x_2).$$

Since $(1/r+x_2)/(1/r+x_1) < (1/\theta_2)/(1/\theta_1)$ and $V_{1,2} < 0$, we have

$$V_{1,1}(w_1,w_2)/\theta_1 + V_{1,2}(w_1,w_2)/\theta_2 \quad > \quad V_{2,2}(f_2(w_2),w_2)/\theta_2.$$

We also know from part (iii),

$$V_{1,1}(w_1,w_2) + V_{1,2}(w_1,w_2) \quad < \quad V_{2,2}(f_2(w_2),w_2).$$

Equations (26) and (31) thus imply that, given a certain level of promised utilities, consumption and output are always lower with report 1. (Of course, consumption in state 1 could be higher than that in 2 if compared at different promised utilities.)

(v) $u'(c)/(v'(y)/\theta_1)$ is the ratio of marginal utility of consumption to marginal disutility of production. With a report 1, since $V_{1,2} < 0$, we have $u'(c)/(v'(y)/\theta_1) = (V_{1,1} + \theta_1/\theta_2 V_{1,2})/(V_{1,1}+V_{1,2}) > 1$. With a report 2, since $V_{2,1} = 0$, we have $u'(c)/(v'(y)/\theta_2) = (\theta_2/\theta_1 V_{2,1} + V_{2,2})/(V_{2,1}+V_{2,2}) = 1$. To show that the distortion increases with report 1, it is sufficient to show that $V_{1,2}/V_{1,1}$ is decreasing along the time path. By convexity, $V_{1,2}/V_{1,1}$ is an increasing function of w_2, because

$$\begin{aligned}\frac{\partial(V_{1,2}/V_{1,1})}{\partial w_2} &= \frac{V_{1,22}V_{1,1} - V_{1,2}V_{1,12}}{V_{1,1}^2} \\ &= \frac{V_{1,22}((1/r+x_1)V_{1,11} + (1/r+x_2)V_{1,12})}{V_{1,1}^2} - \frac{V_{1,12}((1/r+x_1)V_{1,21} + (1/r+x_2)V_{1,22})}{V_{1,1}^2} \\ &= \frac{(1/r+x_1)(V_{1,22}V_{1,11} - V_{1,12}^2)}{V_{1,1}^2} > 0.\end{aligned}$$

For any straight line f that is below but parallel to f_2, the time path can cross f at most once; otherwise suppose the time path crosses f from different directions twice, it contradicts equation (19). Therefore, with a report 1, the time path moves farther and farther away from the line f_2, i.e., $w_2(t) - (1/r + x_2)/(1/r + x_1)w_1(t)$ is decreasing, which implies that

$$\begin{aligned} & \frac{V_{1,2}(w_1(t), w_2(t))}{V_{1,1}(w_1(t), w_2(t))} \\ = & \frac{V_{1,2}(0, w_2(t) - w_1(t)(1/r + x_2)/(1/r + x_1))e^{w_1(t)/(1/r+x_1)}}{V_{1,1}(0, w_2(t) - w_1(t)(1/r + x_2)/(1/r + x_1))e^{w_1(t)/(1/r+x_1)}} \\ = & \frac{V_{1,2}(0, w_2(t) - w_1(t)(1/r + x_2)/(1/r + x_1))}{V_{1,1}(0, w_2(t) - w_1(t)(1/r + x_2)/(1/r + x_1))} \end{aligned}$$

is decreasing in t.

(vi) Equations (24), (25), (29), and (30) imply that

$$\begin{aligned} \frac{d(V_{1,1} + V_{1,2})}{dt} &= q((V_{1,1} + V_{1,2}) - (V_{2,1} + V_{2,2})), \\ \frac{d(V_{2,1} + V_{2,2})}{dt} &= q((V_{2,1} + V_{2,2}) - (V_{1,1} + V_{1,2})). \end{aligned}$$

Define $\Phi(t, w, \iota_t) = V_{\iota_t,1}(w_1, w_2) + V_{\iota_t,2}(w_1, w_2)$. Using the Dynkin's formula stated in the proof of THEOREM 1, since

$$\begin{aligned} A\Phi(t, w, 1) &= \frac{d(V_{1,1} + V_{1,2})}{dt} + q((V_{2,1} + V_{2,2}) - (V_{1,1} + V_{1,2})) = 0, \\ A\Phi(t, w, 2) &= \frac{d(V_{2,1} + V_{2,2})}{dt} + q((V_{1,1} + V_{1,2}) - (V_{2,1} + V_{2,2})) = 0, \end{aligned}$$

we see that $E_t[(V_{\iota_s,1} + V_{\iota_s,2})] = (V_{\iota_t,1} + V_{\iota_t,2})$, i.e., $V_{\iota_t,1}(w_1, w_2) + V_{\iota_t,2}(w_1, w_2)$ is a martingale. By the martingale convergence theorem and equations (26) and (31), consumption converges to 0 a.s. Next we show that output converges to its upper bound a.s. Since $V_{2,1} = 0$,

$$\lim_{t\to\infty} \chi_{\{\iota_t=2\}} V_{2,2}(w_1(t), w_2(t)) = 0, a.s.,$$

where χ is the indicator function. This implies $\lim_{t\to\infty} \chi_{\{\iota_t=1\}} V_{1,1}(w_1(t), w_2(t)) = 0$, a.s., because $V_{1,1}$ equals $V_{2,2}$ on f_2 and falls with the duration of report 1. It follows that $\lim_{t\to\infty} \chi_{\{\iota_t=1\}} V_{1,2}(w_1(t), w_2(t)) = 0$. Therefore,

$$\lim_{t\to\infty} V_{\iota_t,1}(w_1(t), w_2(t))/\theta_1 + V_{\iota_t,2}(w_1(t), w_2(t))/\theta_2 = 0, \text{ a.s.},$$

which, with equations (26) and (31), implies that the output converges to its upper bound.

Q.E.D.

APPENDIX B: AUXILIARY RESULTS

LEMMA 7 $V_{1,11} > 0, V_{1,22} \geq 0, V_{1,12} \leq 0, V_{2,11} \geq 0, V_{2,22} > 0, V_{2,12} \leq 0.$

Proof of Lemma 7: We only prove for V_1. That $V_{1,22} \geq 0$ follows from the convexity of V_1. Differentiating (40) with respect to w_2 yields

$$V_{1,2}(w_1, w_2) = V_{1,12}(w_1, w_2)(1/r + x_1) + V_{1,22}(w_1, w_2)(1/r + x_2). \tag{48}$$

Since $V_{1,2} \leq 0, V_{1,22} \geq 0$, we find that $V_{1,12} \leq 0$. Differentiating (40) with respect to w_1 yields

$$V_{1,1}(w_1, w_2) = V_{1,11}(w_1, w_2)(1/r + x_1) + V_{1,12}(w_1, w_2)(1/r + x_2).$$

Since $V_{1,1} > 0, V_{1,12} \leq 0$, we find that $V_{1,11} > 0$. *Q.E.D.*

Lemma 8 *On efficiency lines,*

$$V_{1,1}(w_1, f_1(w_1)) \leq V_{2,2}(w_1, f_1(w_1)), V_{1,1}(f_2(w_2), w_2) \geq V_{2,2}(f_2(w_2), w_2).$$

Proof of Lemma 8: We only prove the first inequality. At (w_1, w_2), $w_2 > f_1(w_1)$, the optimal control is $\mu_2 = \infty$, and the system immediately jumps. If we restrict $\mu_2 = 0$, we will have an inequality version of the HJB equation (see, for example, Fleming and Soner (2006, equation (7.3), p.132)),

$$\begin{aligned} 0 \leq & \min_c\{c - (V_{1,1} + V_{1,2})u(c)\} + \min_y\{-y + (V_{1,1}/\theta_1 + V_{1,2}/\theta_2)v(y)\} + q_1 V_2(w_1, w_2) \\ & + V_{1,1}((q_1 + r)w_1 - q_1 w_2) + V_{1,2}((q_2 + r)w_2 - q_2 w_1) - (q_1 + r)V_1(w_1, w_2). \end{aligned}$$

When $w_2 = f_1(w_1)$, the right side is 0 (according to the HJB equation) and $V_{1,2} = V_{1,22} = V_{1,12} = 0$. We have

$$0 \leq \frac{\partial(\text{the right side})}{\partial w_2}\Big|_{w_2 = f_1(w_1)} = q_1(V_{2,2}(w_1, f_1(w_1)) - V_{1,1}(w_1, f_1(w_1))).$$

Q.E.D.

Lemma 9 *If $V_{1,1} < V_{2,2}$ on f_1, then $V_1 \geq V_2$ for all $w_2 \leq f_1(w_1)$. Similarly, if $V_{1,1} > V_{2,2}$ on f_2, then $V_1 \leq V_2$ for all $w_1 \leq f_2(w_2)$.*

Proof of Lemma 9: If $V_{1,1} < V_{2,2}$ on f_1, then starting from $(w_1, f_1(w_1))$, the time path with report 1 moves below f_1 because (25) implies

$$\frac{dV_{1,2}}{dt} = q_1 V_{1,1} - q_1 V_{2,2} + (q_1 - q_2)V_{1,2} < 0.$$

This means that on the time path, the slack control variable μ_2 is 0. But V_2 is the cost of controlling the process with one more slack variable (μ_1), and having more control variables always lowers the cost. Thus $V_2(w_1, w_2) \leq V_1(w_1, w_2)$, for $w_2 \leq f_1(w_1)$. *Q.E.D.*

Lemma 10 *On efficiency lines,*

$$V_{1,1}(w_1, f_1(w_1)) < V_{2,2}(w_1, f_1(w_1)), V_{1,1}(f_2(w_2), w_2) = V_{2,2}(f_2(w_2), w_2).$$

Proof of Lemma 10: There are four possibilities, and we will rule out three of them.

(i) If $V_{1,1} = V_{2,2}$ on both f_1 and f_2, since $V_{1,1}(w_1, w_2) - V_{2,2}(w_1, w_2)$ is strictly decreasing in w_2, it is a contradiction to the fact that f_1 is above f_2.

(ii) If $V_{1,1} = V_{2,2}$ on f_1, and $V_{1,1} > V_{2,2}$ on f_2, then Lemma 9 states that $V_2(w_1, w_2) \geq V_1(w_1, w_2)$, for $w_1 \leq f_2(w_2)$. In particular, since f_1 is to the left of f_2, we know that

$$V_1(w_1, f_1(w_1)) \quad \leq \quad V_2(w_1, f_1(w_1)).$$

But the assumption that $V_{1,1} = V_{2,2}$ on f_1 yields

$$\begin{aligned} V_1(w_1, f_1(w_1)) &= V_{1,1}(w_1, f_1(w_1))\,(1/r + x_1) \\ &> V_{2,2}(w_1, f_1(w_1))\,(1/r + x_2) \\ &> V_{2,1}(w_1, f_1(w_1))\,(1/r + x_1) + V_{2,2}(w_1, f_1(w_1))\,(1/r + x_2) \\ &= V_2(w_1, f_1(w_1)), \end{aligned}$$

where the first line follows from (40) and $V_{1,2} = 0$ on f_2, the second line from $x_1 > x_2$, the third line from $V_{2,1} < 0$ on f_2, and the last line from (40) again. Thus there is a contradiction.

(iii) If $V_{1,1} < V_{2,2}$ on f_1, and $V_{1,1} > V_{2,2}$ on f_2, then Lemma 9 implies that $V_1(w_1, w_2) = V_2(w_1, w_2)$ if $f_2^{-1}(w_1) \leq w_2 \leq f_1(w_1)$. But then $V_{2,1} = V_{1,1} > 0$ contradicts part (iii) in Lemma 2.

Q.E.D.

APPENDIX C: Extensions to Other Homogeneous Utilities

In Section 5 we derive the results on the dynamics of the optimal contract with logarithmic utility and disutility functions. These results can be generalized to include functional forms as $\frac{c^{1-\sigma}}{1-\sigma}$ and $-\exp(-\sigma c)$, as long as some form of homogeneity exists. Although the exact form of homogeneity varies in each case, all the interesting results (including efficiency lines, absorbing region, dynamic properties of policy functions, and immiserization) remain unchanged. We will briefly lay out results from the following three extensions, most of which can be proved by the same techniques used for the logarithmic case.

(i) $\sigma > 0$: $u(c) = -\exp(-\sigma c)$, $c \in \mathbb{R}$, and $v(y) = \exp(\sigma y)$, $y \in \mathbb{R}$.

(ii) $\sigma > 1$: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $c > 0$, and $v(y) = -\frac{(-y)^{1-\sigma}}{1-\sigma}$, $y < 0$.

(iii) $0 < \sigma < 1$: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $c > 0$, and $v(y) = -\frac{(-y)^{1-\sigma}}{1-\sigma}$, $y < 0$.

Different from the logarithmic case, here the sets of implementable utilities W are proper subsets of $\mathbb{R}^2$. Although utility and disutility functions are unbounded here, we can still use the four families of contracts discussed in Section 4 to find boundaries of W. (Note that some of the boundaries are degenerate, and W is no longer compact.) For each of the above three cases, we get

(i) $W = \left\{(w_1, w_2) \in \mathbb{R}^2 : w_1 < w_2 < \frac{\theta_1}{\theta_2} w_1 < 0\right\}$.

(ii) $W = \left\{(w_1, w_2) \in \mathbb{R}^2 : w_1 < w_2 < \frac{\theta_1}{\theta_2} w_1 < 0\right\}$.

(iii) $W = \left\{(w_1, w_2) \in \mathbb{R}^2 : 0 < \frac{\theta_1}{\theta_2} w_1 < w_2 \leq w_1\right\} \cup \{(0,0)\}$.

The homogeneity properties for the logarithmic case in Lemma 2 are modified as

(i) For any $\lambda > 0$,

$$V_i(\lambda w_1, \lambda w_2) = -\frac{\log(\lambda)}{\sigma r} + V_i(w_1, w_2), i = 1, 2. \tag{49}$$

(ii) For any $\lambda > 0$,

$$V_i(\lambda w_1, \lambda w_2) = \lambda^{\frac{1}{1-\sigma}} V_i(w_1, w_2), i = 1, 2. \tag{50}$$

(iii) For any $\lambda > 0$,

$$V_i(\lambda w_1, \lambda w_2) = \lambda^{\frac{1}{1-\sigma}} V_i(w_1, w_2), i = 1, 2. \tag{51}$$

As to Lemma 2, the monotonicity properties

$$V_{1,2} \leq 0, V_{1,1} > 0, V_{2,1} \leq 0, \text{ and } V_{2,2} > 0$$

still hold, but the proofs for them need modifications.

(i) To see that $V_{1,1} > 0$, differentiating equation (49) at $\lambda = 1$, we have

$$V_{1,1} w_1 + V_{1,2} w_2 = -\frac{1}{\sigma r}. \tag{52}$$

It follows from $w_1 < 0$, $w_2 < 0$, and $V_{1,2} \leq 0$ that $V_{1,1} > 0$.

(ii) To see that $V_{1,1} > 0$, differentiating equation (50) at $\lambda = 1$, we have

$$V_{1,1} w_1 + V_{1,2} w_2 = \left(\frac{1}{1-\sigma} - 1\right) V_1. \tag{53}$$

It follows from $(\frac{1}{1-\sigma} - 1) < 0$, $V_1 > 0$, $w_1 < 0$, $w_2 < 0$, and $V_{1,2} \leq 0$ that $V_{1,1} > 0$.

(iii) To see that $V_{1,1} > 0$, differentiating equation (51) at $\lambda = 1$, we have

$$V_{1,1} w_1 + V_{1,2} w_2 = \left(\frac{1}{1-\sigma} - 1\right) V_1. \tag{54}$$

It follows from $(\frac{1}{1-\sigma} - 1) > 0$, $V_1 > 0$, $w_1 > 0$, $w_2 > 0$, and $V_{1,2} \leq 0$ that $V_{1,1} > 0$.

The definitions of the two efficiency lines remain unchanged; however, the domains of the lines are restricted to be $(-\infty, 0)$ in case (i), $(-\infty, 0)$ in case (ii), and $[0, \infty)$ in case (iii). They still are straight lines, but are no longer parallel to each other. Instead, f_1 and f_2 will have an intersection point at $(0, 0)$. LEMMAS 3 and 4 still hold and can be proved using the same proofs. The two HJB equations remain unchanged. The ODE system is still valid, except that the optimal policies from $\min_c\{c - (V_{1,1} + V_{1,2})u(c)\}$ and $\min_y\{-y + (V_{1,1}/\theta_1 + V_{1,2}/\theta_2)v(y)\}$ depend on particular functional forms (they are no longer $c = (V_{1,1} + V_{1,2}), y = -(V_{1,1}/\theta_1 + V_{1,2}/\theta_2)$).

Away from the intersection point $(0, 0)$, line f_1 is still strictly above f_2, thus LEMMA 5 remains valid and can be proved using the same proof. The set of W is again split into three regions. LEMMA 6 is still valid, and when the Markov chain is symmetric, the region between f_2 and the lower boundary of W is absorbing. Because the proof of LEMMA 6 uses the homogeneity property of logarithmic utilities, we need to modify the proof to utilize equations (49), (50), and (51). (The procedure is routine and details are omitted.)

Last, all the implications from THEOREM 3 remain valid.

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