Sustainable growth in a resource-based economy: the extraction-saving relationship

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Abstract

The paper presents two new results for the Dasgupta-Heal-Solow-Stiglitz model with an essential nonrenewable resource:

(1) the pattern of resource extraction can be more important for sustainable growth than the pattern of saving when the Hotelling rule modifier is not small enough;

(2) the qualitative behavior of the long-run per capita output can be examined along any smooth enough path of extraction for any variable saving rate using the "index of sustainable extraction" introduced in the paper.

Key words: modified Hotelling rule; imperfect economy; sustainable growth; index of sustainable extraction; Hubbert curve consumption
JEL : O13; O47; Q32; Q38

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1. Introduction

Dasgupta and Heal (1979, pp. 303-306) and Hamilton et al. (2006) showed that the investments exceeding the standard Hartwick investment rule (Hartwick 1977) imply sustainable unbounded growth in per capita consumption under the standard Hotelling rule for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta, Heal 1974; Solow 1974; Stiglitz 1974).

Stollery (1998) considered an externality (global warming) that implied modification of the Hotelling rule and corresponding modification of the path of extraction. Combination of this extraction path with the standard Hartwick rule resulted in bounded growth of per capita consumption. Another example was obtained in Bazhanov (2007b) where I examined the properties of transition paths constructed under the assumption of the modified Hotelling rule. This case also gave the patterns of bounded and unbounded growth of per capita consumption under the standard Hartwick rule.

These examples raise a question about the roles of the patterns of saving and extraction for sustaining the growth of a resource-based economy in the long run. The answer to this question is the first main result of the paper (Proposition 1, Section 3). It shows that the pattern of resource extraction is more important for sustainability of growth than the pattern of saving when the Hotelling rule modifier is not close enough to zero. The pattern of saving defines the level of consumption along the growing or declining path in this case. This result is obtained for the DHSS model with an arbitrary investment rule and with the Hotelling rule modified in a general form. The assumption of the modified Hotelling rule introduces a generalized imperfection in an economy. In this sense, the paper is close to the study of Arrow, Dasgupta and Mäler (2003) because I also consider the question: "How should we evaluate policy change in an imperfect economy?" (Arrow, Dasgupta and Mäler 2003, p. 149). I use the Hotelling rule modifier here as a variant of measure of economic imperfection.

Another question of Arrow, Dasgupta and Mäler (2003) that I examine in this paper is: "How can we check whether intergenerational well-being will be sustained along a projected economic programme?" (p. 149). Arrow, Dasgupta and Mäler offer an approach of calculating accounting prices that show if the resource extraction is sustainable. I analyze here the path of extraction directly. This is the second result of the paper (Proposition 2, Section 4), which allows for estimation of the long-run weak sustainability[1] of resource extraction along any smooth enough path. The importance of this result is connected with the long-standing problem of estimating the properties of the extraction paths with the "maximum possible" and with the optimal or equilibrium rates. For example, Dasgupta and Heal (1979, p. 298-303) showed that the path of extraction under the positively discounted utilitarian criterion is unsustainable (consumption declines to zero in the long run). Another example is a well-known Hubbert curve of oil extraction that estimates the maximum possible rates basing on his-

[1] In a sense of nondecreasing per capita output and consumption.

torical data. Applying the "index of sustainable extraction" (Proposition 2), I show that the long-run consumption along this curve declines to zero regardless of the choice of the curve's parameters and regardless of the pattern of saving. Numerical examples provided in the paper are calibrated on the current world's oil extraction data.

2. The model

The DHSS model was introduced for studying the role of an essential[2] non-renewable resource in economic growth. The reasons for this specification varied from plausibility of this case: "Only the Cobb-Douglas form may be said to have properties that are reasonable at the corner" (Dasgupta, Heal 1974, p. 14), to theoretical interest: "If the elasticity of substitution between resources and other factors exceeds one, then resources are not indispensable to production. If it is less than one, then the average product of resources is bounded. So only the Cobb-Douglas remains" (Solow 1974, p. 34), to technical simplicity: "In a Cobb-Douglas production function, we need not distinguish between labour, capital, and resource augmenting technical progress" (Stiglitz 1974, p. 131),[3] and to orientation on subsequent numerical studies and teaching (Dasgupta, Heal 1974, p. 26). Dasgupta and Heal (1974, p. 26) noted that this narrow specification does not restrict the results from further generalization, however, as Solow (1974, p. 34) put it, "Any extra generality hardly seems worth striving for."

Empirical evidence showed that the elasticity of substitution between natural resources and capital in some cases exceeds unity (e.g., Nordhaus 1972, Pindyck 1979) while other investigations (e.g., Fuss 1977; Magnus 1979; and partly Halvorsen, Ford 1979) indicated that energy and capital are rather strong complements than substitutes (elasticity is less than unity), and some researches found that this value can be rather close to unity (e.g., Griffin, Gregory 1976; Pindyck 1979). This empirical controversy supported the assumption that the use of the Cobb-Douglas production function is not implausible in some problems of resource economics. A review on this question can be found, e.g., in Neumayer (2000, Section 4).

All these studies triggered a substantial body of literature, where the DHSS model was used and is still being used mostly for the analysis of the role of saving (Dixit 1976; Hoel 1977) in behavior of a welfare indicator (Hartwick 1977; Dasgupta, Heal 1979; Pezzey, Withagen 1998; Asheim, Buchholz, Withagen 2003; Buchholz, Dasgupta, Mitra 2005; Hamilton, Withagen 2007; Benchekroun, Withagen 2008) with different patterns of population growth (exponential - Stiglitz 1974; Takayama 1980; quasi-arithmetic - Mitra 1983; Asheim et al 2007), and with different patterns of technical change (exogenous exponential total factor

[2] This term was offered by Dasgupta and Heal (1974, p.14).

[3] Dasgupta and Heal used the same argument: "The Cobb-Douglas case is particularly interesting since the analysis can relatively easily be taken further" (1974, p.17).

productivity (TFP) - Stiglitz (1974); Suzuki (1976), and Solow (1986); endogenous reserve-augmenting and TFP-augmenting - Takayama (1980); exogenous quasi-arithmetic - Pezzey (2004)[4] and Asheim et al (2007)).

However, most of the results for the DHSS model are obtained for a "perfect" economy with the Hotelling rule in its original form. The papers with some deviations from the standard Hotelling rule (e.g., Suzuki (1976); Takayama (1980); Stollery (1998); Bazhanov (2007b)) did not consider the question of relationship between saving and extraction from the point of view of non-declining consumption over time. The motivation of this paper is to show that the role of saving in sustainability could be limited to defining only the level of consumption along sustainable or unsustainable paths in economies with a modified Hotelling rule, while government policies affecting a "resource allocation mechanism" (Arrow, Dasgupta, Mäler 2003, p. 650) can qualitatively define growing or declining path of consumption in the long run. I show this despite the use of the Solow's (1974) assumption that the share of capital in production exceeds the share of natural resource. Classical result for "perfect" economies claims: "Even with no technical change, capital accumulation can offset the effects of the declining inputs of natural resources, so long as capital is 'more important' than natural resources, i.e. the share of capital is greater than that of natural resources" (Stiglitz 1974, p. 131).

It is commonly accepted that there are other essential factors for sustainability such as population growth, technical change, and substitutability between resources and capital. Population growth is considered as the main threat to sustainability starting from the work of T. Malthus in 1798 up to recent papers (e.g., Brander 2007). The debates on this problem are concentrating now around the constant that could be the limit to this growth. For example, the UN estimate that the world's population growth is going to flatten out at the level around 10 billion (UN 1999). Stabilization has already happened in developed countries, which are the main users of nonrenewable resources. Hence, I assume here that population has already stabilized at some level.

Resource-capital substitutability and technical change are the most uncertain factors in sustainability. A review of pessimistic and optimistic positions about the role of these interrelated factors can be found, e.g., in Lecomber (1979, Chapter 2) or in Neumayer (2000, Sections 4 and 5). Pessimists claim that natural resources and man-made capital are rather complements than substitutes while optimists argue that technical change increases substitutability between these two factors with time and resources can become inessential in production. The Cobb-Douglas function, in this sense, is "an average" assumption since it assumes that the natural resource is essential and at the same time this function allows for infinite growth of output with the limited resource.

Uncertainty of technical change is reflected in a wide variety of models used in the literature. Optimistic approaches assume that this factor is exponentially

[4]Pezzey calls it "hyperbolic" because the rate of TFP growth in this case is positive and decreases hyperbolically over time.

growing in a form of TFP (Phelps 1966;[5] Stiglitz 1974; Solow 1986) or exponential endogenous growth of knowledge (Grimaud, Rouge 2005).[6] I mentioned above that there are models with less than exponential, e.g., with a quasi-arithmetic TFP and even with limited TFP.[7] Note also, that in fact, TFP goes up more slowly than knowledge, and in some cases it can even decline in the short run due to raising research expenses and other development costs (Lipsey, Carlaw 2004). The uncertainty of this factor combined with nonrenewability of the resource can be used as an argument against extreme optimistic models of technical change.

Since the main aim of this paper is to compare the roles of saving and extraction in sustainability, I will use below a simple assumption about a form of technical change that is somewhere between optimistic and pessimistic approaches. Some studies assume for simplicity that the technical change exactly compensates for the growing population (e.g., Dasgupta and Heal, 1979; Stollery, 1998). However, as I mentioned above, the assumption about constant population becomes more and more plausible with time. This implies that technical change (or a part of this change) can "compensate for" another negative factor of growth. I assume here that growth of TFP exactly compensates for capital depreciation since, unlike the growth of population, technical change and capital decay presumably will exist until exist human civilization and capital. The convenience of this assumption is linked with the correctness of the use of the basic DHSS model in cases with unlimited growth in consumption.

Hence, I consider the DHSS model for a closed decentralized economy with zero population growth, zero extraction cost, and the production function $q(t) = f(k(t), r(t)) = k^{\alpha}(t)r^{\beta}(t)$, where α, $\beta \in (0, 1)$, $\alpha + \beta < 1$ are constants. Population equals to labor and the lower-case variables are in per capita units, q - output, k - produced capital, r - current resource use. Then $r = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$). Prices of capital and the resource are $f_k = \alpha q/k$, $f_r = \beta q/r$, where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$.

The TFP $A(t)$ exactly compensating for capital depreciation δk implies that $q(t) = A(t)f(k(t), r(t)) - \delta k = f(k(t), r(t))$ that gives in the Cobb-Douglas case $A(t) = 1 + \delta k f^{-1} = 1 + \delta k^{1-\alpha} r^{-\beta}$. It can be shown (Bazhanov 2007a), that in some cases $A(t)$ is asymptotically linear with rather small slope and "stronger" than linear in some other cases. In AK-model without resource this approach gives identically constant TFP.

I assume that

(1) there are some phenomena in the economy such as simple externalities,

[5] Phelps considered factor-augmenting exponential technical progress in a general model that included the Cobb-Douglas case.

[6] A review of the models with endogenous technical change and an essential nonrenewable resource can be found in Bretschger (2005).

[7] Nordhaus and Boyer use TFP in the form of $A(t) = A_0 \exp\left[\int_0^t g_0^A e^{-\delta\xi} d\xi\right]$ calling the behavior of this factor "a major uncertainty" in their DICE-99 and RICE-99 models (Nordhaus, Boyer 2000, p. 17). This TFP is asymptotically constant ($\lim_{t\to\infty} A(t) = A_0 g_0^A/\delta$) and fast-growing in the short run.

government regulations and taxes/subsidies, which modify the Hotelling rule and which combined effect can be expressed in terms of changes in tax $\nu(t)$ or interest rate $\tau(t)$. This implies that if p is the "standard equilibrium Hotelling" price then the ratio $(\dot{p}+\dot{\nu})/(p+\nu)$ is not already equal to the rate of interest $f_k(t)$ when $\nu(t) \neq 0$. Denoting the observable price by $f_r = p + \nu$, it can be written as follows:

$$\dot{f}_r(t)/f_r(t) = f_k(t) + \tau(t), \tag{1}$$

where the modifier $\tau(t) = \tau[\nu(t)] = 0$ when $\nu(t) = 0$.[8] This generalized form of the asset equilibrium condition allows for various feasible scenarios of the resource extraction $r_\tau(t)$. The assumption is essential for the goals of the paper because it implies that all the examined paths of extraction are realizable. Realization of the specific extraction path depends on the concrete paths of the phenomena modifying the Hotelling rule, including government policies. A review of these phenomena and a review of papers providing empirical evidence of distortions between the "standard Hotelling" price paths and data for various nonrenewable resources can be found in Gaudet (2007). I imply in this assumption that the government can use all the instruments of influence on the externalities and, by modifying the equilibrium condition (1), on the corresponding path of extraction. For example, the government can use taxes (Karp, Livernois 1992), regulations (Davis, Cairns 1999), and environmental policy in a form of tax (Grimaud, Rouge 2005), in order to change a path of extraction in a desirable way. A large body of empirical research in the "oil peak" theory (e.g., Laherrere 2000)[9] support plausibility of the assumption that the paths of extraction considered in the paper can be realizable. Following Arrow, Dasgupta and Mäler (2003) I consider here only equilibrium extraction resulted from the corresponding changes in $\tau(t)$ and I do not assume that social planner maximizes a welfare criterion. For simplicity of notation, I will omit below the subindex τ that denotes the dependence of the path of extraction on the specific combination of phenomena modifying the Hotelling rule.

(2) The initial conditions $r_0, \dot{r}_0$ and s_0 are given and they are consistent with the imperfection[10] expressed in $\tau(t)$. This means that imperfection of an economy can be expressed either by the specific modifier of the Hotelling rule

[8] A specific variant of the Hotelling rule arrives usually from the first-order conditions in an optimal-control problem of a welfare maximization. The standard form of this rule results from maximizing the present value of profits of the owner of the resource stock by choosing the path of extraction (Dasgupta and Heal 1979, p. 157-158). A variant of the modified rule was obtained, e.g., in Levhari and Liviatan (1977) with $\tau(t) = -[\partial C/\partial(s_0 - s)]/M\pi(t)$, where $C(r(t), s_0 - s(t))$ - cost function ($\partial C/\partial(s_0 - s) > 0$) and $M\pi(t)$ - marginal profit.

[9] The main goal of this theory is to explain the behavior of historical extraction data by some empirical curves, and then use these paths as a forecasting tool. The patterns of consumption along some of these curves are examined in this paper.

[10] The Hotelling rule can be modified not only by imperfections. For example, technical change in extracting industry implies an additional summand in this rule (see, e.g., Takayama (1980) or Gaudet (2007)). For simplicity, I will use sometimes the word "imperfection" following Arrow, Dasgupta and Mäler (2003), but I will imply here, by this term, all the phenomena modifying the Hotelling rule.

or by the specific initial conditions. In the former case, the specific modifier implies equilibrium initial state, which could be used for policy recommendation *before* introducing an "imperfection" in reality. In this case, the initial state is considered as "the future." In the later case, the given initial conditions imply the initial value of the modifier $\tau(t)$ and these conditions are treated already as "the past" because these data presumably must be available from the last issues of some journals or from the last rows of some databases. In a mixed case, the effect of a specific imperfection can be considered on an example of a real economy, which *is already extracting* the resource under a set of "imperfections." In this case, a generalized modifier τ could be added as an extra modifier to the Hotelling rule in order to absorb all other imperfections, which presumably exist in the economy and which are reflected in the "imperfect" initial conditions.[11] Then the given initial conditions define the initial value of τ.

As a simple example, consider N resource owners extracting from a common pool divided by porous barriers with a diffusion coefficient λ (Arrow, Dasgupta, Mäler 2003, p. 664).[12] The present-value utility maximization $(U(r) = -r^{-(\eta-1)})$ implies for this imperfection that the Hotelling rule modifier is $\tau = (N-1)\lambda \geqslant 0$ and the path of extraction is $r = (\mu/\eta)s_0 e^{-\mu t/\eta}$, where $\mu = \rho + \tau$, ρ - the social discount rate, $\eta > 1$ - the elasticity of demand. One can see that the larger is N, the higher is the initial rate of extraction $r_0 = (\mu/\eta)s_0$ and the faster is the decline in the rate of extraction. This simple model captures the qualitative effect of the common property situation when N owners have just obtained the rights on their resource stocks and they are going to start extraction. However, if this approach is used (say, for constructing a forecast) in a case where the resource owners are already extracting, then the model can explain only the fast decline in the rates of extraction because the initial conditions are given and they reflect already all imperfections in the economy.

In this paper I consider an economy with a set of known and unknown imperfections expressed in τ. These imperfections result in the available historical path of the states of the economy. The last point of this path can be used as an initial condition in order to construct a forecast. In this framework I compare the roles of saving and imperfections, expressed in deviations of the path of extraction. Hence, I do not introduce here a specific imperfection explicitly and consider the initial state as given. This assumption implies the following

Definition. The path of extraction $r(t)$ is *feasible* if

(a) $r(t) > 0$ for all $t \geqslant 0$;

(b) total extraction does not exceed the reserve: $\int_0^\infty r(t)dt \leqslant s_0$;

(c) $r(t)$ is consistent with the initial conditions $r(0) = r_0; \dot{r}(0) = \dot{r}_0$;

(d) $r(t)$ is twice continuously differentiable for large enough t.

(3) The economy follows an investment rule in the form of $\dot{k} = wq$, where

[11] Note, however, that in this case τ can absorb also the imperfections of the model.

[12] This situation in the general case is referred to as "the tragedy of the commons" after the paper of G. Hardin (1968). In more detail this problem is described at http://en.wikipedia.org/wiki/Tragedy_of_the_commons (December 2008).

$w(t) \in (0,1)$. This rule includes the Hartwick investment rule for $w \equiv \beta$. "There is no presumption though that the saving rate is optimum; rather, it is a behavioural characteristic of consumers, reflecting their response to an imperfect credit market" (Arrow, Dasgupta, Mäler 2003, p. 657).

The following sections examine the influence of extraction and saving on growth of output. The corresponding results for the path of consumption follow from substitutions $q = c/(1-w)$ and $\dot{q} = \dot{c}/(1-w) + \dot{w}c/(1-w)^2$, which give $\dot{c} = \dot{q}(1-w) - \dot{w}q$. The results for output and consumption qualitatively coincide when w is constant.[13] In the general case, boundedness of w implies that $\dot{w}$ can be either monotone and very close to zero or exhibit the short-run shocks that can influence the results provided below only in some specific cases. These cases seem to be of solely theoretical interest (I consider this question in Section 5) since there is empirical evidence that the patterns of world's saving oscillate around some constants (15% – 25% of GDP) for rather long period of time (Maddison 1992).

3. The roles of extraction and saving in sustainability

The modified Hotelling rule (1) implies $\dot{f}_r/f_r = \alpha\dot{k}/k + (\beta-1)\dot{r}/r = \alpha q/k + \tau$. After substitution of the saving rule $\dot{k} = wq$, it becomes $\alpha qw/k + (\beta - 1)\dot{r}/r = \alpha q/k + \tau$. This gives the generalized equation in $r(t) : \dot{r}/r = -(1-w)\alpha q/\left[(1-\beta)k\right] - \tau/(1-\beta)$ or

$$\dot{r}/r = -\left[(1-w)f_k + \tau\right]/(1-\beta). \tag{2}$$

Note that given the initial value r_0 and the path of $w(t)$, there is one-to-one correspondence between r and τ, where the latter includes government interventions. Therefore, the path of extraction can be considered as a control variable in some social-planner problems bearing in mind that $r(t)$ is just a result of dynamically changing equilibrium and the "optimal" r uniquely defines the optimal path of tax. Equation (2) can be rewritten using (1) as follows:

$$\dot{r}/r = -\left[\dot{f}_r/f_r - wf_k\right]/(1-\beta). \tag{3}$$

Equation (3) can be derived from the production function by combining expressions for $\dot{q}/q$ and $\dot{f}_r/f_r$. However, the rate of extraction $\dot{r}/r$, as it can be seen from this equation, depends on the relationship between the rate of change of the observable price $\dot{f}_r/f_r$ and the rate of interest f_k or, in other words, on the deviations from the standard Hotelling rule $\tau(t)$ what is explicitly expressed in (2). Equation (3) does not contain the modifier $\tau(t)$ explicitly because according to (1) the distortions caused by the imperfections are reflected in the price changes $\dot{f}_r/f_r$. This means that if $\tau(t)$ includes all known and unknown effects,

[13] Constant w corresponds to Dixit-Hammond-Hoel (Dixit et al. 1980) rule when the resource rent is a constant share of output (βq) and $\dot{q} = 0$ (e.g., in the Cobb-Douglas case) since then genuine investment equals $(w-\beta)q$, which is a constant.

distorting the Hotelling rule in the real economy, then we must take for numerical examples the changes in the real market price for the term $\dot{f}_r/f_r$. Then, formula (3) is interpreted as follows: *"actual" rates of extraction are defined by the rates of the "actual" price changes $\dot{f}_r/f_r$ diminished by the interest rate f_k weighted by the saving rate w.*[14]

Substitution of formula (2) into the expression for the output per cent change implies $\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r = \alpha q/k\,[(w-\beta)/(1-\beta)] - \tau\beta/(1-\beta)$ that can be rewritten as follows:

$$\dot{q}(1-\beta)/(\beta q) = f_k(w/\beta - 1) - \tau.$$

This implies the following

Proposition 1. *In the DHSS economy, with the investment rule $\dot{k} = wq$, where $w(t) \in (0,1)$ and with the modified Hotelling Rule $\dot{f}_r/f_r = f_k + \tau(t)$, the sign of the change in per capita output satisfies the following condition:*

$$\dot{q} \gtreqless 0 \text{ iff } \tau \lesseqgtr f_k\left[w/\beta - 1\right],$$

which means that the path of investment (defined by w) can qualitatively influence the pattern of growth only when $-f_k < \tau < f_k\left[1/\beta - 1\right]$.

The specific cases of this result are:

1. The necessity and sufficiency of the Hartwick rule ($w \equiv \beta$) for sustaining the constant per capita consumption in the DHSS model with the standard ($\tau(t) \equiv 0$) Hotelling rule (Hartwick 1977; Dixit et al. 1980).

2. The necessity and sufficiency of the modified Hartwick rule ($w > \beta$) for sustaining per capita growth of consumption in the DHSS model with the standard Hotelling rule ($\tau(t) \equiv 0$) (Dasgupta, Heal 1979, p. 303-306, formula (10.33); Hamilton et al. 2006).

3. The growing per capita consumption in the DHSS economy with the modified Hotelling rule ($\tau(t) < 0$) and the standard Hartwick rule ($w = \beta$) (Stollery 1998; Bazhanov 2007b, 2009).

Note that these results were obtained under the different welfare criteria (the maximin in cases 1 and 3, and the utilitarian with zero felicity discounting in Dasgupta and Heal (1979)). In the latter case the growth was obtained for the felicity function $u(c) = -c^{-(\eta-1)}$ with the social rate of time preferences $\eta > (1-\beta)/(\alpha-\beta) > 1$. The rate of growth declined to zero with $\eta \to \infty$.

The result of Proposition 1 can be, of course, expressed without introducing τ explicitly (see Corollary 1 below). However, the relationship between $\dot{q}$ and τ can be convenient in some cases, e.g., when τ is connected with the specific phenomena. In these cases, Proposition 1 shows the link between these phenomena

[14] Since all variables in formula (3) are observable, it can be used for estimation of accuracy of the model for the real economy. I use the word "actual" in quotation marks because an aggregate model can reflect only qualitative behavior of a real economy with some level of inaccuracy in numbers.

and the growth in the economy. Formulation of Proposition 1 implicitly provides a mechanism of this link via the tie of τ with the rates of extraction (Eq. (2)). For example, in the case of irreversible global warming (Stollery 1998) the properties of the global atmosphere affect the consumption and /or the production not only directly but by influencing the extraction as well. The Hotelling rule takes the form: $\dot{f}_r/f_r = f_k + (f_T + u_T/u_c)T_{s_0-s(t)}/f_r$ (Hartwick 2009). Here, the utility $u(c,T)$ is negatively affected ($u_T < 0$) by the growing atmospheric temperature $T(s_0 - s(t))$, and the temperature is rising due to oil use in the economy. In the case with the temperature in the utility alone, the modifier is $\tau(t) = u_T T_{s_0-s(t)}/(u_c f_r)$, which is negative when the resource is being extracted. Formula (2) shows that negative τ increases the rate of extraction and it can result in growing extraction even for rather small deviations, namely, for $\tau < -(1-w)f_k$.

A number of reasons can influence the paths of resource price and extraction (Gaudet 2007). These reasons can include an endogenous technical change, e.g., in the form of labor allocated into R&D sector of the oil industry in order to expand the resource stock: $\dot{s} = -r + s\phi(L_R/L)$, where L_R/L - research share of the total labor and $\phi(0) = 0, \phi' > 0$ implying $\phi \geqslant 0$ (Takayama 1980, Section 2). Maximization of $\int_0^\infty u(c)e^{-\rho t}dt$, with utility from a class defined by $u' = c^{\eta-1}, \eta \in [0,1)$, gives in this case $\tau = -\phi$. This again means that the reserve-expanding research follows increase in the rate of extraction, which is qualitatively consistent with the world's dynamics of resource reserves and extraction. As a result, Proposition 1 implies the growth of output, which also conforms with the data.

Resource allocation mechanism, resulting from deviations in the Hotelling rule, can be the main source of growth or decline in a resource-based economy regardless of the pattern of saving. This interpretation of Proposition 1 is intuitive since there is well-known empirical evidence (Pearce and Atkinson 1993; Proops, Atkinson and Schlotheim 1999; Hamilton et al 2006) that some of the resource-based economies have negative genuine investments while their conventionally estimated GDP are growing due to the high rates of resource extraction. In sustainability literature negative genuine investment is associated with unsustainability of the economy. The novelty of the result expressed in Proposition 1 is that rather small influence of externalities and/or government interventions imply that the saving rule by itself can not be already neither necessary nor sufficient condition for maintaining constant or growing per capita output over time. Depending on the combined effect of all deviations (τ), the economy can be (weak) sustainable despite the negative genuine investment, and vice versa, per capita output can decline to zero even if almost all output is being invested into capital. The examples of these different outcomes for various saving rates are provided in sections 5 and 6.

Investment rule, of course, is still very important in an economy with externalities. Stollery (1998) showed that the Hartwick rule is optimal despite the influence of global warming[15] and it is sufficient for maintaining constant utility over time. This does not contradict with the conventional notion of weak sustainability since constant utility means growing consumption in this case. However, the value of τ can be positive in some problems. Just for the sake of argument, assume that the temperature is a normal good as it is assumed, e.g., by Nordhaus and Boyer (2000, p. 14) for Russia, Canada, and some other countries. Then, in the Stollery's framework, the utility will be still constant while the per capita consumption will be decreasing, which is already not consistent with regular notions of sustainability. Hence, in the general case, the specific modification of the Hotelling rule qualitatively defines sustainability of a resource-based economy while the saving rule specifies the level of consumption along the sustainable or unsustainable path (see Figures 4 and 5 in the following section).

In some applications it could be convenient to aggregate in τ "the rest of deviations" even if a specific externality is explicitly introduced in the model. The aggregate meaning of τ can be used in empirical works, e.g., on valuing a resource reserve. The values of τ are not observed directly in the economy, however, they can be easily estimated given the resource prices and the interest rates. Appendix A provides a simple example on estimation of τ for the U.S.A. economy. Although it could be questionable, if the simplest variant of the DHSS model can be adequate for this complicated economy, there is a visible negative correlation between the changes of τ for the crude oil and GDP percent change (Fig. 7). Proposition 1 implies in this case that the ratio w/β is significantly greater than unity that coincides with the known empirical evidence (Pearce and Atkinson 1993; Proops, Atkinson and Schlotheim 1999; Hamilton et al 2006) that genuine investment is positive in the U.S.A.

The result of Proposition 1 can be expressed in terms of the rates of change of the observable market prices $\dot{f}_r/f_r$ by substitution for $\tau(t)$, using (1). The result of this substitution is formulated below as

Corollary 1. *Under the conditions of Proposition 1, the sign of change in per capita output is defined as follows:*

$$\dot{q} \gtreqless 0 \text{ iff } \left[\dot{f}_r/f_r \right]/f_k \lesseqgtr w/\beta,$$

which implies that the pattern of saving (w) can qualitatively influence the pattern of growth iff $0 < \left[\dot{f}_r/f_r\right]/f_k < 1/\beta$.

One of the practical implications of this result is that the DHSS model, given oil as the resource input, has some empirical support from the qualitative behavior of the world's economy, depending on major changes in (market) oil prices. One can recall the prosperity of the economy when the price of oil was declining ($\dot{f}_r/f_r < 0$) or the rate of change was very small, before the spikes in

[15] Stollery (1998, p. 734) used the Bellman-Jacobi-Hamilton equation to show that zero net investment maximizes a welfare function $W \equiv \int_0^\infty \overline{u}\delta \exp\{-\delta t\}dt = \overline{u} = const$ implying that the Hartwick rule is still optimal in this framework.

1973 and 1979 followed by recessions.[16] More recent empirical support can be found, e.g., in IEA (2004): "World GDP would be at least half of one percent lower … in the year following a $10 oil price increase." Although, of course, the dependence of the world's output on oil prices is much more complicated than can be described by a simple aggregate model (see, e.g., IMF 2007, p. 17; Elekdag et al. 2008).

The examples from the world's history and a large body of empirical research on testing the Hotelling rule[17] support the assumption about the strong influence of different phenomena modifying the Hotelling rule in the real economy and about relatively large absolute values of the modifier τ . Therefore, government policies with respect to extracting industries could be primary for sustainable economic development. Concentration only on the patterns of saving could not be enough. Dasgupta and Heal (1979, p. 309) wrote on this matter: "Governments of most countries ... have in the past been concerned with the rate of investment and, more recently, with the rate of utilization of the world's exhaustible resources." This implies the importance of estimating the qualitative behavior of the long-run output along some "program" paths of extraction. A solution of this problem is in the following section.

4. Sustainability of growth for any feasible scenario of extraction

Assume that the government is going to rely on a forecast of extraction in the framework of the long-term energy program. The government can influence the extraction in various ways: "even though the government does not optimize, it can bring about small changes to the economy by altering the existing resource allocation mechanism in minor ways. The perturbation in question could be small adjustments to the prevailing structure of taxes for a short while, or it could be minor alterations to the existing set of property rights for a brief period, or it could be a small public investment project" (Arrow, Dasgupta, Mäler 2003, p. 655). Then, sustainability of growth in the economy will depend on (a) the possibility of realization of this path (reliability of the forecast) and (b) the consequences for the economy when the path is realizable. I concentrate here only on the second question, namely, on the analysis of the long-run per capita output and consumption along some paths $r(t)$, assuming that the paths are realizable.

The change of output is $\dot{q} = f_k \dot{k} + f_r \dot{r}$, which in the DHSS economy equals $\alpha q^2 w/k + \beta q \dot{r}/r$. Then $\dot{q}$ can be expressed as follows: $\dot{q} = (\alpha q^2 w/k) \left[1 + (\beta/(w\alpha))(k\dot{r}/(rq))\right]$. For simplicity, assume that $r(t)$ is monotone in the long run; in other words, that condition $\int_0^\infty r(t)dt = s_0$ implies that $\dot{r} < 0$ for large enough t. Then $\dot{q} \geqslant 0$, in the long run, iff

$$k\left|\dot{r}\right|/(rq)/w \leqslant \alpha/\beta. \tag{4}$$

[16] A book of D. Yergin (1991) is a good guide on the qualitative dependence of the world's economy on oil.

[17] The review is in (Gaudet 2007).

This inequality contains an unknown path of capital $k(t)$ that can be defined from the differential equation $\dot{k} = wk^{\alpha}r^{\beta}$ (the saving rule). In the general case, (for any feasible $r(t)$ and $w(t)$) the solution cannot be expressed in elementary functions. However, the qualitative behavior of per capita output in the long run can be examined by considering the left hand side of inequality (4) in the limit with $t \to \infty$. The L'Hôpital's rule yields[18]

$$\begin{aligned}
\lim_{t\to\infty} k\,|\dot{r}|\,/(rq)/w &= \lim_{t\to\infty} k^{1-\alpha}r^{-1-\beta}\,|\dot{r}|\,/w = \infty\cdot 0 = \lim_{t\to\infty}\left[k^{1-\alpha}\right]/\left\{1/(r^{-1-\beta}\,|\dot{r}|)\right\}/w \\
&= \infty/\infty = \lim_{t\to\infty} d\left[\cdot\right]/dt\,/\,d\left\{\cdot\right\}/dt\cdot(1/w) \\
&= (1-\alpha)\lim_{t\to\infty} k^{-\alpha}\dot{k}/\left\{\left[(1+\beta)r^{\beta}\dot{r}\,|\dot{r}| - r^{1+\beta}d\,|\dot{r}|\,/dt\right]/\dot{r}^{2}\right\}/w.
\end{aligned}$$

After substitution of the saving rule $\dot{k} = wq$ for $\dot{k}$ it becomes

$$\begin{aligned}
&(1-\alpha)\lim_{t\to\infty} r^{\beta}\dot{r}^{2}/\left[(1+\beta)r^{\beta}\dot{r}\,|\dot{r}| + \ddot{r}r^{1+\beta}\right] \\
=\;&(1-\alpha)\lim_{t\to\infty} \dot{r}^{2}/\left[\ddot{r}r - (1+\beta)\dot{r}^{2}\right],
\end{aligned}$$

since unknown function $k(t)$ cancels out and since for our case $\dot{r} \to -0$ (increasing) with $t \to \infty$ and therefore $-d\,|\dot{r}|\,/dt = \ddot{r} > 0$. This implies that condition (4) can be reformulated as follows: $\dot{q} \geqslant 0$ iff $\lim_{t\to\infty}\dot{r}^{2}/\left[\ddot{r}r - (1+\beta)\dot{r}^{2}\right] \leqslant \alpha/\left[\beta(1-\alpha)\right]$. After dividing the numerator and denominator of the left hand side by $\dot{r}^{2}$, this condition becomes: $\lim_{t\to\infty}\ddot{r}r/\dot{r}^{2} \geqslant 1+\beta/\alpha$. The following Proposition[19] summarizes the result.

Proposition 2. *The growth of output q is sustainable in the long run* ($\lim_{t\to\infty}\dot{q} \geqslant 0$) *in the DHSS economy with the investment rule $\dot{k} = wq$ where $w(t) \in (0,1)$ and with the modified Hotelling Rule $\dot{f}_r/f_r = f_k + \tau(t)$ iff*

$$\lim_{t\to\infty}\ddot{r}_{\tau}r_{\tau}/\dot{r}_{\tau}^{2} \geqslant 1+\beta/\alpha,$$

where $r_{\tau}(t)$ is smooth enough.

Henceforth, for simplicity, I will use the notation $\Phi\left[r_{\tau}(t)\right] \equiv \lim_{t\to\infty}\ddot{r}_{\tau}r_{\tau}/\dot{r}_{\tau}^{2}$, and I will call the value of $\Phi\left[r_{\tau}(t)\right]$ *the index of sustainable extraction* for the curve $r_{\tau}(t)$.

Proposition 2 shows that an economy with unity elasticity of factor substitution maintains sustainable growth when and only when the index of sustainable extraction exceeds unity plus the ratio of the resource share to the share of capital.

Note that the saving rate $w(t)$ has canceled out in the process of derivation of this result, which followed that $\Phi\left[r_{\tau}(t)\right]$ does not depend on w explicitly.

[18] Assume that $r^{-1-\beta}\,|\dot{r}| \to 0$. Otherwise, the LHS of formula (4) goes to infinity (at least with infinitely growing capital) and then output goes to zero

[19] Proposition 2 generalizes the results obtained for specific paths of extraction in (Bazhanov 2007b; Andreeva, Bazhanov 2007).

However, this does not mean that the sign of $\dot{q}$ does not depend on the pattern of saving at all in the long run. It would contradict the known results, e.g., when $\tau \equiv 0$ and the sign of $\dot{q}$ is completely defined by w. Equation (2) implies that any path $r(t)$ is the result of combined influence of the pattern of investment defined by $w(t)$ and the path of the modifier $\tau(t)$.

The result of Proposition 2 looks interesting since it can be used for comparative estimation of weak sustainability for any forecasted paths of extraction. Moreover, it can be easily used in order to compare sustainability of extraction under the conditions of the DHSS model with the one obtained in the models with different assumptions. Examples are provided in the next section.

Proposition 2 gives only qualitative result, does not saying anything about the behavior of output along the path of extraction. This point-to-point connection between the extraction, saving, and output can be obtain with the use of Proposition 1 or Corollary 1. Indeed, formula (2) implies that $\tau = -f_k(1-w) - (1-\beta)\dot{r}/r$, which is linked to the behavior of $\dot{q}$ by Proposition 1. Substitution for τ and expression of $\dot{r}/r$ give the result, which I formulate as

Corollary 2. *Under the conditions of Proposition 1 the sign of change in per capita output is defined as follows:*

$$\dot{q} \gtreqless 0 \text{ iff } \dot{r}/r \gtreqless -f_k w/\beta.$$

Note that this result, in general, does not imply sustainability since the short-run growth in output can be obtained in a resource-based economy exclusively by maintaining the rate of change in extraction $\dot{r}/r$ at a high positive level until the resource is extracted with the subsequent collapse of the economy. Note also that the interest rate $f_k = \alpha k^{\alpha-1} r^{\beta}$ goes to zero with $t \to \infty$. These facts yield, for a feasible monotonically decreasing path of extraction, the following

Corollary 3. *Under the conditions of Proposition 1, if, in the long run, per capita output is not decreasing (*$\lim_{t\to\infty} \dot{q} \geqslant 0$*) and r is monotonically declining (there is $\bar{t} \geqslant 0$ such that $\dot{r}(t) < 0$ for any $t \geqslant \bar{t}$) then* $\lim_{t\to\infty} \dot{r}/r = 0$.

Proof is straightforward since the assumptions of Corollary 3 and the claim of Corollary 2 imply that the limit superior and the limit inferior for $\dot{r}/r$ coincide and are both zero.

Corollary 3 gives only a necessary condition for the path of extraction to be sustainable in a sense of not decreasing output. The reverse could be wrong when $\dot{r}/r$ goes to zero with time slower than $-f_k w/\beta$. The result of Corollary 3 can be useful for defining unsustainable paths of extraction when the ratio $\dot{r}/r$ does not go to zero (see examples in the following section).

It is easy to check these results for a classical case with $\tau \equiv 0$ (the standard Hotelling rule) and with $w \equiv \beta$ (the standard Hartwick rule). In this case, the path of extraction (see, e.g., Bazhanov 2007b) is $r_{Hart}(t) = r_0 \left[1 + At\right]^{-\alpha/\beta}$, where $A = r_0\beta / \left[s_0(\alpha - \beta)\right]$. The first derivative is $\dot{r}_{Hart}(t) = -r_0 A\alpha \left[1 + At\right]^{-\alpha/\beta - 1} / \beta$ and the second one is $\ddot{r}_{Hart}(t) = -r_0 A^2 \alpha(\alpha+\beta) \left[1 + At\right]^{-\alpha/\beta-2} / \beta^2$ that follows $\ddot{r}_{Hart} r_{Hart} / \dot{r}^2_{Hart} \equiv 1 + \beta/\alpha$. Proposition 2 implies that per capita output and consumption are constant over time along r_{Hart}. Corollary 2 gives the same conclusion (since $\dot{r}/r = -f_k$ in this case), which coincides with the well-known result of J.M. Hartwick (1977).

The path r_{Hart} is monotonically decreasing starting with $\dot{r}_{Hart}(0) = -\alpha r_0^2 / [s_0(\alpha - \beta)]$ that is not observed yet in the real economy. Further evidence is that the prices of nonrenewable resources do not grow exponentially, as they should according to the standard Hotelling rule (Gaudet, 2007). This implies that the more realistic assumption would be $\tau(t) \neq 0$. The following section provides the examples of calculating the indices of sustainable extraction for some known paths that are compatible with the data from real economy.

5. Consumption along the Hubbert and some other curves

There is a long-standing question about defining the "physical" peak of a nonrenewable resource extraction. M.K. Hubbert (1956) and his followers (e.g., Laherrere 2000) use a specific function with a single maximum (Hubbert curve) or a set of these functions, whose parameters are to be calibrated on historical data of oil extraction and new fields discoveries. The curve(s) uniquely define the peak(s) and the rates of the future extraction. Laherrere (2000) defines the Hubbert curve as follows:

$$r_H(t) = 2r_{\max} / \{1 + \cosh [b(t - t_{\max})]\},$$

where $r_{\max}$ is the peak of extraction in the year $t_{\max}$. Numerical example with the world's oil extraction data[20] gives here $t_{\max} = 8.73$ (peak in 2016) and $r_{\max} = 3.7985$ (Fig. 1, solid line). Parameter b defines the shape of the curve.

This curve proved to be the most accurate approximation describing historical data of oil extraction in the oil-peak literature (e.g., Laherrere 2000). That is why it is extensively used in order to predict the time and the shape of oil peak. It is known that in general, historical data of oil extraction do not follow decreasing path implied by the model of Hotelling. This means that the modifier $\tau(t)$ was not identically equal to zero during the periods of the observations. Assume that the combination of phenomena modifying the Hotelling rule will cause such a path of $\tau(t)$ that extraction will continue to follow the Hubbert curve in the long run as it is assumed by the oil-peak theory. Then qualitative behavior of consumption can be estimated along this path using the result of Proposition 2.

Derivatives $\dot{r}_H$ and $\ddot{r}_H$ are

$$\dot{r}_H = -2br_{\max} \sinh [b(t - t_{\max})] / \{1 + \cosh [b(t - t_{\max})]\}^2$$

and

$$\ddot{r}_H = 2b^2 r_{\max}(\cosh [b(t - t_{\max})] - 2) / \{1 + \cosh [b(t - t_{\max})]\}^2 .$$

[20] All the paths of extraction below are calibrated on the current world's oil extraction data (World 2007). The details of calibration are in Appendix B.

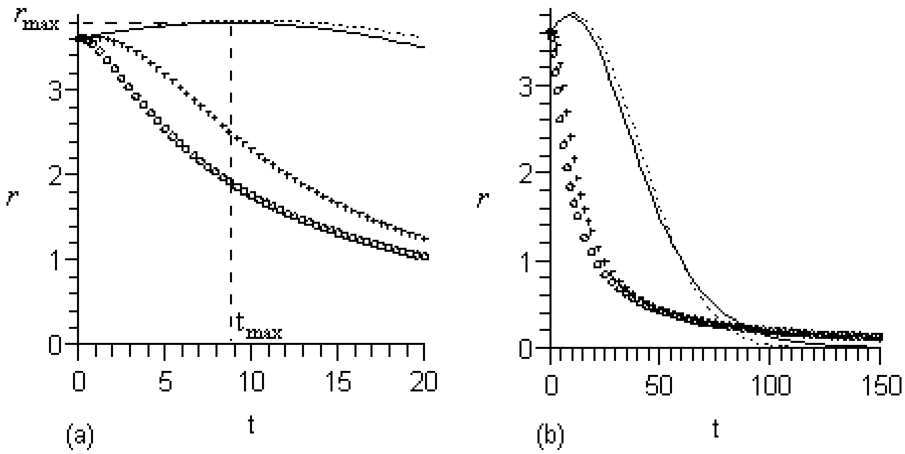

Figure 1: Scenarios of the world's oil extraction [bln t per year; time t in years starting from 2008], (a) in the short run, (b) in the long run: the Hubbert curve (solid); the Gauss curve (dotted); the Cauchy curve (crosses); the rational curve (circles)

Then,

$$\ddot{r}_H r_H / \dot{r}_H^2 = \{\cosh[b(t - t_{\max})] - 2\} / \{\cosh[b(t - t_{\max})] - 1\},$$

implying the index of sustainable extraction for the Hubbert curve: $\Phi[r_H(t)] \equiv \lim_{t\to\infty} \ddot{r}_H r_H / \dot{r}_H^2 = 1$, which is always less than $1 + \beta/\alpha$. This means that the path of output along this curve declines to zero in the long run regardless of the pattern of saving and the choice of the curve parameters. This "negative" conclusion can be obtained even simpler by using the result of Corollary 3. Indeed, the ratio $\dot{r}_H / r_H$ does not go to zero with time since $\lim_{t\to\infty} \dot{r}_H / r_H = -b$ implying that output eventually declines to zero.

One can ask: if there is a specific variable saving rate $w(t)$ that can prevent the path of consumption from decreasing while output goes to zero? The answer on this question can be easily obtained under the more general framework than the DHSS model. Given the relationship $\dot{c} = (1 - w)\dot{q} - \dot{w}q$, assume that there is a saving rate $w(t)$ that maintains the constant per capita consumption implying that $qdw/dt = (1 - w)dq/dt$ or $dw/(1 - w) = dq/q$. Integration gives $w(t) = 1 - (1 - w_0)q_0/q(t)$. The same result trivially follows from equation $q(t) = c(t) - w(t)q(t)$ assuming that consumption is constant ($c = c_0$) over some period. Hence, a closed economy with per capita output $q(t) = c(t) - w(t)q(t)$ maintains constant per capita consumption over any period $[t_0, t_1] \subseteq [0, \infty)$ iff $w(t) = 1 - c_0/q(t)$,where $w(t) \in (0, 1)$ is a feasible saving rate. The boundedness of feasible saving rate implies the following

Corollary 4. *A closed economy with per capita output* $q(t) = c(t) - w(t)q(t)$ *can maintain constant per capita consumption for* $t \geqslant t_0$ *only if* $c_0 < q(t) < \infty$,

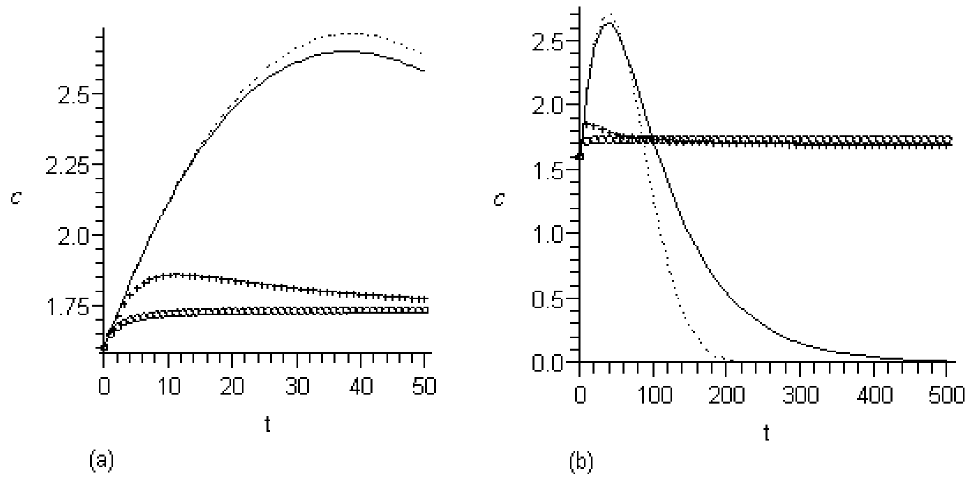

Figure 2: Paths of per capita consumption [time t in years starting from 2008], (a) in the short run, (b) in the long run along: the Hubbert curve (solid); the Gauss curve (dotted); the Cauchy curve (crosses); the rational curve (circles)

where $c_0 = c(t_0)$.

This result means that there is no feasible variable saving rate $w(t) \in (0, 1)$ that can provide constant or growing consumption along the Hubbert curve. Hence, the path of consumption along this curve declines to zero (Fig. 2, solid line[21]) for any saving rate; therefore the government should do its best using taxes, environmental policies, and regulations in order to shift the extracting industry from following the Hubbert curve. As Arrow, Dasgupta and Mäler put it: "it can be that although the economy is in principle capable of achieving a sustainable development path, social welfare is unsustainable along the path that has been forecast because of bad government policies" (2003, p. 654).

Another pattern of extraction considered in (Laherrere 2000) is a well-known Gauss curve (Fig. 1, dotted):

$$r_G(t) = r_{\max} \exp\left[-(t_{\max} - t)^2/2b^2\right],$$

where the roles of parameters are the same: $t_{\max}, r_{\max}$ are the year and the amount of the maximum extraction and b describes the deviation. The derivatives are $\dot{r}_G = (t_{\max} - t)r_G/b^2$ and $\ddot{r}_G = \left[(t_{\max} - t)^2 - b^2\right] r_G/b^4$. Then the

[21] The paths of consumption are obtained for all cases by solving numerically the differential equation for capital with $\alpha = 0.3$, $\beta = 0.25$ and $w = \beta$. The same qualitative results were obtained for some curves in (Andreeva, Bazhanov 2007) for $\alpha = 0.3$, $\beta = 0.05$, and $\dot{r}_0 = 0.08$. The difference was in the lower level of consumption (e.g. asymptote for the consumption path along the rational curve was 1.539 in comparison with 1.736 in the current paper).

index of sustainable extraction is

$$\Phi\left[r_G(t)\right] \equiv \lim_{t\to\infty} \ddot{r}_G r_G/\dot{r}_G^2 = \lim_{t\to\infty} \left[(t_{\max}-t)^2 - b^2\right]/(t_{\max}-t)^2 = 1$$

that implies the same pessimistic outcome as for the Hubbert curve (Fig.2, dotted). Corollary 3 gives the same result since $\lim_{t\to\infty} \dot{r}_G/r_G = -\infty$.

Note that one more pattern of exponential extraction $r_{GR}(t) = r_0 e^{-gt}$ derived as optimal in (Grimaud, Rouge 2005, Proposition 1, p. 115) leads to the same qualitative result in the DHSS economy as the Gauss and Hubbert curves. Indeed, $\Phi\left[r_{GR}(t)\right] \equiv 1$ that also means declining to zero per capita consumption in the DHSS economy. However, in the framework of Grimaud and Rouge (GR) this extraction path implies exponential growth of per capita consumption. This contrast results from the difference in the models. The simplest variant of the DHSS economy (no capital decay and no growth in TFP) can be interpreted as a TFP exactly compensating for capital depreciation, which is "weaker" than exponential. The GR model does not contain physical capital explicitly and, of course, it does not contain capital decay. This assumption can be also reformulated as the assumption about the implied technical progress exactly compensating for capital depreciation. In addition to this implicit technical change in the GR model, there is endogenous exponential technical progress in the form of growing knowledge and there is also a specific externality caused by a polluting resource. The "extra" technical change certainly gives the GR model additional opportunities for growth in comparison with the variant of the DHSS model considered in this paper.

The "optimistic" alternatives to the curves above can be found among the densities of fat-tailed distributions. These paths of extraction can be compatible with the Cobb-Douglas production function in a sense that they give the opportunity to sustain non-decreasing per capita consumption in the long run. This property is connected with the fat tail that provides more resources to the future generations. These patterns of resource extraction make it possible to adjust capital adequately to the rate of shrinking of an essential resource. This means that the rate of extraction along the Hubbert and Gauss curves approaches zero too quickly in the long run, so that the Cobb-Douglas technological properties do not allow to compensate for that fast decline of r by growing capital regardless of the saving rate. In terms of Proposition 1 it means that τ becomes positive (or, in terms of Corollary 1, the price-interest rate changes become unfavorable) along these curves and exceeds the bound, below which the growth of output is possible (see the next section). In this sense, the following patterns of extraction can be compatible with the production function, depending on their parameters. For example, the curve

$$r_C(t) = b^d r_{\max}/\left[b + (t_{\max}-t)^2\right]^d$$

is the probability density function for the Cauchy distribution for $d = 1$, where $t_{\max}$ is the location parameter and b is the scale parameter.[22] The generalizing

[22] $t_{\max}$ is not the expectation because the expectation and all other higher moments do not

parameter d is introduced here as a control variable for the index of sustainable extraction of this curve, which is

$$\begin{aligned}\Phi\left[r_C(t)\right] &\equiv \lim_{t\to\infty} \ddot{r}_C r_C/\dot{r}_C^2 \\ &= 0.5 \lim_{t\to\infty}\left[(t_{\max}-t)^2 + 2d(t_{\max}-t)^2\right] / \left[d(t_{\max}-t)^2\right] \\ &= (1+2d)/2d.\end{aligned}$$

Proposition 2 implies that in the long run $\dot{q} \geqslant 0$ iff $(1+2d)/2d \geqslant 1+\beta/\alpha$ or $d \leqslant \alpha/(2\beta)$.[23] This curve with $d = \alpha/(2\beta)$ is depicted in Fig. 1 in crosses. The corresponding path of consumption is asymptotically constant for this value of d (Fig. 2, in crosses).

Another example of the extraction path, allowing for sustainable economic growth, is a variant of the transition path that I called "rational" and examined in (Bazhanov 2007b). The first derivative of this curve is $\dot{r}_R(t) = (\dot{r}_0 + bt)/(1+ct)^d$, the curve itself is $r_R(t) = r_0(1+b_r t)/(1+ct)^{d-1}$, where $b_r = c(d-1)+\dot{r}_0/r_0$ and $\ddot{r}_R(t) = \left[b(1+ct) - dc(\dot{r}_0+bt)\right]/(1+ct)^{d+1}$. The initial conditions imply $b = -c(d-2)\left[r_0 c(d-1) + \dot{r}_0\right]$ and then the index is

$$\begin{aligned}\Phi\left[r_R(t)\right] &\equiv \lim_{t\to\infty} \ddot{r}_R r_R/\dot{r}_R^2 = r_0(1+b_r t)\left[b(1+ct) - dc(\dot{r}_0+bt)\right]/(\dot{r}_0+bt)^2 \\ &= 1 + 1/(d-2),\end{aligned}$$

implying that the paths of consumption and production are not declining iff $d \leqslant \alpha/\beta + 2$. This coincides with the result of Corollary 1 in Bazhanov (2007b). For comparison with the curve r_C, I considered r_R with $d = \alpha/\beta+2$, which also gives asymptotically constant consumption (Fig. 1 and Fig. 2, in circles).

In the following section I will use some of these examples in order to illustrate numerically the result of Proposition 1, namely, the roles of saving rates and imperfections in sustainability and in the level of consumption.

6. Extraction versus saving

Proposition 1 implies that an economy will follow decreasing (or increasing) path of output regardless of the saving rate $w(t) \in (0,1)$ when the Hotelling rule modifier $\tau(t)$ is not close enough to zero. This section illustrates how this result works in specific cases with the extraction curves analyzed in the previous section. I consider here only constant saving rates in order not to overcrowd the plots, demonstrating how the consumption path along an extraction curve changes with the saving rate. Variable saving rates imply in these cases only wandering of the resulting consumption path among the constant-rate paths and does not change the qualitative result (growing, declining or asymptotically constant) in the long run.

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exist for this distribution due to the divergence of the corresponding integrals.

[23] The result coincides with the one obtained in (Andreeva and Bazhanov 2007).

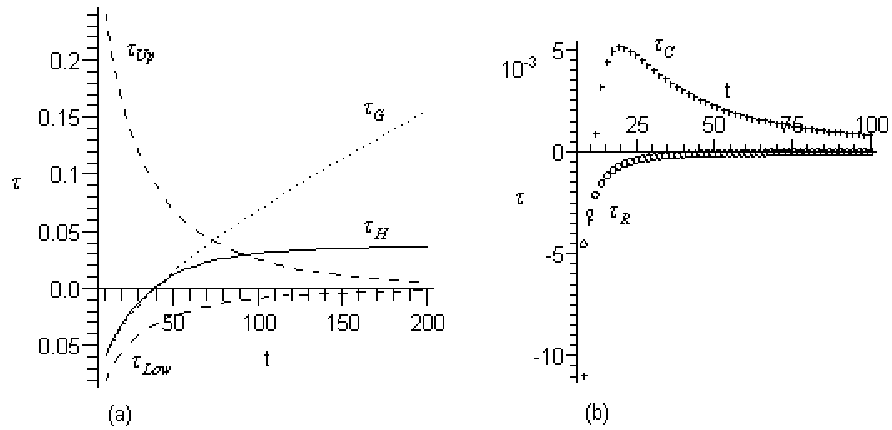

Figure 3: The paths of the Hotelling rule modifiers for various extraction curves: (a) Hubbert τ_H (solid); Gauss τ_G (dotted); (b) Cauchy τ_C (crosses); rational τ_R (circles)

It was shown that the long-run consumption declines to zero along the Hubbert and the Gauss curves for any patterns of saving. This means (Proposition 1) that in the long run the modifier $\tau(t)$ for these curves is greater than $f_k\left[1/\beta - 1\right] = \tau_{Up}$. One can see it in Fig. 3a where τ_H (τ for the Hubbert curve, solid line) asymptotically approaches a positive constant and τ_G (τ for the Gauss curve, dotted) goes to infinity while the upper (τ_{Up}) and the lower ($\tau_{Low} = -f_k$) bounds asymptotically converge to zero (dashed lines).[24] The paths of consumption, declining to zero along the Hubbert curve, are depicted in Fig. 4a for different values of w.

The paths of $\tau(t)$ for the Cauchy and the rational curves are rather deep inside the bounds τ_{Up} and τ_{Low} (Fig. 3b, the bounds are not depicted) and they converge with the bounds to zero regardless of the saving rate. This implies asymptotically constant consumption for all cases (Fig. 4b). The role of w is to define the level of the asymptote for the consumption path that one can see in Fig. 4b where the paths for $w_1 = 0.05$ and $w_3 = 0.8$ are the patterns of overconsumption and overinvestment correspondingly.

The paths of consumption that must grow in the long run according to Proposition 2 are depicted in Fig. 5 for the Cauchy curve with $d = \alpha/(2\beta) - 0.02$. The properties of this curve causing the long-run growth are illustrated in Fig. 6 in terms of observable variables (Corollary 1). Even in the case of overconsumption ($w = 0.05$), there is a moment of time ($t_{\min} \approx 5000$ years,

[24]The bounds τ_{Up} and τ_{Low} are depicted only for the Hubbert curve in order to not overcrowd the figure. The behavior of these values for the Gauss curve is the same with the only difference that they approach zero faster.

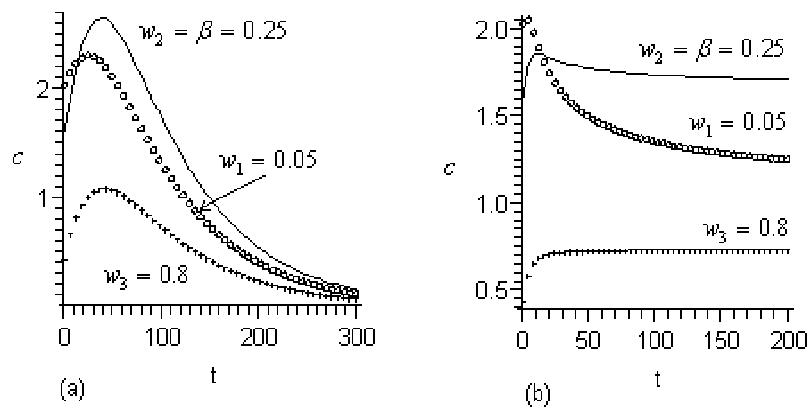

Figure 4: Paths of per capita consumption for different saving rates along: (a) Hubbert curve; (b) Cauchy curve with $d = \alpha/(2\beta)$

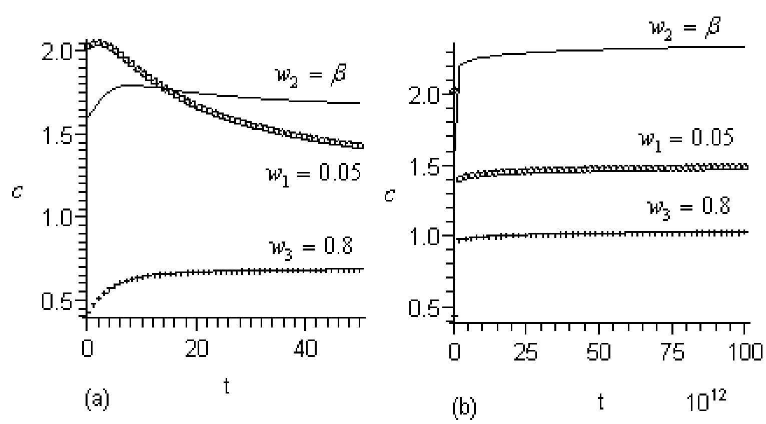

Figure 5: Paths of per capita consumption for different saving rates along the Cauchy curve with $d = \alpha/(2\beta) - 0.02$ (long-run unbounded growth): (a) in the short run; (b) in the long run

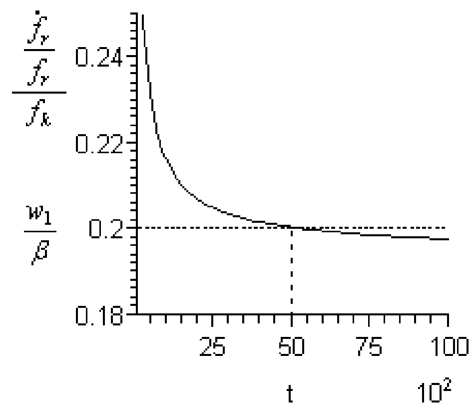

Figure 6: The path of the ratio of the rate of change of the price to the interest rate for the Cauchy path of extraction with unbounded growth of consumption in the long run (saving coefficient $w_1 = 0.05$)

Fig. 6) when the ratio of the rate of the price change to the rate of interest $(\left[\dot{f}_r/f_r\right]/f_k)$ becomes equal w_1/β that corresponds to the minimum of per capita consumption and implies a slow but unbounded growth for $t > t_{\min}$.

Hence, a resource-based economy can be either unsustainable despite non-negative genuine saving (Fig. 4a, $w = 0.25$ and $w = 0.8$) or it can be growing in the long run even with negative genuine saving (Fig. 5b, $w = 0.05$). Sustainability in these cases is qualitatively defined by deviation of the path of extraction, which in turn is the result of combined effect of externalities and government interventions. This implies that a policy with respect to an essential resource can play the primary role in sustainability. The policy can adequately compensate for negative externalities and introduce the positive ones affecting the path of extraction, while the saving rule selects the level of consumption from the resulting family of sustainable or unsustainable paths.

7. Concluding remarks

The paper has presented two new results for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model with the modified Hotelling rule.

(1) Proposition 1 and Corollary 1 (Section 3) have shown that a resource-based economy is growing if and only if the Hotelling rule modifier is less than the interest rate weighted by the factor $w/\beta - 1$ where $w = w(t)$ is the saving rate and β is the resource share; or, in terms of observable variables (Corollary 1), an economy is growing if and only if the ratio of the rate of the price change to the rate of interest is less than w/β. This result implies that the qualitative pattern of the economy's output (growth, stagnation, or decline) is defined by the path

of the resource extraction, which in turn is defined by the phenomena modifying the Hotelling rule (including government policy). Saving rate specifies the level of consumption along the growing, constant, or declining path and defines the pattern of output in the cases when the Hotelling rule modifier is close to zero.

(2) Proposition 2 (Section 4) provides a tool for estimating (weak) economic sustainability of forecasted paths of a nonrenewable resource extraction. The "index of sustainable extraction" offered in Proposition 2 can be easily calculated for any smooth enough feasible path of depletion. The index shows whether the path of output is growing, constant, or declining in the long run along this path of extraction. Corollary 4 showed that a closed economy with decreasing output can maintain not declining consumption over time only if output decreases no less than the initial value of consumption. As an example, it was shown that the path of per capita consumption always declines to zero in the long run along the well-known Hubbert curve regardless of the patterns of saving and the choice of parameters for this curve. This result is a warning sign appealing to the government's attention because the Hubbert curve is recognized in a large body of empirical research as the best tool for estimating the historical data of oil extraction. I considered the examples of the curves that allow for the oil-peak estimation and that imply nondecreasing consumption in the long run.

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9. Appendix A. The Hotelling rule modifier for the crude oil in the U.S.A. economy in 1955 - 2007

The path of the resource price with possible deviations can be written, e.g., in a form: $p(t) = p_0 \exp\{it + \Upsilon(t)\}$, where the interest rate i is piecewise-constant and $\Upsilon(t)$ (integral modifier) is the aggregate influence of phenomena modifying the Hotelling rule. Then $\dot{p}/p = i + \dot{\Upsilon}$ or $\tau = \dot{\Upsilon} = \dot{p}/p - i$, where $\Upsilon(t) = \ln[p(t)/p_0] - it$. The path of distortion τ can be estimated either directly or via $\Upsilon(t)$ using data for the resource price and for the interest rate. For the U.S.A. economy, both variants give a medium significant (at the level 0.01) negative correlation between τ and GDP percent change ($\dot{q}/q$) with the correlation coefficient -0.4168 (Fig. 7). The plot in Figure 7 is constructed for $\tau(t) = \Upsilon(t+1) - \Upsilon(t)$ with the data from the following sources: the crude oil prices are from (Potter and Christy 1962, p. 319, Colomn L) and from (EIA 2008); the interest rates are from (FR 2008).

One can see that sometimes there are small time lags between the maxima (minima) of τ and minima (maxima) of $\dot{q}/q$. That is natural result of mutual

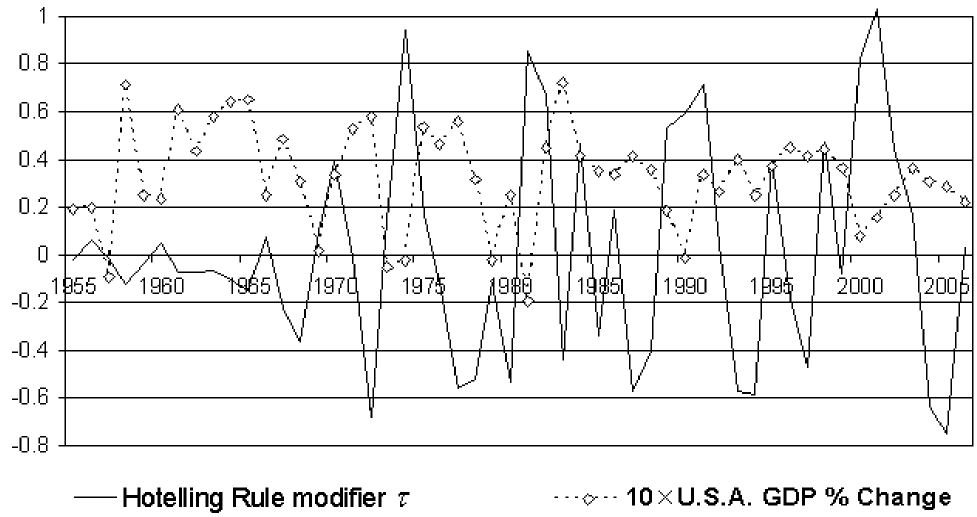

Figure 7: The Hotelling rule modifier (crude oil) and GDP percent change for the U.S.A. economy

dependence (with different lags) between economic growth and the price of oil in the real economy as well as dependence of both variables on other phenomena like monetary policies (1957), wars, embargoes (1973, 1974, 1979-1981) etc. The plot shows that even the simplest variant of the DHSS model, being applied to a real economy, can capture some qualitative effects examined in the paper. For example, according to formula (2), growing τ follows decline in the rates of extraction, which, for sufficiently large τ, can cause decline in output due to the "too fast" decrease in the essential input and inability of the economy's technology to compensate for this decrease by growing capital.

"Vertical" analysis shows that τ is essentially positive when $\dot{q}$ is negative (1973, 1974, 1979, 1981, 1990) except 1957. This implies (Proposition 1) that the ratio w/β is more than unity for the U.S.A. economy; in other words, the genuine investment is positive, which qualitatively coincides with the result of Hamilton et al (2006, table III).

10. Appendix B. Calibration of the extraction curves

The parameters of the curves are calibrated on the current world's oil reserve and extraction data (World 2007). The initial rate of extraction (on January 1, 2008) is $r(0) = r_0$ =3.618 bln t /year (1 t = 7.3 barrel), the paths are assumed to satisfy the necessary condition of efficiency $\int_0^\infty r(t)dt = s_0$ =182.424 bln t (reserve estimate on January 1, 2008), and I took $\dot{r}(0) = \dot{r}_0 = 0.04$, which is the average $\dot{r}$ since 1984 (the methodology of estimation of $\dot{r}$ for historical data is in (Bazhanov 2006)). This way of calibration gives feasible patterns of extraction unlike the conventional calibration on historical data, which can lead to inconsistency with reserve estimates (Bazhanov, Vyscrebentsev 2006).

a) *The Hubbert curve* (Andreeva, Bazhanov 2007). The initial value r_0 implies $r_{H\max} = 0.5r_0(1+\cosh[-b_H t_{H\max}])$. The value of $\dot{r}(0)$ gives us $t_{H\max} = (1/b_H)\ln\left[(b_H r_0 + \dot{r}_0)/(b_H r_0 - \dot{r}_0)\right]$. Coefficient $b_H = (2r_0^2 + s_0\dot{r}_0)/(s_0 r_0)$ is obtained from the efficiency condition $\int_0^\infty r(t)dt = s_0$.

b) *The Gauss curve.* Using the condition $r(0) = r_0$ the curve can be expressed as follows: $r_G(t) = r_0\exp[t_{G\max}^2/(2b_G^2)]\exp[-(t_{G\max}-t)^2/(2b_G^2)]$ that gives us $r_{G\max} = r_0\exp[t_{G\max}^2/(2b_G^2)]$. The initial condition for $\dot{r}_0$ implies $t_{G\max} = \dot{r}_0 b_G^2/r_0$ and the efficiency condition for s_0 gives a nonlinear equation in b_G

$$(\sqrt{2\pi}/2)r_0 b_G \exp\left[\dot{r}_0^2 b_G^2/(2r_0^2)\right]\left[1+\operatorname{erf}\left\{\dot{r}_0 b_G/(r_0\sqrt{2})\right\}\right] = s_0$$

with a single relevant root that can be found numerically.

c) *The Cauchy curve.* The peak of extraction $r_{C\max} = r_0(b_C + t_{\max}^2)^d/b_C^d$ is expressed via r_0, the initial condition for $\dot{r}_0$ is more convenient to use in this case for obtaining b_C, which gives us $b_C = 2r_0 t_{C\max} d/\dot{r}_0 - t_{C\max}^2$; and $t_{C\max}$ can be found from a nonlinear equation $\int_0^\infty r_C(t, t_{C\max})dt - s_0 = 0$.

d) Calibration of *the rational curve* is in (Bazhanov 2007b).

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