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**SOLVING NONLINEAR SYSTEMS OF EQUATIONS
AND NONLINEAR SYSTEMS OF DIFFERENTIAL
EQUATIONS BY THE MONTE CARLO METHOD
USING QUEUEING NETWORKS
AND GAMES THEORY**

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In this paper we will solve some nonlinear systems of equations and nonlinear systems of differential equations by the Monte Carlo method using queueing networks and some results from games theory.

Keywords: Monte Carlo, queueing networks, symmetric games.

AMS Subject Classification: 60K25, 65C05, 91A05.

1. Introduction

Stochastic models for solving some problems were used among others by Ermakov (see [8,9]) and Văduva (see [18,20]).

For solving the linear system

$$x = A \cdot x + f \tag{1}$$

Ermakov uses an ergodic Markov chain with n states, where n is the dimension of the system. The transition probabilities of this Markov chain are 0 for the null elements of A and non-zero values in the contrary case. We consider an arbitrary vector h and an initial distribution $(p_i)_{i=1,n}$ with non-zero values at the same positions. Using a trajectory of this ergodic Markov chain Ermakov estimates the scalar product $\langle h, \bar{x} \rangle$, where \bar{x} is the solution of the system.

Văduva (see [18,20]) uses, opposite Ermakov, an absorbing Markov chain with $n+1$ states instead of an ergodic Markov chain with n states. The values P_{ij} for $1 \leq i, j \leq n$ are built in the same way as in Ermakov, but the sums of the transition probabilities from the state $i = \overline{1,n}$ to the states from 1 to n become less than 1. The differences to 1 are the probabilities to move to the

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state $n + 1$ (absorption). Using N independent trajectories $\gamma_1, \dots, \gamma_N$ with the initial transient state i_0 , Văduva estimates x_{i_0} .

In fact between the ergodic Markov chains and the absorbing Markov chains there exists a connection (see [13]).

In [5] we use a Jackson queueing network (see [10]) to solve some linear systems of equations. A Jackson queueing network is an open network with k nodes such that the inter-arrival time in the node i from outside the network is $\exp(\lambda_i)$, the service time at the node i is $\exp(\mu_i)$ and after he finishes its service at the node i , a customer goes to the node j with the probability P_{ij} or lives the network with the probability P_{i0} . The arrivals from outside network are set according to the right sides, and the transition probabilities P_{ij} are set according to the system matrix A (see [5]). If the right sides have elements with different signs we have to generate two Jackson queueing networks: one for positive values and one for the negative ones. Using the average numbers of customers in the nodes we can find the solution of the system.

In [6] we have solved by the Monte Carlo method the nonlinear equation in σ

$$A^*(\mu(1 - \sigma)) = \sigma, \quad (2)$$

where A^* is the moments generating function of the inter-arrival times density function a (i.e. its Laplace transform):

$$A^*(\xi) = \int_0^{\infty} e^{-\xi \cdot t} \cdot a(t) dt. \quad (2')$$

We generate (see [6]) a $G/M/1$ queueing system with the inter-arrival times density function a and the service times $\exp(\mu)$. We divide the simulation period t into m periods when we have no arrival and no service finalization. We estimate the average number of customers in the system and, using this estimation we estimate a solution of the nonlinear equation if we know an analytical formula for A^* . In the contrary case, we can solve an integral equation in the same way (see [6]).

A Gordon and Newell queueing network (see [11]) is a closed queueing network with k nodes and N customers. The service time in the node i has the distribution $\exp(\mu_i)$, and after the service in this node the customer goes to the node j with the probability P_{ij} . We notice that the matrix P as above is the transition matrix of an ergodic Markov chain (see [13]). If we denote by $(p_i)_{i=1, \dots, k}$ the ergodic probability we know that

$$P(N_1 = n_1, \dots, N_k = n_k) = \alpha \cdot \prod_{j=1}^k x_j^{n_j}, \quad (3)$$

where x_j is proportional with $\frac{p_j}{\mu_j}$, and α is computed such that

$$\sum_{\sum_{j=1}^k n_j = N} P(N_1 = n_1, \dots, N_k = n_k) = 1. \quad (3')$$

A Buzen queueing network (see [3]) is a generalization of a Gordon and Newell queueing network where the service in the node i depends on the number n_i of customers in that node: its distribution is $\exp(a_i(n_i) \cdot \mu_i)$, where $\mu_i > 0$ and a_i is a given function. In this case we have

$$P(N_1 = n_1, \dots, N_k = n_k) = \alpha \cdot \prod_{j=1}^k \frac{x_j^{n_j}}{A_j(n_j)}, \quad \text{where} \quad (4)$$

$$A_j(n_j) = \begin{cases} 1, & \text{if } n_j = 0 \\ \prod_{i=1}^{n_j} a_j(i), & \text{if } n_j \neq 0 \end{cases} \quad (4')$$

and α is given by (3').

We will solve some nonlinear systems of equations by the Monte Carlo method using closed queueing networks and some results from games theory that we will present in the following.

Definition 1

A game is in the normal form if $\Gamma = (I, D_i, \pi_i)$, where I is the set of the players, D_i is the set of pure strategies of the player i , and $\pi_i : \prod_{i=1}^n D_i \rightarrow \mathbf{R}$ is a function with $\pi_i(d)$ being the utility of the player i if the player j uses the strategy d_j and $d = (d_1, \dots, d_i, \dots)$.

Definition 2

The game is finite if $I = \{1, 2, \dots, n\}$ is finite. The game is with complete information if D_i are finites.

Definition 3

A mixed strategy of the player i is a probability distribution x_i on D_i .

We denote by Δ_i the set of mixed strategies of the player i and by $\Theta = \times_{i=1}^n \Delta_i$. We denote also by $x_i(d_i)$ the probability that the player i uses the pure strategy d_i . Because in the case of non-cooperative games the mixed strategies x_j are independent, we obtain the following formula for the average utility of the player i :

$$u_i(x) = u_i(x_1, \dots, x_n) = \sum_{d \in \Theta} \prod_{j=1}^n x_j(d_j) \cdot \pi_i(d). \quad (5)$$

Definition 4

$x = (x_i, x_{-i}) \in \Theta$ is a Nash equilibrium if for any $y_i \in \Delta_i$ we have $u_i(x) \geq u_i(y_i, x_{-i})$.

Theorem 1([22,16,17])

For any finite game with complete information the set Θ^{NE} of Nash equilibria is not empty.

Definition 5

A two-player game with complete information is called bimatriceal.

In this case $D_1 = \{1, \dots, m\}$, $D_2 = \{1, \dots, n\}$, $\pi_1(i, j) = A_{ij}$ and $\pi_2(i, j) = B_{ij}$.

Definition 6

The game is symmetric if $m = n$ and $A^T = B$. The game is doubly symmetric if $A^T = B$.

In a symmetric game $\Delta_1 = \Delta_2 = \Delta$. We denote by $\Delta^{NE} = \{x \in \Delta \mid (x, x) \in \Theta^{NE}\}$.

Theorem 2([22])

For any symmetric two-player game Δ^{NE} is not empty.

2. Solving Nonlinear Systems of Equations

Consider the symmetric two-persons game given by the $k \times k$ matrix A . We denote by $e^{(i)}$ the vector with k components $e_j^{(i)} = \delta_j^i$. We denote by Δ^0 the set of the solutions from Δ of the nonlinear system

$$\begin{cases} (e^{(i)} - x)^T Ax \cdot x_i = 0, i = \overline{1, k} \\ \sum_{i=1}^k x_i = 1 \end{cases} \quad (6)$$

and by $\Delta^{00} = \Delta^0 \cap \text{Int}(\Delta)$.

Theorem 3([22])

In the above conditions we have $\Delta^{00} = \Delta^{NE} \cap \text{Int}(\Delta)$.

Consider now the function $g : D \rightarrow \mathbf{R}^k$, where D is a domain such that $\Delta \subset D \subset \mathbf{R}^k$. We will use the same notations as above for the system

$$\begin{cases} g_i(x) \cdot x_i = 0, i = \overline{1, k} \\ \sum_{i=1}^k x_i = 1 \end{cases} \quad (6')$$

Definition 7([22])

The function g from (6') is a regular growth-rate function if it is Lipschitz continuous on D and $g^T(x) \cdot x = 0$ for any $x \in \Delta$.

Definition 8([22])

Let g be a regular growth-rate function. g is payoff monotonic if for any $x \in \Delta$ and $i, j \in \{1, \dots, k\}$ we have $u(e^{(i)}, x) > u(e^{(j)}, x) \Leftrightarrow g_i(x) > g_j(x)$. g is payoff positive if for any $x \in \Delta$ and $i \in \{1, \dots, k\}$ we have $\text{sgn}(g_i(x)) = \text{sgn}(u(e^{(i)} - x, x))$.

Definition 9([22])

A regular growth-rate function is weakly payoff positive if for any $x \in \Delta$ we have $B(x) = \{i | u(e^{(i)}, x) > u(x, x)\} \neq \Phi \Rightarrow g_i(x) > 0$ for at least one $i \in B(x)$.

We know (see [22]) that if a regular growth-rate function g is payoff positive or payoff monotonic then g is weakly payoff positive. If g from (6') is a regular growth-rate function we know that $\sum_{i=1}^k x_i$ is constant.

Theorem 4([22])

If g is a weakly payoff positive growth-rate function and $x \in \text{Int}(\Delta)$ x is a solution of (6') if and only if $x \in \Delta^{\text{NE}}$.

Therefore solving the nonlinear system of equations (6') in the case that we know the matrix A such that g is a weakly payoff positive growth-rate function is equivalent with finding the interior Nash equilibrium.

Suppose that at a given moment the distribution of the pure strategies in the population is given by x . A player who uses the pure strategy i at that moment decide to revise his strategy with the average rate $r_i(x)$. At the revision moment the player continues with the strategy i with the probability $p_i^j(x)$, or he changes his strategy with $j \neq i$ with the probability $p_i^j(x)$.

For some particular cases of r_i and p_i^j Lipschitz-continuous on an open domain X that contains Δ and any $x^0 \in \text{Int} \Delta$, if the time moments tends to infinity x tends to an interior Nash equilibrium (see [22]).

The first model considered by Weibull is the imitation driven by dissatisfaction. In this case we have

$$\begin{cases} p_i^j(x) = x_j \\ r_i(x) = \rho(u(e^{(i)}, x), x) \end{cases} \quad (7)$$

where ρ is a Lipschitz-continuous function, strictly decreasing in its first argument. Particularly, if $\rho(a, b) = \alpha - \beta \cdot a$, where $\beta > 0$ and $\alpha \geq \beta \cdot u(e^{(i)}, x)$ we obtain

$$\begin{cases} p_i^j(x) = x_j \\ r_i(x) = \alpha - \beta \cdot (e^{(i)})^T Ax \end{cases} \quad (7')$$

For instance we can take in (7') $\beta = 1$ and $\alpha \geq \max_{i,j} |A_{ij}|$, and the above conditions are fulfilled. In the C++ program we consider $\alpha > \max_{i,j} |A_{ij}|$ to avoid the average rate 0 (i.e. with $\mu = \infty$).

Another model is the imitation of the successful agent, model I (see [22]). We have

$$\begin{cases} r_i(x) = 1 \\ p_i^j(x) = x_j \cdot \Phi(u(e^{(j)} - e^{(i)}, x)) \text{ if } u(e^{(j)} - e^{(i)}, x) > 0 \end{cases} \quad (8)$$

where Φ is a probability distribution on the strategies that involve a greater payoff strictly increasing on the payoffs' difference $u(e^{(j)} - e^{(i)}, x)$. A simple modality to compute the above probabilities is to divide the positive differences of payoffs by their sum. Because $\Phi(u(e^{(j)} - e^{(i)}, x))$ is multiply by x_j the sum of the above probabilities $p_i^j(x)$ is less than 1. The rest to 1 is uniform distributed to all the strategies (including i) such that $u(e^{(j)} - e^{(i)}, x) = 0$.

Another model is the imitation of the successful agent, model II (see [22]). We have

$$\begin{cases} r_i(x) = 1 \\ p_i^j(x) = \frac{\omega_i(u(e^{(j)}, x), x) x_j}{\sum_{h=1}^k \omega_i(u(e^{(h)}, x), x) x_h} \end{cases}, \quad (9)$$

where $\omega_i(\cdot, \cdot) > 0$ is Lipschitz-continuous in the first argument. If we take $\mu > 0$ and $\lambda > -\mu \cdot u(e^{(i)}, x)$ for any i we can consider the above function ω_i linear: $\omega_i(z, x) = \lambda + \mu \cdot z$. If the payoffs are positives (we can make them positives by adding a positive constant to each element of the matrix A) we can take $\lambda = 0$ and we obtain

$$\begin{cases} r_i(x) = 1 \\ p_i^j(x) = \frac{(e^{(j)})^T A x}{x^T A x} \end{cases}. \quad (9')$$

We solve the nonlinear system of equations (6') by generating a closed queueing network with k nodes and N customers as follows.

- 1) We consider at each moment $x_i = \frac{N_i}{N}$, where N_i is the number of customers in the node i .
- 2) After the end of a service at the node i the customer goes to the node j with the probability $P_{ij} = p_i^j(x)$ from the above models.
- 3) We generate the next service in the node i having the distribution $\exp(r_i(x))$.

The above queueing network is generated as a Gordon and Newell queueing network or a Buzen queueing network during a maximum simulation period $tsim$, but after any service in the node i we have to refresh the values of P_{ij} and μ_i . P_{ij} are computed just before the customer lives the node i , and μ_i is computed just after these. For starting from an interior x we take initially $N_i = N_1$ and $N = k \cdot N_1$ and we generate the first service in each node.

Example 1

Consider the system of equations (6) with

$$A = \begin{pmatrix} 5 & 1 & -3 & 2.5 & -1 \\ 1 & 4 & -2 & 9 & 0 \\ -3.25 & 1 & 3 & -2 & 1 \\ 4 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 7 & -2.5 \end{pmatrix}.$$

First we generate the exponentials by the inverse method. We obtain

$$x = (0.211, 0.192, 0.209, 0.203, 0.185)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00339, 0.02467, -0.02489, -0.00116, 0.00478)^T$ (these values are theoretically equal to 0) if we consider the imitation driven by dissatisfaction,

$$x = (0.498, 0.189, 0.12, 0.112, 0.081)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (0.03487, 0.00386, -0.03636, 0.00209, -0.00445)^T$ if we consider the imitation of the successful agent, model I, and

$$x = (0.198, 0.201, 0.217, 0.192, 0.192)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00466, 0.02477, -0.02272, -0.00122, 0.00383)^T$ if we consider the imitation of the successful agent, model II.

Next we generate the exponentials by the rejection method. We obtain

$$x = (0.203, 0.19, 0.196, 0.203, 0.208)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.0038, 0.0246, -0.02319, -0.00157, 0.00396)^T$ if we consider the imitation driven by dissatisfaction,

$$x = (0.45, 0.183, 0.163, 0.112, 0.092)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i\right)_i = (0.02983, 0.00723, -0.03898, 0.00451, -0.00259)^T$ if we consider the imitation of the successful agent, model I, and

$$x = (0.207, 0.183, 0.185, 0.27, 0.155)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i\right)_i = (-0.00229, 0.0312, -0.02985, -0.00834, 0.00928)^T$ if we consider the imitation of the successful agent, model II.

Finally we generate the exponentials by the mixture method. We obtain

$$x = (0.198, 0.202, 0.204, 0.202, 0.194)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i\right)_i = (-0.00441, 0.0263, -0.02353, -0.00254, 0.00417)^T$ if we consider the imitation driven by dissatisfaction,

$$x = (0.482, 0.173, 0.133, 0.12, 0.092)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i\right)_i = (0.03341, 0.005, -0.03779, 0.00337, -0.00399)^T$ if we consider the imitation of the successful agent, model I, and

$$x = (0.22, 0.203, 0.201, 0.171, 0.205)^T$$

and $\left((e^{(i)} - x)^T Ax \cdot x_i\right)_i = (-0.00272, 0.02279, -0.02195, 0.00063, 0.00126)^T$ if we consider the imitation of the successful agent, model II.

3. Solving Nonlinear Systems of Differential Equations

First we denote for any vector $y = (y_1, \dots, y_k)^T$ by $\exp(y) = (e^{y_1}, \dots, e^{y_k})^T$. Consider the $k \times k$ matrices B and $A = \alpha \cdot B$, and the Cauchy problem

$$\begin{cases} \tilde{y}_i'(\tau) = \left(e^{(i)} - \frac{\exp(\tilde{y})}{\alpha} \right) \cdot B \cdot \exp(\tilde{y}) \cdot \varphi'(\tau), \\ \tilde{y}_i(\tau_0) = \tilde{y}_i^{(0)} \end{cases}, \quad (10)$$

where φ is a monotonic function on the interval bordered by τ_0 and τ_1 with $\varphi(\tau_0) = 0$ and $\lim_{\tau \rightarrow \tau_1} \varphi(\tau) = \infty$ ($\tau_1 \neq \tau_0$), and $\alpha = \sum_{i=1}^k \exp(\tilde{y}_i^{(0)})$. If we take $y = \exp(\tilde{y})$ and $y^{(0)} = \exp(\tilde{y}^{(0)})$ we obtain the equivalent Cauchy problem

$$\begin{cases} y_i'(\tau) = \left(e^{(i)} - \frac{y}{\alpha}\right)^T \cdot B \cdot y \cdot y_i \cdot \varphi'(\tau) \\ y_i(\tau_0) = y_i^{(0)} \end{cases} \quad (11)$$

Consider a weak positive payoff growth-rate function g if A is the payoff matrix of a symmetric two players game. Consider also the systems of differential equations

$$\begin{cases} \tilde{y}_i'(\tau) = g_i\left(\frac{\exp(\tilde{y})}{\alpha}\right) \cdot \varphi'(\tau) \\ \tilde{y}_i(\tau_0) = y_i^{(0)} \end{cases}, \quad \text{respectively} \quad (10')$$

$$\begin{cases} y_i'(\tau) = g_i\left(\frac{y}{\alpha}\right) \cdot y_i \cdot \varphi'(\tau) \\ y_i(\tau_0) = y_i^{(0)} \end{cases} \quad (11')$$

If we use in (11) and (11') the substitutions $z = \frac{y}{\alpha}$ and $z^{(0)} = \frac{y^{(0)}}{\alpha}$, we obtain the systems of differential equations

$$\begin{cases} z_i'(\tau) = \left(e^{(i)} - z\right)^T A z \cdot z_i \cdot \varphi'(\tau) \\ z_i(\tau_0) = z^{(0)} \end{cases} \quad \text{and} \quad (12)$$

$$\begin{cases} z_i'(\tau) = g_i(z) \cdot z_i \cdot \varphi'(\tau) \\ z_i(\tau_0) = z^{(0)} \end{cases}, \quad (12')$$

where $\sum_{i=1}^k z_i^{(0)} = 1$.

Finally we use the substitutions $x(t) = (z \circ \varphi^{-1})(t)$ and $x^{(0)} = z^{(0)}$. We obtain the systems of differential equations

$$\begin{cases} x_i'(t) = \left(e^{(i)} - x\right)^T A x \cdot x_i \\ x_i(0) = x^{(0)} \end{cases} \quad \text{and} \quad (13)$$

$$\begin{cases} x_i'(t) = g_i(x) \cdot x_i \\ x_i(0) = x^{(0)} \end{cases} \quad (13')$$

We know that for $t \rightarrow \infty$ in (13) or (13') x tends to an interior Nash equilibrium x^* if $x^{(0)}$ is interior (see [22]).

Therefore we obtain x^* as in the previous section, and $z(\tau_1) = x^*$. Obviously $y(\tau_1) = \alpha \cdot x^* = \exp(\tilde{y}(\tau_1))$. In the algorithm we do not need to multiply B by α because if we apply a linear increasing function to the payoffs we obtain an equivalent game. We notice that the results do not depend on φ , τ_0 and τ_1 . The only conditions are that φ is a monotonic function on the interval bordered by τ_0 and τ_1 with $\varphi(\tau_0) = 0$ and $\lim_{\tau \rightarrow \tau_1} \varphi(\tau) = \infty$ ($\tau_1 \neq \tau_0$). From a given φ monotonic on an interval we can find $\tau_0 = \varphi^{-1}(0)$ and $\tau_1 = \varphi^{-1}(\infty)$. If we have only φ we need the value of τ_0 .

Example 2

Consider the system of differential equations

$$\tilde{y}_i'(\tau) = \left(\sum_{j=1}^5 B_{ij} e^{\tilde{y}_j} - \frac{1}{\alpha} \sum_{j=1}^5 \sum_{k=1}^5 B_{jk} e^{\tilde{y}_j + \tilde{y}_k} \right) \cdot \frac{1}{6 - \tau}$$

with

$$B = \begin{pmatrix} 5 & 1 & -3 & 2.5 & -1 \\ 1 & 4 & -2 & 9 & 0 \\ -3.25 & 1 & 3 & -2 & 1 \\ 4 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 7 & -2.5 \end{pmatrix}$$

and the initial condition $\tilde{y}^{(0)} = \tilde{y}(1) = (1, -1, 0, 1, 3)^T$.

We obtain $\alpha = \sum_{i=1}^5 \exp(\tilde{y}_i^{(0)}) = 29.60826$. Because the primitive of $\frac{1}{6-\tau}$ is $-\ln(6-\tau) + \mathbf{c}$ we obtain $\mathbf{c} = \ln 5$, $\tau_1 = 6$ and $\varphi(\tau) = \ln \frac{5}{6-\tau}$.

First we generate the exponentials by the inverse method. We obtain

$$\tilde{y}(6) = (1.76857, 1.7533, 1.7533, 1.82263, 1.7935)^T$$

if we consider the imitation driven by dissatisfaction,

$$\tilde{y}(6) = (2.6909, 1.70604, 1.34783, 1.08547, 0.93465)^T$$

if we consider the imitation of the successful agent, model I, and

$$\tilde{y}(6) = (1.82263, 1.4159, 1.71674, 1.75841, 1.74816)^T$$

if we consider the imitation of the successful agent, model II. For the computation of the theoretical solution we make the substitutions to obtain the system (13) and, from the above considerations we take $A = B$ instead of $A = \alpha \cdot B$. We obtain

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00417, 0.02611, -0.02356, -0.00282, 0.00444)^T$$

if we consider the imitation driven by dissatisfaction,

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (0.03623, 0.00297, -0.03743, 0.00293, -0.00476)^T$$

if we consider the imitation of the successful agent, model I, and

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00357, 0.02744, -0.0237, -0.00305, 0.00288)^T$$

if we consider the imitation of the successful agent, model II.

Next we generate the exponentials by the rejection method. We obtain

$$\tilde{y}(6) = (1.81784, 1.77862, 1.79842, 1.71674, 1.77862)^T$$

if we consider the imitation driven by dissatisfaction,

$$\tilde{y}(6) = (2.69491, 1.64508, 1.27609, 1.1988, 1.00209)^T$$

if we consider the imitation of the successful agent, model I, and

$$\tilde{y}(6) = (1.83688, 1.75841, 1.84159, 1.75841, 1.68978)^T$$

if we consider the imitation of the successful agent, model II. The theoretical solution is

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00371, 0.02431, -0.02278, -0.00076, 0.00295)^T$$

if we consider the imitation driven by dissatisfaction,

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (0.03561, 0.00312, -0.03638, 0.0028, -0.00514)^T$$

if we consider the imitation of the successful agent, model I, and

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00346, 0.02443, -0.0245, -0.00077, 0.0043)^T$$

if we consider the imitation of the successful agent, model II.

Finally we generate the exponentials by the mixture method. We obtain

$$\tilde{y}(6) = (1.73779, 1.7736, 1.7836, 1.80311, 1.7935)^T$$

if we consider the imitation driven by dissatisfaction,

$$\tilde{y}(6) = (2.6006, 1.62196, 1.4639, 1.3555, 1.05501)^T$$

if we consider the imitation of the successful agent, model I, and

$$\tilde{y}(6) = (1.90525, 1.86937, 1.83216, 1.60426, 1.64508)^T$$

if we consider the imitation of the successful agent, model II. The theoretical solution is

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00472, 0.02626, -0.02292, -0.00283, 0.00421)^T$$

if we consider the imitation driven by dissatisfaction,

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (0.0296, 0.00749, -0.03829, 0.00398, -0.00279)^T$$

if we consider the imitation of the successful agent, model I, and

$$\left((e^{(i)} - x)^T Ax \cdot x_i \right)_i = (-0.00253, 0.02443, -0.02372, -0.00004, 0.00188)^T$$

if we consider the imitation of the successful agent, model II.

4. Conclusions

For the problems solved in this paper by the Monte Carlo method we use queueing networks as in [7] for solving nonlinear equations and linear systems of equations. We use for a nonlinear equation a service system with one server and exponential service (G/M/1 service system as it is denoted by Kleinrock [15]) as in [6,7], and for linear systems of equations we use Jackson queueing networks (see [10]) as in [7,5]. For nonlinear systems of equations (and of differential equations) we use a closed network (a Jackson network is an open network). The difference between this kind of network and a Gordon and Newell network (see [11]) is that in our case the values of μ_i and P_{ij} are not constant.

We use also some results from games theory. A relation between queueing theory, inventory theory and games theory is presented in [7,4].

In the case of the imitation driven by dissatisfaction the service in the node i depends on the number of customers in the node, n_i (in fact it depends on $x_i = \frac{n_i}{N}$). This is true as well for the Buzen queueing networks (see [3]). But in our model this service depends also on the other x_j , and on the payoffs matrix A . In the case of the imitation driven by dissatisfaction we have $p_i^j(x) = x_j$, hence even the probabilities p_i^j are not constant and depend on n_j . In the cases of the imitation of successful agent (model I and model II) these probabilities depend also on the payoffs matrix A , but the services remain constant 1.

For solving the above problems we must know the matrix A in the case of a nonlinear system of equations and the matrix B in the case of nonlinear system of a nonlinear system of differential equations. An open problem is to find the matrix if the system of equations is in the form (6') and if the system of differential equations is in the form (10') (of course, we must find first the weakly payoff positive growth-rate function g). Other open problems are to use the Monte Carlo method and queueing networks to solve other problems, as the linear programming problem and quadratic programming problem.

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