Complex equilibrium dynamics in a simple OLG model of neoclassical growth with endogenous retirement age and public pensions

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Abstract We analyse the steady-state equilibrium dynamics of the conventional overlapping generations economy à la Diamond (1965) with pay-as-you-go public pensions and second period of life divided between working and retirement time in a proportion dependent on the individual health status (a rather realistic assumption especially in the current world with high longevity). In contrast to an economy without public health spending – which is always stable with monotonic trajectories –, an economy with tax-financed health care services (which in turn affect the individual health status and hence the length of the retirement time) may experience complex equilibrium dynamics with deterministic chaotic business cycles and, in particular, complicated dynamical phenomena, such as multiple "bubblings" may occur when crucial economic parameters change. Interestingly, it is shown that increasing the size of PAYG pensions, although initially may trigger chaotic cycles, eventually works for stability.

Keywords Health; Old-age workers; OLG model; Perfect foresight; Public PAYG pensions

JEL Classification C62; H55; I18; O41

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1. Introduction

The problems concerning the public provision of health care services and pension benefits for retired mature workers are currently high on the political agendas in several industrialised countries. As is known, health status may affect the individual economic behaviours through multiple channels. One of them is surely represented by the effects that health may exert on both the ability to work and the productivity of work.[1] The link between health status and labour productivity has been early recognised by the pioneering Grossman (1972). Moreover, several studies have empirically found that health status has a significant effect on labour force of older people (e.g. Chirikos, 1993; Currie and Madrian, 1999; Campolieti, 2002; Cai and Kalb, 2006; Disney et al., 2006). However, the role played by both public health spending and retirement age on the equilibrium dynamics of a neoclassical growth model has not so far been deeply investigated. In this paper we aim to fill this gap in the framework of the famous Diamond's (1965) model extended with the following three simple realistic assumptions: (*i*) old people may supply labour for a fraction of their time endowment when old depending on their health status, (*ii*) the individual health status, in turn, depends in non linear form by the size of the public health spending, and (*iii*) public PAYG pensions exist to support the old-aged unable to work and thus retired.

In order to concentrate on the effects of health spending and pension benefits on the length of the age of retirement we abstract from considering adult mortality and the utility effect of health per se, and moreover we also neglect the role of leisure.

In particular, we assume that the length of the retirement period (conversely, the working period), which is of course a fraction of the second period of life span of people, only depends on the individual health status when old. This means that the date of retirement is chosen neither

[1] The other main channel through which health may affect economic growth and that has been more largely investigated is that between health and longevity rates (e.g. Chakraborty, 2004; de la Croix and Ponthiere, 2010; Leung and Wang, 2010).

voluntarily by mature workers nor in a mandatory way by governments, while being "objectively" determined by the health status of mature individuals: if health is low, then mature workers are "authorised" to retire and thus they are "entitled" to a pension benefit (alternatively, it may be assumed that for the period of sickness, in which old individuals cannot work, they receive a pay-as-you-go health insurance bonus).

The main finding of the present paper is the following: when the conventional Diamond's model is extended with the three assumptions above mentioned, the equilibrium dynamics may be oscillatory. Moreover, both regular and chaotic business cycles seem to be the rule rather than the exception for this economy. In particular, it is worth noting that deterministic chaotic motions emerge in a simple one-dimensional model with Cobb-Douglas utility and production functions, which a rather rare occurrence in the OLG context with production. In fact, so far the literature has shown that complex outcomes typically occur either in higher dimension models for low values of the elasticity of substitution in production, in particular for values well below unity (i.e., in the case of the Leontief technology),[2] or abandoning the rational expectation paradigm and then assuming myopic expectations, where deterministic chaos can emerge at high values of the inter-temporal elasticity of substitution (e.g., Michel and de la Croix, 2000; de la Croix and Michel, 2002; Chen et al. 2008).

Moreover, it is noteworthy that the equilibrium dynamics of the present model shows a rather high complexity – i.e., multiplicity of "bubbling" phenomena. The bubbling phenomenon shows that the regular bifurcation pattern reverses itself, undergoes period halving via flip bifurcations, and eventually returns to a unique steady state for larger parameter values.[3]

[2] For instance, Reichlin (1986) discusses the Leontief case, and in Farmer's (1986) example for the CES technology, endogenous cycles occur only when the production function exhibit lower factor substitutability than the Cobb-Douglas function.

[3] Pioneering discussions about this dynamical feature are Bier and Bountis (1984) and Stone (1993).

In addition to the feature of multiplicity of bubblings, for which a continuum of regularities and irregularities in the economic behaviour appear for the most part of the feasible policy changes (we used the contribution rate to the pension system as the bifurcation parameter), it is interesting to note that relatively high values of the contribution rate eventually tend to smooth economic fluctuations, thus properly working for the global "stability" of the economic system. This twofold role of such a policy parameter is remarkable from an economic point of view: on the one hand, it may concur to explain the observed irregular business fluctuations – showing that an endogenous deterministic origin of economic cycles may be complementary to the stochastic origin which is at the core of the real business cycle theory – but, on the other hand, and maybe more important, the policy variable may even be used to control and potentially to suppress undesirable business fluctuations. Thus, we may conclude that the equilibrium dynamics of this simple economy may embody two undisputable stylised facts: the existence of both irregular business cycles and the self-evident persistence of the economic system.

The paper proceeds as follows. Section 2 presents the model. In Section 3 the dynamics of the economy is analysed and discussed. In Section 4 the complexity of the dynamic behaviour of the economy is shown. Section 5 discusses the results and concludes.

2. The economy

2.1. Individuals

Consider a general equilibrium overlapping generations (OLG) closed economy populated by a continuum of identical individuals of measure one. Life is divided into youth (first period) and old-age (second period). In each period, individuals are endowed with one unit of time.

When young, agents of generations t work and inelastically offer their whole time endowment on the labour market, while receiving wage income at the competitive rate w_t. This income is used to consume ($c_{1,t}$) and to save (s_t). Moreover, the government levies wage income taxes on the young's labour income, at the constant rates $0<\tau<1$ and $0<\theta<1$, to separately finances at a balanced budget health care services and public PAYG pensions, respectively. Therefore, the budget constraint of a young agent born at t reads as

$$c_{1,t}+s_t=w_t(1-\tau-\theta). \tag{1.1}$$

The unitary time endowment of the old-aged is divided between working time (d_{t+1}) and retirement time ($1-d_{t+1}$). For the fraction of time supplied on the labour market, they receive earnings equal to $d_{t+1}w^{e}{}_{t+1}$, where $w^{e}{}_{t+1}$ is the wage individuals expect to earn at time $t+1$, while as regards the retirement time, they expect to be entitled to an amount of public pensions $(1-d_{t+1})p^{e}{}_{t+1}$ (see, amongst others, Hu, 1979; Momota, 2003). The budget constraint at $t+1$ of an old person born at t, therefore, is

$$c_{2,t+1}=(1+r^{e}{}_{t+1})s_t+d_{t+1}w^{e}{}_{t+1}+(1-d_{t+1})p^{e}{}_{t+1}, \tag{1.2}$$

that is, consumption when old $c_{2,t+1}$ is supported by savings plus expected interests from t to $t+1$, accrued at the rate $r^{e}{}_{t+1}$, the expected wage income for the working time when old, d, and the expected pension benefit for the retirement time, $1-d$.

In this paper we assume the coefficient d_{t+1} is endogenous and determined by the individual health status when old, which, in turn, is assumed to positively depend on the public health spending the government provides at t, h_t, i.e. the healthier an old individual, the larger the fraction of time devoted to supply labour. The relationship between the working time in the old-age period (i.e. the old-age labour supply) and health expenditure is assumed to be captured by the generic *non-decreasing-bounded* function:[4]

[4] Alternatively, this amounts to assuming that the health technology is a non-decreasing bounded function of health expenditure and the working fraction of the second period of life is proportional to the individual health status with a proportionality coefficient equal to one.

$$d_{t+1} = d(h_t), \tag{2}$$

where $d(0) = d_0 \geq 0$, $\lim_{h\to\infty} d(h) = d_1$ and $0 \leq d_0 < d_1 < 1$.

Preferences of individuals of generation t over young-aged consumption and old-aged consumption are described by the following lifetime logarithmic utility function:

$$U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}), \tag{3}$$

where $0 < \beta < 1$ is the degree of individual (im)patience to consume over the life cycle.

The representative individual wishes to choose how much to save out of her disposable income in order to maximise the lifetime utility index Eq. (3) subject to Eqs. (1), where actual and expected factor prices, the coefficient d_{t+1} and the expected pension benefit $p^e{}_{t+1}$ are taken as given. Therefore, the saving rate is:

$$s_t = \frac{\beta w_t (1 - \tau - \theta)}{1+\beta} - \frac{1}{(1+\beta)(1 + r^e{}_{t+1})}\left[d_{t+1} w^e{}_{t+1} + (1 - d_{t+1}) p^e{}_{t+1}\right]. \tag{4}$$

2.2. Firms

At time t firms produce a homogeneous good, Y_t, by combining capital and labour, K_t and L_t, respectively, through the constant returns to scale Cobb-Douglas technology $Y_t = AK_t{}^{\alpha} L_t^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. Labour supply is $L_t = \overline{L}(1 + d_t)$, where $\overline{L}$ is the constant number of workers in each cohort (young and old); then, we set $\overline{L} = 1$ without loss of generality. Therefore, output per efficient worker (y_t) as a function of capital per efficient worker (k_t) is

$$y_t = Ak_t{}^{\alpha}, \tag{5}$$

where $y_t := Y_t / L_t$ and $k_t := K_t / L_t$.

Firms maximise profits[5] and perfect competition guarantees that factor inputs are paid their marginal products, that is

$$r_t = \alpha A k_t^{\alpha - 1} - 1, \tag{6.1}$$

$$w_t = (1-\alpha) A k_t^{\alpha}. \tag{6.2}$$

2.3. Government

The government separately finances with labour income taxes both health care services (e.g., vaccines, hospitals, scientific research and so on) to ameliorate the individual health status, and unfunded PAYG pensions to transfer resources from the young to the old, at a balanced budget.

As regards health care expenditure at t, it is constrained by the following per capita government budget:

$$h_t = \tau w_t, \tag{7.1}$$

the left-hand side being the health expenditure and the right-hand side the tax receipt (see Chakraborty, 2004).

The per capita pension accounting rule followed at t, instead, reads as:

$$(1-d_t) p_t = \theta w_t (1+d_t), \tag{7.2}$$

the left-hand side being the pension expenditure and the right-hand side the tax receipts.

2.4. General equilibrium

Exploiting the one-period forward pension accounting rule Eq. (7.2) to substitute out for the pension expenditure into Eq. (4) and using Eqs. (2), the saving rate can be rearranged as

[5] Without loss of generality, we assume the price of final output is normalised to unity and capital totally depreciates at the end of each period.

$$s_t = \frac{\beta w_t(1-\tau-\theta)}{1+\beta} - \frac{d(h_t)w^e{}_{t+1}}{(1+\beta)(1+r^e{}_{t+1})} - \frac{\theta w^e{}_{t+1}[1+d(h_t)]}{(1+\beta)(1+r^e{}_{t+1})}. \tag{8}$$

Given the government budgets Eqs. (7), market-clearing in goods and capital markets is determined as the equality between investments and savings $K_{t+1} = S_t$, that can also be expressed as:

$$k_{t+1}[1+d(h_t)] = s_t. \tag{9}$$

Now, combining Eqs. (8) and (9), assuming individuals are perfect foresighted (i.e. the future values of both the wage and interest rate is expected to depend on the future value of the capital stock), and using Eqs. (6.2) and (7.1) to obtain a relationship between the fraction of working time when old and capital per efficient worker, i.e. $d(k_t)$, the dynamic evolution of capital is simply described by the following first order non-linear difference equation with constant coefficients:

$$k_{t+1} = \frac{I\,y(k_t)}{L + Md(k_t)}, \tag{10}$$

where $I := \beta\alpha(1-\alpha)(1-\tau-\theta)$, $L := \alpha(1+\beta)+\theta(1-\alpha)$ and $M := 1+\alpha\beta+\theta(1-\alpha)$ are positive constants and $y(k_t) = Ak_t^{\alpha}$ from Eq. (5).

3. Dynamics

Analysis of the phase map Eq. (10) gives the following propositions as regards the existence and uniqueness of the steady state, monotonic and oscillatory dynamics and the possibility of endogenous fluctuations depending on the length of the retirement time.

Proposition 1. *(Existence and uniqueness of the steady state). A unique non-trivial steady state* $k^* > 0$ *of the dynamic system described by Eq. (10) exists.*

Proof. Define the right-hand side of Eq. (10) as $J(k)$. Since $d(0)=d_0 \geq 0$, then it is straightforward to verify that $J(0)=0$. Now, fixed points of Eq. (10) are determined as interior solutions to $k = J(k)$, that can also be rearranged as $Z_1(k)=Z_2(k)$, where $Z_1(k) := k^{1-\alpha}$ and $Z_2(k) := \frac{IA}{L+Md(k)}$. Therefore, since $Z_1(k)$ and $Z_2(k)$ are continuous functions and:

(*i*) $Z_1(0)=0$, $Z'_{1,k}(k)=(1-\alpha)k^{-\alpha} > 0$ for any $k > 0$ and $\lim_{k\to+\infty} Z_1(k) = +\infty$, and

(*ii*) $Z_2(0)=\frac{IA}{L+Md(0)}=\frac{IA}{G}>0$, where $G := L+Md_0 > 0$, $Z'_{2,k}(k)=\frac{-MIAd'_k(k)}{[L+Md(k)]^2}<0$ for any $k>0$ since $d'_k(k)>0$ and $\lim_{k\to+\infty} Z_2(k)=\frac{IA}{L+Md(\infty)}=\frac{IA}{E}>0$, where $E := L+Md_1 > 0$, $\frac{IA}{E}<\frac{IA}{G}$, since $E > G$ because $0 \leq d_0 < d_1 < 1$.

Then for any $k > 0$ we get $Z_1(k)=Z_2(k)$ only once at k^*. **Q.E.D.**

Proposition 2. *(Non-monotonic behaviour). If*

$$\varepsilon_d > \varepsilon_y, \tag{11}$$

and

$$d(k^*) > \hat{d}(k^*) := \frac{\varepsilon_y}{\varepsilon_d - \varepsilon_y} \cdot Q, \tag{12}$$

then the law of motion in Eq. (10) is non-monotonic, where

$\varepsilon_d := d'_k(k^*)\frac{k^*}{d(k^*)}$ *and* $\varepsilon_y := y'_k(k^*)\frac{k^*}{y(k^*)}$ *are, respectively, the elasticity of the supply labour when old and the elasticity of GDP per efficient worker with respect to the stock of capital per efficient worker evaluated at* k^**, and* $Q := \frac{L}{M} < 1$.

Proof. Differentiating $J(k)$ with respect to k gives and evaluating it at k^* gives:

$$J'_k(k^*) = \frac{IA[L\varepsilon_y - M(\varepsilon_d - \varepsilon_y)d(k^*)]}{(k^*)^{1-\alpha}[L + Md(k^*)]^2}. \tag{13}$$

Since $\mathrm{sgn}\{J'_k(k^*)\} = \mathrm{sgn}\{L\varepsilon_y - M(\varepsilon_d - \varepsilon_y)d(k^*)\}$, then $\varepsilon_d > \varepsilon_y$ is necessary and $d(k^*) > \hat{d}(k^*)$ sufficient to have $J'_k(k^*) < 0$. If $\varepsilon_d < \varepsilon_y$ then $J'_k(k^*) > 0$ always holds. **Q.E.D.**

Proposition 3. *(Endogenous fluctuations). If (11) and (12) hold and*

(1) if $\hat{d}(k^*) < d(k^*) < \underline{d}(k^*)$*, then the law of motion in Eq. (10) is non-monotonic and convergent to* k^**;*

(2) if $d(k^*) = \underline{d}(k^*)$*, then a flip bifurcation may generically occur;*[6]

(3) if $\underline{d}(k^*) < d(k^*) < \overline{d}(k^*)$*, then the law of motion in Eq. (10) is non-monotonic and divergent from* k^**;*

(4) if $d(k^*) = \overline{d}(k^*)$ *a reverse flip bifurcation may generically occur;*

(5) if $\overline{d}(k^*) < d(k^*) < 1$*, then the law of motion in Eq. (10) is non-monotonic and convergent to* k^**, where*

$$\underline{d}(k^*) := \frac{-2L(k^*)^{1-\alpha} + IA(\varepsilon_d - \varepsilon_y) - \sqrt{P(k^*)}}{2M(k^*)^{1-\alpha}}, \tag{14}$$

[6] In the numerical example in the next section we show that the flip bifurcation is super-critical and, hence, attractive (i.e., the bifurcation points are symmetrical and stable).

$$\bar{d}\left(k^*\right) := \frac{-2L\left(k^*\right)^{1-\alpha} + IA\left(\varepsilon_d - \varepsilon_y\right) + \sqrt{P\left(k^*\right)}}{2M\left(k^*\right)^{1-\alpha}}, \tag{15}$$

where $P\left(k^*\right) := (IA)^2\left(\varepsilon_d - \varepsilon_y\right)^2 - 4IAL\varepsilon_d\left(k^*\right)^{1-\alpha}$.

Proof. From Eq. (13) we find that $J_k'\left(k^*\right) \leq -1$ if and only if

$$M^2\left(k^*\right)^{1-\alpha}\left[d\left(k^*\right)\right]^2 + M\left[2L\left(k^*\right)^{1-\alpha} - IA\left(\varepsilon_d - \varepsilon_y\right)\right]d\left(k^*\right) + L\left[L\left(k^*\right)^{1-\alpha} + IA\varepsilon_y\right] \leq 0. \tag{16}$$

If $2L\left(k^*\right)^{1-\alpha} - IA\left(\varepsilon_d - \varepsilon_y\right) > 0$, then (16) can never be verified and the steady state is stationary through oscillations. If $2L\left(k^*\right)^{1-\alpha} - IA\left(\varepsilon_d - \varepsilon_y\right) < 0$, then when $P\left(k^*\right) > 0$ two positive real solutions exist of (16), as given by Eqs. (14) and (15) where $\bar{d}\left(k^*\right) > \underline{d}\left(k^*\right) > 0$, such that $-1 < J_k'\left(k^*\right) < 0$ if $\hat{d}\left(k^*\right) < d\left(k^*\right) < \underline{d}\left(k^*\right)$, $J_k'\left(k^*\right) = -1$ if $d\left(k^*\right) = \underline{d}\left(k^*\right)$, $J_k'\left(k^*\right) < -1$ if $\underline{d}\left(k^*\right) < d\left(k^*\right) < \bar{d}\left(k^*\right)$, $J_k'\left(k^*\right) = -1$ if $d\left(k^*\right) = \bar{d}\left(k^*\right)$ and $-1 < J_k'\left(k^*\right) < 0$ if $\bar{d}\left(k^*\right) < d\left(k^*\right) < 1$. **Q.E.D.**

In order to elucidate how the main steady-state macroeconomic variables react to a change in the size of the pension system, θ, as well as to grasp the economic intuition behind the results, we now perform a sensitivity analysis by modelling the relationship between the working fraction of the old-age period and health expenditure with the following rather general function, following for instance Blackburn and Cipriani (2002), Blackburn and Issa (2002),and de la Croix and Ponthiere (2010):[7]

$$d_{t+1} = d\left(h_t\right) = \frac{d_0 + d_1\,\Delta h_t^{\ \delta}}{1 + \Delta h_t^{\ \delta}}, \tag{17}$$

[7] Although in the analyses by Blackburn and Cipriani (2002), Blackburn and Issa (2002), Chakraborty (2004), de la Croix and Ponthiere (2010) and Leung and Wang (2010) the dependent variable is the longevity rate rather than the fraction of working (retirement) period in the old-age, the line of reasoning to justify this formulation may be the same.

where $\delta, \Delta > 0$, $0 \le d_0 < 1$, $d_1 > d_0 > 0$, $d(0) = d_0 \ge 0$, $d'_h(h) = \frac{\delta \Delta h^{\delta-1}(d_1 - d_0)}{(1+\Delta h^{\delta})^2} > 0$,

$\lim_{h \to \infty} d(h) = d_1 \le 1$, $d''_{hh}(h) < 0$ if $\delta \le 1$ and $d''_{hh}(h) \gtreqless 0$ for any $h \lesseqgtr h_T := \left[\frac{\delta - 1}{(1+\delta)\Delta} \right]^{\frac{1}{\delta}}$ if $\delta > 1$.

Eq. (17) is able to capture several different features of the ability to work when old of the typical agent as a function of the health measure h: it encompasses, in fact, (*i*) the "saturating" function used in the numerical examples by Chakraborty (2004) and Leung and Wang (2010) when $\delta = \Delta = 1$ and $d_0 = 0$, while also preserving (different from Chakraborty, 2004, and in accord with Leung and Wang, 2010) a positive exogenously given level of old age working period regardless of whether public health spending exists or not; (*ii*) the S-shaped function when $\delta > 1$ (i.e., threshold effects of public health investments exist) used in the numerical examples by Blackburn and Cipriani (2002), Blackburn and Issa (2002), and de la Croix and Ponthiere (2010).

The relationship between health status and income (that, in this model, can be considered as a proxy of the public health spending, the latter being a constant fraction of income), has been shown to be S-shaped by Ecob and Davey Smith (1999)[8] which argue that "these indices of morbidity, both self-reported and measured, are approximately linearly related to the logarithm of income, in all except very high and low incomes (this means that increasing income is associated with better health, but that there are diminishing returns at higher levels of income)." (p. 693). Moreover, Strauss and Thomas (1998) contrasted an index that captures nutritional status and health (the Body Mass Index) with the logarithm of wage income, finding an S-shaped relationship for Brazil (see Strauss and Thomas, 1998, Figure 3, p. 774).

In order to clarify the interesting dynamical properties of the model, we now resort to some numerical exercises taking the following configuration of parameters (exclusively chosen for illustrative purposes): $A = 22$, $\alpha = 0.45$ (which is an average between the values usually referred to

[8] See Ecob and Davey Smith (1999, Figures 1 and 2, p. 698–700), that plot some indices of morbidity as a function of the logarithm of income for England, Wales and Scotland in 1984 and 1985.

developed countries, i.e. $\alpha = 0.36$, see e.g., Kehoe and Perri, 2002, and those usually used for developing countries, i.e. $\alpha = 0.50$, see Purdue University's Global Trade Analysis Project 2005 database – GTAP), $\beta = 0.60$ (see Žamac, 2007), $\tau = 0.06$, $d_0 = 0$, $d_1 = 0.9$, $\delta = 50$, while also assuming $\Delta = 1$ without loss of generality, given the purely technical (and not economically interpretable) nature of such a parameter. Note that the choice of the parameter $\delta = 50$ [9] amounts to assume that health investments have a more intense effect in reducing the old age sickness state (that is, enhancing the old age health status) when a certain threshold level of public health expenditure is achieved, while becoming scarcely effective when the ability to work (the good health state) is close to its saturating value.[10]

[9] Note that if $d(h)$ were concave, i.e. $\delta \leq 1$, $\varepsilon_d > \varepsilon_y$ cannot hold and, hence, non-monotonic trajectories can never appear in that case.

[10] As an example, we may think about the existence of threshold effects in the accumulation of knowledge required for new medical advances and discoveries in the treatment of diseases (e.g. vaccines) to be effective: the public health expenditure to finance new research projects may be high and apparently useless until a certain degree of knowledge is achieved. Beyond such a threshold, however, a "jump" effect exists that allows to trigger and bring to light the beneficial effects of the new discoveries, to make them efficient, usable and operative across population and eventually transformed into higher health status for old aged individuals.

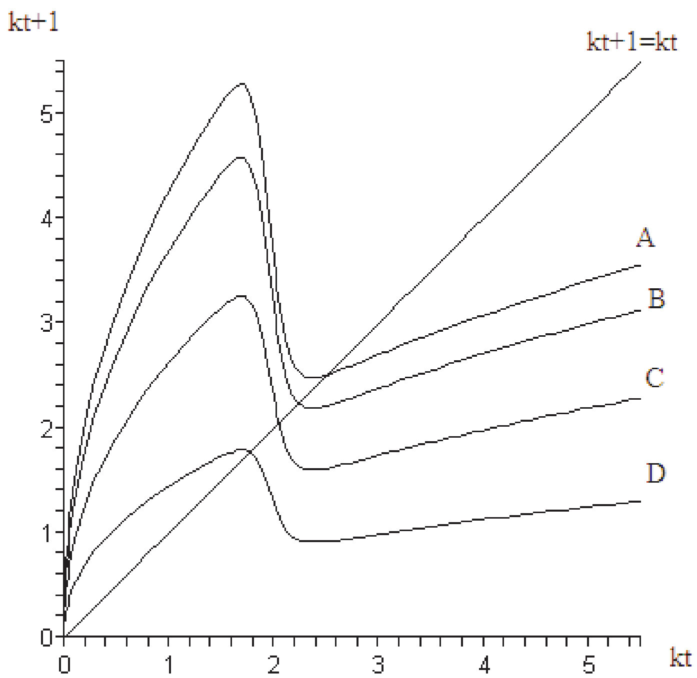

Figure 1. The phase map Eq. (10) with the corresponding steady state when θ varies.

Table 1 illustrates Figure 1 as regards the effects of a rise in θ on both the evolution and stability of the steady state (to this purpose Table 1 reports the values of the first derivative of Eq. 10 with respect to k_t evaluated at the steady state, namely $J_k'(k^*)$). Moreover, we also report the values of other macroeconomic variables of interest at the steady state, such as the per capita health expenditure, h^*, the length of the time spent working when old, d^*, the level of per capita GDP, $Y^* = A(k^*)^{\alpha}(1+d^*)$, the ratio of per capita health spending to per capita GDP, h^*/Y^*, and the ratio of the per capita pension expenditure to per capita GDP, $\frac{(1-d^*)p^*}{Y^*}$.

Table 1. Macroeconomic variables at the steady state when θ varies.

θ	0	0.04	0.07695	0.15	0.25	0.40	0.50
k^*	2.5	2.32	2.23	2.13	2.04	1.92	1.76

$J_k'(k^*)$	0.32	-0.20	-1.00	-2.65	-4.22	-3.71	-0.69
h^*	1.09	1.061	1.04	1.02	1.001	0.97	0.93
d^*	0.891	0.856	0.80	0.66	0.47	0.19	0.035
Y^*	62.93	59.70	56.85	51.49	44.62	35.3	29.45
h^*/Y^*	0.017	0.017	0.018	0.019	0.022	0.027	0.031
$\frac{(1-d^*)p^*}{Y^*}$	0	0.02	0.042	0.082	0.137	0.22	0.275

From Table 1 the following results hold.

Result 1. *The steady state stock of capital per efficient worker, k^*, the per capita health expenditure, h^*, the length of the working time when old, d^*, and the per capita GDP, Y^*, monotonically reduce when θ raises.*

Result 2. *The ratio of the per capita health expenditure to per capita GDP at the steady state, h^*/Y^*, and the ratio of the capita pension expenditure to per capita GDP, $\frac{(1-d^*)p^*}{Y^*}$ monotonically increase when θ raises.*

The economic intuition behind Results 1 and 2 is simple. First, we recall that the length of the retirement period $(1-d^*)$ (conversely, the working period, d^*) depends only on the health status when old. Increasing PAYG pensions causes an expected negative effect on capital accumulation and wages, and this in turn reduces health expenditure. Although the reduction in health spending is rather small and the ratio between health spending to GDP slightly increases,[11] the worsening of the health status, that is the reduction in the ability to work, is rather strong: for example, while in the absence of public pensions mature workers spent almost 89 per cent of their second period of life working, when the size of the pension system is assumed in line with that of several European countries (i.e. a payroll tax rate about the 15 per cent of wage income), the working fraction of the second period of life drops to about 66 per cent, and when the contribution rate further raises to about 25 per cent of wage income, as predicted by many economists for the near future, the time spend working in the old-aged shrinks to almost 47 per cent of the whole time endowment. Thus, the higher is the size of the PAYG system, the lower per capita GDP. This negative effect on the (neoclassical) economic growth is due to a twofold channel: not only the well-known crowding out effect of PAYG pensions on savings and capital accumulation properly works, but also a negative effect on the labour supply of mature workers exists because of an indirect crowding out effect of PAYG pensions on health spending. This mechanism, therefore, resembles rather to a vicious circle: the higher the size of the PAYG system, the larger the number of pensioners (i.e. the number of mature individuals with bad health, unable to work, raises).

4. Chaotic cycles under perfect foresight and multiple "bubbling" phenomena

In the previous section we analysed the equilibrium dynamical properties of the phase map Eq. (10) and illustrated (*i*) the behaviour of the unique equilibrium, (*ii*) the switch to instability and the re-switch to stability with respect to changes in the contribution rate to the pension system.

In this section we investigate – with numerical simulations – the local and global dynamics of the model using the same parameter θ as the key bifurcation parameter.

[11] Note that the ratio of per capita health spending and pension expenditure to per capita GDP are rather realistic, passing from 1.7 per cent to 2.2 per cent and from 0 per cent to 13.7 per cent, respectively, when the payroll tax rate raises from zero up to 0.25.

It is shown that the non-linear dynamics described by the time map Eq. (10) may be chaotic, as the following example reveals. The parameter set is the same as in the previous section. Moreover, we used $k_0 = 1$ as the initial value of the stock of capital.

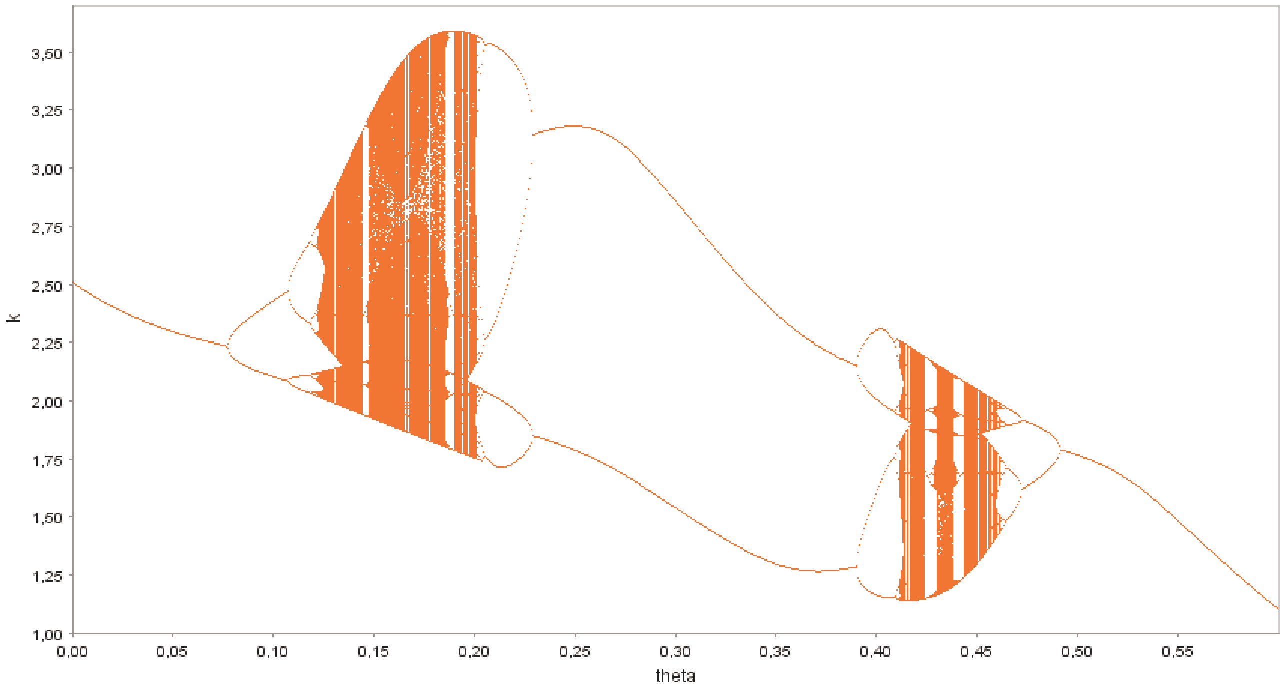

Figure 2. Bifurcation diagram for θ (an enlarged view for $0 < \theta < 0.6$ and $1 < k^* < 3.7$).

Figure 2 clearly reveals a noteworthy dynamical phenomenon: the presence of a twofold "bubbling" with a persistent two-period cycle between the two "bubbles" and overall a significant inclusion of all trajectories in a bounded region, meaning that the economic system is "resilient".

In order to better understand the appearance of multiple "bubblings", it is useful to examine the picture of the time map Eq. (10), Figure 1, together with the corresponding bifurcation diagram, Figure 2, when θ changes, showing how a period-doubling cascade initiates and then reverses. The map has a "bimodal" shape with an increasing concave branch for large capital stocks. The effect of increasing the policy variable θ is both to translate the phase map vertically downwards and flatten out the increasing branch of it for large capital stocks. Therefore when θ is continuously increased from zero, the fixed point[12] steadily reduces and the slope of the curve at the intersection point shrinks until the value minus one (see the curves A and B when θ increases from 0 to 0.1 in Figure 1) when a period-doubling bifurcation occurs (at $\theta = 0.07695$, see Figure 2): the fixed point becomes unstable and a stable two-cycle is born, followed, as usual, by a succession of period-doublings as θ is gradually raised (see the curve C when $\theta = 0.3$ in Figure 1, and the bifurcation diagram Figure 2 in the interval $0.0795 < \theta < 0.199$). In contrast, if we consider the case when θ is large, the equilibrium point is stable because it lies in the region of the map where the slope is only slightly negative or even positive (see, for example, the curve D in Figure 1 where $\theta = 0.5$ and the slope at the intersection point is -0.69). As θ is gradually reduced, the slope of the phase map continuously decreases until it reaches the value minus one when a period-doubling bifurcation emerges ($\theta = 0.49$); again, similar to the case discussed above, the fixed point becomes unstable, a stable two-cycle emerges, and for further reductions in θ a succession of period-doublings results.

However, it is easy to observe that at $\theta = 0.1995$ a reversal of period-doublings initially occurs, leading to a stable two-period cycle in the interval $0.1995 < \theta < 0.39$, and this is followed at $\theta = 0.3995$ by the appearance of new period-doubling cascades. It is interesting to note that, although in the interval $0.1995 < \theta < 0.39$ the slope of the time map at the fixed point is "more negative" than outside of that interval (as indicated both by "eye" in Figure 1 and values in the third row of Table 1), a simple two-period cycle prevails in such an interval, while outside of it chaotic behaviour emerges: this is due to the fact that bifurcations are governed by the "interaction" between the two features of the "bimodal" map: the "hump", which is largely responsible for period-doubling, and by the concavity as well as the height above the abscissa-axis of the increasing branch for large capital stocks, which are responsible for period-halving. Therefore, it happens that,

[12] As is known, the fixed point k^* may be graphically found at the intersection of the 45° line $k_{t+1} = k_t$ and the curve described by Eq. (10). The stability of k^* is ensured if the slope of the time map at the intersection point is (in absolute value) less than unity.

for making a “heuristic” example, when θ decreases from 0.5 to 0.3 (curve D and curve C in Figure 1, respectively), on the one hand the slope at the fixed point is always more negative, so that period-doublings occur more likely, but, on the other hand, both the concavity and the height of the increasing branch for large capital stocks tend to be larger, so that period halving also occur. Therefore in the overall, the former effect initially prevails and the latter one eventually dominates (as Figure 2 shows when θ decreases from 0.5 to 0.4).

Therefore we may sum up this process, saying that as θ raises from zero (*i*) the phenomenon of period-doubling, once appeared, must necessarily be followed, at least for large values of θ, by period-halving, and (*ii*) the “bubbling” phenomenon may appear even more than once in the present model.

Noteworthy, in any case the equilibrium dynamics remains bounded in an economically meaningful region although it may realistically display irregular business cycles.

5. Conclusions

We analyzed the standard OLG model with capital accumulation and PAYG pensions when the retirement period is dependent on the state of individual health when old, which is, in turn, determined by the provision of public health care services.

Our focus is on the equilibrium dynamics and our results are the following: (*i*) equilibrium dynamics differ considerably between economies with and without the link between retirement period and state of health; (*ii*) when the link does exist and the relationship between health spending and health technology is S-shaped, as is often assumed in literature, we show that the positive equilibrium is unique but it may be periodic and unstable prompting endogenous business cycles.

Moreover, we found the existence of complicated dynamical phenomena, such as multiple “bubblings”, when the contribution to the PAYG system change. However, large PAYG pensions are found to work for stability.

The conclusion is that the equilibrium dynamics in this rather simple economy, which is completely deterministic, may reconcile the existence of both irregular business cycles as well as the self-evident persistence of the economic system.

Since the retirement age as well as disability pension and health spending problems currently figure prominently on both the pension and public health system reform agendas, then our analysis indicates that addressing those issues in a theoretical model might be worthwhile as regards the comprehension of dynamical properties of the economy.

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