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Abstract

Auctioneers who have an indivisible object for sale and believe that bidders are risk neutral can find the recipe for an optimal auction in Myerson (1981); auctioneers who believe that bidders are loss averse can find it here: An optimal auction is an all pay auction with minimum bid, and any optimal mechanism is all pay.

Keywords: Auctions, Loss Aversion, All Pay Mechanisms, Mechanism Design, Revenue Equivalence (JEL C70, D03, D44, D81, D86)

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1 Introduction

In standard auction settings with independent private values (IPV) and risk neutral bidders, traditional auction theory (e.g. Vickrey (1961)) gives a clear prediction on which auction formats deliver higher expected revenue: So long as across auction formats, each bidder receives the object with the same probability and the lowest possible valuation type never wins, it does not matter. Experimental work (e.g. Schramm and Onderstal (2008)) has shown, however, that the auction format *does* matter for the revenue of the auctioneer. Schramm and Onderstal (2008) report that the all pay auction (APA) yields higher revenue than the first price auction (FPA). As an application of the theoretical work of Kahneman and Tversky (1979) and Köszegi and Rabin (2006) on loss aversion and reference dependent preferences, Lange and Ratan (2010) incorporate gain loss preferences into the standard IPV auction framework and find that the FPA dominates the Vickrey auction if bidders are loss averse. In this paper, I show that the FPA is sub optimal for the auctioneer and that the APA is preferred if bidders are loss averse. Moreover, I extend the theory of mechanism design with risk neutral agents to settings with loss averse agents and provide a version of the revelation principle and the revenue equivalence theorem. It is shown that a revenue maximizing mechanism is an APA with minimum bid, and that any mechanism with, from an interim view, uncertain ex post payments is sub optimal. In addition, the results are compared to various notions of risk aversion. In general, the revenue ranking across auction formats is ambiguous with risk averse bidders. However, risk aversion in the money dimension in conjunction with strong separability of preferences in the good and in the money dimension leads to the same results regarding the revenue ranking across auction formats. Additionally, every risk averse bidder participates in the auction, whereas not every loss averse bidder necessarily does.

1.1 What this Paper is and is not about

The main goal of this paper is to establish the link between two popular and widely studied areas of economic theory: behavioral economics and mechanism design. Both of these areas have received increased attention in the recent past, so that it seems as a next natural step to consider auction and mechanism design with boundedly rational agents. As explained below, all familiar results from the extensively studied risk neutral case follow as limiting cases of the theory presented here. Additionally, the theory presented in what follows may provide an alternative theoretical framework for understanding empirical regularities. Finally, if bidders are loss averse, for which there is a plethora of field and experimental evidence (e.g. DellaVigna (2009)), the optimal sales mechanism is derived. Recent work on incorporating models of loss aversion (Heidhues and Köszegi (2008), Herweg, Müller, and Weinschenk (2010) (2010)) into market settings has helped to understand some 'puzzles' that cannot be explained based on orthodox economic theory. However, rather than attempting to convince the reader that loss aversion alone is what governs bidders' decisions under risk, the objective of this paper is to inform about the implications of the presence of loss averse bidders in auction settings.

The paper is organized as follows. Section 2 provides an overview of the assumptions made and a description of the auction environment. In section 3, I compare the FPA and the APA; the derivation of the optimal auction can be found in section 4. Section 5 relates the results to the existing literature, before section 6 concludes.

2 Assumptions and Auction Environment

As proposed in Köszegi and Rabin (2006), I assume that bidders' preferences are given by

$$u(c^g, c^m | r^g, r^m, \theta) := \underbrace{\theta c^g + c^m}_{\text{intrinsic utility}} \quad \underbrace{+ \eta^g \mu^g(\theta(c^g - r^g)) + \eta^m \mu^m(c^m - r^m)}_{\text{gain loss utility}},$$

where $c^g, r^g \in \{0, 1\}$ captures the good dimension, $c^m, r^m \in \mathbb{R}$ captures the money dimension, $\eta^l > 0$, $l \in \{g, m\}$ measures the weight attached to gain loss utility in dimension l, and $\theta \geq 0$ is the bidder's intrinsic valuation for the good (also referred to as 'type'). Moreover,

$$\mu^l(x) := \begin{cases} x, & \text{if } x \geq 0 \\ \lambda^l x, & \text{if } x < 0, \end{cases}$$

where $\lambda^l > 1$, $l \in \{g, m\}$. In contrast to the original formulation in Köszegi and Rabin (2006), I allow gain loss utility to have a different impact in the two dimensions. These preferences capture loss aversion through the Kahneman and Tversky (1979) value function[1], μ^l, $l \in \{g, m\}$. A deviation from the reference point is disliked more if it is a loss than it is liked if it is a gain.

I consider auctions of a single, indivisible object with $N \geq 2$ loss averse bidders who share the same η^l and λ^l, $l \in \{g, m\}$, and whose valuations, $\{\theta_i\}_{i=1}^N$, are the realizations of N independent draws from the continuous distribution function, $F : \Theta := [\theta_{\min}, \theta_{\max}] \subseteq \mathbb{R}_+ \to [0, 1]$ with strictly positive density, $f = F'$, everywhere. The valuation of bidder i, θ_i, is assumed to be private information to bidder i. In addition, I assume that the bidders and the auctioneer share the same prior beliefs. While applying Köszegi and Rabin (2007)'s (2007) solution concept of *choice acclimating personal equilibrium* (CPE), I focus on symmetric, strictly increasing equilibrium bidding functions only.

Using Köszegi and Rabin (2006)'s notation, if the distribution of reference points is G, and the distribution of actual consumption outcomes is H, the decision maker's ex ante expected utility is given by

$$U(H|G) := \int_{\{(c^g,c^m),\theta\}} \int_{\{(r^g,r^m),\theta\}} u(c^g, c^m | r^g, r^m, \theta) dG(r^g, r^m, \theta) dH(c^g, c^m, \theta).$$

In the above described auction setting, each bidder learns his valuation before submitting his bid and therefore, maximizes his interim expected utility,

$$U(H|G, \theta) := \int_{\{(c^g,c^m)\}} \int_{\{(r^g,r^m)\}} u(c^g, c^m | r^g, r^m, \theta) dG(r^g, r^m | \theta) dH(c^g, c^m | \theta).$$

Definition 1 *(Köszegi and Rabin (2007)) Conditional on the realization of the type, θ, for any choice set, D, $H \in D$ is an interim CPE if $U(H|H, \theta) \geq U(H'|H', \theta)$, for all $H' \in D$.*

[1]This piece wise linear specification does not satisfy all the properties identified by Kahneman and Tversky (1979) and chosen for expositional convenience. In appendix A.2, I allow for a more general form of μ^l by introducing diminishing sensitivity. It is shown that the results derived below are robust to this extension.

In the auction setting described above, fixing all other bidders' behavior, each bidder's bid, b_i, induces a distribution, $H_i(\mathcal{A}|b_i, b_{-i})$, over the set of alternatives, $\mathcal{A} := \{0,1\}^N \times \mathbb{R}^N$. Therefore, the definition can be modified in the following way to match the auction setting under consideration.

Definition 2 *Conditional on the realization of the type, θ_i, $b : \Theta \to \mathbb{R}_+$ is a symmetric interim CPE bidding function if for all $i, \theta_i, \theta_{-i}, b' \geq 0$,*

$$U\left(H_i(\mathcal{A}|b(\theta_i), b_{-i} = b(\theta_{-i}))|H_i(\mathcal{A}|b(\theta_i), b_{-i} = b(\theta_{-i})), \theta_i\right)$$
$$\geq U\left(H_i(\mathcal{A}|b', b_{-i} = b(\theta_{-i}))|H_i(\mathcal{A}|b', b_{-i} = b(\theta_{-i})), \theta_i\right).$$

As laid out in Köszegi and Rabin (2007), CPE is the most appropriate solution concept for decisions under risk, whose uncertainty is resolved long after the decision is made. An alternative solution concept, *choice unacclimating personal equilibrium* (UPE), requires the decision to be optimal, given the expectations at the time the decision is made. In some auction settings, the outcome is announced long after the bids are submitted; in others, the time elapsed between the submitted bid and the announcement of the winner is comparatively short. Nevertheless, because of two reasons, I consider CPE the most appropriate solution concept for the analysis of auctions in the setting described above. First, what is important is that the submission of the bid and the announcement of the outcome of the auction are temporally separated. The uncertainty associated with an auction is not exactly comparable to offering the decision maker a lottery, as e.g. a coin toss, with immediate resolution of uncertainty and reception of pay offs. In an auction, when making their decision, bidders are strategic about their submitted bid. An interpretation of CPE is that bidders are required to also be strategic about the reference dependence in their preferences. Second, bidders are not able to arrive at the auction site with meaningful expectations about the resolution of uncertainty, because the exact value of θ is not known until the object has been inspected, and none of the bidders can be exactly certain about the number of opponents they are competing with. Instead, once all relevant information has been collected, each bidder is assumed to choose a bid that maximizes his interim expected utility, taking into account the feelings he has after the auction is over. Besides loss aversion, this modeling approach captures aspects of regret, disappointment, and relief.

For the following analysis, it is convenient to define $\Lambda^l := \eta^l(\lambda^l - 1) > 0$, $l \in \{g, m\}$, which can be viewed as an overall measure of the degree of loss aversion in the respective dimension. The following condition guarantees that all bidders participate in the auction for any realization of their own type, and that their equilibrium bidding functions derived below are actually strictly increasing.

Condition 1 ***(No Dominance of Gain Loss Utility)*** $\Lambda^g \leq 1$.

This condition places, for a given η^g (λ^g), an upper bound on λ^g (η^g). In Herweg, Müller, and Weinschenk (2010), this condition is referred to as *no dominance of gain loss utility.*

3 First Price vs. All Pay Auctions

In this section, I consider the following class of auctions with all pay component, $\alpha \in [0, 1]$. Bidders simultaneously submit their bid, and the bidder with the highest bid wins the object and pays his entire bid. All other bidders walk away without the object but have to pay α of their bid. For $\alpha = 0$, we have the FPA and for $\alpha = 1$, the APA. Since winning ties happen with probability 0, any tie breaking rule is applicable. Consider the ex post utility of bidder i when his bid is x, and x_{-i} is the vector of all other bidders' bids. Let $P_i(x) = P(i \text{ wins } | x, x_{-i}) = P(x > \max_{j \neq i}\{x_j\})$ be the probability that bidder i wins the auction, conditional on his own and all other bidders' bids. When he ends up with the object and pays x, his utility is

$$\underbrace{\theta_i - x}_{\text{intrinsic utility}} + \underbrace{\eta^g \left(1 - P_i(x)\right) \theta_i - \eta^m \lambda^m \left(1 - P_i(x)\right) (1 - \alpha) x}_{\text{gain loss utility}}.$$

The first term represents intrinsic utility, and the second term captures gain loss utility. Compared to the situation in which the bidder does not win the auction, which happens with probability, $1 - P_i(x)$, he experiences a gain in the good dimension and a loss in the money dimension. In case bidder i ends up without the object and his bid is x, his utility is

$$\underbrace{-\alpha x}_{\text{intrinsic utility}} \quad + \underbrace{\eta^g \lambda^g P_i(x)(-\theta_i) + \eta^m P_i(x)(1-\alpha)x}_{\text{gain loss utility}},$$

since, compared to the situation in which he wins the auction, which happens with probability, $P_i(x)$, this is considered a loss in the good dimension and a gain in the money dimension. Therefore, bidder i's interim expected utility is

$$\begin{aligned} &P_i(x)\left(\theta_i - x + \eta^g\left(1-P_i(x)\right)\theta_i - \eta^m\lambda^m\left(1-P_i(x)\right)(1-\alpha)x\right) \\ &\quad + \left(1-P_i(x)\right)\left(-\alpha x - \eta^g\lambda^g P_i(x)\theta_i + \eta^m P_i(x)(1-\alpha)x\right). \end{aligned}$$

I look for strictly increasing, symmetric equilibrium bidding functions only. Hence, dropping the i subscript, the bidder's program is

$$\begin{aligned} V_\alpha(\theta) \quad &:= \quad \max_{x\in\mathbb{R}_+}\{P(x)\left(\theta - x + \eta^g\left(1-P(x)\right)\theta - \eta^m\lambda^m\left(1-P(x)\right)(1-\alpha)x\right) \\ &\qquad\qquad + \left(1-P(x)\right)\left(-\alpha x - \eta^g\lambda^g P(x)\theta + \eta^m P(x)(1-\alpha)x\right)\} \\ &= \quad F^{N-1}(\theta)\left(1-\Lambda^g\left(1-F^{N-1}(\theta)\right)\right)\theta \\ &\qquad -F^{N-1}(\theta)\left(1+\Lambda^m\left(1-F^{N-1}(\theta)\right)\right)(1-\alpha)b_\alpha(\theta) - \alpha b_\alpha(\theta), \end{aligned}$$

where the ultimate equality follows from independence of the types, b_α being strictly increasing (and hence, invertible), and the definition of Λ^l, $l \in \{g, m\}$. For economy of notation, let $F^{N-1}(\theta) =: F_\theta$. By the envelope theorem[2],

$$V'_\alpha(\theta) = F_\theta\left(1-\Lambda^g\left(1-F_\theta\right)\right) =: \Gamma(\theta),$$

which implies that

$$V_\alpha(\theta) = V_\alpha(\theta_{\min}) + \int_{\theta_{\min}}^{\theta}\Gamma(s)ds.$$

Since in equilibrium, every bidder is playing a monotone strategy, it follows that $V_\alpha(\theta_{\min}) = 0$, for all $\alpha \in [0,1]$. Inserting this expression for V_α into the above definition of V_α and solving for b_α yields the following expression for the equilibrium bidding function with loss averse bidders.

$$b_\alpha(\theta) = \frac{\Gamma(\theta)\theta - \int_{\theta_{\min}}^{\theta}\Gamma(s)ds}{(1-\alpha)F_\theta\Delta(F_\theta)+\alpha},$$

[2]Although $\mu^l, l \in \{g, m\}$ is not differentiable everywhere, because of the kink at the reference point, the bidder's value function is differentiable in θ.

where

$$\Delta(F_\theta) := (1 + \Lambda^m(1 - F_\theta)) \geq 1.$$

Proposition 1 *Suppose condition 1 holds. Then, $b_\alpha(\theta)$ is strictly increasing, for almost all θ.*

Proof: As shown in lemma 2 and lemma 3 in appendix A.3, condition 1 guarantees that the bidder's objective satisfies strictly increasing differences in (θ, x), so that $b_\alpha(\theta)$, as the unique maximizer of this objective, is strictly increasing in θ. $\square$

Let b_α^{RN} be the equilibrium bidding function when bidders are risk neutral ($\Lambda^g = \Lambda^m = 0$). Then, the equilibrium bidding function with loss averse bidders can be written as

$$b_\alpha(\theta) \quad = \quad \frac{1 - \Lambda^g}{\psi_\alpha(\theta)} b_\alpha^{RN}(\theta) + \frac{\Lambda^g}{\psi_\alpha(\theta)} \kappa_\alpha(\theta),$$

where

$$\psi_\alpha(\theta) := \frac{(1-\alpha)F_\theta \Delta(F_\theta) + \alpha}{(1-\alpha)F_\theta + \alpha} \geq 1 \quad \text{and} \quad \kappa_\alpha(\theta) := \frac{F_\theta^2 \theta - \int_{\theta_{\min}}^{\theta} F_s^2 ds}{(1-\alpha)F_\theta + \alpha}.$$

b_α is a distorted convex combination of b_α^{RN} and κ_α. The coefficients are distorted by gain loss considerations in the money dimension, measured by ψ_α. In order to study the equilibrium bidding behavior of loss averse bidders, it is instructive to first consider the case, in which bidders are only loss averse in the money dimension ($\Lambda^m > 0$) and risk neutral in the good dimension ($\Lambda^g = 0$). In this case, the equilibrium bidding function reads

$$b_\alpha(\theta) \quad = \quad \frac{1}{\psi_\alpha(\theta)} b_\alpha^{RN}(\theta).$$

If bidders only have gain loss considerations in the money dimension, then the equilibrium bid is the distorted risk neutral bid. Regarding the comparative statics results with respect to the parameter, Λ^m, the following holds.

Proposition 2 *$b_\alpha(\theta)$ is strictly decreasing in Λ^m, for almost all θ, $\alpha \in [0, 1]$.*

Proof: Immediate by inspection. □

As gain loss considerations in the money dimension become more important, the equilibrium bid is reduced. To see the intuition behind this result, consider a bidder in auction $\alpha \in [0, 1)$. In case he wins the auction, he experiences a loss in the money dimension; if he loses the auction he experiences a gain of equal magnitude. Since losses weigh more than gains, the overall effect reduces the equilibrium bid. Another insight about the effects of loss aversion in the money dimension can be obtained by considering the distortion coefficient,

$$\psi_\alpha(\theta) = \frac{(1-\alpha)F_\theta\Delta(F_\theta) + \alpha}{(1-\alpha)F_\theta + \alpha} = \frac{(1-\alpha)\left(F_\theta + \Lambda^m F_\theta(1-F_\theta)\right) + \alpha}{(1-\alpha)F_\theta + \alpha},$$

which is increasing in $F_\theta(1-F_\theta)$, the variance of the Bernoulli distributed outcome of winning or losing the auction. Additionally, ψ_α is decreasing in α, and $\psi_1 = 1$. If an increased fraction of the bid is paid for sure, gain loss considerations in the money dimension become less important in distorting the equilibrium bid.

In order to examine how loss aversion in the good dimension affects the bidding behavior, consider the case in which $1 \geq \Lambda^g > 0$ and $\Lambda^m = 0$. Then, the equilibrium bidding function is given by

$$b_\alpha(\theta) \quad = \quad (1-\Lambda^g)b_\alpha^{RN}(\theta) + \Lambda^g \kappa_\alpha(\theta).$$

Regarding the effect of an increase in Λ^g, we have

$$\begin{aligned} & \frac{\partial}{\partial \Lambda^g} b_\alpha(\theta) = -b_\alpha^{RN}(\theta) + \kappa_\alpha(\theta) \leq 0 \\ \iff \quad & b_\alpha(\theta) \leq b_\alpha^{RN}(\theta) \\ \iff \quad & F_\theta^2\theta - \int_{\theta_{\min}}^{\theta} F_s^2 ds \leq F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds. \end{aligned}$$

For $\theta = \theta_{\min}$, both the LHS and the RHS of the above expression are equal to 0. The derivative of the expression on the RHS is greater than the derivative of the expression on the LHS if and only if $F_\theta \leq 1/2$, which implies that the bid of the lowest types is always reduced by an increase in Λ^g, whether the bid of the highest types is increased depends on the distribution, F, only. It is instructive to consider a concrete example. Figure 1 depicts the equilibrium bidding functions for

$N = 2$ and $\theta \sim U[0,1]$, compared to the same situation with risk neutral bidders ($\Lambda^g = \Lambda^m = 0$).

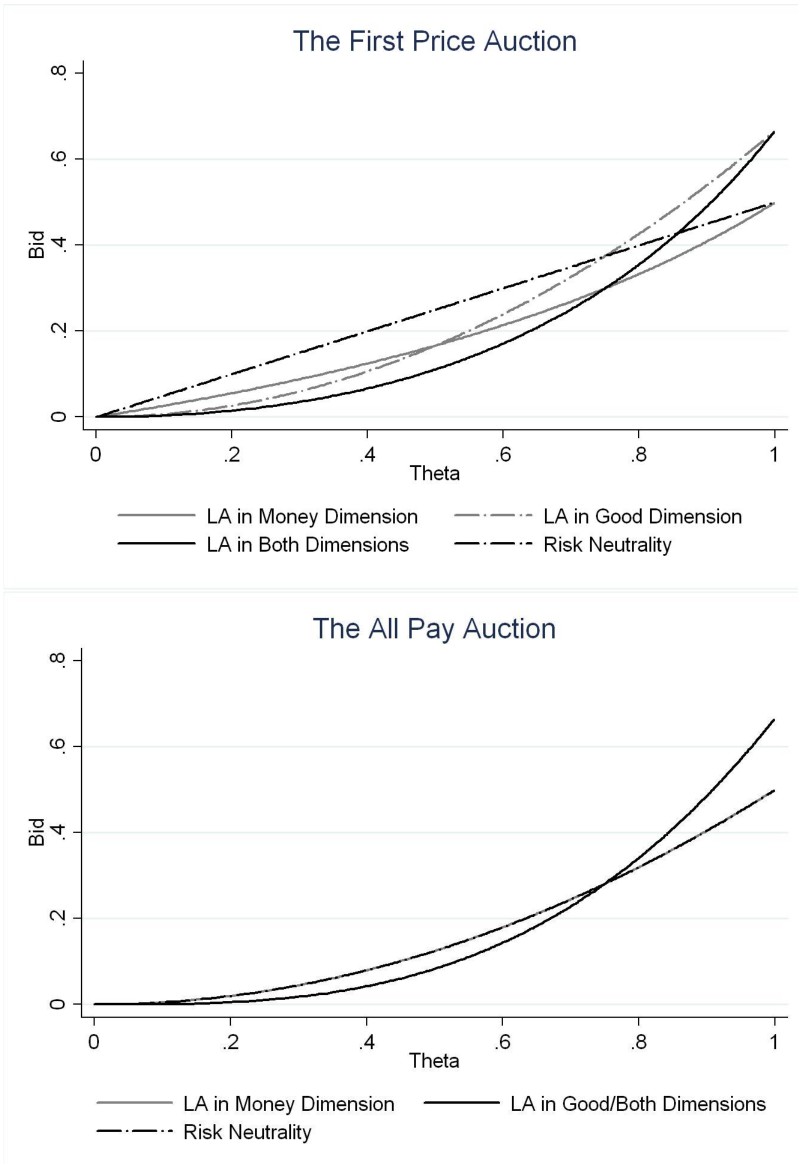

Figure 1: FPA and APA with $\Lambda^g = 1$, $\Lambda^m = 1$, $N = 2$, and $\theta \sim U[0,1]$.

In order to see the intuition behind the above results and the example, consider the equilibrium interim expected utility of a bidder of type θ, when following the above prescribed strategy, b_α,

$$V_\alpha(\theta) = \Gamma(\theta)\theta - F_\theta(1-\alpha)b_\alpha(\theta) - \alpha b_\alpha(\theta) - \Lambda^m F_\theta \left(1 - F_\theta\right)(1-\alpha)b_\alpha(\theta).$$

Also,

$$V_\alpha(\theta) = \int_{\theta_{\min}}^{\theta} \Gamma(s)ds,$$

so that the interim expected pay off of a bidder of type θ is identical, for all α. Now, consider the case in which $\Lambda^m > 0$ and $\Lambda^g = 0$. Then,

$$V_\alpha(\theta) = F_\theta\theta - F_\theta(1-\alpha)b_\alpha(\theta) - \alpha b_\alpha(\theta) - \Lambda^m F_\theta \left(1 - F_\theta\right)(1-\alpha)b_\alpha(\theta).$$

If $\alpha = 1$, gain loss considerations have no effect on the interim pay off. As α decreases, gain loss utility becomes more and more important and reduces the interim expected pay off. This is compensated for by reducing the equilibrium bid. Next, consider the case where $1 \geq \Lambda^g > 0$ and $\Lambda^m = 0$. In this case, a bidder of type θ seeks to maximize

$$P(x)\left(\theta - (1-\alpha)x\right) - \alpha x - \Lambda^g P(x)\left(1 - P(x)\right)\theta.$$

That is, his objective is the objective of a risk neutral bidder less a penalty term for the variance of the Bernoulli distributed outcome of winning or losing the auction. If only gain loss utility matters, utility maximizing behavior, in the absence of any constraints imposed by equilibrium play, implies that x ought to be chosen such that $P(x)$ is either maximized or minimized. This translates into submitting either a very high or a very low bid. Since in equilibrium, every bidder plays a strictly increasing bidding strategy, low (high) types have to sacrifice more of their intrinsic utility when submitting a high (low) bid than when submitting a low (high) bid, which induces the same variance. Therefore, in the above example, the lowest types submit a lower bid than in the risk neutral case, and the highest types bid more than under risk neutrality. In general, whether and where the bidding function with loss averse bidders intersects the one of risk neutral bidders depends on the distribution, the number of bidders, the auction format, and the degree of loss aversion. Köszegi and Rabin (2007) mention that loss averse decision makers are drawn towards certain outcomes. This translates into low types submitting low bids to be more certain to lose the auction, and high types submitting high bids to be more certain to win the auction.

So far, it has been assumed that condition 1 is satisfied. As Lange and Ratan (2010) show, if condition 1 is not met, there are some bidders who choose to not participate in the auction. This is also true in the auction setting studied here. More specifically, the implications of a violation of condition 1 are the following.

Proposition 3 *Suppose condition 1 does not hold, i.e.* $\Lambda^g > 1$*. Then, there is a unique interior threshold,* $\tilde{\theta} \in (\theta_{\min}, \theta_{\max})$*, such that* $b_\alpha(\theta) = 0$*, for all* $\theta < \tilde{\theta}$*, and*

$$b_\alpha(\theta) = \frac{\Gamma(\theta)\theta - \int_{\tilde{\theta}}^{\theta} \Gamma(s)ds}{(1-\alpha)F_\theta \Delta(F_\theta) + \alpha},$$

for all $\theta \geq \tilde{\theta}$*, for all* $\alpha \in [0,1]$*. Additionally,* $\tilde{\theta}$ *is strictly increasing in* Λ^g *and the number of bidders,* N*.*

Proof: If condition 1 does not hold, then, using that $\Lambda^g > 1$,

$$V'_\alpha(\theta) \quad = \quad F_\theta \left(1 - \Lambda^g \left(1 - F_\theta\right)\right) < 0 \Longleftrightarrow \frac{\Lambda^g - 1}{\Lambda^g} > F_\theta = F^{N-1}(\theta).$$

The LHS of the above inequality is independent of θ, and $(\Lambda^g - 1)/\Lambda^g \in (0,1)$, for $\Lambda^g > 1$. The RHS is strictly increasing in θ. By continuity and monotonicity of F, there is a unique interior value, $\hat{\theta} \in (\theta_{\min}, \theta_{\max})$, satisfying

$$\left(\frac{\Lambda^g - 1}{\Lambda^g}\right)^{1/(N-1)} = F(\tilde{\theta}),$$

such that for all $\theta < \tilde{\theta}$, $V'_\alpha(\theta) < 0$ and for all $\theta > \tilde{\theta}$, $V'_\alpha(\theta) > 0$, so that $V'_\alpha(\tilde{\theta}) = 0$. Since the LHS of the above expression is strictly increasing in Λ^g and N, $\tilde{\theta}$ is strictly increasing in Λ^g and N. Hence, all bidders with types $\theta < \tilde{\theta}$ receive strictly negative interim expected utility from participating in the auction. These bidders can secure themselves an interim expected pay off of 0 by submitting a bid of 0. Given the behavior of the bidders with types $\theta < \tilde{\theta}$, it is optimal for all types $\theta \geq \tilde{\theta}$ to submit non negative bids, since for them, $V'_\alpha(\theta) \geq 0$. Given the behavior of these types, it is optimal for types $\theta < \tilde{\theta}$ to submit a bid of 0 and to not participate in the auction.
$\square$

This result indicates that when loss aversion in the good dimension is too pronounced, there is a set of types of strictly positive measure, for which it is not optimal to participate in the auction at all and submit a positive bid. The cut off point, $\tilde{\theta}$, is identical across all auction formats, $\alpha \in [0,1]$. In order to see the intuition behind this, suppose that all bidders participate. The interim expected pay off of bidder of type $\theta < \tilde{\theta}$ is

$$V_\alpha(\theta) = \int_{\theta_{\min}}^{\theta} \left(F_s - \Lambda^g F_s(1 - F_s)\right) ds.$$

Again, the variance of the Bernoulli distributed outcome of winning or losing the auction reduces the information rents. The above result says that, depending on the value of Λ^g, this reduction can be too pronounced to make participation worth while for the lowest types, since they have the lowest information rents to start with. Figure 2 depicts the equilibrium bidding function for the APA in the setting of the previous example in figure 1 if condition 1 is violated ($\Lambda^g = \Lambda^m = 2$). If loss aversion is very pronounced ($\Lambda^g > 1$), it is not profitable for the bidders at the bottom of the distribution to take the risk of participating in the auction by submitting a positive bid. Loss averse bidders prefer certain outcomes. If gain loss utility dominates intrinsic utility ($\Lambda^g > 1$), then the lowest types have to be compensated for taking the risk associated with participating in the auction, which translates into the non negativity constraint on the submitted bid to be binding for these types.

3.1 Revenue Non Equivalence

A well known result from classical auction theory with risk neutral bidders is that the expected ex ante revenue for the auctioneer is identical, for all $\alpha \in [0,1]$. As shown in this subsection, this property fails to hold if bidders are loss averse. The following propositions summarize the revenue ranking across different auction formats.

Proposition 4 *If bidders are loss averse in the money dimension ($\Lambda^m > 0$), the expected revenue for the auctioneer is strictly increasing in α.*

Figure 2: APA with $\Lambda^g = \Lambda^m = 2$, $N = 2$, and $\theta \sim U[0, 1]$.

Proof: The expected payment, $p_\alpha(\theta)$, of a bidder of type, θ, conditional on the other bidders' behavior, is

$$p_\alpha(\theta) = \alpha b_\alpha(\theta) + F_\theta(1 - \alpha)b_\alpha(\theta),$$

i.e. αb_α with certainty, and $(1-\alpha)b_\alpha$ only if he wins, which happens with probability F_θ. Differentiating the above expression with respect to α yields

$$\frac{\partial}{\partial \alpha} p_\alpha(\theta) = \frac{F_\theta(\Delta(F_\theta) - 1)}{(1 - \alpha)F_\theta\Delta(F_\theta) + \alpha} b_\alpha(\theta),$$

which is non negative, for all θ and strictly positive, for all $\theta > \tilde{\theta}$. Since the interim expected payment is non decreasing for all types and strictly increasing for a set of types of strictly positive measure, this implies that the ex ante expected revenue for the auctioneer, $N \int p_\alpha(\theta)dF(\theta)$, is strictly increasing in α. $\square$

The following is now an immediate corollary.

Proposition 5 *If bidders are loss averse in the money dimension, the expected revenue for the auctioneer strictly higher in the APA than in the FPA.*

Proof: By proposition 4. □

As seen above, gain loss considerations in the money dimension distort the equilibrium bid downwards. By requiring bidders to pay their bid regardless of whether they win the object or not, gain loss distortions in the money dimension are minimized. If $\alpha < 1$, loss averse bidders realize gains in the money dimension if they lose, and losses if they win. Since, under loss aversion, losses weigh more than gains, bidders bid more hesitantly in any auction with $\alpha < 1$ than in the APA. Therefore, among all auctions with fixed all pay component, α, the APA maximizes the auctioneer's expected revenue. As further elaborated on in the next section, the all pay nature of a mechanism is an important ingredient in designing the optimal auction when bidders are loss averse, while any uncertainty in the payments is sub optimal from a revenue maximizing perspective. Also, as $\Lambda^m \to 0$, $\Delta(F_\theta) \to 1$, so that revenue equivalence holds in the limit, as loss aversion in the money dimension vanishes. Loss aversion in the good dimension is irrelevant for the revenue ranking across auction formats, as summarized in the following proposition.

Proposition 6 *If bidders are loss averse in the good dimension and risk neutral in the money dimension ($\Lambda^g > 0, \Lambda^m = 0$), the expected revenue for the auctioneer is the identical, for all $\alpha \in [0, 1]$.*

Proof: By proposition 4. □

This result confirms that the revenue ranking across auction format is solely driven by loss aversion in the money dimension, whereas loss aversion in the good dimension is responsible for the limited participation results.

3.2 Risk Aversion or Loss Aversion?

A natural question to ask is whether the results derived above are driven by risk aversion rather than loss aversion. Auctions with risk averse bidders are studied in Riley and Samuelson (1981), Maskin and Riley (1984), and Matthews (1987), where bidders' preferences take the form $u(\theta, -x)$, and u is strictly increasing and strictly concave in both arguments. As a special case of this formulation, which is studied in Fibich, Gavious, and Sela (2006), bidders' preferences take the form $u(\theta - x)$, where u is strictly increasing and strictly concave. Fibich, Gavious, and Sela (2006) compare the expected revenue in the APA and the FPA. Their finding is that the revenue ranking is ambiguous in the sense that there are utility functions and distributions for which either the APA or the FPA yields higher expected revenue for the auctioneer. Maskin and Riley (1984) derive the optimal auction for risk averse bidders. They find that a *perfect insurance auction* is optimal with homogeneously risk averse bidders, who differ only in their type, θ. A perfect insurance auction is an auction with two payment schemes, one for bidders who win the auction, x^W, and one for bidders who lose the auction, x^L, that depend on the reported type, but are deterministic otherwise, and have the property that for highest type, the marginal utility of money is identical in each state, that is

$$u_2(\theta_{\max}, -x^W(\theta_{\max})) = u_2(0, -x^L(\theta_{\max})),$$

or, as studied in Fibich, Gavious, and Sela (2006),

$$u'(\theta_{\max} - x^W(\theta_{\max})) = u'(-x^L(\theta_{\max})) \Longrightarrow \theta_{\max} - x^W(\theta_{\max}) = -x^L(\theta_{\max}).$$

The APA is nested in the class of perfect insurance auctions, for $x^W = x^L$, and the FPA is nested for $x^L = 0$. The results in Maskin and Riley (1984) imply that a necessary condition for the APA ($x^W = x^L$) to yield the highest expected revenue for the auctioneer is that the marginal utility of money is independent of the valuation, θ. Furthermore, the insights obtained by Maskin and Riley (1984) rationalize the ambiguous revenue ranking between the APA and the FPA reported in Fibich, Gavious, and Sela (2006). If bidders' preferences take the form

$$u(\theta, -x) = \theta - m(x),$$

where $m : \mathbb{R}_+ \to \mathbb{R}_+$, is continuous, strictly increasing, strictly convex, and $m(0) = 0$, the marginal utility of money does not depend on θ, and therefore, the APA yields the highest expected revenue for the auctioneer, which leads to the following result.

Proposition 7 *Suppose bidders are risk averse in the money dimension Then, (i) the symmetric, strictly increasing equilibrium bidding strategy is given by* $b_\alpha^{RA}(\theta) =$

$m^{-1}\left(b_\alpha^{RN}(\theta)\right)$, (ii) the expected revenue for the auctioneer is strictly higher in the APA than in the FPA, and (iii) every bidder participates in the auction and submits a positive bid.

Proof: In appendix A.1. □

The intuition behind this result is that risk averse bidders like consumption smoothing in the money dimension. In the FPA, a given increase in the expected payment is more costly to the bidders than in the APA, and therefore, in expected terms, they bid more aggressively in the APA. Furthermore, every risk averse bidder with the above preferences is locally risk neutral. This implies that every risk averse bidder participates in the auction and submits a positive bid, because he obtains non negative expected pay off from doing so. As seen above, this is not necessarily the case if bidders are loss averse. This raises the question whether the limited participation results derived above for high degrees of loss aversion in the good dimension can be explained by first order risk aversion. In order to further study the effects of risk preferences on equilibrium bidding behavior and the revenue ranking across auction formats, I consider bidders with rank dependent expected utility (RDEU) preferences as in Yaari (1987). Consider a lottery with monetary outcomes, $x_1 \leq x_2 \leq \ldots \leq x_K$ which occur with probabilities, $p_1, p_2, \ldots, p_K$, respectively. A decision maker with RDEU preferences evaluates this lottery according to

$$U(p,x) = \sum_{k=1}^{K} u(x_k)\left(g\left(\sum_{j=1}^{k} p_j\right) - g\left(\sum_{j=1}^{k-1} p_j\right)\right),$$

where $g:[0,1] \to [0,1]$ is a strictly increasing probability distortion function, with $g(0) = 0$ and $g(1) = 1$. For a suitable choice of g (e.g. g concave), these preferences generate first order risk aversion at the certainty line. In order to focus on first order risk aversion, I assume that $u(\theta, -x) = \theta - x$ and that g is strictly concave. This leads to the following result regarding the revenue ranking across auction formats when bidders have RDEU preferences.

Proposition 8 *Suppose bidders have RDEU preferences of the form specified above, with g concave. Then, (i) the symmetric, strictly increasing equilibrium bidding* *function,* b_α^{RDEU}*, is given by*

$$b_\alpha^{RDEU}(\theta) = \frac{g\left(F_\theta\right)\theta - \int_{\theta_{\min}}^{\theta} g\left(F_s\right)ds}{(1-\alpha)g\left(F_\theta\right)+\alpha},$$

(ii) the expected revenue for the auctioneer is strictly higher in the APA than in the FPA, and (iii) every bidder participates in the auction and submits a positive bid.

Proof: In appendix A.1. □

The utility in case of winning the auction is $\theta - b_\alpha^{RDEU}(\theta)$, and the utility in the event of losing is $-b_\alpha^{RDEU}(\theta)$. Since $\theta \geq b_\alpha^{RDEU}(\theta) \geq 0$, the decision maker only considers lotteries on one side of the certainty line. Therefore, the reason for the APA yielding higher expected revenue for the auctioneer than the FPA does not follow an insurance argument. Instead, the above result is driven by bidders overweighting the probability of winning, since the assumptions on g imply that $g(F_\theta) \geq F_\theta$, for all θ. This implies that a bidder with RDEU preferences is overoptimistic or overconfident of winning the auction. To see why this leads to revenue ranking across auction formats, consider the interim expected pay off of a bidder of type θ,

$$V_\alpha^{RDEU}(\theta) = g(F_\theta)\left(\theta - (1-\alpha)b_\alpha^{RDEU}(\theta)\right) - \alpha b_\alpha^{RDEU}(\theta).$$

Since $g(F_\theta) \geq F_\theta$, a bidder with RDEU preferences attaches a too high probability to winning the auction, in which case he has to pay the remaining $(1-\alpha)$ of his bid. If $\alpha = 1$, the bidder has to pay his bid with certainty, and therefore, there is no room for overweighting probabilities (since $g(1) = 1$). Also, as in the case of ordinary risk aversion, every bidder with RDEU preferences participates in the auction and submits a positive bid. These results suggest that the revenue ranking across auction formats is also obtained with second order risk aversion and RDEU preferences with probability overweighting, in conjunction with strong separability of the Bernoulli utility function in the good and in the money dimension. Figure 3 illustrates the bidding behavior when bidders are risk averse and risk loving, first and second order.

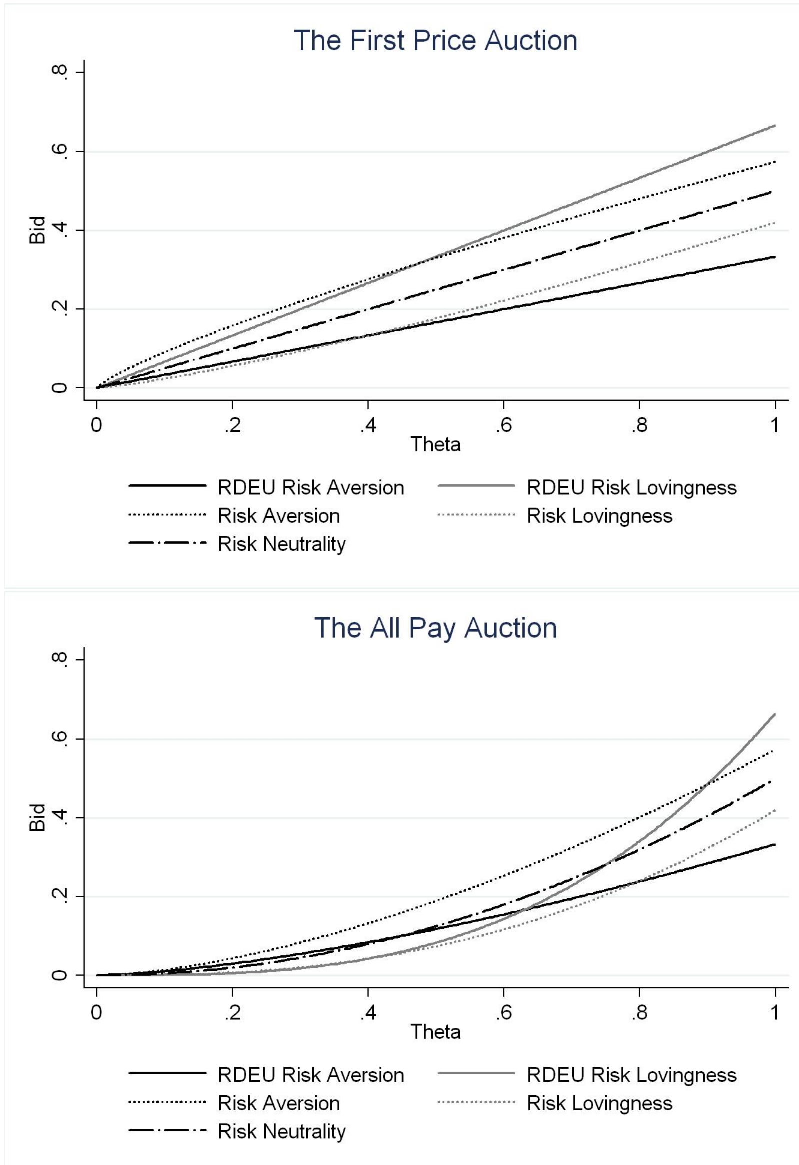

Figure 3: $N = 2$, $g^{RA}(x) = \sqrt{x}$, $g^{RL}(x) = x^2$, $m(x) = x^\gamma$, $\gamma^{RL} = 4/5$, $\gamma^{RA} = 5/4$, and $\theta \sim U[0, 1]$.

4 The Optimal Auction

In the preceding analysis, only a restricted class of sales mechanisms has been considered. In the spirit of Myerson (1981), I now investigate what sales mechanism the auctioneer optimally announces when he can choose freely. Before turning to the derivation of the optimal auction, some tools from the theory of mechanism design with risk neutral agents are needed[3].

4.1 Mechanism Design with Loss Averse Agents

Definition 3 *A mechanism, $\mathcal{M}$, consists of a collection of allowable strategies, S_i, for each agent i and an outcome function, $\mathcal{G} : \times_{i=1}^{N} S_i \to \mathcal{A}$, where $\mathcal{A}$ is the set of alternatives.*

Together with the description of priors, pay offs, and the type space, $\times_{i=1}^{N}\Theta$, a mechanism, $\mathcal{M}$, describes a Bayesian game, in which a strategy for agent i is a mapping, $\sigma_i : \Theta \to S_i$.

Definition 4 *A social choice function, $\mathcal{F}$, is a mapping from the type space to the set of alternatives, $\mathcal{F} : \times_{i=1}^{N}\Theta \to \mathcal{A}$.*

Definition 5 *A mechanism, $\mathcal{M}$, implements $\mathcal{F}$ in CPE if there is a CPE strategy profile, $\{\sigma_i\}_{i=1}^{N}$, of the game induced by $\mathcal{M}$, such that*

$$\mathcal{G}\left(\sigma_1(\theta_1), \ldots, \sigma_N(\theta_N)\right) = \mathcal{F}(\theta_1, \ldots, \theta_N),$$

for all $(\theta_1, \ldots, \theta_N) \in \times_{i=1}^{N}\Theta$.

Definition 6 *A direct mechanism, $\mathcal{M}^D$, is a mechanism in which $S_i = \Theta$, for all i.*

Definition 7 *A social choice function, $\mathcal{F}$, is CPE incentive compatible (CPEIC) if the strategy, $\sigma_i(\theta_i) = \theta_i$, is a CPE strategy in the direct mechanism, $\mathcal{M}^D = \{\cdot, \mathcal{G} = \mathcal{F}\}$, for all i, θ_i.*

[3] Apart from Myerson (1981), the following discussion draws primarily from Krishna (2009), ch. 5, as well as material taught in lectures by Jeffrey Ely, Ron Siegel, Rakesh Vohra, and Asher Wolinsky, whom I thank for this.

Using the above definitions, the following proposition can be stated.

Proposition 9 ***(Revelation Principle for CPE)*** *There is a mechanism implementing social choice function, $\mathcal{F}$, in CPE if and only if it is CPEIC.*

Proof: For sufficiency, choose the direct mechanism. In order to show necessity, suppose that there is a mechanism, $\mathcal{M} = \{S_1, \ldots, S_N, \mathcal{G}\}$, that implements $\mathcal{F}$ in CPE and let $H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i, \sigma_{-i})$ be the distribution over $\mathcal{A}$ when agent i plays according to σ_i, the rules of the mechanism are given by $\mathcal{G}$, and all other agents play according to σ_{-i}. If $\{\sigma_i(\theta_i)\}_{i=1}^N$ is part of a CPE, then for all $i, \sigma' \in S_i, \theta_i$,

$$\begin{aligned} &U\left(H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}))|H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})), \theta_i\right) \\ &\quad \geq U\left(H_i^{\mathcal{G}}(\mathcal{A}|\sigma', \sigma_{-i}(\theta_{-i}))|H_i^{\mathcal{G}}(\mathcal{A}|\sigma', \sigma_{-i}(\theta_{-i})), \theta_i\right). \end{aligned}$$

In particular, this is true for $\sigma' = \sigma_i(\hat{\theta}_i)$, for all $i, \hat{\theta}_i$. Therefore, if $\{\sigma_i(\theta_i)\}_{i=1}^N$ is part of a CPE, then for all $i, \theta_i, \hat{\theta}_i$,

$$\begin{aligned} &U\left(H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}))|H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})), \theta_i\right) \\ &\quad \geq U\left(H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\hat{\theta}_i), \sigma_{-i}(\theta_{-i}))|H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\hat{\theta}_i), \sigma_{-i}(\theta_{-i})), \theta_i\right). \end{aligned}$$

Since $\mathcal{G}(\sigma_1(\theta_1), \ldots, \sigma_N(\theta_N)) = \mathcal{F}(\theta_1, \ldots, \theta_N)$, $H_i^{\mathcal{F}}(\mathcal{A}|\theta_i, \theta_{-i}) = H_i^{\mathcal{G}}(\mathcal{A}|\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})))$, for all i, θ_i. This implies that for all $i, \theta_i, \hat{\theta}_i$,

$$U\left(H_i^{\mathcal{F}}(\mathcal{A}|\theta_i, \theta_{-i})|H_i^{\mathcal{F}}(\mathcal{A}|\theta_i, \theta_{-i}), \theta_i\right) \geq U\left(H_i^{\mathcal{F}}(\mathcal{A}|\hat{\theta}_i, \theta_{-i})|H_i^{\mathcal{F}}(\mathcal{A}|\hat{\theta}_i, \theta_{-i}), \theta_i\right).$$

□

As usual, this result implies that it is without loss of generality to restrict attention to direct mechanisms. Consider a direct mechanism and suppose that $Q_i(\theta_i, \theta_{-i}) \in \{0, 1\}$ is the ex post allocation[4] to agent i when he is of type θ_i. The interim expected allocation to agent i is thus

$$q_i(\theta_i) := \int_{\{\theta_{-i}\}} Q_i(\theta_i, \theta_{-i}) dF(\theta_{-i}|\theta_i).$$

[4]For simplicity, I assume that the object is indivisible, so that ex post, the auctioneer either gives the object to an agent or not. Allowing the auctioneer to allocate the object to a bidder with some probability would only complicate matters, and ultimately lead to the same result.

Let $T_i(\theta_i, \theta_{-i})$ be agent i's ex post payment to the mechanism when his report is θ_i, and the vector of his opponents' reports is θ_{-i}. Then, the ex post (after the realization of θ_{-i}) indirect utility of an agent with type θ_i in this mechanism is

$$\underbrace{\left(Q_i(\theta_i,\theta_{-i}) + \eta^g Q_i(\theta_i,\theta_{-i})(1-q_i(\theta_i)) - \eta^g \lambda^g (1-Q_i(\theta_i,\theta_{-i}))q_i(\theta_i)\right)}_{=:\mathcal{B}_i(\theta_i,\theta_{-i})} \theta_i \\ -T_i(\theta_i,\theta_{-i}) - \Omega_i(\theta_i,\theta_{-i}),$$

where

$$\begin{aligned} \Omega_i(\theta_i,\theta_{-i}) \quad := \quad & \underbrace{\eta^m \lambda^m \int_{\{\theta'_{-i}:T_i(\theta_i,\theta'_{-i})<T_i(\theta_i,\theta_{-i})\}} \left(T_i(\theta_i,\theta_{-i}) - T_i(\theta_i,\theta'_{-i})\right) dF(\theta'_{-i}|\theta_i)}_{\text{losses}} \\ & - \underbrace{\eta^m \int_{\{\theta'_{-i}:T_i(\theta_i,\theta'_{-i})>T_i(\theta_i,\theta_{-i})\}} \left(T_i(\theta_i,\theta'_{-i}) - T_i(\theta_i,\theta_{-i})\right) dF(\theta'_{-i}|\theta_i)}_{\text{gains}}. \end{aligned}$$

Letting $\mathcal{B}_i(\theta_i,\theta_{-i})$ be the *perceived* ex post allocation rule and $\mathcal{T}_i(\theta_i,\theta_{-i}) := T_i(\theta_i,\theta_{-i}) + \Omega_i(\theta_i,\theta_{-i})$ be the *perceived* ex post payment to the mechanism, the ex post indirect utility of agent i with type θ_i in a CPEIC mechanism can be written as

$$V_i(\theta_i|\theta_i,\theta_{-i}) = \mathcal{B}_i(\theta_i,\theta_{-i})\theta_i - \mathcal{T}_i(\theta_i,\theta_{-i}),$$

which is of the same form as the ex post indirect utility of a risk neutral agent in a Bayesian Nash IC mechanism. The interim expected utility of an agent of type θ_i reporting to be type $\hat{\theta}_i$ is

$$V_i(\hat{\theta}_i|\theta_i) = \beta_i(\hat{\theta}_i)\theta_i - \tau_i(\hat{\theta}_i),$$

where[5]

$$\tau_i(\hat{\theta}_i) := \int_{\{\theta_{-i}\}} \mathcal{T}_i(\hat{\theta}_i,\theta_{-i})dF(\theta_{-i}|\theta_i)$$

and

$$\beta_i(\hat{\theta}_i) \quad := \quad \int_{\{\theta_{-i}\}} \mathcal{B}_i(\hat{\theta}_i,\theta_{-i})dF(\theta_{-i}|\theta_i) = q_i(\hat{\theta}_i)(1-\Lambda^g(1-q_i(\hat{\theta}_i))).$$

[5]Here, I make use of the independence assumption.

By the revelation principle, the social choice function, $\mathcal{F}(\theta_1, \ldots, \theta_N) = (\{Q_i\}_{i=1}^N, \{T_i\}_{i=1}^N)$ is CPEIC if and only if

$$V_i(\theta_i|\theta_i) = \beta_i(\theta_i)\theta_i - \tau_i(\theta_i) \geq \beta_i(\hat{\theta}_i)\theta_i - \tau_i(\hat{\theta}_i) = V_i(\hat{\theta}_i|\theta_i), \qquad \text{(CPEIC)}$$

for all $i, \theta_i, \hat{\theta}_i$. These considerations taken together imply an analogue of the characterization result for Bayesian Nash incentive compatibility for risk neutral agents, as well as a version of the revenue equivalence theorem for perceived payments in all CPEIC mechanisms. All results from the theory of mechanism design with risk neutral agents apply to this setting with loss averse agents to the perceived allocation rule and the perceived payments, so that the following propositions hold.

Proposition 10 ***(Characterization of CPEIC)*** *(i) Let* $\beta_i : [\theta_{\min}, \theta_{\max}] \to [-(\Lambda^g - 1)^2/ (4\lambda^g), 1]$*. Then, there exist functions,* $\tau_i : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}$*, such that* $(\{\beta_i\}_{i=1}^N, \{\tau_i\}_{i=1}^N)$ *satisfy (CPEIC) if and only if* β_i *is non decreasing, for all* i, θ_i*. (ii) If* $(\{\beta_i\}_{i=1}^N, \{\tau_i\}_{i=1}^N)$ *satisfy (CPEIC), then*

$$V_i(\theta_i|\theta_i) = V_i(\theta_{\min}|\theta_{\min}) + \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds$$

and

$$\tau_i(\theta_i) = \beta_i(\theta_i)\theta_i - V_i(\theta_{\min}|\theta_{\min}) - \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds.$$

Proof: If $\Lambda^g \leq 1$, condition 1 is satisfied, and $\beta_i(\theta) \in [0, 1]$, for all $q_i(\theta) \in [0, 1]$. If $\Lambda^g > 1$, then β_i is a convex function of q_i, minimized at $q^* = (\Lambda^g - 1)/(2\Lambda^g) \in (0, 1)$ with

$$\beta^* := \min_{q \in [0,1]} \{q(1 - \Lambda^g(1 - q))\} = -\frac{(\Lambda^g - 1)^2}{4\Lambda^g} < 0.$$

CPEIC requires that for all $i, \theta_i, \hat{\theta}_i$,

$$V_i(\theta_i|\theta_i) \quad \geq \quad V_i(\hat{\theta}_i|\theta_i) = \beta_i(\hat{\theta}_i)\theta_i - \tau_i(\hat{\theta}_i) = V_i(\hat{\theta}_i|\hat{\theta}_i) + \beta_i(\hat{\theta}_i)(\theta_i - \hat{\theta}_i).$$

Reversing the roles of θ_i and $\hat{\theta}_i$,

$$V_i(\hat{\theta}_i|\hat{\theta}_i) \geq V_i(\theta_i|\theta_i) + \beta_i(\theta_i)(\hat{\theta}_i - \theta_i).$$

Without loss of generality, assume that $\theta_i \geq \hat{\theta}_i$. Then, the above two inequalities combined imply that

$$\beta_i(\hat{\theta}_i) \leq \frac{V_i(\theta_i|\theta_i) - V_i(\hat{\theta}_i|\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq \beta_i(\theta_i).$$

Therefore, β_i has to be monotone in the reported type, proving necessity in (i). Taking the limit as $\theta_i \to \hat{\theta}_i$ shows that

$$\frac{\partial}{\partial \theta_i} V_i(\theta_i|\theta_i) = \beta_i(\theta_i),$$

so that

$$V_i(\theta_i|\theta_i) = V_i(\theta_{\min}|\theta_{\min}) + \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds.$$

Also,

$$V_i(\theta_i|\theta_i) = \beta_i(\theta_i)\theta_i - \tau_i(\theta_i).$$

The last two equations combined imply that

$$\tau_i(\theta_i) = \beta_i(\theta_i)\theta_i - V_i(\theta_{\min}|\theta_{\min}) - \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds,$$

which proves (ii). For sufficiency in (i), suppose that β_i is non decreasing and let

$$\tau_i(\theta_i) = \beta_i(\theta_i)\theta_i - \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds.$$

This implies that

$$V_i(\theta_i|\theta_i) = \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds.$$

Since $\theta_i \geq \hat{\theta}_i$,

$$V_i(\theta_i|\theta_i) - V_i(\hat{\theta}_i|\hat{\theta}_i) = \int_{\hat{\theta}_i}^{\theta_i} \beta_i(s)ds \geq \beta_i(\hat{\theta}_i)(\theta_i - \hat{\theta}_i),$$

where the ultimate inequality follows from monotonicity of β_i. Together with the definition of τ_i, this yields

$$V_i(\theta_i|\theta_i) \geq \beta_i(\hat{\theta}_i)\theta_i - \left(\beta_i(\hat{\theta}_i)\hat{\theta}_i - V_i(\hat{\theta}_i|\hat{\theta}_i)\right) = \beta_i(\hat{\theta}_i)\theta_i - \tau_i(\hat{\theta}_i) = V_i(\hat{\theta}_i|\theta_i),$$

so the incentive compatibility constraints are satisfied, for all $i, \theta_i, \hat{\theta}_i$. $\square$

Proposition 11 ***(Perceived Revenue Equivalence)*** *If* $(\{\beta_i\}_{i=1}^N, \{\tau_i\}_{i=1}^N)$ *and* $(\{\beta_i\}_{i=1}^N,\ \{\tilde{\tau}_i\}_{i=1}^N)$ *satisfy (CPEIC), then,* $\tau_i(\theta_i) - \tilde{\tau}_i(\theta_i) = h_i$*, for some number,* h_i*, for all* i, θ_i*.*

Proof: By proposition 10, since τ and $\tilde{\tau}$ implement the perceived allocation rule, β_i, in CPEIC, they can only be different through V_i and $\tilde{V}_i$. Hence, $h_i = \tilde{V}_i(\theta_{\min}|\theta_{\min}) - V_i(\theta_{\min}|\theta_{\min})$. $\square$

The expected *physical* payment to the mechanism of an agent of type θ_i is

$$t_i(\theta_i) := \tau_i(\theta_i) - \omega_i(\theta_i),$$

where

$$\omega_i(\theta_i) := \int_{\{\theta_{-i}\}} \Omega_i(\theta_i, \theta_{-i}) dF(\theta_{-i}|\theta_i).$$

For the remaining discussion, it is useful to define the following concept.

Definition 8 *An ex post CPEIC payment schedule,* $\{T_i\}_{i=1}^N$ *is said to satisfy the 0 probability property (0PP) if it satisfies the following.*

$$\begin{aligned} & \exists i, \theta_i : \exists \theta_1, \theta_2 : T_i(\theta_i, \theta_1) \neq T_i(\theta_i, \theta_2) \\ \implies \ & P\left(\theta_{-i} \in \{\theta' : T_i(\theta_i, \theta') \neq T_i(\theta_i, \theta_1)\}\right) = 0 \qquad \text{or} \\ & P\left(\theta_{-i} \in \{\theta' : T_i(\theta_i, \theta') \neq T_i(\theta_i, \theta_2)\}\right) = 0. \end{aligned}$$

In words, if there are at least two different ex post payments, then they may only be different with probability 0, from an interim perspective. Using this definition, the following lemma can be stated.

Lemma 1 $\omega_i(\theta_i) \geq 0$*, for all* i, θ_i*. If payments to the mechanism are non degenerate and do not satisfy the 0PP, the inequality holds strict.*

Proof: By definition of Ω_i and ω_i,

$$\begin{aligned} & \omega_i(\theta_i) = \int_{\{\theta_{-i}\}} \Omega_i(\theta_i, \theta_{-i}) dF(\theta_{-i}|\theta_i) \\ = \ & \eta^m \lambda^m \int_{\{\theta_{-i}\}} \int_{\{\theta'_{-i} : T_i(\theta_i, \theta'_{-i}) < T_i(\theta_i, \theta_{-i})\}} \left(T_i(\theta_i, \theta_{-i}) - T_i(\theta_i, \theta'_{-i})\right) dF(\theta'_{-i}|\theta_i) dF(\theta_{-i}|\theta_i) \\ & - \eta^m \int_{\{\theta_{-i}\}} \int_{\{\theta'_{-i} : T_i(\theta_i, \theta'_{-i}) > T_i(\theta_i, \theta_{-i})\}} \left(T_i(\theta_i, \theta'_{-i}) - T_i(\theta_i, \theta_{-i})\right) dF(\theta'_{-i}|\theta_i) dF(\theta_{-i}|\theta_i). \end{aligned}$$

Consider the range of integration of the inner integral in the penultimate line of the above expression. For every $\theta_1 \in \{\theta'_{-i} : T_i(\theta_i, \theta'_{-i}) < T_i(\theta_i, \theta_{-i})\}$, there is a $\theta_2 \in \{\theta_{-i} : \exists\theta'_{-i} : T_i(\theta_i, \theta'_{-i}) > T_i(\theta_i, \theta_{-i})\}$ with $\theta_1 = \theta_2$. In words, comparing the current state to a better state implies that when the better state occurs, comparing it to the current state is considered a loss. This implies that ω_i can be rewritten as

$$\omega_i(\theta_i) = \Lambda^m \int_{\{\theta_{-i}\}} \int_{\{\theta'_{-i}:T_i(\theta_i,\theta'_{-i})<T_i(\theta_i,\theta_{-i})\}} \left(T_i(\theta_i, \theta_{-i}) - T_i(\theta_i, \theta'_{-i})\right) dF(\theta'_{-i}|\theta_i) dF(\theta_{-i}|\theta_i).$$

The range of integration implies that the inner integrand is strictly positive if the transfer payments to the mechanism are non degenerate as a function of θ_{-i}. If, in addition, the non degenerate transfer payments do not satisfy the 0PP, also the outer integrand is strictly positive, which implies that $\omega_i(\theta_i) > 0$, for all i, θ_i. If payments are deterministic or satisfy the 0PP, then $\omega_i(\theta_i) = 0$, for all i, θ_i. □

For a given perceived allocation and payment rule, the mechanism designer can choose functions t_i and ω_i, so long as for all i, θ_i, their sum is exactly $\tau_i(\theta_i)$. Since the expected physical payment of an agent with type θ_i is strictly decreasing in ω_i, for all θ_i, the physical payments to the mechanism are maximized when setting $\omega_i(\theta_i) = 0$, for all i, θ_i. This translates into having no uncertainty (except, possibly, at probability 0 events) in the payments. Furthermore, this implies revenue equivalence for the physical payments among all CPEIC mechanisms that have perceived allocation rule β_i, deterministic payments, or payments satisfying the 0PP.

4.2 The Auctioneer's Problem

Consider now the optimization problem of the auctioneer. When designing the optimal mechanism, he has to ensure that the agents participate in the mechanism, and do not prefer to get their outside option instead, which is normalized to 0. Therefore, the individual rationality (IR) constraints are given by

$$V_i(\theta_i|\theta_i) = \beta_i(\theta_i)\theta_i - \tau_i(\theta_i) \geq 0, \qquad \text{(IR)}$$

for all i, θ_i. In the absence of opportunity and production costs, the auctioneer's program is

$$\max_{\{\beta_i,q_i,t_i,\omega_i,Q_i,V_i\}_{i=1}^N} \left\{ \sum_{i=1}^{N} \int_{\theta_{\min}}^{\theta_{\max}} t_i(\theta_i) f(\theta_i) d\theta_i \right\} \qquad \text{(P1)}$$

$$
\begin{aligned}
\text{subject to} \quad & \text{(CPEIC)}, \\
& \text{(IR)}, \\
& \omega_i(\theta_i) \geq 0, \\
& q_i(\theta_i) = \int_{\{\theta_{-i}\}} Q_i(\theta_i, \theta_{-i}) dF(\theta_{-i}|\theta_i), \\
& \beta_i(\theta_i) = q_i(\theta_i) \left(1 - \Lambda^g(1 - q_i(\theta_i))\right), \\
& Q_i(\theta_i, \theta_{-i}) \in \{0, 1\}, \\
& \sum_{i=1}^{N} Q_i(\theta_i, \theta_{-i}) \leq 1, \text{ for all } i, \theta_i, \theta_{-i}.
\end{aligned}
$$

That is, he seeks to maximize the sum of ex ante expected physical payments to the mechanism, subject to the various feasibility constraints. Using the characterization result in proposition 10 and substituting for $t_i(\theta_i)$, (P1) can be written as

$$
\begin{aligned}
\max_{\{\beta_i, q_i, \omega_i, Q_i, V_i\}_{i=1}^N} \quad & \left\{ \sum_{i=1}^{N} \int_{\theta_{\min}}^{\theta_{\max}} \left(\beta_i(\theta_i)\theta_i - \omega_i(\theta_i) - V_i(\theta_{\min}|\theta_{\min}) \right. \right. \\
& \left. \left. - \int_{\theta_{\min}}^{\theta_i} \beta_i(s) ds \right) f(\theta_i) d\theta_i \right\} \qquad \text{(P2)} \\
\text{subject to} \quad & \text{Monotonicity of } \beta_i \\
& \text{(IR)}, \\
& \omega_i(\theta_i) \geq 0, \\
& q_i(\theta_i) = \int_{\{\theta_{-i}\}} Q_i(\theta_i, \theta_{-i}) dF(\theta_{-i}|\theta_i), \\
& \beta_i(\theta_i) = q_i(\theta_i) \left(1 - \Lambda^g(1 - q_i(\theta_i))\right), \\
& Q_i(\theta_i, \theta_{-i}) \in \{0, 1\}, \\
& \sum_{i=1}^{N} Q_i(\theta_i, \theta_{-i}) \leq 1, \text{ for all } i, \theta_i, \theta_{-i}.
\end{aligned}
$$

By lemma 1, $\omega_i(\theta_i) \geq 0$, for all i, θ_i, which enters with a negative sign in the auctioneer's objective, for all i. Besides there, it only appears in the IR constraints. Lowering $\omega_i(\theta_i)$, for all i, θ_i, relaxes the IR constraints and increases the value of the objective. Therefore, the auctioneer will optimally choose $\omega_i(\theta_i) = 0$, for all i, θ_i. He can do this in two different ways. First, he can offer a payment schedule with deterministic payments as a function of the reported type. Second, he can offer payments that depend also on the report of the other agents, but in this case, each but one value in the support of payments of the so induced distribution may only occur with probability 0, from an interim perspective. Using these insights and integration by parts, (P2) can be written as

$$\begin{aligned}\max_{\{\beta_i,q_i,Q_i,V_i\}_{i=1}^N} \quad & \left\{ \sum_{i=1}^{N} \int_{\theta_{\min}}^{\theta_{\max}} \beta_i(\theta_i) \underbrace{\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)}_{=:\nu(\theta_i)} f(\theta_i) d\theta_i \right. \\ & \qquad \left. - \sum_{i=1}^{N} V_i(\theta_{\min}|\theta_{\min}) \right\} \\ \text{subject to} \quad & \text{Monotonicity of } \beta_i, \\ & \text{(IR)}, \\ & q_i(\theta_i) = \int_{\{\theta_{-i}\}} Q_i(\theta_i,\theta_{-i}) dF(\theta_{-i}|\theta_i), \\ & \beta_i(\theta_i) = q_i(\theta_i)\left(1-\Lambda^g(1-q_i(\theta_i))\right), \\ & Q_i(\theta_i,\theta_{-i}) \in \{0,1\}, \\ & \sum_{i=1}^{N} Q_i(\theta_i,\theta_{-i}) \le 1, \text{ for all } i, \theta_i, \theta_{-i}.\end{aligned} \tag{P3}$$

The expression in the objective is identical to the one obtained in Myerson (1981), except β_i is replaced by q_i. As the analysis in Myerson (1981) suggests, it is a useful step in finding the optimal mechanism to consider the point wise maximization of the expression in the integrand while dropping the monotonicity constraint and making the following technical assumption.

Assumption 1 ***(Regularity)*** *ν is strictly increasing in θ_i, for all θ_i.*

Define $\theta^{**} := \max\{\tilde{\theta}, \theta^*\}$, where $\nu(\theta^*) = 0$, and $\tilde{\theta}$ is the same as in proposition 3. If condition 1 is satisfied ($\Lambda^g \le 1$), $\beta_i \ge 0$, for all i, $q_i \in [0,1]$, so that, as in the original analysis of Myerson (1981) with risk neutral agents, $\tilde{\theta} = \theta_{\min}$ and, by implication, $\theta^{**} = \theta^*$. In this case, an optimal auction gives the object to the bidder with the highest type, provided that it is at least θ^*. Suppose now that condition 1 is violated ($\Lambda^g > 1$). In this case, the auctioneer maximizes the objective in (P3), by setting $\beta_i(\theta_i) = \beta^* < 0$, for all $i, \theta_i < \theta^*$ with $v(\theta_i) < 0$. However, the IR constraints require that

$$V_i(\theta_i|\theta_i) = V_i(\theta_{\min}|\theta_{\min}) + \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds \geq 0 \Longrightarrow V_i(\theta_{\min}|\theta_{\min}) \geq -\int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds,$$

for all $i, \theta_i < \theta^*$. Part (ii) of proposition 10 implies that

$$t_i(\theta_i) \quad = \quad \beta_i(\theta_i)\theta_i - \int_{\theta_{\min}}^{\theta_i} \beta_i(s)ds - V_i(\theta_{\min}|\theta_{\min}) \leq \beta_i(\theta_i)\theta_i < 0.$$

Clearly, this is not revenue maximizing, since the auctioneer can strictly improve by ex ante committing to set $\beta_i(\theta_i) = 0$, for all i, $\theta_i < \theta^*$. Furthermore, for $\theta_i \geq \theta^{**}$, he wants to give the object to the bidder with the highest type. Another distinction has to be made whether $\theta^{**} = \tilde{\theta}$ or $\theta^{**} = \theta^*$. Suppose first that $\tilde{\theta} < \theta^*$. In this case, the result in Myerson (1981) applies. Since $\nu(\theta_i) < 0$, for all $\theta_i < \theta^*$, it is never ex ante optimal for the auctioneer to give the object to a bidder with $\theta_i < \theta^*$. Suppose now that $\tilde{\theta} > \theta^*$ and suppose that the auctioneer ex ante commits to give the object to a bidder with $\tilde{\theta} > \theta_i > \theta^*$. By proposition 10, for all i, θ_i, in any optimal mechanism, such an agent's interim expected utility is

$$V_i(\theta_i|\theta_i) = V_i(\theta_{\min}|\theta_{\min}) + \int_{\theta^*}^{\theta_i} \beta_i(s)ds = V_i(\theta_{\min}|\theta_{\min}) + \int_{\theta^*}^{\theta_i} \Gamma(s)ds.$$

Since $\Gamma(\theta_i) < 0$, for all $i, \theta_i < \tilde{\theta}$, the IR and IC constraints imply that $t_i(\theta_i) < 0$, for these types. Again, this is not revenue maximizing for the auctioneer, because he can strictly improve by ex ante committing to not allocate the object to any bidder with $\theta_i < \tilde{\theta}$. Additionally, $V_i(\theta_{\min}|\theta_{\min})$ enters with a negative sign in the auctioneer's objective, and appears only in the IR constraint of the lowest type. This is because monotonicity of β_i together with the fact that $\beta_i < 0$ is never optimal for the auctioneer implies that if the IR constraint is satisfied for $\theta_{\min}$, then it is satisfied for all $i, \theta_i \geq \theta_{\min}$. Therefore, the auctioneer optimally sets $V_i(\theta_{\min}|\theta_{\min}) = 0$, for all i. Consequently, an optimal mechanism gives the object to highest type, provided that is at least $\theta^{**} = \max\{\tilde{\theta}, \theta^*\}$. This rule satisfies monotonicity of β_i, for all i. Hence, the relaxed problem yields the same value of the objective as the constrained problem. Since the value of the objective in the constrained problem is bounded above by its value in the relaxed problem, this rule is revenue maximizing. Therefore, an optimal auction gives the object to the bidder with highest type, has deterministic payments or non degenerate payments only if they satisfy the 0PP, and never gives the object to a bidder with $\theta < \theta^{**}$. The above analysis readily extends to situations in which the auctioneer has multiple units of the same object to sell, and each bidder only wants one of them. In this case, the optimal auction gives the $K \geq 1$ available objects to the bidders with the K highest values, provided they are at least θ^{**}.

4.3 Implementing an Optimal Auction

The auctioneer can design an APA with minimum bid as an indirect revenue maximizing mechanism. The minimum bid needs to be chosen so that it is never optimal for a bidder with $\theta < \theta^{**}$ to participate in the auction. Suppose the auctioneer has set the minimum bid, $b_{\min}$, such that only bidders with $\theta \geq \theta^{**}$ participate in the auction. Then, using the equilibrium bidding function derived above, for $\theta \geq \theta^{**}$,

$$b_1(\theta) = \Gamma(\theta)\theta - \int_{\theta^{**}}^{\theta} \Gamma(s)ds.$$

This gives $b_1(\theta^{**}) = \Gamma(\theta^{**})\theta^{**}$. Suppose the auctioneer chooses $b_{\min} = \Gamma(\theta^{**})\theta^{**}$ as the minimum bid. Then, any bidder with $\theta < \theta^{**}$ receives strictly negative interim expected pay off from submitting a bid of $b_{\min}$ or above, a bidder with type θ^{**} receives interim expected utility of exactly 0, and any bidder with $\theta > \theta^{**}$ earns strictly positive interim expected utility from submitting a bid above $b_{\min}$. Therefore, setting $b_{\min} = \Gamma(\theta^{**})\theta^{**}$ achieves the revenue maximizing goal of the auctioneer. If $\theta^* > \tilde{\theta}$, $b_{\min}$ is strictly decreasing in Λ^g, so that the participation threshold is reduced in the revenue maximizing mechanism; if $\tilde{\theta} > \theta^*$, $b_{\min} = 0$. As argued above, risk averse bidders are locally risk neutral and therefore, the optimal minimum bid is positive. Even in the case of RDEU preferences, the optimal minimum bid is never 0. The following proposition summarizes the optimal sales mechanism.

Proposition 12 *Suppose assumption 1 holds. Then, in an optimal mechanism, the auctioneer allocates the object to the bidder with the highest report, provided that it is at least* $\theta^{**} = \max\{\tilde{\theta}, \theta^*\}$*. Payments can only depend on the other bidders' type in a non degenerate way if the 0PP holds. An optimal indirect mechanism can be constructed by announcing an APA with minimum bid,* $b_{\min} = \Gamma(\theta^{**})\theta^{**}$*.*

Proof: In text. □

The striking insight is that any revenue maximizing mechanism is of the all pay nature. The reason is that the interim pay off of an agent of type θ_i is the same across all CPEIC mechanisms but the physical payment is not. By introducing uncertainty in the ex post payments to the mechanism, the auctioneer only lowers the amount he can extract from the agents. In a mechanism with a binary ex post payment schedule based on the reported type as in the FPA, if a bidder wins the auction, he compares this probabilistically to the situation in which he looses the auction and considers the bid paid a loss. If he looses the auction, he compares this probabilistically to the cases in which he wins the auction and considers the bid saved as a gain. Since bidders like gains less than they dislike losses, the overall effect on the interim pay off is negative. A similar argument proves the sub optimality of the Vickrey auction. Lange and Ratan (2010) show that the expected revenue in the FPA is higher than in the Vickrey auction. This insight can be obtained in the framework of the above analysis by comparing the values of ω_i in the two auction formats. If a bidder in the Vickrey auction wins the auction, he does not only have gain loss considerations in the money dimension due to payments conditional on winning, but also due to the fact that what he pays depends on what the opponents have reported. This leads to additional distortions, resulting in lower revenue for the auctioneer. By offering all pay mechanisms, the auctioneer can shut these effects down and extract the surplus generated through this. Consequently, all pay mechanisms dominate all mechanisms with, from an interim view, uncertain ex post payments. Additionally, not every optimal mechanism is efficient in the classical sense, i.e. if $\max_i\{\theta_i\} < \theta^{**}$, the auctioneer keeps the object. When determining the optimal minimum bid, the auctioneer trades off two effects. If he increases the minimum bid, he increases the risk of not selling the object at all. However, in the cases that he still sells the object, he sells it at a higher price, because a higher minimum bid induces bidders to bid more aggressively. The optimal minimum bid is chosen such that these two effects exactly outweigh each other. The following example illustrates the construction of an optimal indirect mechanism.

Example 1 *Suppose that* $N = 2$ *and* $\theta \sim U[0,1]$. *In this case,* $\theta - (1 - F(\theta))/f(\theta) =$

$2\theta - 1$ and $\theta^* = 1/2$. Figure 4 shows the equilibrium bidding function in the APA with minimum bid, as an indirect optimal mechanism, for the cases $\Lambda_1^g = 1$, $\Lambda_2^g = 2$ (i.e. $\theta^{**} = \tilde{\theta} = \theta^* = 1/2$), and risk neutrality ($\Lambda^g = 0$).

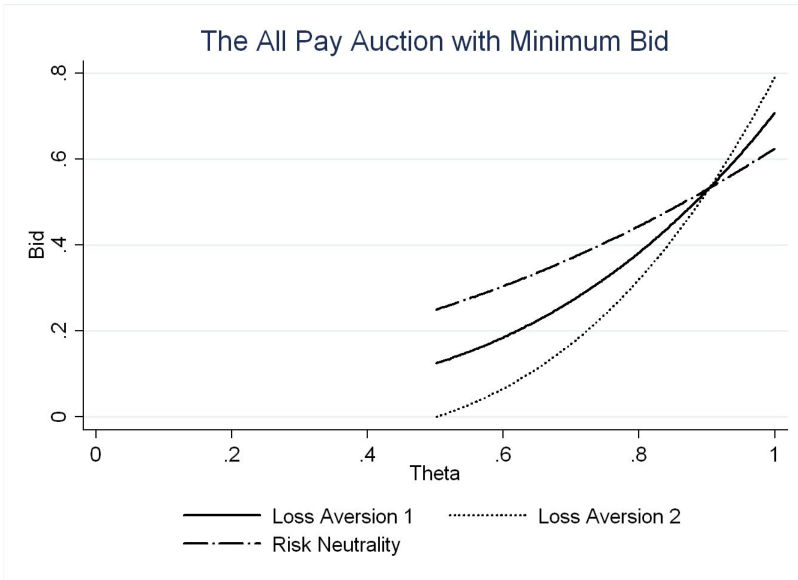

Figure 4: All Pay Bids in an Optimal Auction for $\Lambda_1^g = 1$, $\Lambda_2^g = 2$, $N = 2$, and $\theta \sim U[0, 1]$.

4.4 Legal Constraints

In some legal systems, the above described optimal auction format with $\alpha = 1$ and a minimum bid of $b_{\min} = \Gamma(\theta^{**})\theta^{**}$ may not be implementable, since it is considered a form of gambling. However, the auctioneer can always circumvent this constraint by announcing the following mechanism. Bidders are told that the mechanism is an FPA with a suitably chosen minimum bid, so that no bidder with $\theta < \theta^{**}$ participates. Each bidder writes his bid on a piece of paper and submits it to the auctioneer. Assuming the bidders play a symmetric, strictly increasing CPE strategy, the bids submitted are generated according to $b_0(\theta)$. In doing this, bidders essentially reveal their type to the auctioneer, who can then offer them insurance of the following form. They are offered to pay $b_1(\theta) = b_0(\theta)F_\theta\Delta(F_\theta)$, and save their bid, $b_0(\theta)$, *in case they win the auction.* According to the above analysis, bidders are indifferent between accepting the insurance or not, since it leads to the same interim expected pay off. Assuming this indifference is broken in favor of the insurance, this auction format is also optimal.

5 Related Literature

The present paper and the results derived above are related to three areas of economic research: auction theory and mechanism design, experimental work in economics, and behavioral IO. The connection to each of these is established separately.

5.1 Auction Theory and Mechanism Design

To a large extent, classical theory of auctions and mechanism design is governed by the paradigm of risk neutral agents. Riley and Samuelson (1981), Maskin and Riley (1984) and Matthews (1987) study the implications of risk averse bidders in auction settings. Lange and Ratan (2010) extend this body of economic research to the case of loss averse bidders and show that the FPA yields higher expected revenue than the Vickrey auction. Shunda (2008) shows that under a different notion of reference dependence, the auctioneer can increase his expected revenue by introducing a buy now price. Nakajima (2010) considers bidders with preferences allowing for the Allais Paradox and finds that the Dutch auction generates higher expected revenue for the auctioneer than the FPA. In the present paper, I focus in particular on all pay mechanisms and provide a generalization the work of Lange and Ratan (2010). Furthermore, all results derived above for loss averse agents are a generalization of the existing theory of risk neutral agents, since the well known results with risk neutral agents obtain in the limit, as loss aversion vanishes. In the theoretical literature on auctions, the same revenue ranking between the FPA and the APA has been proven for the case of affiliated values (Amann (1995)) and common value auctions with budget constrained bidders (Che and Gale (1996)). Riley and Samuelson (1981) and Maskin and Riley (1984) consider optimal auctions with risk averse bidders. The general insight obtained is that the expected revenue is higher in the FPA than in the Vickrey auction, a result also obtained by Lange and Ratan (2010) in the case of loss averse bidders. Maskin and Riley (1984) derive the optimal auction for a wide range of preferences exhibiting risk aversion. In their analysis, the optimal auction is a perfect insurance auction, which is an auction in which the marginal utility of the highest type is equated across states.

5.2 Experimental Work in Economics

The results derived above have sharp testable implications. This involves testing attitudes towards decisions under risk such as risk aversion, risk lovingness, loss aversion, as well as the theory of Köszegi and Rabin (2007). As part of the existing experimental literature on auctions, Schramm and Onderstal (2008) bring empirical proof for the failure of revenue equivalence across different auction formats and the departure from risk neutral equilibrium bidding in controlled IPV settings. They report that for the APA, most types overbid compared to the risk neutral equilibrium bid, while there are some types that underbid. Schramm and Onderstal (2008) also find that the lowest types do not participate and submit a bid of 0. Their empirical findings for the FPA suggest that all types underbid compared to the risk neutral case. Filiz-Ozbay and Ozbay (2007) study the FPA and propose a theory of winner and looser regret to support their experimental findings. A more philosophical question is whether empirical findings depart from the predictions of classical theory because of non standard attitudes towards decisions under risk or because decision makers outside economic models are simply not capable of playing equilibrium strategies. However, the present paper provides an alternative approach of studying the matter and a partial explanation for laboratory evidence.

5.3 Behavioral IO

Recently, increased interest of the profession in models of non standard preferences and non standard decision making has led to the development of models of how a firm or a principal provides the optimal contract to agents which are e.g. loss averse (Heidhues and Köszegi (2008), Herweg, Müller, and Weinschenk (2010)), time inconsistent (Eliaz and Spiegler (2009), Yilmaz (2009), Heidhues and Köszegi (2010)), or overconfident (Grubb (2009)). The present paper provides a general framework for analyzing mechanisms with loss averse agents, with applications to contracting, allocation problems, etc. Of the existing work, the present paper is most closely related to the recent work of Herweg, Müller, and Weinschenk (2010), who study an agency relationship with moral hazard in which the principal is risk neutral and the agent is loss averse in the sense of Köszegi and Rabin (2007). The main finding is that a binary payment schedule is the optimal contract offered by the principal. The intuition behind this is similar to the one provided above. Uncertainty in the money dimension only causes inefficiencies when agents are loss averse. They receive identical interim pay offs across all CPEIC mechanisms. Because of the moral hazard problem, the principal in Herweg, Müller, and Weinschenk (2010) is constrained to offer at least some performance pay component as part of the optimal contract. In their framework, this translates into offering a binary payment scheme with a jump only where it matters most for the principal. Related is also the main finding of Heidhues and Köszegi (2008). They show that, under certain conditions, imperfect competition of firms in a differentiated product industry facing loss averse consumers induces a pricing equilibrium in which all firms charge the same price, despite the fact that they may face different distributions of production costs. Taken these and the above findings together, loss aversion appears like *uncertainty aversion*, at least from a theoretical point of view.

6 Limitations

The above results rely on strong separability of preferences in the good and in the money dimension and the solution concept used. Strong separability is used extensively in the literature on the theory of auctions. The choice of the solution concept has been justified above. In appendix A.2, the piece wise linear specification of the Kahneman and Tversky (1979) value function is relaxed to allow for diminishing sensitivity in gains and losses. The revenue ranking across auction formats continues to hold in this case. A shortcoming of the theory is the assumption that bidders are homogeneous with respect to their degree loss aversion ($\Lambda^l, l \in \{g, m\}$ is the same across bidders) and that this is common knowledge among the auctioneer and all bidders. Possible extensions of the theory presented may try to relax this assumption in the light of mechanism design with multi dimensional types.

7 Conclusion

The results derived above have sharp implications testable in the laboratory and in the field. The theory developed above raises new questions in experimental work and may provide a theoretical foundation for bidding behavior observed in experiments. If bidders are loss averse, all pay mechanisms dominate any other form of sales mechanism, from a revenue maximizing view. In reality, however, actual all pay mechanism for allocation purposes are rare. Auctions for charitable purposes are sometimes characterized by all pay components. Recently, some on line auctioneers have implemented auctions with significant all pay components, in which each bidder has to pay a fee for raising the current bid by a fixed increment. This mechanism has the flavor of an APA with a minimum bid of essentially 0. For most sales mechanisms however, auction formats with payments conditional on winning (FPA, Vickrey auction) are much more common. As shown above, the APA yields the highest expected revenue for the auctioneer under a wide range of risk attitudes, yet loss aversion is the only one of the above for which an APA with a minimum bid of 0 can be optimal.

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A Appendix

A.1 Omitted Proofs:

Proof of second order risk aversion: A bidder of type θ solves

$$\begin{aligned}\hat{V}_\alpha(\theta) &:= \max_{x\in\mathbb{R}_+}\{P(x)(\theta - m((1-\alpha)x)) - m(\alpha x)\}\\ &= F_\theta\left(\theta - m\left((1-\alpha)b_\alpha^{RA}(\theta)\right)\right) - m\left(\alpha b_\alpha^{RA}(\theta)\right).\end{aligned}$$

Application of the envelope theorem shows that b_α^{RA} is the solution to the following equation.

$$\int_{\theta_{\min}}^{\theta} F_s ds + m\left(\alpha b_\alpha^{RA}(\theta)\right) = F_\theta\left(\theta - m\left((1-\alpha)b_\alpha^{RA}(\theta)\right)\right).$$

Using the fact that m is monotone, the equilibrium bidding functions in the FPA ($\alpha = 0$) and the APA ($\alpha = 1$) are given by

$$b_0^{RA}(\theta) = m^{-1}\left(\theta - \frac{\int_{\theta_{\min}}^{\theta} F_s ds}{F_\theta}\right) = m^{-1}\left(b_0^{RN}(\theta)\right)$$

and

$$b_1^{RA}(\theta) = m^{-1}\left(F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds\right) = m^{-1}\left(b_1^{RN}(\theta)\right).$$

The expected payment of a bidder of type θ in the two auction formats are given by

$$p_0^{RA}(\theta) = F_\theta m^{-1}\left(\theta - \frac{\int_{\theta_{\min}}^{\theta} F_s ds}{F_\theta}\right) = F_\theta m^{-1}\left(b_0^{RN}(\theta)\right)$$

and

$$p_1^{RA}(\theta) \quad = \quad m^{-1}\left(F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds\right) = m^{-1}\left(b_1^{RN}(\theta)\right) = m^{-1}\left(F_\theta b_0^{RN}(\theta)\right).$$

Since m is strictly increasing and strictly convex, m^{-1} is strictly increasing and strictly concave. By definition of m, $m^{-1}(0) = 0$. Comparing the two expected payments,

$$\begin{aligned}& p_0^{RA}(\theta) \le p_1^{RA}(\theta)\\ \iff\quad & F_\theta m^{-1}\left(b_0^{RN}(\theta)\right) + (1-F_\theta)\underbrace{m^{-1}(0)}_{=0} \le m^{-1}\left(b_0^{RN}(\theta)\right) + \underbrace{m^{-1}((1-F_\theta)0)}_{=0},\end{aligned}$$

which is always true, since m^{-1} is strictly concave. Also, the inequality holds strict on a set of measure one. $\square$

Proof of first order risk aversion: A bidder with RDEU preferences of type θ solves

$$\begin{aligned} V(\theta) &= \max_{x\in\mathbb{R}_+} \left\{ g\left(P(x)\right)\left(\theta-(1-\alpha)x\right)-\alpha x\right\} \\ &= g\left(F_\theta\right)\left(\theta-(1-\alpha)b_\alpha^{RDEU}(\theta)\right)-\alpha b_\alpha^{RDEU}(\theta). \end{aligned}$$

By the envelope theorem,

$$b_\alpha^{RDEU}(\theta)=\frac{g\left(F_\theta\right)\theta-\int_{\theta_{\min}}^{\theta} g\left(F_s\right)ds}{(1-\alpha)g\left(F_\theta\right)+\alpha}.$$

The expected payment of a bidder of type θ in auction α is given by

$$p_\alpha(\theta)=\left(\alpha+F_\theta(1-\alpha)\right)b_\alpha(\theta).$$

Differentiating with respect to α yields

$$\frac{\partial}{\partial\alpha}p_\alpha(\theta)=\left[\left(g\left(F_\theta\right)-1\right)\frac{F_\theta(1-\alpha)+\alpha}{g\left(F_\theta\right)(1-\alpha)+\alpha}+(1-F_\theta)\right]b_\alpha(\theta).$$

Since $b_\alpha(\theta)\geq 0$, for all θm the above expression is non negative if and only if

$$1-F_\theta\geq\left(1-g\left(F_\theta\right)\right)\frac{F_\theta(1-\alpha)+\alpha}{g\left(F_\theta\right)(1-\alpha)+\alpha}.$$

A concave g will make the above inequality true because $g(0)=0$, $g(1)=1$, and g concave imply that $g\left(F_\theta\right)\geq F_\theta$. $\square$

A.2 Diminishing Sensitivity

For the value function, I now assume that

$$\tilde{\mu}^g(x):=\begin{cases} d(x), & \text{if } x\geq 0 \\ -\lambda^g d(-x), & \text{if } x<0, \end{cases}$$

where $d:\mathbb{R}_+\to\mathbb{R}_+$ is continuous, strictly increasing, non convex, and $d(0)=0$. Hence, a bidder of type θ solves

$$\begin{aligned} V_\alpha(\theta) &= \max_{x\in\mathbb{R}_+}\left\{P(x)\left(\theta-x-\lambda^g\eta^g\left(1-P(x)\right)d\left((1-\alpha)x\right)\right)\right. \\ &\qquad \left.+\left(1-P(x)\right)\left(-\alpha x+\eta^g P(x)g\left((1-\alpha)x\right)\right)\right\} \\ &= F_\theta\theta-\left((1-\alpha)F_\theta+\alpha\right)b_\alpha(\theta)-\Lambda^g(1-F_\theta)F_\theta d\left((1-\alpha)b_\alpha(\theta)\right). \end{aligned}$$

By the envelope theorem, the equilibrium bidding function is the solution to the following equation

$$b_\alpha(\theta) = \frac{F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds - \Lambda^g(1-F_\theta)F_\theta d\left((1-\alpha)b_\alpha(\theta)\right)}{(1-\alpha)F_\theta + \alpha}.$$

For the FPA ($\alpha = 0$) and the APA ($\alpha = 1$), this reads

$$b_0(\theta) = \frac{F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds - \Lambda^g(1-F_\theta)F_\theta d\left(b_0(\theta)\right)}{F_\theta}$$

and

$$b_1(\theta) = F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds.$$

The interim expected payment of a bidder of type θ, in auction α, p_α, is given by

$$p_0(\theta) = F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds - \Lambda^g(1-F_\theta)F_\theta d\left(b_0(\theta)\right)$$

and

$$\begin{aligned} p_1(\theta) &= F_\theta\theta - \int_{\theta_{\min}}^{\theta} F_s ds \\ &= p_0(\theta) + \underbrace{\Lambda^g(1-F_\theta)F_\theta d\left(b_0(\theta)\right)}_{\geq 0}, \end{aligned}$$

which implies that $p_1(\theta)[>] \geq p_0(\theta)$, for [almost] all θ, so that the ex ante expected revenue for the auctioneer is strictly higher in the APA than in the FPA.

A.3 Strictly Increasing Differences

Lemma 2 *Suppose condition 1 is satisfied. Then,* $F_\theta\left(1 - \Lambda^g(1-F_\theta)\right)$ *is strictly increasing in* θ*.*

Proof: Differentiating the above expression with respect to θ yields

$$\begin{aligned} &\frac{\partial}{\partial\theta}\left\{F_\theta\left(1 - \Lambda^g(1-F_\theta)\right)\right\} > 0 \\ \iff\ & 1 - \Lambda^g > -2\Lambda^g F_\theta. \end{aligned}$$

If the above condition does not hold, then

$$1 - \Lambda^g \leq -2\Lambda^g F_\theta,$$

for some θ. Since $\Lambda^g > 0$, the RHS of this inequality is strictly negative, for $\theta \in (\theta_{\min}, \theta_{\max}]$. Therefore, for the above inequality to hold, it has to also be that the LHS is strictly negative, so that $1 < \Lambda^g$, proving the contra positive of the claim. $\square$

Lemma 3 *Under condition 1, b_α is strictly increasing in θ.*

Proof: I will show that, under condition 1, the bidder's objective satisfies strictly increasing differences in (θ, x), so that $b_\alpha(\theta)$, as the unique maximizer of this objective, is strictly increasing in θ. Consider

$$\begin{aligned} W_\alpha(\theta, x) \quad := \quad & P_i(x)\left(\theta_i - x + \eta^g\left(1 - P_i(x)\right)\theta_i - \eta^m \lambda^m \left(1 - P_i(x)\right)(1-\alpha)x\right) \\ & + \left(1 - P_i(x)\right)\left(-\alpha x - \eta^g \lambda^g P_i(x)\theta_i + \eta^m P_i(x)(1-\alpha)x\right). \end{aligned}$$

Differentiating this with respect to θ yields

$$\frac{\partial}{\partial\theta} W_\alpha(\theta, x) = F_{b_\alpha^{-1}(x)}\left(1 - \Lambda^g(1 - F_{b_\alpha^{-1}(x)})\right),$$

which, since b_α (and hence, b_α^{-1}) is strictly increasing, is strictly increasing in x, so long as $F_\theta(1 - \Lambda^g(1 - F_\theta))$ is strictly increasing in θ. Condition 1, in conjunction with lemma 2, guarantees that this is so. $\square$