Public expenditure on health and private old-age insurance in an OLG growth model with endogenous fertility: chaotic cycles under perfect foresight

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Abstract This paper analyses the dynamics of a simple overlapping generations economy with endogenous longevity, endogenous fertility and private transfers from children to parents. In this context, it is shown that both the public provision of health care services, which determines the individual length of life, and the size of the intra-family transfer may be a source of chaotic cycles when individuals are perfect foresighted. However, such economic factors also have the potential to ultimately suppress undesirable chaotic fluctuations. This suggests that the equilibrium dynamics of an OLG growth model may endogenously reconcile the existence of both irregular business cycles and the global stability of the economic system.

Keywords Endogenous fertility; OLG model; Perfect foresight; Private old-age support; Public health care services

JEL Classification C62; H55; I18; J14; J18

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1. Introduction

A typical feature of all societies is that young people have always contributed to the support of the elderly through voluntary intra-family transfers. This currently occurs typically in less developed countries, that is in societies with less developed financial markets and small social security programs. The fact according to which this group of countries experiences higher population growth rates than developed one, suggests that children may represent a partial substitute for other saving opportunities.

The aim of this paper is to study the interrelationship between the size of the intra-family transfer and endogenous longevity on (*i*) capital accumulation and economic growth, and (*ii*) the stability of the economy. To this purpose we incorporate the following three hypothesis into an otherwise conventional overlapping generations (OLG) model with production (Diamond, 1965). First, we assume the existence of a "gift" privately transferred from young to old members. Second, we consider fertility as an endogenous determined variables, with individuals being altruistic towards children and not towards parents (i.e. forward altruism instead of backward altruism). Third, we introduce endogenous longevity as determined by public health spending.

In particular, (*i*) following many authors, such as Bental (1989), Raut and Srinivasan (1994), Chakrabarti (1999), Morand (1999), it is assumed that individuals voluntary transfer a constant fraction of their income (loosely speaking, a "gift") to their parents to support them in the old age; (*ii*) the cost of raising children is constant, as often assumed in literature (e.g. Raut and Srinivasan, 1994; van Groezen et al, 2003; van Groezen and Meijdam, 2008); (*iii*) it is assumed that parents derive utility from having children (Becker et al., 1990; Galor and Weil, 1996), in accord with the view of the new home economics. In particular the third assumption means that, while several papers that deal with private intra-family old-age support (e.g., Azariadis and Drazen, 1993; Nishimura and Zhang, 1993; Raut and Srinivasan, 1994) assume that children are only valued as a source of old age support – i.e., as an investment good – in this paper children represent, rather realistically, both consumption and investment goods.

The choice of a constant gift from children to parents when old deserves two comments. First, the issues of why it is in the interest of children to provide the same gift to their parents is neglected here for the sake of simplicity.[1] Second, we also abstract from the possibility of endogenously determine the size of the gift.[2]

Many authors have investigated the features of OLG models with endogenous fertility and private transfers, either in the context of backward altruism (e.g. Nishimura and Zhang, 1993; Zhang and Zhang, 1995) or with both backward and forward altruism (Wigger, 1999), or, more in accord with the new home economics, only with forward altruism (Fanti and Gori, 2010). However, all these authors have abstracted from the other cornerstone of the demographic dynamics, that is adult mortality.

In this paper we attempt to fill this gap by incorporating adult mortality as an endogenous variable. As is known the rate of longevity may depend on the individual private and public expenditures on health (Blackburn and Cipriani, 2002; Chakraborty, 2004; Bhattacharya and Qiao, 2007; Fanti and Gori, 2009). In particular, in this paper we investigate the case of the provision of public health financed by a wage tax in line with Chakraborty (2004), by assuming that the relationship longevity-health spending may be sufficiently non-linear with threshold effects, as

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[1] For instance Azariadis and Drazen (1993) assume that the division of income between generations is determined as part of a bargaining process.

[2] This could be made by assuming that agents care about their parents. In particular, the lifetime utility function of an agent when young depends also on their parents' old-age consumption. Moreover, they also choose the amount of old age support to be provided to parents (Nishimura and Zhang, 1993). The assumption that children are altruistic towards parents is crucial to endogenize the "gift". However, these assumptions imply a rise in the dimension of the dynamical features of the model, that, a part from the increased analytical complexity, would obscure the neat dynamical results of this paper.

assumed by Blackburn and Cipriani (2002), Blackburn and Issa (2002), de la Croix and Ponthiere (2009) and argued by numerous empirical studies (e.g. Fioroni, 2009).[3]

In this paper we pose the following question: which are the effects of the level of "gift" for both economic growth and economic stability under endogenous adult mortality?

The answers are rather intriguing. In fact, the results by Chakraborty (2004) when extended with endogenous fertility (see Fanti and Gori, 2009) and private old-age backing, are dramatically modified. Indeed, although the introduction of public expenditure on health may favour, as in Chakraborty (2004), the appearance of a second high equilibrium and thus the possibility of escaping from poverty, a rise in the contribution rate to the public health program may also: (*i*) decrease the level of the high equilibrium, pushing down it to a level lower than the preceding unique poor equilibrium, and (*ii*) act as an economic de-stabiliser and trigger complex business cycles. However, a high enough increase in the health tax rate acts, in contrast, as an economic stabiliser, triggering a re-switch of stability.

More specific to the present model, however, it is the dynamical role of the level of the fraction of wage devoted by the young to support consumption of the old. To this purpose, it is shown that generically the economy may be stable under either low or high levels of the "gift", while for intermediate-sized transfers we initially observe a loss in stability with consequent complex dynamics and a subsequent re-switch towards stability.

These are rather new results as regards the business cycles issue in the economic literature with overlapping generations. As is known, in fact, a strand within the OLG literature exists that tried to explain the observed economic cycles, in addition to the modern equilibrium business cycle theory that instead believe that business cycle reflects transitional dynamics around balanced growth paths (e.g., Kydland and Prescott, 1982). Indeed the appearance of periodic orbits and chaos when individuals are perfect foresighted is due to distortions in the production side of the economy, such

[3] However, it should be noted that the shape of the relationship is rather controversial in empirical works, for instance Leung and Wang (2010) found a simple concave shape.

as non perfect competition, positive externalities and so on (see e.g. Reichlin, 1986; Cazzavillan et al., 1998; Grandmont et al., 1998; Cazzavillan, 2001),[4] and generally it requires at least a two-dimensional model which is notoriously more prone to instability. In contrast, in our model these dynamic regimes can be observed in the equilibrium dynamics of a one-dimensional map assuming a simple double Cobb-Douglas economy and perfect competition among firms. Moreover, it is noteworthy the richness of our dynamical regimes to be a rather novel result in the literature framed in the OLG context.

In particular, the equilibrium dynamics of the present one-dimensional map shows "bubbling" phenomena,[5] according to which as the parameters of interest – such as the health tax rate and the "gift" rate – increases from zero a period doubling bifurcation process is triggered and subsequently curtailed and reversed, giving rise to period-halving bifurcations. This means, for instance, that the generosity of either the public health care program or the private inter-generational transfer system may be responsible of the observed business cycles, but ultimately it tends to stabilize the economic fluctuations, thus "warranting" the global stability of the economic system. This twofold role of the policy variables is remarkable from an economic point of view: in fact, on the one hand, it may be an explanation of the observed irregular business fluctuations – showing that an endogenous deterministic origin of economic cycles may be complementary to the stochastic origin at the core of the real business cycle theory – but, on the other hand, and more important, such parametric changes may even be used to control, and potentially to suppress, undesirable fluctuations. Thus, it may be concluded that the equilibrium dynamics of the present economy may embody two undisputable stylised facts: the existence of both irregular business cycles and the global stability of the economic system.

[4] It is worth noting that similar production externalities are also, as recently investigated (e.g. Benhabib and Farmer, 1994; Benhabib and Wen, 2004), channels of amplifying the propagation mechanism capable of explaining many business cycle puzzles that the standard "real business cycles" models fail to explain.

[5] See Bier and Bountis (1984) and Stone (1993).

The paper is organised as follows. Section 2 presents the model. In Section 3 the dynamics of the model is analysed and in Section 4 chaotic behaviours are shown. Section 5 contains a discussion of our findings as well as concluding remarks.

2. The model

Consider a general equilibrium OLG closed economy populated by identical individuals, identical firms and a government that runs a public health policy.

The lifetime of the typical agent is divided into childhood and adulthood, the latter period being, in turn, divided into youth (working period) and old-age (retirement period). As a child she does not make economic decisions. When adult she draws utility from consumption over the life cycle and the number of children.[6]

Young individuals of generation t (N_t) are endowed with one unit of time supplied inelastically on the labour market, while receiving wage income at the competitive rate w_t.

It is assumed that the probability of surviving from youth to old age is endogenous and determined by the individual health level, that is, in turn, augmented by the public provision of health investments such as, for instance, hospitals, vaccination programmes and so on (see Chakraborty, 2004). The survival probability at the end of youth of an individual started working at t, π_t, depends upon her health capital, h_t, and is given by a non-decreasing – though bounded – function $\pi_t = \pi(h_t)$. Following Blackburn and Cipriani (2002) and Blackburn and Issa (2002), we model this relationship with the following rather general function of health capital:[7]

[6] We follow, for instance, Eckstein and Wolpin (1985) and Galor and Weil (1996).

[7] Although the independent variable of the longevity function in the analysis by Blackburn and Cipriani (2002) is human capital instead of public health capital, the line of reasoning to justify this formulation may be the same. To this purpose, in fact, and to capture the idea that life expectancy is positively correlated with the level of development, Blackburn and Issa (2002) argued, in the first part of their paper, that the probability of surviving at the end of youth is

$$\pi_t = \pi(h_t) = \frac{\pi_0 + \pi_1 \Delta(h_t)^\delta}{1 + \Delta(h_t)^\delta}, \tag{1}$$

where $\delta, \Delta > 0$, $0 \leq \pi_0 < 1$, $\pi_1 > \pi_0 > 0$, $\pi(0) = \pi_0 \geq 0$, $\pi'_h(h) = \frac{\delta \Delta h^{\delta-1}(\pi_1 - \pi_0)}{(1 + \Delta h^\delta)^2} > 0$,

$\lim_{h\to\infty} \pi(h) = \pi_1 \leq 1$, $\pi''_{hh}(h) < 0$ if $\delta \leq 1$ and $\pi''_{hh}(h) \gtreqless 0$ for any $h \lesseqgtr h_T := \left[\frac{\delta - 1}{(1+\delta)\Delta}\right]^{\frac{1}{\delta}}$ if $\delta > 1$.

Eq. (1) allows us to capture various aspects of the evolution of the length of life of the typical agent as a function of the health measure h: it encompasses, in fact, the "saturating" function used in the numerical examples by Chakraborty (2004) when $\delta = \Delta = 1$ and $\pi_0 = 0$ as well as the S-shaped function when $\delta > 1$ (i.e. threshold effects of public health investments exist), while also preserving (different from Chakraborty, 2004) a positive constant rate of longevity regardless of public health spending. The empirical evidence shows a rather significant non-linearity in the relationship between longevity rates and either per capita income (which may be a proxy of the public health spending) or the per capita health spending: for instance in the respective Figure 1 and 2 in the paper by Fioroni (2009) drawn by World Development Indicators CD-ROM, World Bank (2008) there is evidence of this non-linearity at the aggregate international level.[8] Also at the micro-

non-decreasing bounded function of the stock capital installed in the economy (and used a stepwise function of the stock of capital to show the possibility of multiple development regimes), while introducing, in the second part of the paper, an S-shaped function to justify the relationship between longevity and public health spending.

[8] "… the relationship between survival rate and income in Fig. 1 is not clearly linear: indeed, in low income countries, increases in per capita GDP are strongly associated with increases in life expectancy; as income per head rises the relationship flattens out. This path reflects the influence of a country's own level of income on mortality through such factors as nutrition, education, leisure and health spending" and later "However, as in the Preston curve, Figure 2 shows that countries with low levels of health expenditure tend to gain more in life expectancy than countries starting with high level of health spending." (Fioroni, 2009, p. 3).

level, although the evidence on the shape of the association between income and mortality as well as morbidity is controversial, there are many cases of a S-shaped relationship.[9]

We assume, following Chakraborty (2004), that at time t per worker health capital is augmented by public investments financed at a balanced budget with a (constant) proportional wage income tax $0<\tau<1$, that is:

$$h_t = \tau w_t . \tag{2}$$

As regards child care activities, we assume that raising children is costly and, in particular, the amount of resources needed to take care of them is given by a fixed cost $e>0$ per child, so that the cost of raising n_t descendants is $e n_t$.

Following Bental (1989), Raut and Srinivasan (1994), Chakrabarti (1999), Morand (1999), it is assumed that each young agent gives an exogenous fraction $0<d<1$ of wage income to their parents as a means of old-age insurance, so that $\pi_{t-1} d w_t$ is the expected cost to each young and $d w^e{}_{t+1} n_t$ is the expected benefit received to the old-aged (see Ehrlich and Lui, 1991).

Therefore, in period t the budget constraint faced by an individual of the younger generation reads as:

$$c_{1,t} + s_t + e n_t = w_t (1 - \tau - \pi_{t-1} d), \tag{3.1}$$

i.e. wage income – net of both the contribution paid to finance health expenditure and the fraction voluntarily lavished to the old-aged – is divided into material consumption when young, $c_{1,t}$, savings, s_t, and the cost of raising n_t children.

Old individuals are retired and live on the proceeds of savings plus expected interests ($r^e{}_{t+1}$) as well as on the amount of resources privately transferred when young. The existence of a perfect annuities market (where savings are intermediated through mutual funds) implies that old survivors

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[9] For instance, Martikainen et al. (2009) found a logistic-like shape of the relationship of hazard ratios of mortality by different measures of income for Finnish men and women aged 30-64 years (see Figure 1, p. 152).

will benefit not only from their own past saving plus interest, but also from the saving plus interest of those who have deceased. Hence, the budget constraint of an old retired individual started working at t can be expressed as

$$c_{2,t+1} = \frac{1+r^{e}{}_{t+1}}{\pi_t} s_t + d\, w^{e}{}_{t+1} n_t \, . \tag{3.2}$$

where $c_{2,t+1}$ is old-aged consumption.

The representative individual entering the working period at time t must choose how much to save out of her disposable income as well as how many children to raise in order to maximise the lifetime utility function

$$U_t = \ln(c_{1,t}) + \pi_t \beta \ln(c_{2,t+1}) + \gamma \ln(n_t), \tag{4}$$

subject to Eqs. (3), where $0 < \beta < 1$ is the subjective discount factor and $\gamma > 0$ captures the parents' taste for children.

Therefore, the demand for children and the saving rate are respectively given by:

$$n_t = \frac{\gamma w_t (1 - \tau - \pi_{t-1} d)}{(1 + \pi_t \beta + \gamma)\left(e - \pi_t d \dfrac{w^{e}{}_{t+1}}{1 + r^{e}{}_{t+1}} \right)}, \tag{5.1}$$

$$s_t = \frac{w_t (1 - \tau - \pi_{t-1} d)}{(1 + \pi_t \beta + \gamma)\left(e - \pi_t d \dfrac{w^{e}{}_{t+1}}{1 + r^{e}{}_{t+1}} \right)} \left[\pi_t \beta e - (\pi_t \beta + \gamma) \pi_t d \frac{w^{e}{}_{t+1}}{1 + r^{e}{}_{t+1}} \right]. \tag{5.2}$$

Firms are identical and act competitively on the market. Aggregate production at t (Y_t) takes place by combining capital (K_t) and labour ($L_t = N_t$ in equilibrium) according to the constant returns to scale Cobb-Douglas technology $Y_t = A K_t^{\alpha} L_t^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$ is the output elasticity of capital. Defining $k_t := K_t / N_t$ and $y_t := Y_t / N_t$ as capital and output per worker, respectively, the intensive form production function may be written as $y_t = A k_t^{\alpha}$.

Assuming total depreciation of capital at the end of each period and normalising the price of final output to unity, profit maximisation implies that factor inputs are paid their marginal products, that is:

$$r_t = \alpha A k_t^{\alpha-1} - 1, \tag{6.1}$$

$$w_t = (1-\alpha) A k_t^{\alpha}. \tag{6.2}$$

3. Dynamics under perfect foresight

Given the presence of the future expected values of factor prices in Eqs. (5), it is indeed necessary to specify the type of expectations formation of individuals with respect to the future values of both the wage and interest rate.

The two extreme cases used to study the dynamics of a (deterministic) general equilibrium economy are (*i*) myopic expectations and (*ii*) rational expectations (see, e.g., de la Croix and Michel, 2002).

In this paper we exclusively analyse the dynamics of the economy under perfect foresight for the very interesting dynamical features that the model presents in that case.

Therefore, under perfect foresight individuals expect the future values of both the interest and wage rates to depend on the future value of the stock of capital per worker, that is

$$\begin{cases} 1 + r^e{}_{t+1} = \alpha A k_{t+1}{}^{\alpha-1} \\ w^e{}_{t+1} = (1-\alpha) A k_{t+1}{}^{\alpha} \end{cases}. \tag{7}$$

Knowing that population evolves according to $N_{t+1} = n_t N_t$, equilibrium in goods and capital markets is given by the equality between investments and savings, that can be written in per worker terms as

$$n_t k_{t+1} = s_t. \tag{8}$$

Combining Eqs. (1), (2), (5.1), (5.2), (7) and (8), the dynamic path of capital accumulation is described by the following non-linear difference equation:

$$k_{t+1} = \frac{\pi_t(k_t)\alpha\beta e}{\alpha\gamma + d(1-\alpha)\pi_t(k_t)[\pi_t(k_t)\beta + \gamma]}, \tag{9}$$

that can also be written as

$$k_{t+1} = \frac{H\left(1+Bk_t^{\alpha\delta}\right)\left(\pi_0+\pi_1 Bk_t^{\alpha\delta}\right)}{\alpha\gamma\left(1+Bk_t^{\alpha\delta}\right)^2 + d(1-\alpha)\left(\pi_0+\pi_1 Bk_t^{\alpha\delta}\right)\left[\beta\left(\pi_0+\pi_1 Bk_t^{\alpha\delta}\right)+\gamma\left(1+Bk_t^{\alpha\delta}\right)\right]}, \tag{10}$$

where $B := \Delta[\tau(1-\alpha)A]^{\delta} > 0$ and $H := \alpha\beta e > 0$ are used to simplify notation.

From Eq. (10) we have the following results about the existence of steady states can be derived.

Result 1. *(Steady states). The dynamic system described by Eq. (10) admits either one positive steady state $\{\bar{k}\}$, which may locally stable or unstable, or three positive steady states $\{\bar{k}_1, \bar{k}_2, \bar{k}_3\}$, the first is locally stable, the second is locally unstable, the third may be locally stable or unstable.*

Moreover, the following propositions show that depending on the size of the privately provided inter-generational transfer, the dynamics of capital may be non-monotonic (Proposition 1) and deterministic endogenous business cycles may even emerge (Proposition 2).

Proposition 1. *(Non-monotonic dynamics). Let*

$$d > \tilde{d}\left(\bar{k}\right) \tag{11}$$

hold. Then the law of motion in Eq. (10) is non-monotonic, where

$$\tilde{d}\left(\bar{k}\right) := \frac{\alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2}{(1-\alpha)\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)^2}. \tag{12}$$

Proof. Define the right-hand side of Eq. (10) as $J(k)$. Then differentiating $J(k)$ with respect to k and evaluating it at the steady state, $\bar{k}$, we get:

$$J_k'(\bar{k}) = \frac{M\bar{k}^{\alpha\delta-1}\left[\alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2 - d(1-\alpha)\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)^2\right]}{\left\{\alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2 + d(1-\alpha)\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)\left[\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)+\gamma\left(1+B\bar{k}^{\alpha\delta}\right)\right]\right\}^2}. \tag{13}$$

Since $\operatorname{sgn}\left\{J_k'(\bar{k})\right\} = \operatorname{sgn}\left\{\alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2 - d(1-\alpha)\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)^2\right\}$, then (11) is necessary and sufficient to have $J_k'(\bar{k})<0$. **Q.E.D.**

Proposition 2. *(Endogenous fluctuations). If (11), $T_2(\bar{k})<0$ and $D(\bar{k})>0$ hold, and*

(1) if $\tilde{d}(\bar{k})<d<\underline{d}(\bar{k})$, then the law of motion in Eq. (10) is non-monotonic and convergent to $\bar{k}$;

(2) if $d=\underline{d}(\bar{k})$, then a flip bifurcation generically occurs;

(3) if $\underline{d}(\bar{k})<d<\bar{d}(\bar{k})$, then the law of motion in Eq. (10) is non-monotonic and divergent from $\bar{k}$;

(4) if $d=\bar{d}(\bar{k})$ a reverse flip bifurcation generically occurs;

(5) if $\bar{d}(\bar{k})<d<1$, then the law of motion in Eq. (10) is non-monotonic and convergent to $\bar{k}$, where

$$\underline{d}(\bar{k}) := \frac{-T_2(\bar{k})-\sqrt{D(\bar{k})}}{2T_1(\bar{k})}, \tag{14}$$

$$\bar{d}(\bar{k}) := \frac{-T_2(\bar{k})+\sqrt{D(\bar{k})}}{2T_1(\bar{k})}, \tag{15}$$

where $T_1(\bar{k}) := (1-\alpha)^2\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)^2\left[\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)+\gamma\left(1+B\bar{k}^{\alpha\delta}\right)\right]^2>0$,

$T_2(\bar{k}) := (1-\alpha)\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)\left\{2\alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2\left[\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)+\gamma\left(1+B\bar{k}^{\alpha\delta}\right)\right]-\beta\left(\pi_0+\pi_1 B\bar{k}^{\alpha\delta}\right)M\bar{k}^{\alpha\delta-1}\right\}$,

$T_3(\bar{k}) := \alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2\left[M\bar{k}^{\alpha\delta-1}+\alpha\gamma\left(1+B\bar{k}^{\alpha\delta}\right)^2\right]>0$ $D(\bar{k}) := \left[T_2(\bar{k})\right]^2-4T_1(\bar{k})T_3(\bar{k})$.

Proof. From Eq. (13) we find that $J_k'(k^*)\le -1$ if and only if

$$T_1(\bar{k})d^2+T_2(\bar{k})d+T_3(\bar{k})\le 0. \tag{16}$$

If $T_2(\bar{k})>0$, then (16) can never be verified and the steady state is stationary through oscillations. If $T_2(\bar{k})<0$, then when $D(\bar{k})>0$ two positive real solutions exist of (16), as given by Eqs. (14) and (15) where $\bar{d}(\bar{k})>\underline{d}(\bar{k})$, such that $-1<J_k'(\bar{k})<0$ if $\tilde{d}(\bar{k})<d<\underline{d}(\bar{k})$, $J_k'(\bar{k})=-1$ if $d=\underline{d}(\bar{k})$, $J_k'(\bar{k})<-1$ if $\underline{d}(\bar{k})<d<\bar{d}(\bar{k})$, $J_k'(\bar{k})=-1$ if $d=\bar{d}(\bar{k})$ and $-1<J_k'(\bar{k})<0$ if $\bar{d}(\bar{k})<d<1$. **Q.E.D.**

The above propositions shows that as the size of the "gift" increases from zero a loss of stability through a flip bifurcation occurs, followed at least for a large size transfer by a re-switch towards stability through a reversal flip bifurcation.

The economic intuition is simple: when the size of the inter-generational transfer increases, the disposable income of the young shrinks, and this in turn implies that savings and eventually the capital accumulation locus are pushed downward as a general effect. However, interestingly, for intermediate levels of the capital stock the threshold effect (see Eq. 1) of health spending on longevity is triggered and, thus, in correspondence to the increased longevity the depressing role exerted by the gift is relatively more intense, and this ultimately reduce capital accumulation. This in turn is due to both an augmented expected cost to each young, caused by the larger period of old-age support, and a reduced intrinsic "value" of the expected benefit received to the old age, caused by the larger old age period and, hence, by the need of a higher consumption when old.

Finally capital accumulation will become again increasing because of the fact that, for further increases in the capital stock and thus in health spending, the individual life span tends to stay constant so that the depressing role of the gift is reduced because the two depressive channels above discussed are sterilised by the trend of the longevity towards a constant floor.

In order to illustrate Result 1 and the above propositions we run numerical simulations and display the evolution of the phase map Eq. (10) and the process of birth and death of the steady states when the parameters of interest change. Since the main specific features of the present model are, namely, endogenous longevity and intra-family transfers for old-age support we focus on the dynamical role of the parameters d and τ (Figures 1 and 2, respectively).[10]

The parameter set chosen to plot Figure 1 and 2 is the following: $A = 10$, $\alpha = 0.33$ (as is usual in the economic growth literature), $\beta = 0.60$ (see, e.g., Žamac, 2007), $\gamma = 0.05$, $\pi_0 = 0.20$, $\pi_1 = 0.95$, $\Delta = 1$, $\delta = 60$, $e = 1$ (with $\tau = 0.12$ and $d = 0.28$ in Figures 1 and 2, respectively).

First, we concentrate on the case of the existence of a unique positive equilibrium, in order to highlight the role of the gift to parents on the economic stability, and analyse how the topological features of such a unique equilibrium evolves when d varies.

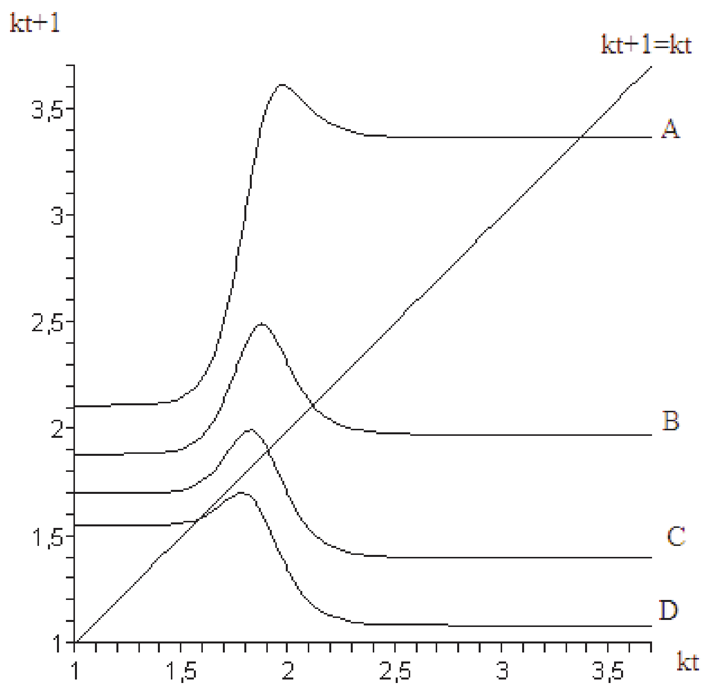

Figure 1. The evolution of the phase map Eq. (10) and the steady stases when d raises ($\tau = 0.12$).

As regards the level of the equilibrium point, we note that, as expected, the higher the gift rate, the lower savings and the capital accumulation. As regards economic stability, Figure 1 shows that for

[10] See the next section for a detailed illustration of the equilibrium dynamics in the cases of the numerical examples displayed in Figures 1 and 2.

either low or high enough values of the gift rate economic stability is preserved, while intermediated-sized transfers to parents may stimulate regular as well as chaotic business cycles, provided that the relationship longevity-health spending is sufficiently non-linear with threshold effects. However, and more important, as d further increases the chaotic phenomenon must be followed, at least for large enough values of d, by a re-switch to stability.

While as regards the role of the gift rate we examined a case of a unique positive equilibrium, as regards the role of the health tax rate, τ, we illustrate how this parameter is responsible of the process of birth and death of multiple equilibria as well as loss of stability, chaotic cycles and re-switch to stability.

Therefore, we now examine the picture of the map (eq. 10) for different values of the parameter τ (with the parameter set described above and $d = 0.28$).

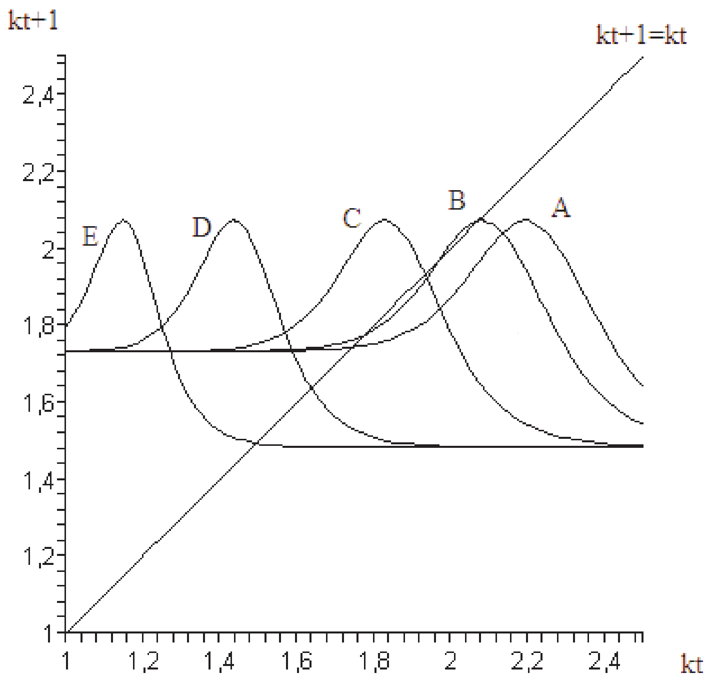

Figure 2. The evolution of the phase map Eq. (10) and the steady states when τ raises ($d = 0.28$).

As is evident from Figure 2 (the introduction and the rise of) a public health program, tuned by increasing τ from zero, is responsible for the appearance and disappearance of multiple equilibria as well as for dramatically different slopes of the phase map at the equilibrium points, as is better detailed in the next section.

4. Chaotic cycles under perfect foresight

In the previous section, we made an analytical discussion of the equilibrium dynamics of implied by Eq. (10) while also illustrating the processes of (*i*) birth and death of equilibria, and (*ii*) switching to instability and re-switching to stability, under the evolution of two key parameters of this economy, namely, the gift rate (which determines the size of the private inter-generational transfer system) and the health tax rate (which, instead, determined the size of the public health care system).

In this section we investigate through numerical simulations both the local and global dynamics of the model by using the gift rate and the health tax rate as key bifurcation parameters. It is shown that the non-linear dynamics described by Eq. (10) gives rise to deterministic chaotic cycles determined by either the size of the private inter-generational transfer or that of the public health system, as the following examples reveal. The parameter set is the same as in the previous section. Moreover, we used $k_0 = 1$ as the initial value of the stock of capital.

To explain how a period-doubling cascade initiates and then reverses, it is useful to examine together the graph of the phase map (Eq. 10) – see Figure 1 – and the corresponding bifurcation diagram Figure 3 for different values of the parameter d. The map has a flattened plateau region for small capital stocks and a plateau region that flattens out for large capital stocks. The effect of increasing d is that both to translate the plateau vertically downwards and to lower relatively more the plateau region for large capital stock than that for small capital stocks. Therefore, when the gift rate is continuously increased from zero, the fixed point[11] is lowered and the slope of the curve at the intersection point is reduced until the value minus one (see curve A and B in Figure 1 when d increases from 0.1 to 0.2) when a period-doubling bifurcation occurs (at $d = 0.195$): the fixed point becomes unstable and a stable two-cycle is born, followed, as usual, by a succession of period-doublings as d is gradually increased (see curve C in Figure 1 when $d = 0.3$, and the bifurcation diagram in the interval $0.195 < d < 0.315$). Conversely, if we consider now the case when d is large, the equilibrium point is stable because it lies in the plateau region for large capital stocks where the slope is approximately zero (see for example curve D in Figure 1 where $d = 0.4$). As d is gradually reduced from 0.4, the slope of the map Eq. (10) continuously decreases until it reaches the value minus one when a period-doubling bifurcation emerges (at $d = 0.338$); again, similar to the case discussed above, the fixed point becomes unstable, a stable two-cycle emerges and for further reductions of d a succession of period-doublings results.

We may therefore sum up this process saying that as the gift rate increases from zero the phenomenon of period-doubling must eventually be followed, at least for large enough values of d, by period-halving, that is the "bubbling" phenomena should be expected in the present model.

Now we turn to the analysis of dynamical effects of the health tax rate τ, and we examine together the graph of the phase map Eq. (10) – see Figure 2 – and the corresponding bifurcation diagram Figure 4 for different values of the parameter τ.

As shown by Figure 2, the effect of increasing τ is that to translate the phase map – and thus also the plateau – horizontally leftwards. Therefore, when the policy variable τ is continuously increased from zero, the unique fixed point as well as the slope of the curve at the intersection point are initially increased, while subsequently multiple (three) equilibria appear (which may termed as

[11] As is known, the fixed point $\bar{k}$ can be found graphically at the intersection of the 45° degree line $k_{t+1} = k_t$ and the curve Eq. (10), and the stability of the steady state is ensured if the slope of the curve at the intersection point is less than unity in absolute value.

low, medium, and high, respectively, allowing for the possibility to escape from the low equilibrium towards the high equilibrium), and ultimately only the high equilibrium persists with decreasing values, while the slope of the curve at this equilibrium point is reduced until the value minus one, when a period-doubling bifurcation occurs: the fixed points becomes unstable and a stable two-cycle is born, followed, as usual, by a succession of period-doublings as τ is gradually increased. Increases τ further on causes a horizontal shift of the map with the fixed point located in the plateau region, where the slope of the map is approximately zero, and thus necessarily a period-halving and a subsequent re-switch to stability are occurred. In fact, if the rise in τ beyond a certain threshold (in the example beyond $\tau = 0.1145$, when three equilibria appear, with a high equilibrium at $\bar{k}_3 = 2.075$, see curve B in Figure 2, while the previous poor equilibrium was $\bar{k}_1 = 1.78$, see curve A in Figure 2 and also see the corresponding "jump" in the bifurcation diagram Figure 4), permits the possible escapement from the poverty, however, increasing, even slightly, the health tax rate τ further on causes the disappearance of the low and medium equilibria, while also triggering regular and chaotic business cycles around the "survived" high equilibrium, which is now reduced (see curves C and D in Figure 2, when τ is 0.12 and 0.13, respectively): in fact, increases in τ causes deterministic economic fluctuations with large amplitude until another threshold value ($\tau = 0.123$) is reached; beyond such a critical value, higher values of τ imply that the period-doubling bifurcation process curtails and reverses, giving rise to period-halving bifurcations and ultimately (at $\tau = 0.134$) to a re-switch of stability of the high equilibrium $\bar{k}_3$ which is, however, at a lower level than the preceding unique low equilibrium ($\bar{k}_3 = 1.57$, while the unique equilibrium $\bar{k}_1$ before the rise in τ had caused the appearance of the high equilibrium $\bar{k}_3$, was $\bar{k}_1 = 1.78$). For further increases in the health tax rate τ, the equilibrium level is reduced (see for instance curve E in Figure 2 corresponding to which $\tau = 0.14$ and $\bar{k}_3 = 1.40$).

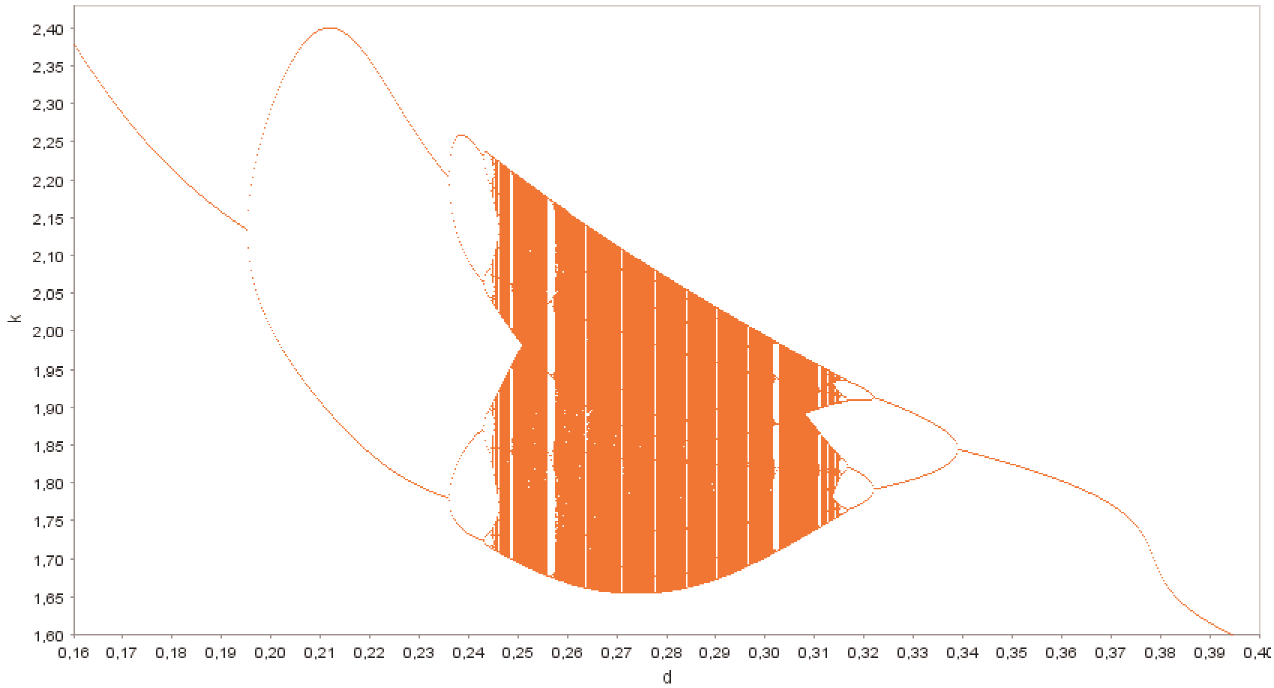

Figure 3. Bifurcation diagram for d ($\tau = 0.12$). An enlarged view for $0.16 < d < 0.4$ and $1.6 < k < 2.43$ ($k_0 = 1$).

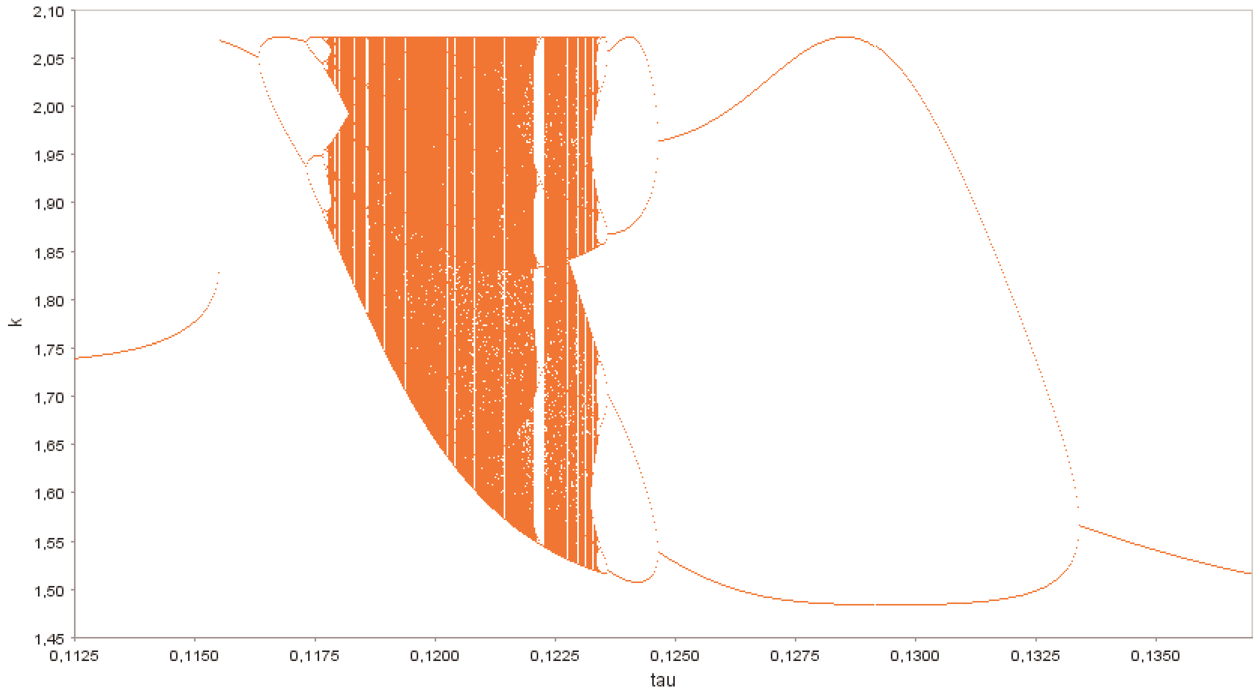

Figure 4. Bifurcation diagram for τ ($d = 0.28$). An enlarged view for $0.1125 < \tau < 0.1375$ and $1.45 < k < 2.10$ ($k_0 = 1$).

This analysis has shown the remarkable complicated role of either a public health policy, through their effects on longevity, or behaviours as regards the "gift" from children to parents, and suggested that the equilibrium dynamics through the generic "bubbling" phenomenon may successfully explain jointly both irregular business cycles, multiplicity of equilibria and the observed global stability of the economic system.

5. Conclusions

This paper examines the dynamical role played by both the intra-family transfers for old-age backing and the public health spending on the equilibrium dynamics in the standard OLG growth model extended with endogenous fertility and mortality behaviour. It is shown that under perfect foresight of individuals, the public provision of health care services, which determines the individual length of life, as well as the size of the intra-family transfer may determine complex dynamics, and, more interestingly, a rich qualitative dynamical behaviour such as the "bubbling" phenomenon. In particular, it is shown that (*i*) the introduction and the rise of public health spending may initially destabilise the economy and thus trigger complex business cycles, while eventually acting act as an economic stabiliser, thus triggering a return to stability, and (*ii*) the economy may be stable under either low or high levels of the "gift" rate, while intermediate-sized values of it initially produces a loss of stability with consequent complex dynamics while and subsequently a re-switch of stability occur.

This means that such economic factors may be a source of chaotic cycles, but ultimately they also have the potential to suppress undesirable economic fluctuations.

The findings of this paper also suggest that the equilibrium dynamics of an OLG growth model may endogenously reconcile the existence of both irregular business cycles and the global stability of the economic system under the evolution of either a policy or a behavioural parameter. Finally we note that, since in this paper we have performed a one-parameter bifurcation analysis, by investigating the dynamical role of changing either the policy parameter (the health tax rate) or an exogenously given behavioural parameter (the "gift" rate), the extension of the present model to another public policy dimension typical of a changing demographic context, such as, for instance, either a family policy or a social security policy, may allow for a richer two-parameters bifurcations analysis, shedding thus new light on the potentially complex dynamical effects of policies.

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