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Abstract

People think by analogies and comparisons. Such way of thinking, termed coarse thinking by Mullainathan et al [Quarterly Journal of Economics, May 2008] is intuitively very appealing. We derive a new option pricing formula based on the assumption that the market consists of coarse thinkers as well as rational investors. The new formula, called the behavioral option pricing formula is a generalization of the Black-Scholes formula. The new formula not only provides explanations for the implied volatility skew and term structure puzzles in equity index options but is also consistent with the observed negative relationship between contemporaneous equity price shocks and implied volatility.

Keywords: *Coarse Thinking, Option Pricing, Implied Volatility, Implied Volatility Skew, Implied Volatility Smile, Implied Volatility Term Structure*

JEL Classification: G12, G13

Coarse Thinking, Implied Volatility, and Valuation of Call and Put Options

People think by analogies. In fact, comparisons are so important that our language is filled with metaphors and analogies. Perhaps, analogies enable us to construct mental models which are useful in generating new inferences.
In an interesting paper, Mullainathan, Schwartzstein & Shleifer (2008) formalize "thinking by analogy" in the context of a model of persuasion. Their model is based on the notion that agents use analogies for assigning values to attributes (the attribute valued in their model is "quality"). The idea is that people co-categorize situations that they consider analogous and assessment of attributes in a given situation is affected by other situations in the same category. This way of drawing inferences, which is termed coarse thinking, is in contrast with rational (Bayesian) thinking in which each situation is evaluated logically (often deductively), in isolation, and according to its own merit. Coarse thinking appears to be a natural way of modeling how humans process information. See Kahneman & Tversky (1982), Lakoff (1987), Edelmen (1992), Zaltman (1997), and Carpenter, Glazer, & Nakamoto (1994) among others.

Anecdotal evidence of the role of coarse thinking is all around us. In fact, Mullainathan et al (2008) use the advertising theme of Alberto Culver Natural Silk Shampoo as a motivating example to explain coarse thinking. The shampoo was advertised with a slogan "We put silk in the bottle." The company actually put some silk in the shampoo. However, as conceded by the company spokesman, silk does not do anything for hair (Carpenter et al (1994)). Then, why did the company put silk in the shampoo? Mullainathan et al (2008) write that the company was relying on the fact that consumers co-categorize shampoo with hair. This co-categorization leads consumers to value "silk" in shampoo because they value "silky" in hair (clearly not a rational response). That is, a positive trait from hair is transferred to shampoo by adding silk to it. Such transfer of the perceived informational content of an attribute across co-categorized situations is termed *transference*.

In this article, we raise the following question. Given undeniable evidence of the role of coarse thinking in almost everything we do, what are the implications for options pricing if some investors are coarse thinkers? Intuitively, an in-the-money call option is similar to its underlying stock. So rather than investing in the underlying outright, some investors prefer to buy in-the-money calls instead. An in-the-money call option offers the same dollar-for-dollar increase or decrease in payoff as the underlying; however, it only requires a fraction of investment. However, this (leveraging) advantage comes at a cost. Of course, an in-the-money call is riskier than the underlying as one can lose all of his investment in the event of an adverse price change, whereas a fraction of investment can (almost) always be recovered if one invests in the underlying. A rational investor, consequently deduces, that an in-the-money call, even though similar to the underlying, is riskier. Hence, he demands a higher expected return than what he demands for holding the underlying. A coarse thinker, on the other hand, co-categorizes an in-the-money call with the underlying and equates (mistakenly) the expected return of the two. That is, the price he is willing to way is determined in *transference* with the underlying stock by equating the expected returns. In other words, a coarse thinker is willing to pay a higher price for an in-the-money call option than a rational investor. If market frictions prevent rational investors from making arbitrage profits at the expense of coarse thinkers, both types will survive, and the price dynamics of in-the-money call options (and corresponding out-of-the-money put options via put-call parity) will be affected.

In this article, we formalize the intuition described above and derive closed form solutions for call and put options. We call these formulae the behavioral option pricing formulae. We then investigate the implications for implied volatility if actual price dynamics are determined according to the behavioral formula and the Black-Scholes formula is used to back-out implied volatility. Our findings are consistent with the observed implied volatility skew pattern in equity index options and with the observed term structure of implied volatility. So implied volatility skew puzzles are resolved if coarse thinking is incorporated into option pricing formulae. Furthermore, the behavioral approach provides an alternative explanation for the observed negative relationship between contemporaneous equity price shocks and implied volatility.

Despite early recognition of a key problem with the Black-Scholes formula (implied volatility skew), the formula remains perhaps one of the most widely used in the world; reasons being its ease of use (existence of a closed form solution) and lack of an alternative. The behavioral formula is a promising alternative since it is also easy to implement (closed from solution exists) and is essentially a generalization of the original Black-Scholes formula.

Coarse thinking or analogy based reasoning is likely to play an important role in understanding financial market behavior. Many researchers have pointed out that there appears to be clear departures from Bayesian thinking (Babcock & Loewenstein (1997), Babcock, Wang, & Loewenstein (1996), Hogarth & Einhorn (1992), Kahneman & Frederick (2002), Kahneman, Slovic, & Tversky (1982)). Such departures from rational thinking have been measured both at the individual as well as the market level (Siddiqi (2009a), Kluger & Wyatt (2004)). However, the question of what type of behavior to allow for if non-Bayesian behavior is admitted is a difficult one to address in the absence of an alternative which is amenable to systematic analysis. Coarse thinking may provide such an alternative especially when the intuitive appeal of analogy based reasoning is undeniable.

This paper is organized as follows: In section 2, we explain the hypothesis of coarse thinking in the context of a simple three-state world, and derive a price prediction, which can be experimentally tested against alternatives. In fact, if one scans the vast experimental literature, one finds that a similar test has already been conducted under a different name and results reported in

Rockenback (2004). As the hypothesis of coarse thinking is formalized in Mullainathan et al (2008), a few years after the experiment, the results were interpreted slightly differently. We discuss the similarities and differences. In section 3, the new option pricing formula is derived. In section 4, its implications for implied volatility skew are discussed. Section 5 discusses the limits to arbitrage that may stop rational investors from arbitraging coarse thinkers out of the market. Section 6 concludes.

2. Coarse Thinking: A Simple Example

Consider a simple three state world. The equally likely states are Red, Blue, and Green. There is a stock with payoffs $X_1, X_2, and\ X_3$ corresponding to states Red, Blue, and Green respectively. The state realization takes place at time 1. The current time is time 0. For simplicity, we assume the discount rate to be 0. The current price of the stock is P. There is another asset, which is a call option on the stock. By definition, the payoffs from the call option in the three states are:

$$C_1 = max\{(X_1 - K), 0\}, C_2 = max\{(X_2 - K), 0\}, C_3 = max\{(X_3 - K), 0\} \quad (1)$$

where K is the striking price, and $C_1, C_2, and\ C_3$ are the payoffs from the call options corresponding to Red, Blue, and Green states respectively.

As can be seen, the payoffs in the three states depend on the payoffs from the stock in corresponding states. Furthermore, by appropriately changing the striking price, the call option can be made more or less similar to the underlying stock with the similarity becoming exact as K approaches zero (all payoffs are constrained to be non-negative). For simplicity, assume:

$X_1 - K > 0, X_2 - K > 0, and\ X_3 - K > 0.$

How much is a coarse thinker willing to pay for this call option?

A coarse thinker co-categorizes this call option with the underlying and values it in *transference* with the underlying stock. In other words, a coarse thinker values the option in such a way so as to equate the expected return on the call option with the expected return on the underlying.

We denote the return on an asset by $q \in Q$, where Q is some subset of $\Re$ (the set of real numbers). In calculating, the return of an asset, a coarse thinker faces two similar, but not identical, observable situations, $s \in \{0,1\}$. In $s = 0$, "return demanded on the call option" is the attribute of interest and in $s = 1$, "actual return available on the underlying stock" is the attribute of interest. The coarse thinker has access to all the information described above. We denote this public information by r.

The actual expected return available on the underlying stock is given by,

$$E[q \mid r, s = 1] = \frac{\{X_1 - P\} + \{X_2 - P\} + \{X_3 - P\}}{3 \times P} \tag{2}$$

For a coarse thinker, the expected return demanded on the call option is:

$$E[q|r, s = 0] = E[q|r, s = 1] = \frac{\{X_1 - P\} + \{X_2 - P\} + \{X_3 - P\}}{3 \times P} \tag{3}$$

So, the coarse thinker infers the price of the call option, P_c , from:

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\} + \{C_3 - P_c\}}{3 \times P_c} = \frac{\{X_1 - P\} + \{X_2 - P\} + \{X_3 - P\}}{3 \times P} \tag{4}$$

It follows,

$$P_c = \frac{C_1 + C_2 + C_3}{X_1 + X_2 + X_3} \times P \tag{5}$$

Given co-categorization of the call option with the underlying stock, coarse thinkers choose a price for the option that equates the expected return on the option with the expected return on the underlying stock (*transference*). A coarse thinker prices the call option in analogy with the underlying stock. The underlying stock has a certain link between the payoffs and price, which is captured by the concept of expected return. While pricing with analogy, the same link is transferred to the asset being priced.

2.1 Experimental Evidence on Coarse Thinking

Rockenbach (2004) presents an experiment in which individuals' willingness to pay for an in-the-money call option is measured. The main finding is that a hypothesis that says "a call option is priced in a manner that equates the expected return on the underlying with the expected return on the option" outperforms other hypotheses. The results are interpreted as supporting a particular form of mental accounting hypothesis in which the underlying and the call option are placed in the same mental account (hence, the equality of expected returns). This is the hypothesis in this article also; however, there is a crucial difference. According to the coarse thinking hypothesis, in order for there to be an equality of expected returns in the mind of a coarse thinker, the call option must be similar to the underlying. That is, the call option must be in-the-money. Rockenbach (2004) happens to use a deep in-the-money call option but the significance of the similarity (due to the option being deep-in-the-money) is not emphasized. So, the hypothesis in Rockenbach (2004) is presented as being applicable to all call options whereas, according to the coarse thinking hypothesis, it is only applicable to in-the-money call options. So, coarse thinking is equivalent to conditional mental accounting, the condition being similarity of the call option with the underlying. Hence, as a call option becomes less and less in-the-money, the performance of the hypothesis of equality of expected returns (mental accounting/coarse thinking) should weaken.

Siddiqi (2009b) replicates the experiment in Rockenbach (2004) and varies the similarity systematically. The finding is that indeed as a call option becomes less and less in-the-money (and goes out-of-the money in one state), the performance of the hypothesis of equality of returns (mental accounting/coarse thinking) worsens.

Next, we show how the Black-Scholes formula changes if instead of assuming rational investors, both rational investors and coarse thinkers are assumed to co-exist. We will see that the new formula, which can be considered a generalization of the original Black-Scholes formula, provides a potential solution to the volatility skew puzzle as well as explains the term structure of implied volatility. The new approach also provides an alternative explanation for the observed negative relationship between contemporaneous equity price shocks and implied volatility.

3. The Coarse Thinking Option Pricing Formula

Black. F, and Sholes, M. (1973), and Merton R. (1973), independently put forward an option pricing model that paved the way for numerous advances in finance. Specifically, they came up with a way to price a financial option without appealing to the risk preferences of investors. The model revolutionized the world of finance and is now famously known as the Black-Scholes option pricing model.

Here, we first briefly sketch the standard derivation of the Black-Scholes formula so that the nature of the puzzling behavior of implied volatility becomes clear to the reader.[1] Dividends are assumed to be zero throughout this article for simplicity. All options are European.

[1] A reader interested in the formal derivation can consult any standard graduate text on derivative pricing.

In deriving the Black-Scholes formula, it is assumed that the price of the underlying follows a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dZ \tag{6a}$$

where S is the stock price, μ is a constant denoting the expected return on the underlying stock, σ is a constant denoting the standard deviation of return, and dZ is a random variable which is an accumulation of a large number of independent random effects over an interval dt. dZ has a mean of zero. It can be shown that variance of dZ scales with the length of the time interval under consideration.

That is,

$$Var[dZ] \propto dt$$

$$\Rightarrow \sqrt{Var[dZ]} \propto \sqrt{dt}$$

It follows,

$$dZ \sim n\sqrt{dt}$$

where n is a standard normal variable with a mean equal to zero and a standard deviation equal to one.

The price of a European call option (C) is then considered as a function of the underlying stock price (S) and time (t), that is, $C = f(S,t)$. Ito's lemma leads to

$$dC = \{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\}dt + \{\frac{\partial C}{\partial S}\sigma S\}dZ \tag{6b}$$

By using a portfolio replication argument, the Black-Scholes PDE is then derived:

$$\frac{\partial C}{\partial t} + 1/2 \times \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{6c}$$

Equation (6c), with some variable transformations can be converted to a homogeneous heat equation, which can be solved with an appropriate boundary condition to yield the famous Black-Scholes formula for a European call option:

$$C = SN(d_1) - e^{-r(T-t)} KN(d_2) \tag{6d}$$

where K is the striking price, r is the risk-free interest rate, N(.) is cumulative standard normal distribution, $d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, and $d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$.

From Put-Call parity, the price of a European put option follows:

$$P = e^{-r(T-t)} K \cdot N(-d_2) - S \cdot N(-d_1) \tag{6e}$$

The only unobservable in equations (6d) & (6e) is σ, the standard deviation of stock returns. By plugging in the observables, the value of σ as implied by the observables can be backed out. One expects that if a number of call options are considered, each written on the same underlying, and differing only in their striking prices, then their implied standard deviations should be identical. After all, standard deviation of stock returns is a property of the underlying stock and similar call options written on the same underlying (differing only in striking prices) must reflect this fact. The implied volatility when plotted against the striking price must be a constant according to the Black-Scholes model as σ is a constant in the model.

When σ as implied by the market price of options written on the same equity index is plotted against the striking price, an interesting pattern is observed. In-the-money call options (and corresponding out-of-the-money put

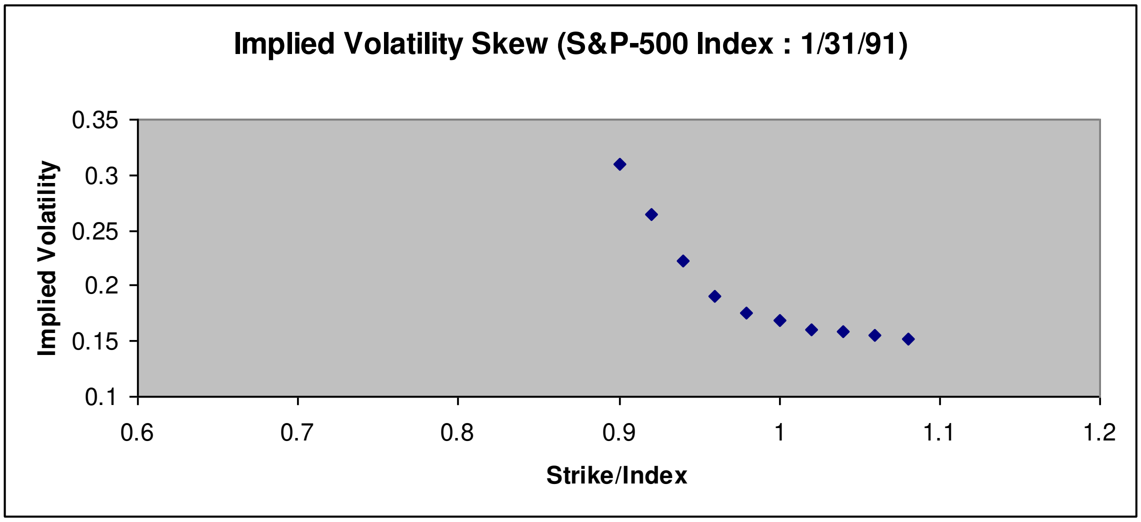
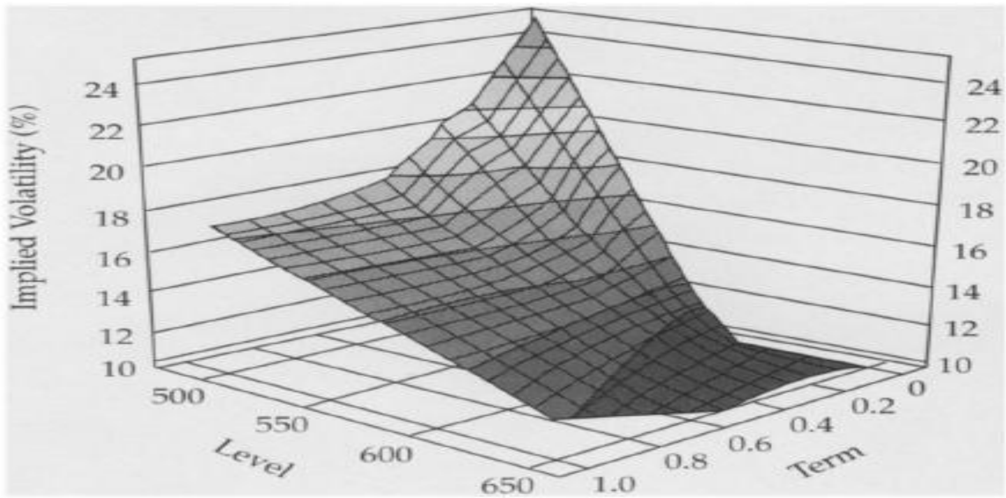


Figure 1

options) are found to have a higher implied volatility compared to at-the-money and out-of-the-money call options (corresponding at-the-money and in-the – money puts respectively). Figure (1) shows a typical pattern for S&P-500 equity index options. Similar patterns are observed for other equity index options (such as Nikkei and Dow Jones). The shape is that of a smile skewed to the left, hence, the name volatility skew. Why do we observe this pattern? Clearly, this pattern is indicating a problem with the Black-Scholes model as σ is a constant in the model.

There is an additional interesting pattern. Implied volatilities vary with time to expiry also. Often, implied volatilities tend to slope upwards with expiry, however, for deep-in-the-money calls (and corresponding deep-out-of-the-money puts), implied volatilities typically slope downwards with expiry, whereas according to the Black-Scholes model, implied volatility should not change with expiry. This phenomenon is known as the term structure of implied volatility.

Often, both the skew and the term structure are plotted together to create an implied volatility surface. Figure 2 shows a typical surface for S&P-500 index options. In contrast with the prediction of the Black-Scholes model (a flat plane),



The implied volatility surface of S&P-500 index options as a function of strike level and term to expiry on September 27, 1995.[2]

Figure 2

implied volatility surface clearly has a negative skew (for fixed expiry) as well as a term structure (for fixed strike).

The implied volatility surface is no t a constant. It changes with time. However, there are certain constant features. Firstly, as mentioned earlier, the negative skew for fixed expiry is a permanent feature. Secondly, as expiry increases, the volatility skew tends to flatten. This second feature is the result of the mostly upward slope of implied volatility for in-the-money calls (for fixed strike and increasing expiry) and the downward slope of deep-in-the-money calls.

Clearly, the implied volatility surface indicates a problem with the Black-Scholes model. Like any model, the Black-Scholes model is also a simplification of reality. The information contained in the implied volatility surface is the total

[2] Source: Derman, E., Kani, I., & Zou, J. (1996), "The local volatility surface: unlocking the information in index option prices", *Financial Analysts Journal*, 52, 4, 25-36

impact of factors that the Black-Scholes model ignores or simplifies away. Perhaps, a key factor ignored here is the presence of coarse thinkers in the market. We show that incorporating coarse thinking provides an explanation for the implied volatility skew as well as the term structure of implied volatility.

3.1 Behavioral Option Pricing with Coarse Thinking

The intuition behind the coarse thinking approach as applied to the pricing of financial options is as follows: Instead of buying the underlying outright, some investors prefer to buy in-the-money calls as in-the-money call options are similar to the underlying and require only a fraction of investment. Due to the similarity, some investors who are coarse thinkers (mistakenly) equate the expected return on the call option with the expected return on the underlying. That is, coarse thinkers co-categorize a call option with its underlying and price it with *transference* from the underlying. A rational investor, on the other hand, realizes that an in-the-money call option is riskier than the underlying and demands a higher expected return. Due to the differences in expected returns demanded, the presence of coarse thinkers alters the price dynamics of in-the-money call options (and corresponding out-of-the money put options via put-call parity). The question we consider is the following: How does option pricing formula change if coarse thinking is allowed in the model?

Let q denote the return on a given asset. In calculating, the return of an asset, investors face, two similar, but not identical, observable situations, $s \in \{0,1\}$. In $s = 0$, "return on the call option" is the attribute of interest and in $s = 1$, "return on the underlying stock" is the attribute of interest. Let I denote the information set.

Suppose the function describing the price of a call option is $C(S,t)$. Initially, assume that the market consists of rational investors only. The price at the next instant $(S + dS, t + dt)$ can be approximated by expanding around (S, t) in a Taylor series expansion:

$$C(S + dS, t + dt) = C(S,t) + \frac{\partial C}{\partial S}(S + dS - S) + \frac{\partial C}{\partial t}(t + dt - t) + \tfrac{1}{2}\frac{\partial^2 C}{\partial S^2}(S + dS - S)^2 + \tfrac{1}{2}\frac{\partial^2 C}{\partial t^2}(t + dt - t)^2 + \tfrac{1}{2}\frac{\partial^2 C}{\partial S \partial t}(S + dS - S)(t + dt - t) + higher\ order\ terms$$

$$C(S + dS, t + dt) = C(S,t) + \frac{\partial C}{\partial S}(dS) + \frac{\partial C}{\partial t}(dt) + \tfrac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 + \tfrac{1}{2}\frac{\partial^2 C}{\partial t^2}(dt)^2 + \tfrac{1}{2}\frac{\partial^2 C}{\partial S \partial t}(dS)(dt) + higher\ order\ terms \qquad (7a)$$

Substituting for dS from equation 6a to 7a:

$$C(S + dS, t + dt) = C(S,t) + \frac{\partial C}{\partial S}(\mu S dt + \sigma S dZ) + \frac{\partial C}{\partial t}(dt) + \tfrac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mu S dt + \sigma S dZ)^2 + \tfrac{1}{2}\frac{\partial^2 C}{\partial t^2}(dt)^2 + \tfrac{1}{2}\frac{\partial^2 C}{\partial S \partial t}(\mu S dt + \sigma S dZ)(dt) + higher\ order\ terms$$

(7b)

We know that,

$dZ \sim n\sqrt{dt}$ where n is a standard normal variable with a mean equal to zero and a standard deviation equal to one.

As $dt \to 0$, $(dt)^n\ (with\, n > 1) \to 0$ at a faster rate. So, 7b becomes (this is Ito's Lemma),

$$C(S + dS, t + dt) = C(S,t) + \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \tfrac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \left(\frac{\partial C}{\partial S}\sigma S\right)dZ \qquad (7c)$$

$$\Rightarrow dC = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \tfrac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \left(\frac{\partial C}{\partial S}\sigma S\right)dZ$$

$$\Rightarrow E[dC] = (\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2)dt$$

$$\Rightarrow E[q \mid I, s=0] = (\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2)dt \tag{7d}$$

Equation (7d) describes the expected return on the call option if the market consists of rational investors only.

Suppose the market consists of coarse thinkers only. By definition, coarse thinkers co-categorize a call option with its underlying stock, and price it in *transference* with the underlying.

Hence, the expected return on the call option if the market consists of coarse thinkers is:

$$E^C[q \mid I, s=0] = E[q \mid I, s=1] = E[dS \mid I] = \mu S dt \tag{8a}$$

$$\Rightarrow C(S+dS, t+dt) = C(S,t) + \mu S dt + c \cdot e \tag{8b}$$

where c is a constant and e has a mean of zero. The superscript c denotes coarse thinkers.

We know that, $\mu S dt \neq (\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2)dt$.[3] So, if the market consists of coarse thinkers, expected return on the call option is different from the expected return with rational investors. In other words, 7d does not hold, rather, it is replaced by 8a.

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[3] $\mu S dt < (\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2)dt$

In other words, the expected return on a call option demanded by rational investors is always larger than the expected return on the underlying as a call option is riskier than its underlying.

If the market consists of both rational investors as well as coarse thinkers, with the intensity of coarse thinking denoted by a factor $(1-a)$ with $0 < a \le 1$, we postulate,

$$E^G[q \mid I, s=0] = (1-a)E^C[q \mid I, s=0] + aE[q \mid I, s=0] = \mu S(1-a)dt + \{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\}adt \tag{9a}$$

where we have used the superscript G to denote a market where both rational investors as well as coarse thinkers are present.

If coarse thinkers and rational investors are simultaneously present, then $C(S,t)$ satisfies (9a). The function that satisfies 9a while being minimally different from 7c is,

$$C(S+dS, t+dt) = C(S,t) + \{\frac{\partial C}{\partial t}a + \frac{\partial C}{\partial S}\mu Sa + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 a + (1-a)\mu S\}dt + \{\frac{\partial C}{\partial S}\sigma S\}dZ \tag{9b}$$

Hence, if coarse thinkers are also present, the resulting stochastic process is,

$$dC = \{\frac{\partial C}{\partial t}a + \frac{\partial C}{\partial S}\mu Sa + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 a + (1-a)\mu S\}dt + \{\frac{\partial C}{\partial S}\sigma S\}dZ \tag{10}$$

A comparison of equation 10 with equation 6b shows that the presence of coarse thinkers alters the deterministic component of the stochastic process. This is exactly what one expects as the deterministic component determines the expected return and the presence of coarse thinkers changes the expected return.

Coarse thinking exists because of the similarity between an in-the-money call and its underlying stock. Similarity increases with the *moneyness* of the option; more in-the-money call options are more similar to the underlying stock.

Hence, the intensity of coarse thinking (the fraction of investors who are coarse thinkers) should increase with the *moneyness* of the call option. Considering this, we assume $a = K/S$

Substituting for a in equation 10,

$$dC = \{\frac{\partial C}{\partial t} \cdot \frac{K}{S} + \frac{\partial C}{\partial S} \mu K + 1/2 \times \frac{\partial^2 C}{\partial S^2} \sigma^2 S \cdot K + (S - K)\mu\}dt + \{\frac{\partial C}{\partial S} \sigma S\}dZ \qquad (11)$$

Equation (11) holds as long as $0 < K/S \leq 1$. For $K/S > 1$, coarse thinking disappears as the similarity disappears, and

$$dC = (\frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2)dt + (\frac{\partial C}{\partial S} \sigma S)dZ \qquad (12)$$

That is, if $K/S > 1$, coarse thinking disappears and we are back to the original Black-Scholes world with the stochastic process given by 6b and the price of a call option given by 6d.

Proposition 1 gives us the associated Partial Differential Equation (PDE) when both coarse thinkers and rational investors are present.

Proposition 1 ***If the stochastic process followed by the price of a call option is given by equation (10), then the associated PDE for option's price is***

$$\frac{\partial C}{\partial t} + 1/2 \times \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \{\frac{\mu S(1-a) - rS}{a}\} \frac{\partial C}{\partial S} + \frac{\mu S(1-a)}{a} - \frac{r}{a} C = 0 \qquad (13)$$

where $0 < a = K/S \leq 1$

Proof: See Appendix A.

Note, if $a = K/S = 1$, there are no coarse thinkers, and as expected, equation (13) reduces to equation (6c). Lower the value of $a = K/S$, greater is the difference between the coarse thinking PDE and the Black-Scholes PDE.

It is well known that the Black-Scholes PDE is reducible to a homogenous heat equation. The behavioral Black-Scholes PDE (equation (13)), on the other hand, is reducible to an inhomogeneous heat equation, as proposition 2 shows.

Proposition 2 ***The behavioral Black-Scholes PDE (equation (13)) is reducible to an inhomogeneous heat equation with appropriate variable transformations.***

Proof. Start by making the following substitutions in (13):

$\Gamma = \frac{\sigma^2}{2}(T-t)$; $x = \ln S - \ln K$; *and* $C = K \cdot V(x,t)$

It follows,

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial V}{\partial x} \cdot \frac{1}{S} \tag{14a}$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{K}{S^2} \cdot \frac{\partial V}{\partial x} + \frac{K}{S^2} \cdot \frac{\partial^2 V}{\partial x^2} \tag{14b}$$

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial V}{\partial \Gamma} \cdot \frac{\partial \Gamma}{\partial t} = -K \cdot \frac{\partial V}{\partial \Gamma} . \frac{\sigma^2}{2} \tag{14c}$$

With these substitutions in equation (13) and replacing S with Ke^x, it follows,

$$-\frac{\partial V}{\partial \Gamma} - \frac{\partial V}{\partial x}\left[\frac{2\{\mu(1-a) - r\}}{\sigma^2 a} + 1\right] + \frac{\partial^2 V}{\partial x^2} - \frac{2r}{a\sigma^2}V + \frac{2\mu(1-a)e^x}{a\sigma^2} = 0 \tag{15}$$

Now, make the substitution, $V = e^{\alpha x + \beta \Gamma} W$ in equation (15) where $\alpha = \frac{2q+1}{2}$,

$$\beta = -\frac{\{2q+1\}^2}{4} - \frac{2r}{a\sigma^2} \text{ , and } q = \frac{\mu(1-a)-r}{a\sigma^2}.$$

It follows,

$$\frac{\partial W}{\partial \Gamma} = \frac{\partial^2 W}{\partial x^2} + \left\{\frac{2\mu(1-a)}{a\sigma^2}\right\} e^{(1-\alpha)x-\beta\Gamma} \tag{16}$$

Equation (16) is similar to an inhomogeneous heat equation.

■

Note that in equation (16) if $a = K/S = 1$, it becomes a homogeneous heat equation.

Of course, this is exactly what we expect since when $a = 1$, there are no coarse thinkers to cause price distortions and the original Black-Scholes equation is recovered.

Proposition 3 describes the behavioral Black-Scholes formula.

Proposition 3 ***The solution to the behavioral PDE (equation (13)) with*** $\mathbf{0} < a = K/S \le 1$ ***is*** $C = Se^{\frac{-\mu(S-K)(T-t)}{K}} \left\{N(d_1) + f \cdot \frac{1}{Q}\left(e^{Q\Gamma} - 1\right)\right\} - e^{\frac{-r(T-t)S}{K}} \cdot K \cdot N(d_2)$

(17)

where,

$$f = \frac{2\mu(S-K)}{K\sigma^2}$$

$$Q = \frac{(2q+1)^2}{4} + \frac{2rS}{K\sigma^2} \quad ; \left(q = \frac{\mu(S-K)-rS}{\sigma^2 K}\right)$$

$$\Gamma = \frac{\sigma^2}{2}(T-t)$$

$$d_1 = \frac{x}{\sqrt{2\Gamma}} - (2q-1)\sqrt{\frac{\Gamma}{2}}$$

$$d_2 = \frac{x}{\sqrt{2\Gamma}} - (2q+1)\sqrt{\frac{\Gamma}{2}}$$

$N(.)$ is cumulative standard normal distribution.

$$x = ln(\frac{S}{K})$$

Proof. Solving equation (16) by using Duhamel's principle and substituting to recover original variables leads to the behavioral Black-Scholes formula (equation (17)). Steps are shown in Appendix B.

Corollary 3.1 ***If*** $\boldsymbol{\frac{K}{S} = 1}$***, the behavioral option pricing formula for a European call option (equation (17)) reduces to the original Black-Scholes formula for a European call option (equation (6d)).***

Proof. By comparison.

The behavioral option pricing formula derived in this paper can be considered a generalization of the original Black-Scholes formula. The original formula (equation (6d)) is a limiting or a special case of the behavioral option pricing formula (equation (17)), which is recovered if $K/S = 1$.

Proposition 4 ***The Price of a European Put Option with*** $\mathbf{0} < K/S \leq 1$ ***is given by,***

$$\boldsymbol{P} = \boldsymbol{S}\left\{\boldsymbol{e}^{\frac{-\boldsymbol{\mu}(\boldsymbol{S}-\boldsymbol{K})(\boldsymbol{T}-\boldsymbol{t})}{\boldsymbol{K}}}\left\{\boldsymbol{N}(\boldsymbol{d_1}) + f \cdot {}^{1}\!/_{Q}\,(e^{Q\tau} - 1)\right\} - \mathbf{1}\right\}$$

$$+\boldsymbol{K}\left\{\boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})} - \boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})\boldsymbol{S}/\boldsymbol{K}} \cdot \boldsymbol{N}(\boldsymbol{d_2})\right\} \qquad \textbf{(18)}$$

Proof. Follows directly from put-call parity.

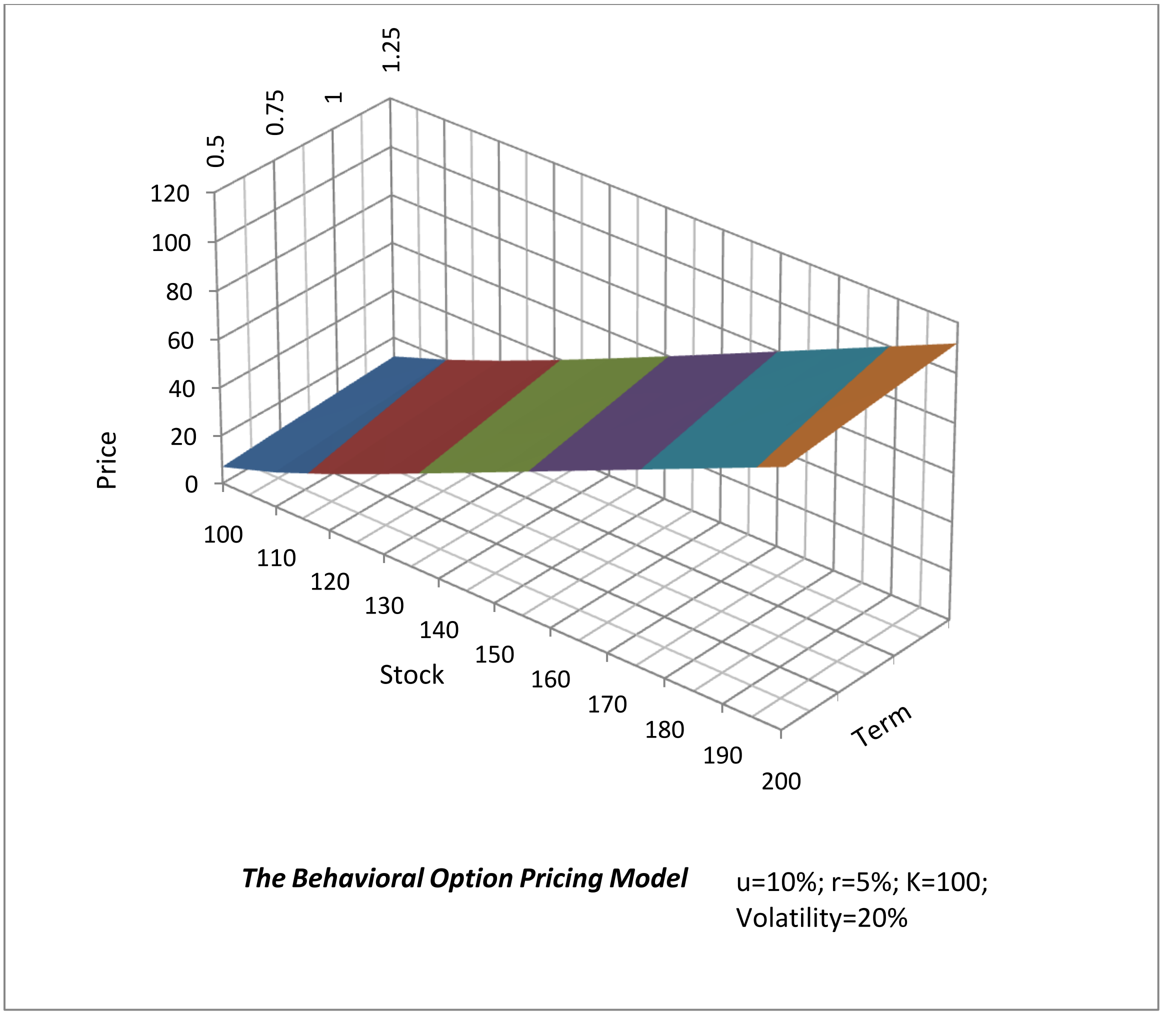


Figure 3

Corollary 4.1 ***If*** $\frac{K}{S} = \mathbf{1}$***, the behavioral option pricing formula for a European put option (equation (18)) reduces to the original Black-Scholes formula for a European put option (equation (6e)).***

Proof. By comparison.

4. Behavioral Option Pricing and Implied Volatility

Figure 3 shows the price of an in-the-money call option according to the behavioral formula as the price of the underlying stock and expiry changes.

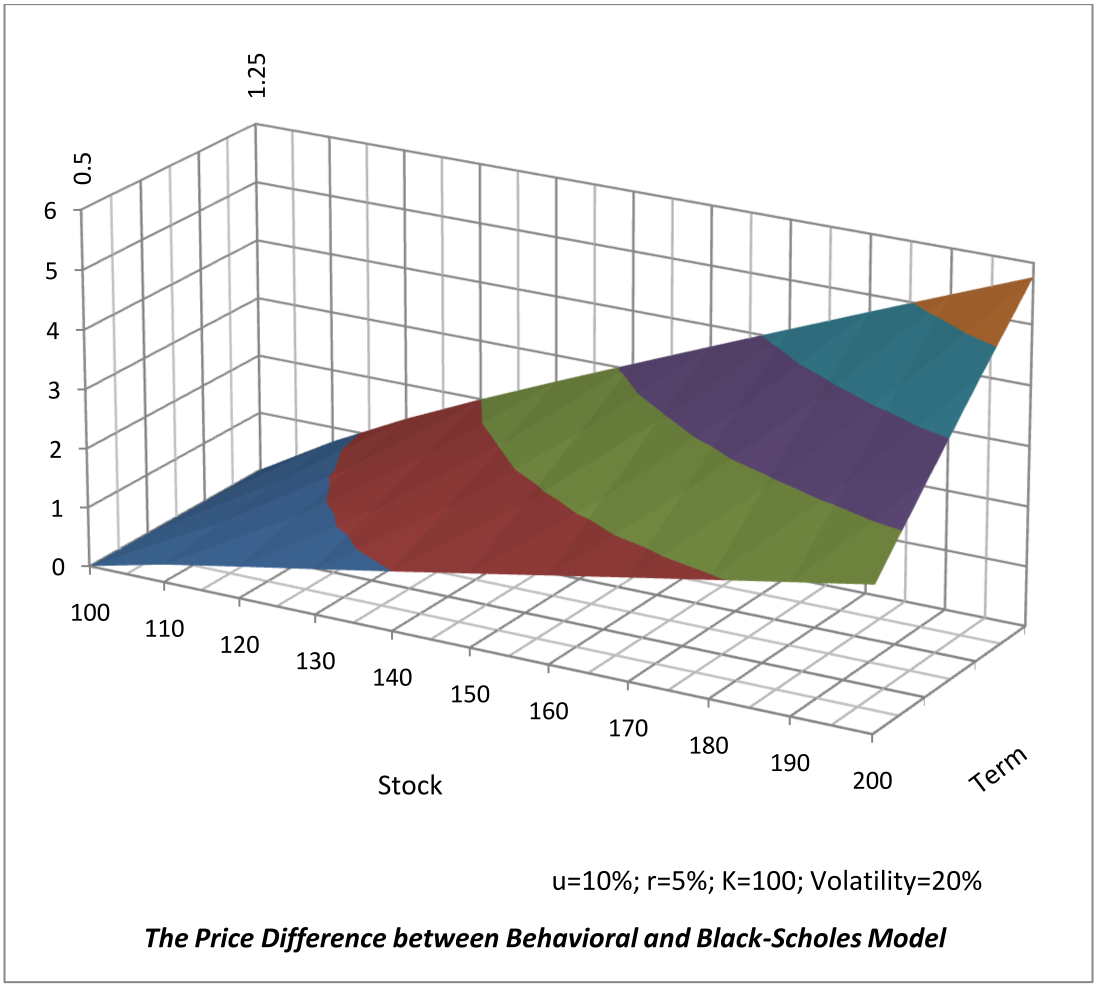


Figure 4

Figure 4 shows the price difference between the behavioral and the Black-Scholes formula as the price of the underlying and expiry increases. As can be seen, the price difference between the two is always positive. The difference is higher for deep-in-the-money options. The price difference also steepens with expiry, even more so when the option is deep-in-the-money. This behavior is consistent with our intuition as the source of the price difference is coarse thinking, which gets stronger as the option becomes more in-the-money.

4.1 Implied Volatility Skew

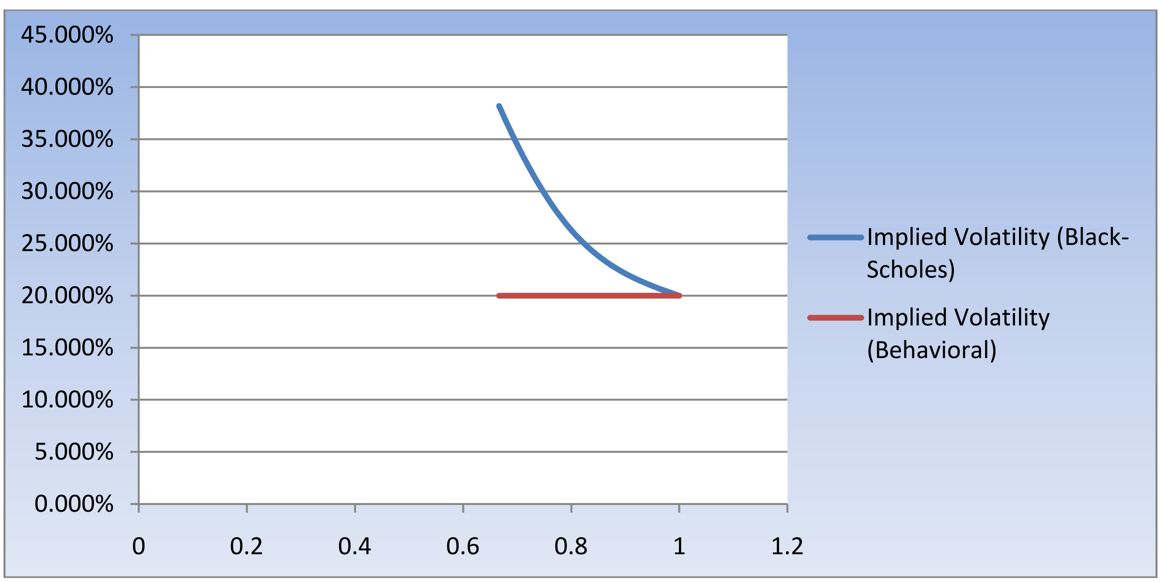
If coarse thinkers are present in the market then the correct option pricing formulae are given by equations 17 and 18. However, if equations 6d and 6e are used instead, to back out implied volatilities, then the implied volatility skew is observed. So, if one accounts for the presence of coarse thinkers and alters the formulae accordingly, implied volatility is a constant. Ignoring the impact of coarse thinking leads to the observed implied volatility skew.

Table 1 shows the prices of a European call option under the two approaches. As can be seen, the price under the behavioral approach is higher than the price under the Black-Scholes model with the prices converging as the stock price approaches the striking price from above. The presence of coarse thinkers changes the price dynamics as they demand a lower expected return than rational investors to hold an in-the-money call option. This pushes up the price of in-the-money call options. Consequently, a deviation between the behavioral and Black-Scholes price arises. Greater the *moneyness* of a call option, higher is the deviation from the Black-Scholes price as table 1 shows.

If the actual price dynamics are given by the behavioral approach and the Black-Scholes model is used to back-out implied volatilities, then a skew is seen as shown in figure 5. In figure 5, the behavioral prices as shown in table 1 (column 2) are used to back-out implied volatilities. That is, figure 5 shows the values of implied volatility if the Black-Scholes model is used to back-out implied volatility when the actual prices are determined by the behavioral formula.

The presence of implied volatility skew is a reflection of an error in the Black-Scholes model. The Black-Scholes model ignores the impact of coarse thinking. Once coarse thinking is taken into account, implied volatility is a constant as it should be.

Table 1 **Price of a European Call Option** $K = 100; (T - t) = 1\ year\ ; \sigma = 20\%; r = 5\%; \mu = 10\%$		
K/S	***Behavioral Option Pricing***	***Black-Scholes Price***
0.67	57.45	54.97
0.69	52.28	50.03
0.71	47.14	45.11
0.74	42.05	40.24
0.77	37.02	35.44
0.80	32.08	30.74
0.83	27.28	26.17
0.87	22.66	21.79
0.91	18.27	17.66
0.95	14.17	13.86
1.00	10.45	10.45
1.05	7.51	7.51



Implied Volatility plotted against Strike/Index.

Figure 5

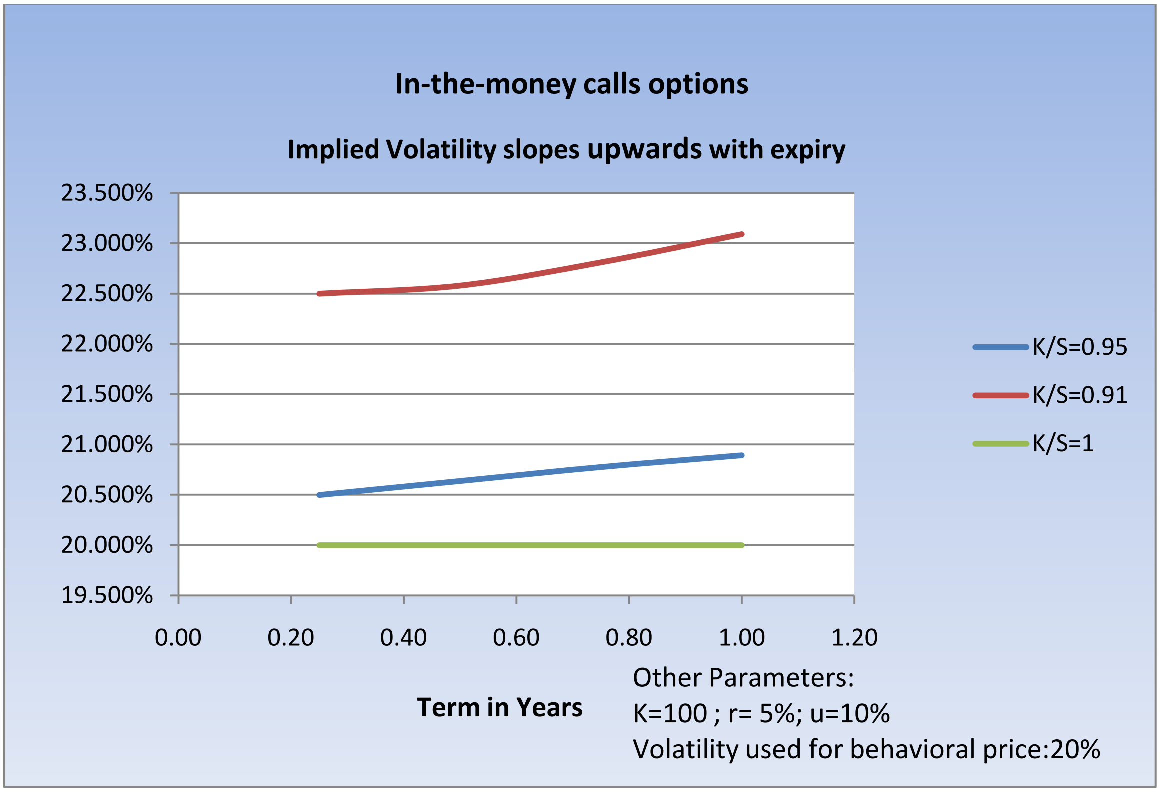


Figure 6

4.2 The Term Structure of Implied Volatility

As mentioned earlier, implied volatility changes with expiry also. That is, it has a term structure. As figure 2 shows, often implied volatility of in-the-money calls slopes upwards with expiry, whereas implied volatility of deep-in-the-money calls typically slopes downwards with expiry. This is reflected in flattening of the skew with expiry.

If the actual prices follow the behavioral formula and the Black-Scholes model is used to back-out implied volatility, then the implied volatility of in-the-money calls slope upwards with expiry as shown in figure 6, and the implied volatility of deep-in-the-money calls slopes downward as shown in figure 7. This is a remarkable match between theory and observation. Incorporating coarse thinking into the model not only explains the negative skew but also

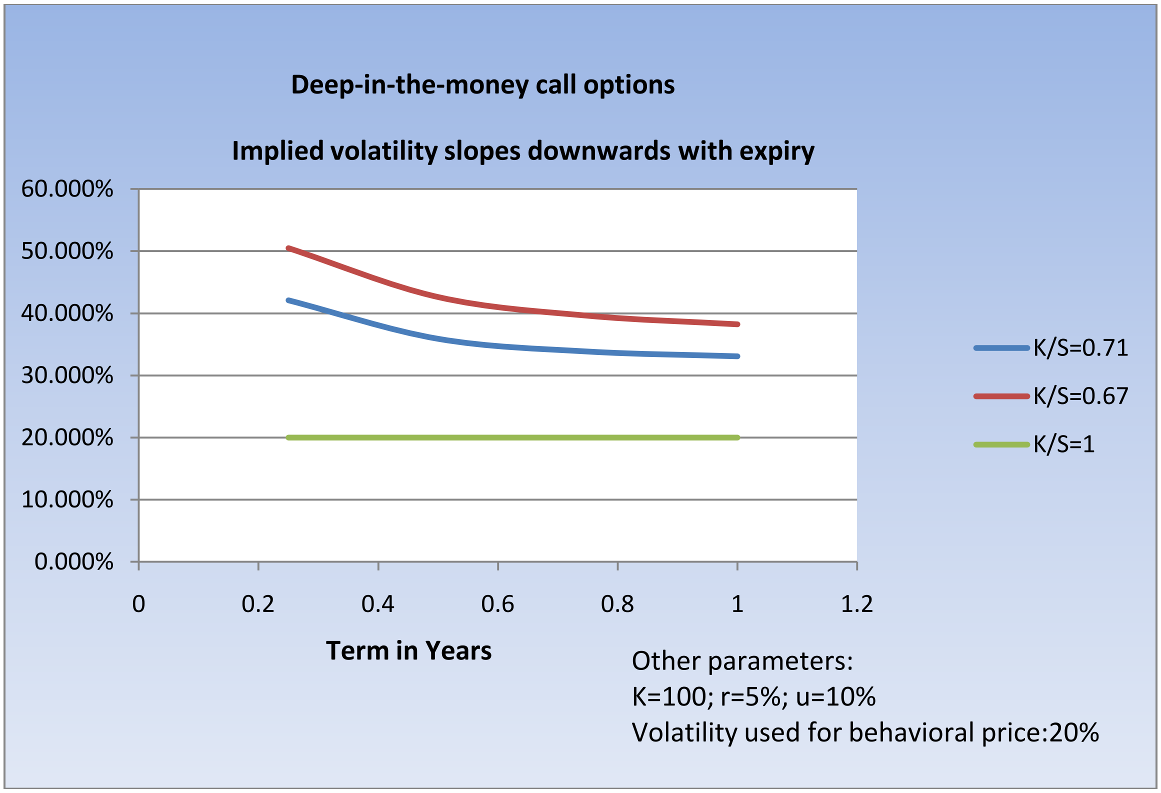
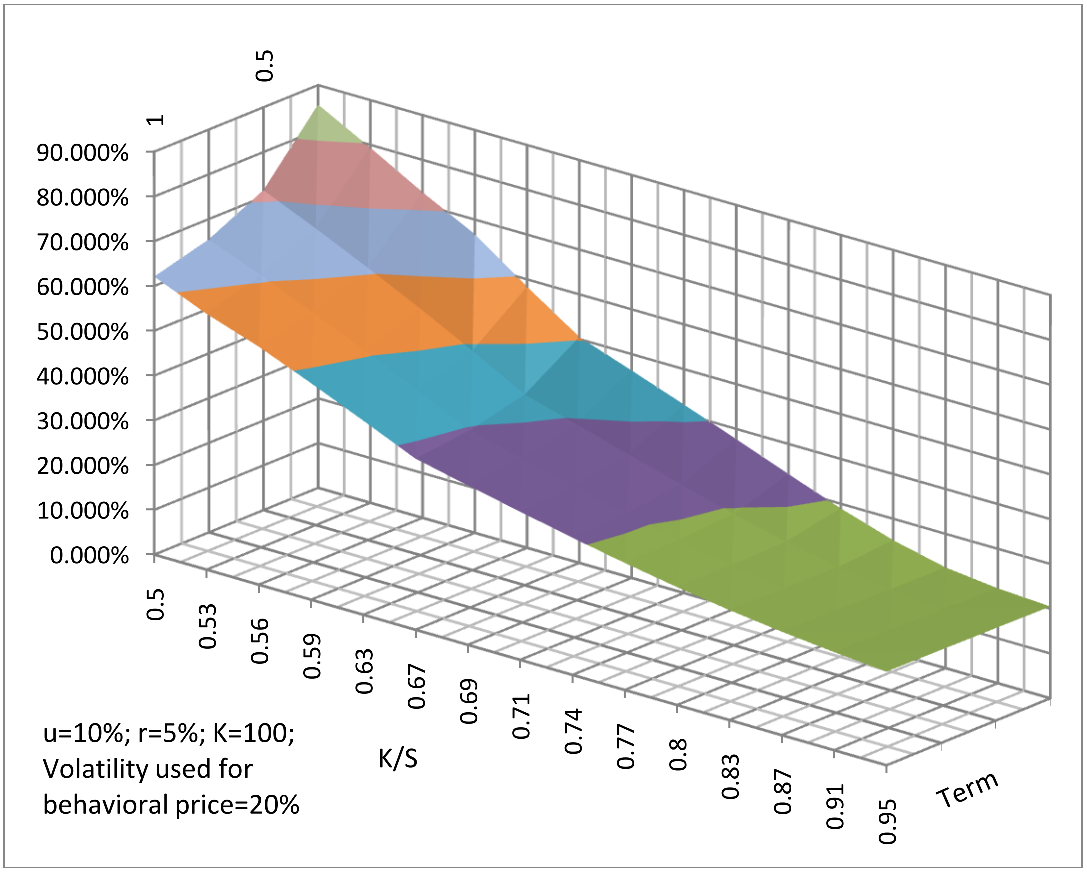


Figure 7

explains the term structure. Figure 8 plots the implied volatility surface. As can be seen, it is similar to figure 2.

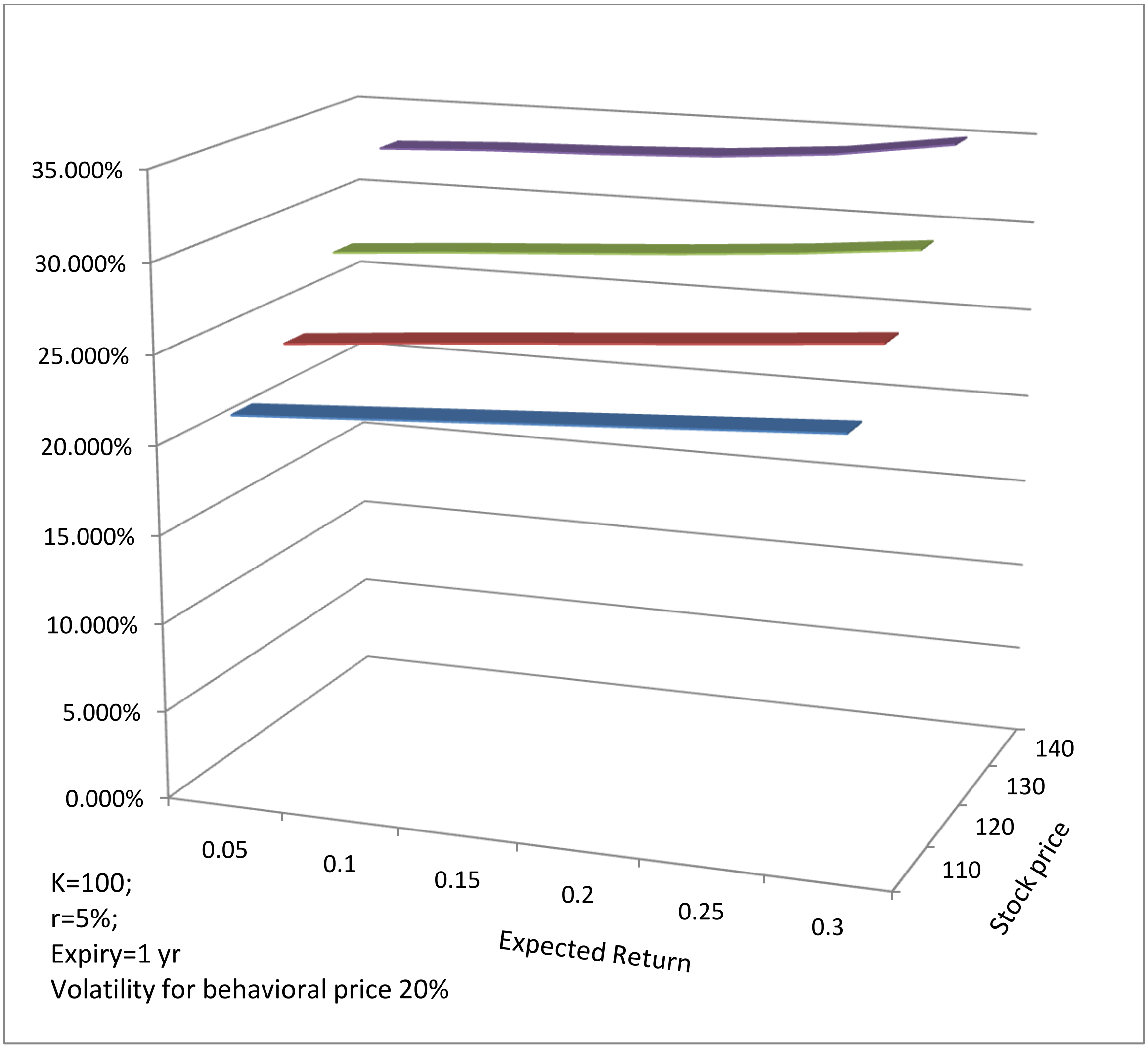


Implied volatility as a function of strike/index and term to expiry

Figure 8

4.3 Expected Return and Implied Volatility

The behavioral option pricing formulae (equations 17 & 18) have one additional parameter when compared with the Black-Scholes formulae. The additional parameter is μ or expected return on the underlying. The expected return has no direct impact on an option's price under the Black-Scholes approach. However, under the behavioral approach, expected return has a direct impact via coarse thinking. Figure 9 shows the relationship between expected return and implied volatility. As before, implied volatility is backed out from the Black-Scholes formula whereas option prices are determined by the behavioral formula.



The plot of Implied Volatility vs. Expected Return
Figure 9

As can be seen, there is a positive relationship between expected return and implied volatility.

The negative relationship between contemporaneous stock price changes and implied volatility is widely documented in the literature. Fleming J., Ostdiek B., and Whaley R. (1995) show that CBOE Market Volatility Index (VIX), an average of S&P 100 option implied volatilities, is inversely related to the contemporaneous S&P 100 index returns. Most studies show a negative correlation between current return shocks and implied volatility. See Schwert (1990), Schwert (1989), Christie (1982), and Black (1976) among others.

A popular theory, often invoked to explain this negative relationship, is the leverage effect hypothesis. According to this theory, as stock price falls, the value of equity as a percentage of total firm value falls. As equity bears the entire risk of the firm, its volatility should subsequently increase. However, Christie (1982) and Schwert (1989) argue that it is difficult to account for the current return – future volatility negative effect given realistic estimates of leverage.

The behavioral approach developed here offers an alternative explanation. Evidence of mean reversion in stock returns has been documented in the literature. See Debondt and Thaler (1985), Summers (1986), Fama and French (1988), and Poterba and Summers (1988) among others. Also, there is undeniable anecdotal evidence of wide-spread market belief in mean reversion. Statements such as "mid cap value has been on a roll, I think it's going to mean revert soon," or "stocks have been falling for a long time, so now is a good time to buy" are very common. A belief in mean reversion lowers expected return after a positive price shock and increases expected return after a negative price shock. Consequently, in accordance with the behavioral formula, implied volatility goes down after a positive price shock and goes up after a negative price shock. Hence, the negative relationship between current price shock and implied volatility is consistent with the behavioral approach.

5. The Limits to Arbitrage

If coarse thinkers and rational investors co-exist, a pertinent question is, can rational investors make arbitrage profits at the expense of coarse thinkers? If yes, then coarse thinkers would be driven out of the market, and coarse thinking would not matter for option pricing.

There are two cases to consider; investment horizon shorter than the expiry of the option, and investment horizon equal to the expiry of the option. If rational investors have a horizon shorter than the expiry of the option, then they can make arbitrage profits if the price distortion caused by the coarse thinkers disappears predictably before the option expires. If their horizon is till the expiry of the option, then they can make arbitrage profits if they can create a replicating portfolio with payoffs equal to that of the call option at expiry, and at a lower cost.

To include the two above mentioned cases, consider a simple scenario with three points in time; 1, 2, and 3. At time 1, the price of the call option according to rational investors is P_r and the price that the coarse thinkers are willing to pay is P_c. For concreteness and in accordance with the behavioral approach, we assume $P_c > P_r$. The actual market price deviates from P_r due to the presence of coarse thinkers to $V_1 = a \cdot P_r + (1-a) \cdot P_c$, where $(1-a)$ captures the intensity of coarse thinking. At time 2, the intensity of coarse thinking may either increase or diminish. If it increases, then the price will further deviate from the rational price. If it diminishes, the price will approach the rational price. Consequently, at time 1, a rational investor with a horizon limited to time 2, cannot be sure about his best strategy. If he thinks, that the intensity of coarse thinking will diminish, it may be optimal for him to sell call options. Otherwise, he may want to hold on till time 2 for further capital gains.

At time 3, both coarse thinkers and rational investors value the in-the-money call option at $V_3 = S - K$. So, a rational investor with a horizon till time 3, needs to do the following to make arbitrage profits: sell a call option at time 1 and buy a replicating portfolio simultaneously. Let $R_1 = P_r$ denote the value of the replicating portfolio at time 1. By definition of a replicating portfolio, its value at time 3 is $R_3 = V_3$. Let c denote the transaction cost of setting up the replicating portfolio, so time 1 payoff is $V_1 - R_1 - c$, and time 3 payoff is

$$-V_3 + R_3 = -V_3 + V_3 = 0.$$

Arbitrage profits exist if,

$$V_1 - R_1 > c.$$

However, at time 3, there are infinitely many payoff states, each corresponding to one particular value of S. Even if we admit a finite number of states, the replicating portfolio must have a large number of assets (number of assets must be equal to the number of states). So, the transaction costs involved in setting up a replicating portfolio are likely be significantly larger than the price deviation rational investor are trying to benefit from. Hence, limits to arbitrage may prevent rational investors from making arbitrage profits at the expense of coarse thinkers.

6. Conclusion

People think by analogies and comparisons. This way of thinking, termed coarse thinking by Mullainathan et al (2008), is intuitively very compelling. In this article, we raised the following question: What are the implications for option pricing if coarse thinking is admitted? In pursuit of an answer to this question, we derived closed form solutions for new option pricing formulae for European call and put options. We find that the new formulae, which can be considered generalizations of the original Black-Scholes formulae, provide an explanation for the implied volatility skew and term structure puzzles in equity index options. The coarse thinking approach also provides an alternative explanation for the observed negative relationship between contemporaneous equity price shocks and implied volatility.

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Appendix A

Consider a trading strategy in which one holds a call option and shorts $\frac{\partial C}{\partial S}$ of the underlying. The value of such a portfolio at a particular point in time t is,

$$\Pi = C - S \cdot \partial C / \partial S$$

At a later time, say, $t + dt$, the value of the portfolio may change. Let $d\Pi$ denote the change in portfolio value over the interval $[t, t + dt]$. That is,

$$d\Pi = dC - dS \cdot \partial C / \partial S \qquad \text{(A1)}$$

From (6a): $dS = \mu S dt + \sigma S dZ$

From (7c): $dC = \{\frac{\partial C}{\partial t} a + \frac{\partial C}{\partial S} \mu S a + 1/2 \times \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a \; + \; (1-a)\mu S\} dt + \{\frac{\partial C}{\partial S} \sigma S\} dZ$

So, $d\Pi = \left\{ \frac{\partial C}{\partial t} a - \frac{\partial C}{\partial S} \mu S(1-a) + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S \right\} dt \qquad \text{(A2)}$

(A2) is risk free since there is no dZ term in (A2). Let r be the risk free rate of return. On the portfolio Π, the return over dt should be $r\Pi dt$ in order to eliminate arbitrage. So,

$$r\Pi dt = \left\{ \frac{\partial C}{\partial t} a - \frac{\partial C}{\partial S} \mu S(1-a) + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S \right\} dt$$

$$\Rightarrow rC - rS \, \partial C / \partial S = \left\{ \frac{\partial C}{\partial t} a - \frac{\partial C}{\partial S} \mu S(1-a) + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S \right\}$$

$$\Rightarrow \frac{\partial C}{\partial t} + 1/2 \times \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \{\frac{\mu S(1-a) - rS}{a}\} \frac{\partial C}{\partial S} + \frac{\mu S(1-a)}{a} - \frac{r}{a} C = 0$$

where $0 < a \leq 1$

Appendix B

Equation (16) is similar to an inhomogeneous heat equation which can be solved by applying the Duhamel's principle. We need to solve,

$$\frac{\partial W}{\partial \Gamma} = \frac{\partial^2 W}{\partial x^2} + \left\{ \frac{2\mu(1-a)}{a\sigma^2} \right\} e^{(1-\alpha)x - \beta\Gamma}$$

Since $\Gamma = \frac{\sigma^2}{2}(T-t)$, the boundary condition $t = T$ is equivalent to the initial condition $\Gamma = 0$. It follows, $W(x,0) = e^{-\alpha x} V(x,0)$.

Since $C = K \cdot V$, therefore $W(x,0) = e^{-\alpha x} \frac{1}{K} \max\{S - K, 0\}$.

$x = \ln S - \ln K$ => $S = Ke^x$,

So, $W(x,0) = e^{-\alpha x} \frac{1}{K} \max\{Ke^x - K, 0\}$

$\Rightarrow W(x,0) = \max\{e^{(1-\alpha)x} - e^{-\alpha x}, 0\}$

$\Rightarrow W(x,0) = \max\left\{ e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0 \right\}$ since $\alpha = \frac{2q+1}{2}$.

So, we need to solve,

$$\frac{\partial W}{\partial \Gamma} = \frac{\partial^2 W}{\partial x^2} + \left\{ \frac{2\mu(1-a)}{a\sigma^2} \right\} e^{(1-\alpha)x - \beta\Gamma} \qquad \text{(B1)}$$

s.t. the initial condition $\Rightarrow W(x,0) = \max\left\{ e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0 \right\}$ (B2)

Duhamel's principle says that the solution to the initial value problem (B1 & B2) is given by

$$W(x,\Gamma) = W^h(x,\Gamma) + G(x,\Gamma) = W^h(x,\Gamma) + \int_0^\Gamma g(x,\Gamma;s)\,ds \qquad \text{(B3)}$$

where $W^h(x,\Gamma)$ is the solution to the homogeneous problem:

$$\frac{\partial W^h}{\partial \Gamma} = \frac{\partial^2 W^h}{\partial x^2}$$

s.t the initial condition $W^h(x,0) = \max\left\{ e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0 \right\}$

and $g(x,\Gamma;s)$ solves :

$$\frac{\partial g}{\partial \Gamma}=\frac{\partial^2 g}{\partial x^2} \quad \text{, for } \Gamma > s$$

$$\text{s.t } g(x,\Gamma;s)=\left\{\frac{2\mu(1-a)}{a\sigma^2}\right\}e^{(1-\alpha)x-\beta s} \quad \text{, for } \Gamma = s$$

<u>*Homogeneous problem*</u>

$$\frac{\partial W^h}{\partial \Gamma}=\frac{\partial^2 W^h}{\partial x^2} \quad s.t.\ W^h(x,0)=\max\left\{e^{\frac{-(2q-1)}{2}x}-e^{\frac{-(2q+1)}{2}x},0\right\}$$

The fundamental solution to the heat equation in one dimension (our case) is given by

$$W^h(x,\Gamma)=\frac{1}{2\sqrt{\pi\Gamma}}\int_{-\infty}^{\infty}e^{-\frac{(x-xo)^2}{4\Gamma}}W^h(x_o,0)dx_o$$

Change of a variable: $z=\dfrac{x_o-x}{\sqrt{2\Gamma}} \Rightarrow dz=\dfrac{dx_o}{\sqrt{2\Gamma}}$

Also, $W^h(x,0)>0$ iff $x>0$, so we can restrict the integration range: $z>-\dfrac{x}{\sqrt{2\Gamma}}$

$$W^h(x,\Gamma)=\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\Gamma}}^{\infty}e^{-z^2/2}dz\left\{e^{-\left(\frac{2q-1}{2}\right)\left(x+z\sqrt{2\Gamma}\right)}-e^{-\left(\frac{2q+1}{2}\right)\left(x+z\sqrt{2\Gamma}\right)}\right\}$$

$$\Rightarrow W^h(x,\Gamma)=\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\Gamma}}^{\infty}e^{-z^2/2}\cdot e^{-\left(\frac{2q-1}{2}\right)\left(x+z\sqrt{2\Gamma}\right)}dz-\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\Gamma}}^{\infty}e^{-z^2/2}\cdot e^{-\left(\frac{2q+1}{2}\right)\left(x+z\sqrt{2\Gamma}\right)}dz$$

$$\Rightarrow W^h(x,\Gamma)=I_1-I_2$$

Complete the square for the exponent in I_1:

$$-\left(\frac{2q-1}{2}\right)\left(x+z\sqrt{2\Gamma}\right)-\frac{z^2}{2}$$

$$=-\frac{1}{2}\left[z^2+z\sqrt{2\Gamma}\cdot(2q-1)\right]-\left(\frac{2q-1}{2}\right)x$$

$$=-\frac{1}{2}\left[z^2+z\sqrt{2\Gamma}\cdot(2q-1)+\frac{\Gamma\cdot(2q-1)^2}{2}\right]-\left(\frac{2q-1}{2}\right)x+\frac{\Gamma\cdot(2q-1)^2}{4}$$

$$=-\frac{1}{2}\left[z+\frac{\sqrt{\Gamma}\cdot(2q-1)}{\sqrt{2}}\right]^2+c \quad where\ c=-\left(\frac{2q-1}{2}\right)x+\frac{\Gamma\cdot(2q-1)^2}{4}$$

$$=-\frac{1}{2}\left[y^2\right]+c \quad where\ y=z+\frac{\sqrt{\Gamma}\cdot(2q-1)}{\sqrt{2}}$$

So, $I_1=\dfrac{e^c}{\sqrt{2\pi}}\displaystyle\int_{-x/\sqrt{2\Gamma}+(2q-1)\sqrt{\Gamma/2}}^{\infty} e^{-y^2/2}\,dy$

$$\Rightarrow I_1=e^c\cdot N(d_1) \quad where\ d_1=\frac{x}{\sqrt{2\Gamma}}-(2q-1)\sqrt{\frac{\Gamma}{2}}$$

Similarly, complete the square for the exponent in I_2 to arrive at

$$I_2=e^d\cdot N(d_2) \quad where\ d_2=\frac{x}{\sqrt{2\Gamma}}-(2q+1)\sqrt{\frac{\Gamma}{2}} \ \ and\ d=-\left(\frac{2q+1}{2}\right)x+\frac{\Gamma\cdot(2q+1)^2}{4}$$

So, $W^h(x,\Gamma)=e^cN(d_1)-e^dN(d_2)$ (B4)

(B4) needs to be adjusted for inhomogeneity in accordance with Duhamel's principle.

We need to solve,

$$\frac{\partial g}{\partial \Gamma}=\frac{\partial^2 g}{\partial x^2} \quad \text{, for } \Gamma>s$$

s.t. $g(x,s;s)=\left\{\dfrac{2\mu(1-a)}{a\sigma^2}\right\}e^{(1-\alpha)x-\beta s}$, $\Gamma=s$

General solution: $g(x,\Gamma)=\dfrac{1}{2\sqrt{\pi\Gamma}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-x_o)^2}{4\Gamma}}\,g(x_o,s)\,dx_o$

Change of a variable: $z = \dfrac{x_o - x}{\sqrt{2\Gamma}} \;\Rightarrow\; dz = \dfrac{dx_o}{\sqrt{2\Gamma}}$

$$f \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \left\{ e^{-\left(\frac{2q-1}{2}\right)\left(x+z\sqrt{2\Gamma}\right)+\left\{\frac{(2q+1)^2}{4}+\frac{2r}{a\sigma^2}\right\}s} \right\} \tag{B5}$$

where $f = \dfrac{\mu(1-a)2}{a\sigma^2}$

Complete the square for the exponent:

$$-\frac{z^2}{2} - \frac{(2q-1)}{2}\left\{x + z\sqrt{2\Gamma}\right\} + \frac{(2q+1)^2 s}{4} + \frac{2rs}{a\sigma^2}$$

$$\Rightarrow -\frac{1}{2}\left(z^2 + (2q-1)z\sqrt{2\Gamma} + \frac{(2q-1)^2\Gamma}{2} \right) - \frac{(2q-1)x}{2} + \frac{(2q+1)^2 s}{4} + \frac{2rs}{a\sigma^2} + \frac{(2q-1)^2\Gamma}{4}$$

$$\Rightarrow -\frac{1}{2}\left(z + (2q-1)\sqrt{\frac{\Gamma}{2}} \right)^2 + h$$

$$where \;\; h = -\frac{(2q-1)x}{2} + \frac{(2q+1)^2 s}{4} + \frac{2rs}{a\sigma^2} + \frac{(2q-1)^2\Gamma}{4}$$

Change of a variable in (B5): $y = z + (2q-1)\sqrt{\dfrac{\Gamma}{2}}$

$$g(x,\Gamma;s) = f \cdot e^h \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$$

$$\Rightarrow g(x,\Gamma;s) = f \cdot e^h$$

$$G(x,\Gamma) = \int_0^{\Gamma} g(x,\Gamma;s)\,ds$$

$$\Rightarrow G(x,\Gamma) = f \cdot e^{-\left(\frac{2q-1}{2}\right)x + \frac{(2q-1)^2\Gamma}{4}} \int_0^{\Gamma} e^{\left\{\frac{(2q+1)^2}{4}+\frac{2r}{a\sigma^2}\right\}s}\, ds$$

$$\Rightarrow G(x,\Gamma) = f \cdot e^{-\left(\frac{2q-1}{2}\right)x + \frac{(2q-1)^2\Gamma}{4}} \cdot \frac{1}{Q}\left[e^{Q\Gamma} - 1\right]$$

$$where\ Q = \frac{(2q+1)^2}{4} + \frac{2r}{a\sigma^2} \tag{B6}$$

Substitute (B4) and (B6) in (B3):

$$W(x,\Gamma) = W^h(x,\Gamma) + G(x,\Gamma) = e^c N(d_1) - e^d N(d_2) + f \cdot e^{-\left(\frac{2q-1}{2}\right)x + \frac{(2q-1)^2\Gamma}{4}} \cdot \frac{1}{Q}\left[e^{Q\Gamma} - 1\right]$$

Substitute for original variables to obtain the behavioral Black-Scholes formula:

$$C = Se^{-\frac{\mu(1-a)(T-t)}{a}} N(d_1) - e^{\frac{-r(T-t)}{a}} \cdot K \cdot N(d_2) + f \cdot S \cdot e^{-\frac{\mu(1-a)(T-t)}{a}} \cdot {}^1\!/_Q\left[e^{Q\Gamma} - 1\right]$$

$$\Rightarrow C = Se^{\frac{-\mu(1-a)(T-t)}{a}} \left\{N(d_1) + f \cdot {}^1\!/_Q\left(e^{Q\Gamma} - 1\right)\right\} - e^{\frac{-r(T-t)}{a}} \cdot K \cdot N(d_2)$$

where,

$$0 < a \le 1$$

$$f = \frac{2\mu(1-a)}{a\sigma^2}$$

$$Q = \frac{(2q+1)^2}{4} + \frac{2r}{a\sigma^2} \quad ;\left(q = \frac{\mu(1-a) - r}{\sigma^2 a}\right)$$

$$\Gamma = \frac{\sigma^2}{2}(T-t)$$

$$d_1 = \frac{x}{\sqrt{2\Gamma}} - (2q-1)\sqrt{\frac{\Gamma}{2}}$$

$$d_2 = \frac{x}{\sqrt{2\Gamma}} - (2q+1)\sqrt{\frac{\Gamma}{2}}$$