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Abstract
In this paper we propose a leader – follower dynamic model of taxation with the government imposing a tax to internalize externalities caused by polluting firms. As expected the Stackelberg games with the government acting as leader yield time inconsistent outcomes. We first show how time inconsistency can be avoided adopting specific utility functions. We then propose a pollution model that uses abatement as the value of accumulated pollution stock and find that the outcome of the proposed Stackelberg model is time consistent with an open – loop informational structure. This yields a tax factor that is time independent. Finally, we show that the result of the game is inefficient compared to the social planner dynamic game.

Keywords: Stackelberg model, dynamic leader–follower games.

JEL Classifications: C61, C62, D43, H32.

1. Introduction

The Stackelberg model is one of the most interesting in Economics. It is common that in most economic branches of activities one or more powerful firms dictate the others react to its (announcing) decisions. In this paper, we propose an a la

Stackelberg dynamic model of taxation, where government imposes at the start of the game a tax rule $\tau(t)$ in order to correct the externalities incurred by polluting firms.

Existing models in the literature (e.g. Long – Benchekroun, 1998) faces the same type problems in a sense of dynamic models where the tax rule announced by the government is an exogenized functional and the tax bill, at time t, is a function of its current output level and of the size of its pollution level. Their model restricts attention to stationary Markov tax rules that are linear in output, though not necessarily linear in the pollution stock.

Long and Soubeyran (2005) demonstrate that optimal tax rates per union of emission are not the same for all firms. They call this property selective penalization and their optimal distortion theorem states that the efficient tax structure requires that high cost firms pay a higher tax rate. They characterize analytically the optimal tax structures of penalties for polluting firms with heterogeneous costs. They show that there is a bias in favor of efficient firms, so in achieving efficiency a structure of systematic biases emerges. Finally their analysis can be extended to study the role of strategic trade policy in a presence of a polluting international oligopoly.

In the present model things are slightly different. Firms, represented by a representative one agent, they produce a disutility caused by the production process. Here the disutility is in the sense of the negative utility; that is, damages from pollution in utility terms. Moreover government pays for the flow of a public good using the tax revenues collected by taxing the polluting agents. Every agent accumulates a stock of pollution depending on the output level that is also subject to taxation.

However in these leader–follower games, where government acts as the leader attempting to influence the other agents (consumers or producers), the followers to the announced government policy, the government's optimal program is often time inconsistent. Here time consistency of the tax policy is intuitive: the effect of the tax on the agent's present discounted value of future utility must be independent of the level of the pollution.

Specifically, it is possible to have open – loop Stackelberg equilibrium that is not only time consistent, but also sub-game perfect. There are at least two cases for which the latter is true. The first and most usual is the case where the leader's control variables cannot not influence the follower's co–state variables, so that implies the leader cannot manipulate the follower's objective functional and the second but not usual one for which the feedback effects are not present caused that all control variables are independent of the state variables.

Models that investigate the problem of time inconsistency can be found in existing works of Xie (1997) and in the recent Karp – Ho Lee (2003) work among others. Xie (1997), using a leader – follower dynamic model of output taxation with capital accumulation, proves that if the boundary (transversality) condition ($\xi = 0$) is necessary for optimality then the government tax policy is time inconsistent due to the tax policy equals to zero, so the zero tax rule can't be an optimal one.

However, imposing two other additional boundary conditions, it shows that the latter is spurious for optimality. Karp and Ho Lee (2003) in the same setting generalize Xie's findings. Assuming, as the basic assumption, that the government tax policy committed to be multiplicatively separable and the primitive production and utility functions are given, there exists the possibility to construct a function $b(t)$ (multiplied by the tax rule $b(t)$), in order to obtain the time consistent tax $b(t)\tau(t)$.

Our proposed model as a leader–follower one, in which the tax policy is endogenous function, faces the same time inconsistency problem. But our model differs from Xie's (1997) and Karp and Lee (2003) as a model of an accumulated pollution stock and the tax policy is imposed in order to correct externalities caused by pollution. So state and control variables are different, but the stress of thought is to find the conditions under which the outcome is time consistent. One our main findings is that as firms emit pollution by production, causing a disutility, it may be possible to have a time consistent government policy announced at the start of the game.

The rest of the paper is organized as follows. Section 2 comments on time inconsistency for the leader – follower games. Section 3 presents the proposed model. Section 4 investigates the boundary conditions under which the open loop tax policy is time consistent. Section 5 sets and solves the social planning optimal control problem; section 6 faces the leader – follower same game with N agents. Section 7 provides a performance index under different regimes and section 8 also concludes.

2. Time inconsistency in Stackelberg games

The issue of time consistency of optimal policy is central in modern economic theory. Since the influential work by Kydland and Prescott (1977) economists have attempted different approaches to resolve the inconsistency problem. One possible strategy is to consider the interaction between the policy maker and the agent in a dynamic game setup. A number of researchers like Cohen and Michel (1988) found that a time consistent outcome corresponds to a feedback Nash equilibrium, while the open loop Stackelberg equilibrium corresponds to a time inconsistent policy. However it is possible in some cases the open loop Stackelberg equilibrium to produce time consistent outcomes.

The most often Stackelberg dynamic models are the models for which the role of the leader is undertaken by the government. The leader announces its optimal strategy (or policy) and adheres or makes the commitment to that announced policy. The follower at first believes the leader's commitment, but as time goes by the leader released from its commitment without the follower's knowledge. There are a few reasonable ways to explain why this deviation is beneficial for the leader. However this modification gives raise to the issue of time inconsistency in Stackelberg games.

In differential Stackelberg games with open loop informational structure time inconsistency hinges on the controllability of the follower's co – state variable. Here controllability of the follower's co – state variable is defined as whether this variable is independent of the leader's control path. Dockner et al (2000) shows that if the follower's co – state variable is uncontrollable by the leader, then the open – loop Stackelberg equilibrium is time consistent. However the time consistency is highly dependent on the game's special structure.

Things seem better in the case where the players use Markovian strategies. As by definition Markovian strategies are time consistent, one would expect that using Markovian strategies in a leader – follower game the equilibrium outcome is expected to be time consistent. However existing literature on Markovian strategies restricts to optimal capital tax and income redistribution models. Optimal capital taxation based on time consistent Markovian strategies has been investigated by Kemp et al (1993) in continuous time and by Krusell (2002) in discrete time. Given Markovian strategies the optimal equilibrium tax rate is generally not zero since without commitment the government will always have an incentive to deviate from the zero tax policy. Strulik (2002) extends the literature of optimal capital taxation and redistribution by discussion of adjustment dynamics.

For the special case of logarithmic utility he provides a qualitative discussion which is supplemented by numerical computations of adjustment paths for the more general case of iso–elastic utility. Lindner and Strulik (2004) investigate a Markovian Stackelberg strategy of taxation and redistribution in a differential game using as basis the referenced Alesina and Rodrik's (1994) median voter static model. In a framework of constant marginal returns on capital they find that optimal tax rates trajectories are time independent. This implies that the Stackelberg solution is time consistent and there exist no adjustment dynamics either in economic growth or in the public sector growth.

3. The proposed model

There is a continuum of producers in an oligopoly market. The above agents produce a single product which is consumed, implying there is no output stock. Production process emits a flow of pollution that is subject to a tax rule imposed by the government to every polluting oligopolist. The tax rule announced by the government at time zero of the game, so is easy to set $\{\tau(t) \in [0,1], t \geq 0\}$ where $\tau(t)$ means the tax rule. As a usual assumption the government expenditures are in the form of public goods.

In order to form the agents' behavior with respect to environmental variables we adopt a neoclassical production function as in augmented Solow's model, with the flow of pollution regarded as an input along the lines of Brock (1973) or Stokey (1996). Adopting these approaches, then actual output can be defined in terms of the flow of pollution.

Moreover it is assumed that production process generate emissions up to a standard emission level (that is, emissions per unit output are constant at a given level), and the disutility from pollution is not taken into account from agents. We denote by s_i the pollution stock accumulated by the i representative agent. As the pollution stock accumulates by production it is reasonable to assume that the pollution stock accumulation is dependent on the net value of the pollution stock and on output, as well.

Here the net value of accumulated pollution stock could be meant the gross pollution value minus taxes imposed to the damage caused by the pollution, but minus the net value of the 'clean' output.

In this way the differential equation that describes the pollution stock evolution is the following (1):

$$\dot{s}_i = \left[1 - \tau(t)\right] s_i - q_i(t)$$

where q_i denotes i's agent 'clean' output and $g(s_i, t) = \tau(t) s_i$ is i's agent tax payment function due to the pollution emission caused by its production.

In order to explain the differential equation of motion of the pollution accumulation we take the position that production function produces a "good" means the clean output and a "bad" representing pollution. For the "good" part of the equation, the second term of the right hand side, the abatement capital, k_a, is used, while for the "bad" production is used the pollutant part of capital k_p.

The equation of pollutants evolution has no meaning in the case of zero production, that is when $q_i(t) = 0$, and this is the standard pollution existence requirement. To make things clear assume for a moment that $q_i(t) = 0$ and $t(t) = 0$ this implies $\dot{s}_i = s_i$ with solution $s(t) = Ce^t$, where C is the integration constant, but

(1) The equation of motion implicitly assumes the technology with respect to pollution that used. Taking for example the steady state level of pollution accumulation, this equation implies that the value of the 'clean' output equals to the net value of pollution which could be meant that no improvement (cleaner) onto technology used in production process. For cleaner technologies $(\dot{s}_i < 0)$ the net value of output exceeds the value of accumulated pollution.

this is impossible, except the arbitrary case for which $C = 0$. So it is reasonable to set $s_i(t) = 0$ in the case of $q_i(t) = 0$ by definition, implying $\frac{ds_i}{dt} = \dot{s}_i = 0$ with solution, to the above equation of pollutants evolution, $s_i(t) = C$, where C is a constant. The latter constant could be meant the initial value of pollutants, as in a usual assumption. In the same way there is no meaning for the equation $\frac{ds_i(t)}{dt} = \dot{s}_i = \left(1 - \tau(t)\right)q_i(t)$ with output absent, so is again reasonable to set by definition of the pollution existence as above $s_i(t) = 0$ when $q_i(t) = 0$.

First, we claim an increase in output to means more output produced in the sense that the increase on output achieved only using cleaner technologies in the production possibilities, with these technologies set against pollution. In this way an increase on output may reduce pollutants and consequently the pollution accumulation. This could be meant that the cleaner technologies usage is obligatory for each polluting firm.

Second, we claim that the accumulation of pollutants to measured in values rather than in quantities. Evaluating pollutants, once an agent fulfills its pollution liability in tax bill (represented by the payment function $g(s_i, t) = \tau(t) s_i$) it is no more responsible for himself to every pollution accumulation, for the simple reason in that the pollutants manipulation is by the government undertaken via the tax revenues devolved to the public good provision. So we can conclude that the payment function $g(s_i, t)$ decays nothing else but the value of the accumulated pollution stock.

Moreover the pollution accumulation is responsible for the disutility incurred in an economy. In this model we denote disutility produced by emission as a negative utility for both agents and government.

As we mention the tax revenues are used by the government for public goods provision, and the public goods produces utility as well. We assume that negative utility caused by production process $E(q_i)$ and the utility caused by the public good provision $V(g)$ are both concave and strictly increasing functions. Furthermore $E'(0) = V'(0) = +\infty$.

After all the representative agent maximization problem is

$$\max \int_0^{\infty} e^{-\rho t} \left[E\left(q_i(t)\right) + V(G) \right] dt$$

subject to $$\dot{s}_i = \left[1 - \tau(t)\right] s_i - q_i(t)$$

Assuming that all agents are identical, so we can remove the subscripts of the above and the problem reduces to

$$\max \int_0^{\infty} e^{-\rho t} \left[E\left(q(t)\right) + V(G) \right] dt$$

subject to $$\dot{s} = \left[1 - \tau(t)\right] s - q(t) \tag{1}$$

4. Open loop Equilibrium

Forming the follower's Hamiltonian current value

$$H_F(G, q, \mu, t) = \left[E(q) + V(G) \right] + \mu_F \left(\left[1 - \tau(t)\right] s - q \right) \tag{2}$$

then the first order conditions (F. O. C) for optimality are:

$$\frac{\partial H_F}{\partial q} = 0 \Rightarrow E'(q) - \mu_F(t) = 0 \tag{3}$$

and $$\dot{\mu}_F(t) = -\frac{\partial H_F}{\partial s} + \rho \mu_F(t) \Rightarrow \dot{\mu}_F(t) = \left(\rho - \left[1 - \tau(t)\right] \right) \mu_F(t) \tag{4}$$

The first condition (3) yields $q_F(t)$ as a function of $\mu(t)$, that is we have $q_F(t) = f(\mu(t))$ where we have set $f(\mu_F) = (E')^{(-1)}(\mu_F)$. Applying the implicit function theorem we can see that $f'(\mu_F) = \dfrac{-1}{E''(f(\mu_F))} > 0$. It follows that if we can find functions $\mu_F(.)$ and $s(.)$ that satisfy the differential equations

$$\dot{s}(t) = f(\mu_F(t))[1-\tau(t)]s(t) \tag{5}$$

$$\dot{\mu}_F(t) = (\rho - [1-\tau(t)])\mu_F(t) \tag{6}$$

and the boundary conditions $s(0) = s_0$ and

$$\lim_{t\to\infty} \mu_F(t)s(t) = 0 \tag{7}$$

then the optimal open loop strategy of the follower is given by $q(.) = f(\mu_F(t))$

Now let us turn into the leader's problem. The government acting as a leader chooses the tax rate $\tau(t)$. The tax revenue is used to provide the government services $g(t) = \tau(t)s(t)$. The utility maximized consists of two parts: the disutility caused by the production that emits pollution and the utility caused by the government services. The leader's maximization problem that appears is subject to both pollution accumulation constraint and to the follower's best response (expressed by μ_F) into the announced government policy constraint as well.

Thus the government problem formulated as follows

$$\max \int_0^{\infty} e^{-\rho t}\left[E(q) + V(g)\right]dt$$

subject to $\qquad \dot{s}(t) = [1-\tau(t)]s(t) - f(\mu_F(t))$

and $$\dot{\mu}_F(t) = \left(\rho - \left[1 - \tau(t)\right]\right)\mu_F(t)$$

the leader's problem treats s, μ_F as state variables and the tax rate, τ, as a control variable.

Then the leader's Hamiltonian is formulated as

$$H_L = E(q) + V(g) + \psi_L\left[\left[1 - \tau(t)\right]s(t) - f\left(\mu_F(t)\right)\right] + \xi_L\left[\left(\rho - \left[1 - \tau(t)\right]\right)\mu_F(t)\right]$$

Where ψ_L, ξ_L are the co-state variables of the states s, μ_F respectively.

The F. O. C. for optimality are:

$$\frac{\partial H_L}{\partial \tau(t)} = 0$$

$$\dot{\psi}_L(t) = -\frac{\partial H_L}{\partial s} + \rho\psi_L = \psi_L\left((1 - \tau) + \rho\right)$$

$$\dot{\xi}_L(t) = -\frac{\partial H_L}{\partial \mu_F} + \rho\xi_L = -\psi_L f'\left(\mu_F(t)\right) - \xi_L\left(\rho - (1 - \tau)\right) + \rho\xi_L$$

From the latter formulation the time inconsistency can be seen as follows. Suppose we have solved the problem for the leader, then the function $\psi_L(.)$ determines the time path of the leader's announced tax rate at time zero of the game. Let the game proceed and the leader adheres to its announcement. Then at some time instant $t_1 > 0$ the state and co–state variables have some values $s^*(t_1), \mu_F^*(t_1), \psi_L^*(t_1), \xi_L^*(t_1)$ with $\xi_L^*(t_1) \neq 0$. Suppose that at time instant t_1 the leader deviates from the announced time path. At the new time starting point the leader imposes the new tax rule $\tau(t_1) = \tau^*(t_1)$, which it must take as given. However, it does not have to take the $\mu_F(t_1) = \mu_F^*(t_1)$ as a given initial condition, but chooses a new initial condition $\mu_F(t_1) \neq \mu_F^*(t_1)$. Therefore the optimal value of the associated co – state variable is $\xi_L = 0$, but this implies that the new solution is not a continuation of the original's problem solution**.**

Let us now consider the case where both the emissions disutility and utility functions $E(.)$ and $V(.)$ respectively are given in logarithmic form and this specification avoids the time inconsistency problem.

Proposition 1.

The Stackelberg pollution game described above yields time consistent strategies provided both the emission and utility functions are given in logarithmic form.

Proof

The Hamiltonian current value is

$$H_F = \ln q(t) + \ln G(t) + \mu_F\left[\left(1-\tau(t)\right)s(t) - q(t)\right]$$

The F. O. C (3) now yields

$$q(t) = \frac{1}{\mu_F(t)} \tag{8}$$

Equation (8) predicts that the follower's co – state variable $\mu_F(t)$ is independent of the leader's control variable $\tau(t)$ and therefore the follower's co – state variable is uncontrollable by the leader's control path. This proves that the strategy is a time consistent one.

Substituting condition (8) into (1) we have

$$\dot{s} = \left[1-\tau(t)\right]s(t) - \frac{1}{\mu_F(t)} \tag{9}$$

Multiplying both sides of (9) by $\mu_F(t)$ the last gives

$$\dot{s}(t)\mu_F(t) = \left[1-\tau(t)\right]\mu_F(t)s(t) - 1 \tag{10}$$

Again multiplying both sides of the F. O. condition(4) by $s(t)$ the latter gives the result

$$\dot{\mu}_F(t)s(t)=\left(\rho-\left[1-\tau(t)\right]\right)s(t)\mu_F(t) \tag{11}$$

Summing up (10) and (11) we have the differential equation

$$\frac{d\left(s(t)\mu_F(t)\right)}{dt}=\rho s(t)\mu_F(t)-1 \tag{12}$$

The solution of (12) is now $$s(t)\mu_F(t)=\frac{1}{\rho}+\Omega_1 e^{\rho t} \tag{13}$$

where Ω_1 is the integration constant. In order to satisfy the tranversality condition (7) is necessary to set $\Omega_1=0$.

The solution (13) turns to

$$s(t)\mu_F(t)=\frac{1}{\rho} \tag{14}$$

Combining equations (8) and (14)

$$q(t)=\rho s(t) \tag{15}$$

Substituting (15) into (1)

$$\dot{s}=\left[1-\tau(t)\right]s(t)-\rho s(t) \tag{16}$$

from which we get the solutions

$$s(t)=s_0 e^{\int_0^t\left[1-\tau(s)-\rho\right]ds} \quad \text{and} \quad q(t)=\rho s_0 e^{\int_0^t\left[1-\tau(s)-\rho\right]ds}$$

Now we can take the leader's position in order to build her Hamiltonian. The policy maker chooses the tax rule $\tau(t)$, so as to maximize the discounted disutility produced by production emissions and to maximize the discounted utility caused by the public good provision, under the constraints represented as:

$$\dot{s} = \left[1-\tau(t)\right]s - q(t)$$

$$\dot{\mu}_F(t) = \left(\rho - \left[1-\tau(t)\right]\right)\mu_F(t)$$

and under the follower's optimal production decision given as

$$q(t) = \frac{1}{\mu_F(t)}$$

In this way the leader's Hamiltonian current value for the problem becomes

$$H_L = \ln\frac{1}{\mu_F} + \ln\left[\tau(t)s(t)\right] + \psi_L\left[\left[1-\tau(t)\right]s(t) - \frac{1}{\mu_F}\right] + \xi_L\left[\left(\rho - \left[1-\tau(t)\right]\right)\mu_F\right]$$

Where ψ_L, ξ_L denote the co–state variables of the states s, μ_F respectively. The F.O.C's for the leader are

$$\frac{\partial H_L}{\partial \tau(t)} = \frac{1}{\tau(t)} - s(t)\psi_L(t) + \xi_L\mu_F = 0 \qquad (17)$$

$$-\frac{\partial H_L}{\partial s(t)} = \dot{\psi}_L(t) - \rho\psi_L(t) \qquad (18)$$

$$-\frac{\partial H_L}{\partial \mu_F(t)} = \dot{\xi}_L(t) - \rho\xi_L(t) \qquad (19)$$

It is worth noting that because the feedback effects absence of either (18) and (19) the government's F. O. Cs are defined for an open loop solution.

Solving equation (17) with respect to the tax rate gives

$$\tau(t) = \frac{1}{s(t)\psi_L(t) - \xi_L\mu_F} \qquad (20)$$

In the same manner (18) and (19) yields respectively

$$\dot{\psi}_L(t) = -\frac{\partial H_L}{\partial s(t)} + \rho\psi_L(t) = -\frac{1}{s(t)} + \psi_L(t)\left[\rho - \left[1-\tau(t)\right]\right] \qquad (21)$$

and $$\dot{\xi}_L(t) = -\frac{\partial H_L}{\partial \mu_F(t)} + \rho\xi_L(t) = \frac{\mu_F^2(t)\left[1-\tau(t)\right]\xi_L(t) + \mu_F(t) - \psi_L(t)}{\mu_F^2(t)} \qquad (22)$$

Substitute into the state dynamics the $\mu_F(t)=\dfrac{1}{\rho s(t)}$, we obtain

$$\dot{s}=\left[1-\tau(t)\right]s-\rho s(t) \tag{23}$$

Multiplying both sides by $\psi_L(t)$ the latter yields

$$\dot{s}(t)\psi_L(t)=\psi_L(t)\left[1-\tau(t)\right]s(t)-\psi_L(t)\rho s(t) \tag{24}$$

Again multiplying both terms of (21) by $s(t)$ yields

$$\dot{\psi}_L(t)s(t)=s(t)\psi_L(t)\left[\rho-\left[1-\tau(t)\right]\right]-1 \tag{25}$$

Summing up (24) and (25) we have

$$\frac{d\left[\psi_L(t)s(t)\right]}{dt}=-1 \tag{26}$$

The differential equation (26) has the solution

$$\psi_L(t)s(t)=\Omega_2-t \tag{27}$$

With Ω_2 to denotes the integration constant. Combine(27) with (14) we have

$$\psi_L(t)=\frac{\Omega_2-t}{s(t)}=(\Omega_2-t)\mu_F(t)\rho \tag{28}$$

Relation (22) now after substitution (28) and multiplication by $\mu_F(t)$ gives

$$\dot{\xi}_L(t)=-\frac{\partial H_L}{\partial\mu_F(t)}+\rho\xi_L(t)=\frac{\mu_F^2(t)\left[1-\tau(t)\right]\xi_L(t)+\mu_F(t)-\psi_L(t)}{\mu_F^2(t)}$$

$$\dot{\xi}_L(t)\mu_F(t)=\frac{\mu_F^2(t)\xi_L(t)\left[1-\tau(t)\right]+(\Omega_2-t)\mu_F(t)\rho-\mu_F(t)}{\mu_F(t)}\Rightarrow$$
$$\dot{\xi}_L(t)\mu_F(t)=\mu_F(t)\xi_L(t)\left[1-\tau(t)\right]-(\Omega_2-t)\rho+1 \tag{29}$$

Now multiplying both sides of (4) by $\xi_L(t)$

$$\dot{\mu}_F(t)\xi_L(t)=\left(\rho-\left[1-\tau(t)\right]\right)\mu_F(t)\xi_L(t) \tag{30}$$

The sum of (29) and (30) yields the differential equation

$$\frac{d\left(\mu_F(t)\xi_L(t)\right)}{dt} = \rho\mu_F(t)\xi_L(t) - (\Omega_2 - t)\rho + 1 \tag{31}$$

with the following solution

$$\mu_F(t)\xi_L(t) = \Omega_3 e^{\rho t} + \Omega_2 - t - \frac{2}{\rho} \tag{32}$$

where Ω_2, Ω_3 are integration constants.

Condition (20) with the aid of $(27),(32)$ can be expressed as

$$\tau(t) = \frac{1}{s(t)\psi_L(t) - \xi_L(t)\mu_F(t)} = \frac{1}{\Omega_2 - t - \Omega_2 + t - \Omega_3 e^{\rho t} + \dfrac{2}{\rho}} = \frac{\rho}{2 - \Omega_3 \rho e^{\rho t}}$$

As the tax rule is always a positive number, it is reasonable to set $\Omega_3 = 0$, so the latter simplifies to $\tau = \dfrac{\rho}{2}$

We find that the tax rule $\tau(t)$ is independent of the pollution stock $s(t)$. This fact combined with the result (8) gives raise that the game is not only time consistent but also sub-game perfect.

Substituting the tax rate into the time paths of pollution and output respectively we have:

$$s(t) = s_0 e^{\int_0^t [1-\tau(s)-\rho]ds} \Rightarrow s(t) = s_0 e^{\int_0^t \left(1-\frac{3\rho}{2}\right)ds} = s_0 e^{\left(1-\frac{3\rho}{2}\right)t} \tag{33}$$

$$q(t) = \rho s_0 e^{\int_0^t [1-\tau(s)-\rho]ds} \Rightarrow q(t) = \rho s_0 e^{\int_0^t \left[1-\frac{3\rho}{2}\right]ds} = \rho s_0 e^{\left[1-\frac{3\rho}{2}\right]t} \tag{34}$$

Both (33) and (34) are monotonically increase over the time provided $\rho < \dfrac{2}{3}$, that is for small discount rates.

The government services provision $g(t)=\tau(t)s(t)$ according to the tax rule and (33) expressed as follows:

$$g(t)=\tau(t)s(t)=\frac{\rho}{2}s(t)=\frac{\rho}{2}s_0e^{\int_0^t\left(1-\frac{3\rho}{2}\right)ds}=\frac{\rho}{2}s_0e^{\left(1-\frac{3\rho}{2}\right)t} \quad (35)$$

for which the increase requirement is the same as above that is for small discount rates $\rho<\frac{2}{3}$.

5. The social planning optimal control problem.

In this section we suppose that the benevolent social planner chooses both output and the tax rate, using the information of the public good provision $g(t)=s(t)\tau(t)$.

Then the problem consists of:

$$\max_{q(t),\tau(t)}\int_0^\infty e^{-\rho t}\left[\ln q(t)+\ln\left(s(t)\tau(t)\right)\right]dt \quad (36)$$

$$s.t \qquad \dot{s}(t)=\left(1-\tau(t)\right)s(t)-q(t) \quad (37)$$

The Hamiltonian current value formulated as follows:

$$H_{sp}=\ln q(t)+\ln\left(s(t)\tau(t)\right)+\lambda(t)\left[\left(1-\tau(t)\right)s(t)-q(t)\right] \quad (38)$$

The F.O.C.s are:

$$\frac{\partial H_{sp}}{\partial q(t)}=0\Rightarrow\frac{1}{q(t)}=\lambda(t) \quad (39)$$

$$\frac{\partial H_{sp}}{\partial \tau(t)}=0\Rightarrow\frac{1}{\lambda(t)s(t)}=\tau(t) \quad (40)$$

$$-\frac{\partial H_{sp}}{\partial s(t)}+\rho\lambda(t)=\dot{\lambda}(t)\Rightarrow\dot{\lambda}(t)=\lambda(t)\left[\rho-\left(1-\tau(t)\right)\right]-\frac{1}{s(t)} \quad (41)$$

and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)s(t)=0 \tag{42}$$

Plugging (39) into the constraint (37) and multiplying both sides by $\lambda(t)$ we have

$$\dot{s}(t)\lambda(t)=\lambda(t)\big(1-\tau(t)\big)s(t)-1 \tag{43}$$

Multiplying (41) by $s(t)$ results to:

$$s(t)\dot{\lambda}(t)=s(t)\lambda(t)\big[\rho-\big(1-\tau(t)\big)\big]-1 \tag{44}$$

Summing up $(43),(44)$ the result is

$$\frac{d\big(\lambda(t)s(t)\big)}{dt}=\rho s(t)\lambda(t)-2 \tag{45}$$

with solution $\lambda(t)s(t)=\dfrac{2}{\rho}+Ce^{\rho t}$.

In order to satisfied the transversality condition (42) we set the integration constant C equal to zero, so the solution turns to $\lambda(t)=\dfrac{2}{\rho s(t)}$ from which making use of (39) and (40), we end-up to

$$q(t)=\frac{\rho s(t)}{2} \tag{46}$$

$$\tau(t)=\frac{\rho}{2} \tag{47}$$

and these result to the next proposition.

Proposition 2.

The Stackelberg game of pollution with logarithmic utility and disutility functions yields the same tax rate as in the benchmark case of social planning, but different levels of output.

We now are able to compute all trajectories of the relevant variables for the social planer problem. Substituting the tax rate into the constraint

$$\dot{s}(t)=(1-\tau(t))s(t)-q(t)=\left(1-\frac{\rho}{2}\right)s(t)-\frac{\rho s(t)}{2}\Rightarrow \dot{s}(t)=(1-\rho)s(t)$$

with solution
$$s(t)=s_0e^{(1-\rho)t} \qquad (48)$$

and with substitution into (46) we take

$$q(t)=\frac{\rho}{2}s_0e^{(1-\rho)t} \qquad (49)$$

And the trajectory of the public good provision is

$$g(t)=\tau(t)s(t)=\frac{\rho}{2}s(t)=\frac{\rho}{2}s_0e^{(1-\rho)t} \qquad (50)$$

Notice that all trajectories are increasing functions provided $\rho<1$, that is for small discount rates.

6. The game with N identical private firms as followers

Suppose now that N identical private firms choose the level of its output with the technology improvement such that described in section 2, that is the more the output the less the pollution will be, due to the cleaner technologies usage. Moreover the firms' share of $t_i(t)$ used by the government in order to provide the public good which is non – rival and non excludable. In this way the amount of public good at time t will be $G(t)=\sum_{i=1}^{n}\tau_i(t)s_i(t)$. So each private firm solves the following problem with respect to the control variables $q_i(t)$ and $\tau_i(t)$.

$$\max_{q_i(t),\tau_i(t)}\int_0^\infty e^{-\rho t}\left[\ln q_i(t)+\ln G(t)\right]dt \tag{51}$$

subject to $$G(t)=\sum_{m=1}^{N}\tau_m(t)s_m(t)$$

$$\dot{s}_m=\left(1-\tau_m(t)\right)s_m-q_m(t) \qquad (m=1,...,N)$$

$$s_m(0)=s_{m_0}>0$$

Forming the Hamiltonian current value we have

$$\begin{aligned}H_i = {} & \ln q_i+\ln\left[\tau_i(t)s_i(t)+\sum_{j=1,j\neq i}^{N}\tau_j(t)s_j(t)\right]+\\ & \lambda_{ii}\left[\left(1-\tau_i(t)\right)s_i(t)-q_i(t)\right]+\sum_{j=1,j\neq i}^{N}\lambda_{ij}\left[\left(1-\tau_i(t)\right)s_i(t)-q_i(t)\right]\end{aligned} \tag{52}$$

and the FOCs:

$$\frac{\partial H_i}{\partial q_i(t)}=\frac{1}{q_i(t)}-\lambda_{ii}(t)=0 \tag{53}$$

$$\frac{\partial H_i}{\partial \tau_i(t)}=\frac{s_i(t)}{s_i(t)\tau_i(t)+\sum_{j=1,j\neq i}^{N}s_j(t)\tau_j(t)}-\lambda_{ii}(t)s_i(t)=0 \tag{54}$$

$$-\frac{\partial H_i}{\partial s_i}=\dot{\lambda}_{ii}(t)-\rho\lambda_{ii}(t)\Rightarrow\dot{\lambda}_{ii}(t)=\lambda_{ii}(t)\left[\rho-\left(1-\tau_i(t)\right)\right]-\frac{\tau_i(t)}{s_i(t)\tau_i(t)+\sum_{j=1,j\neq i}^{N}s_j(t)\tau_j(t)}$$

(55)

$$-\frac{\partial H_i}{\partial s_j}=\dot{\lambda}_{ij}(t)-\rho\lambda_{ij}(t)\Rightarrow\dot{\lambda}_{ij}(t)=\lambda_{ij}(t)\left[\rho-\left(1-\tau_j(t)\right)\right]-\frac{\tau_j(t)}{s_i(t)\tau_i(t)+\sum_{j=1,j\neq i}^{N}s_j(t)\tau_j(t)}$$

(56)

and the transversality condition: $\lim_{t\to\infty} e^{-\rho t}\lambda_{ii}(t)s_i(t)=0$

For the symmetry assumptions let us to write:

$s_i(t)=s_j(t)=s(t), \tau_i(t)=\tau_j(t)=\tau(t)$, therefore the co–state becomes now $\lambda_{ii}(t)=\lambda_{jj}(t)=\lambda(t)$ and the FOCs

(53) becomes:
$$q(t)=\frac{1}{\lambda(t)} \qquad (57)$$

(54) becomes:
$$\tau(t)=\frac{1}{N\lambda(t)s(t)} \overset{(57)}{\Rightarrow} q(t)=Ns(t)\tau(t) \qquad (58)$$

(55) and (56) collapses to $\dot{\lambda}(t)Ns(t)\tau(t)=\lambda(t)\left[\rho-(1-\tau(t))\right]Ns(t)\tau(t)-\tau(t)$ and multiplying both sides by $s(t)$ finally

$$s(t)\dot{\lambda}(t)=s(t)\lambda(t)\left[\rho-(1-\tau(t))\right]-\frac{1}{N} \qquad (59)$$

Substituting the (57) inside the system dynamics yields

$$\lambda(t)\dot{s}(t)=s(t)\lambda(t)\left[1-\tau(t)\right]-1 \qquad (60)$$

The sum of (59) and (60) is: $\frac{d\left(s(t)\lambda(t)\right)}{dt}=\rho s(t)\lambda(t)-\frac{N+1}{N}$, with solution

$$s(t)\lambda(t)=\frac{1+N}{N\rho}+\Omega e^{\rho t} \qquad (61)$$

with Ω to denote the integration constant, which must be set to zero in order to satisfied the transversality condition and the solution now turns to

$$s(t)\lambda(t)=\frac{1+N}{N\rho} \qquad (62)$$

and after substitution (57) into (62) the final output defined as

$$q(t)=\frac{N}{N+1}\rho s(t) \qquad (63)$$

Substituting (62) into equation (58) $t\,(t) = \dfrac{1}{Nl\,(t)s(t)}$ the last yields

$$t\,(t) = \frac{1}{Nl\,(t)s(t)} = \frac{r}{N+1} \qquad (64)$$

Substitution of (63) and (64) into the pollutants equation of motion $\dot{s} = \left(1-\tau(t)\right)s(t) - q(t)$ the latter yields finally:

$$\dot{s}(t) = (1-\ r\,)s(t) \qquad (65)$$

with solution
$$s(t) = s_0 e^{(1-\ r)t} \qquad (66)$$

and (63) finally takes the form:
$$q(t) = \frac{N}{N+1} r\, s_0 e^{(1-\ r)t} \qquad (67)$$

From solutions about output and pollution we can see that these equations are increasing functions over the time provided $r > 1$.

7. A Welfare performance index

Having a measure of the welfare expressed by

$$V = \int_0^{\infty} e^{-\rho t}\left[\ln q(t) + \ln G(t)\right]dt \qquad (68)$$

we can compare the two different regimes of the pollution game.

In the Stackelberg game the measure of welfare is

$$V_S = \int_0^{\infty} e^{-\rho t}\left[\ln\left(\rho s_0 e^{\frac{2-3\rho}{2}t}\right) + \ln\left(\frac{\rho N}{2} s_0 e^{\frac{2-3\rho}{2}t}\right)\right]dt$$

$$V_S = \left[e^{-\rho t}\left[-\frac{\ln\dfrac{\rho N s_0}{2}}{\rho} - \frac{\ln(\rho s_0)}{\rho} - \frac{(2-3\rho)(\rho t+1)}{\rho^2}\right]\right]_0^{\infty}$$

$$V_S = \frac{\ln\frac{1}{2} + 2\ln\left(\rho s_0\right) + \ln N}{\rho} + \frac{2-3\rho}{\rho^2} \qquad (69)$$

Under the social planning regime

$$V_{SP} = \int_0^\infty e^{-\rho t}\left[\ln\left(\frac{\rho s_0}{2}e^{(1-\rho)t}\right) + \ln\left(\frac{\rho N}{2}s_0 e^{(1-\rho)t}\right)\right]dt$$

$$V_{SP} = \left[e^{-\rho t}\left[-\frac{\ln\frac{\rho N s_0}{2}}{\rho} - \frac{\ln\left(\rho s_0\right)}{\rho} - \frac{2\left(1-\rho\right)\left(\rho t+1\right)}{\rho^2}\right]\right]_0^\infty$$

$$V_{SP} = \frac{\ln\frac{1}{4} + \ln N + 2\ln\left(\rho s_0\right)}{\rho} + \frac{2\left(1-\rho\right)}{\rho^2} \qquad (70)$$

Comparing (69) and (70) we have

$$V_{SP} - V_S = \frac{\ln\frac{1}{4} + \ln N + 2\ln\left(\rho s_0\right)}{\rho} + \frac{2\left(1-\rho\right)}{\rho^2} - \frac{\ln\frac{1}{2} + 2\ln\left(\rho s_0\right) + \ln N}{\rho} - \frac{2-3\rho}{\rho^2} =$$

$$= \frac{1-\ln 2}{\rho} > 0$$

Proposition 3.

The Stackelberg game described above, with the government acting as leader, results in an inefficient outcome, comparing to the social planning outcome.

8. Conclusion

It is well known that pollution occurrence is synonymous with the production existence. Consequently pollution externality in production process can be though as the value of the part of output for which the “polluted” part of input capital used, if we assume that it is possible to apart the input capital in the abatement capital and the polluted part of it. The latter could be done in the case of a separable production function for example. So it is easy to distinguish the values of the “net” output from the value of the “polluted” output. This paper takes the position that the value of accumulated pollution is rather a fact of differences, in values, of the two above parts of output. Precisely if the value of the polluted part, at any instant of time, is greater than the abatement part, and this is a prediction what technologies used in the production process, then it is clear that pollution accumulated.

Conversely if the value of the abatement part of output is greater than the polluted part, the pollutants at this instant of time are a negative value, consequently this value reduces the accumulation. But what meaning can be assigned in the value of pollution? In a leader – follower game for which the role of the leader is undertaken by the government, in the role of the follower are N polluting firms with pollution caused by production, and the government announces a tax rule in order to devolve the tax revenues in the public good of antipollution.

We claim in that game once a representative firm fulfils the imposed tax payment liability in money, the above firm is no more responsible for every pollution accumulation in value. It is certain that in this setting the tax payment function represented by the tax revenues collected by the government decays the value of the pollution accumulation. The reasoning of the last adoption is because the load of antipollution is undertaken by the government, as the usual assumption. So the tax rate payment choice is policy discretion by the government undertaken, as well. Here the well known time inconsistency problem arises in the sense that the government is in the place to revise the announced tax policy.

Technically speaking the time inconsistency depends upon the controllability by the leader, on the follower's co – state variable, where controllability of the follower's co – state variable means whether this variable is independent of the leader's control path, in an open loop informational structure of the game. The reason of the open loop choice is that the other type of informational structure, the closed loop or Markovian, is subject to few objectives, one of these is the fact that the leader – follower game solution collapses to one for which the Markovian strategies used by the leader, while open loop strategies used by the follower, as in the *Dockner et al. (2000)* argument.

In these lines, the present paper develops a model of pollution such that for the value of accumulated pollution stock a part of capital, the abatement, is used. In this setting we found, using a well known logarithmic utility functions, the time consistent tax rate and the time paths of the output and pollution of the representative polluted agent, as an expression of the pollution variable. Both the time paths are increasing time functions, for small discount rates. In the social planer optimal control problem the outcomes are in the same time consistent line, with the same tax rate but different level of outputs. Next solving the game with N private agents we found that output is dependent on the value of follower agents N and on the value of pollution as well, as expression (67) reveals. Finally constructing a simple index of performance, in order to compare the various outcomes of the proposed model, we conclude in that the social planning outcome is, as we expect, superior to the Stackelberg game for strategic agents play.

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