A Reference Dependent Representation with Subjective Tastes

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Abstract

Experimental and empirical evidence documents instances where the presence of an inferior option in a menu increases the attractiveness of the better options from that menu and thus distorts the normative ranking across menus. We analyze the case when besides this so called context effects bias there is also a concern for flexibility in the spirit of the literature initiated by Kreps (1979) and Dekel, Lipman and Rustichini (2001). Since the context effects bias and the desire for flexibility both increase the inclination of a decision maker to choose larger menus, the analysis allows the disentangling of the effect of the behavioral bias from the effect of the rational desire from flexibility. We find a weak condition on the set of ex post preferences under which the two effects are identifiable. We show that our representation is essentially unique.

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1 Introduction

Numerous observations from the marketing and psychology literature document the existence of a so called *context-effects bias* as suggested by the following experiment presented in Simonson and Tversky (1992). The participants in the experiment were asked to choose between two substitute products, tissues and towels. Two versions of the experiment were designed. One superior brand of towels and one superior brand of tissues were included in both versions. In addition, in one version of the experiment the participants were offered with one inferior brand of towels, while in the other they were offered with one inferior brand of tissues. The results of the experiment showed that the market share of the superior quality brand was significantly higher when the inferior quality brand belonged to the same category. This example shows how the presence of an inferior option in a menu may make the better bundles of that menu appear more attractive *by comparison,* and thus distort the normative ranking of the available options.[1] This is a pattern of behavior inconsistent with the standard model of rationality, which posits that products which are never chosen for consumption should not influence the decision maker's choices.

In a recent paper, Barbos (2009) studies a model of choice from categories, or menus, consistent with the above experimental evidence. Since menus are the objects of choice that reveal an individual's desire for flexibility, as a natural extension of the certainty model whose axiomatic foundations are provided in that paper, we study here the case when besides the context-effects bias, there is also a concern for flexibility in the spirit of the literature initiated by Kreps (1979) and Dekel, Lipman and Rustichini (2001) (henceforth DLR(2001)). More precisely, we analyze the case in which we allow for the presence of some underlying uncertainty between the moment of the choice of the menu and the moment of the choice of a specific option from within the menu. Allowing for uncertainty between the two stages makes sense especially in those applications in which there is some cost of switching between menus and the choice of the specific element from the menu is made either significantly later or repeatedly over a long period of time. In these cases, when choosing the menu the decision maker has to contemplate various potential realizations of his future preferences and thus, the usual intuition behind the notion of *subjective tastes* (Kreps (1979), DLR (2001)) applies here as well.[2] Since both the context effects bias and the desire for flexibility increase the inclination of a decision maker to choose a particular menu when that menu is expanded, the analysis in this paper allows

[1] There are numerous similar observations in the psychology and experimental literature. See for instance Bhargava, Kim and Srivastava (2000), Hsee and Lecrerc (1998), Huber, Payne and Puto (1982) or Pan, O'Curry and Pitts (1995).

[2] Alternatively, one can think of this model as a study of the behavioral implications of a reference point bias in the model of subjective uncertainty introduced by DLR (2001).

disentangling the effect of the behavioral bias from the effect of the rational desire for flexibility in evaluating preferences over menus. We show that our representation is essentially unique.

As a motivating example, consider an individual choosing between various movie rental services—such as, Netflix, Vongo or Comcast On Demand. Some of these services offer a variety of plans, combining benefits and prices. There is no cost of switching between these plans, but there is a cost of switching between services. Thus, for instance Netflix requires an investment in a DVD player, Vongo requires a broadband internet connection and Comcast On Demand requires a subscription to the cable service. Assume that after some careful consideration, a consumer decides to subscribe to Netflix and then selects one of the available options regarding the number of DVDs he can rent at a time. Now, it may happen that the consumer is uncertain about which of the various options fits him best and needs to test some of them. In this case, since there is no cost of switching between the various plans offered by each service, it makes sense to regard the initial selection of rental service as a choice between menus, and the selection of a plan as a choice from a menu made after some uncertainty is resolved. Thus introducing uncertainty in a model that attempts to capture the agent's behavior is necessary.

For this model, following Kreps (1979), we will identify a menu with an ex ante *observable* action that after some subjective uncertainty is resolved will make a certain set of outcomes available ex post. The observability of these ex ante actions renders the preference over menus a *revealed* preference; thus, we can take this to be our primitive in the uncertainty setting. The *reference dependent representation under uncertainty* that we will axiomatize is the following:

$$V(A) = \int_S \left[\max_{z \in A} U(z,s) \right] \mu(ds) - \theta \min_{x \in A} \left[\int_S U(x,s) \mu(ds) \right] \tag{1}$$

where S is a state space capturing the subjective uncertainty with μ a positive measure over S, $U(z,s)$ is the ex post state utility of option z in state s and θ is a parameter that measures the strength of the behavioral bias.[3] The space S will satisfy an additional condition that will specify that the decision maker does not reverse or almost reverse his ex ante preferences. Thus, while we allow for the presence of uncertainty, we do restrict attention to those applications in which there exists some underlying phenomenon that makes the ex ante preferences relevant for the ex post stage. The behavioral bias is identifiable only in those applications in which this condition is satisfied. We study the behavioral implications of both a finite and an infinite state space S. An infinite state space appears, for instance, in models in which the individual has a continuous distribution of the ex post tastes over the characteristics of the available options.

[3] Note that the reference point is the *ex ante* least preferred option from within the menu.

Now, it is straightforward to see that (1) can be written as:

$$V(A) = (1-\theta)\int_S \left[\max_{z\in A} U(z,s)\right]\mu(ds) + \theta\int_S \left[\max_{z\in A} U(z,s) - \min_{x\in A} U(x,s)\right]\mu(ds) \tag{2}$$

Thus, the preference for a menu is determined by the combination of a normative component and a behavioral bias component. The normative component is the weighted average of the utilities of the normative best options from the menu in each of the possible ex post states. The weighting factors are the subjective probabilities of these ex post states. The behavioral bias component is the weighted average of the difference in utilities between the normatively ex post best options in the menu and the normatively ex ante worst element in the menu. The second component is a measure of the increase in the relative attractiveness of the better options from a menu generated by the presence of the inferior option against which they are compared.

We show that the main axiom that captures the departure from rationality in the certainty model from Barbos(2009) is almost sufficient to deliver the context effects representation under uncertainty when added to the standard axioms from DLR (2001). More precisely, the axiom is sufficient for the case of a finite state space. When the state space is infinite, for the behavioral bias to be identifiable, an additional simple axiom is required. This additional axiom imposes the existence of a pair consisting of a menu A and a lottery y such that y does not provide any ex post flexibility to a decision maker that was faced initially with the menu A.This axiom is equivalent to the condition that the decision maker does not reverse or almost reverse his ex ante preferences. Under this assumption, the effect of the behavioral bias can then be measured by studying the effect of expanding the menu A with the lottery y. As the ex post preferences are not observable, the existence of this pair is imposed through some ex ante behavioral implications.[4]

Preferences over menus were considered for the first time by Kreps (1979). He identified an act with the choice of a set of future options out of which at a later stage the decision maker chooses his most preferred element. He interpreted the agent's preference for the flexibility offered by the menu as being generated by some underlying *subjective* uncertainty that will be resolved between the moment when the choice of the menu is made and the moment when the choice from the menu is made. This allowed him to show that under sufficiently weak conditions, the decision maker behaves as if the uncertainty were described by a subjective state space, where each state is identified with an ex post subjective utility. The preferences that we study in this paper belong to the class of preferences modeled by DLR (2001). DLR (2001) considered menus of lotteries instead of menus of deterministic bundles; this allowed

[4]When the state space is finite, we show that a pair (A, y) with the required property always exists and thus this additional axiom is not necessary.

restricting the ex post state utilities to be of the expected utility form. This addressed the issue of the nonuniqueness of the subjective state space characteristic to the Kreps representation. Also, unlike Kreps(1979), DLR (2001) allowed for subjective states of negative measure to capture not only a preference for flexibility but also a preference for commitment. There is a large body of literature that built on the class of preferences introduced by DLR (2001). Gul and Pesendorfer(2001) were the first to give meaning to the abstract subjective states derived in earlier papers. Thus, they imposed conditions on preferences such that the resulting state space consists of one state of negative measure representing a temptation preference and one state of positive measure representing the second period preferences which combine a normative preference and the temptation preference. This combination of normative and temptation preferences has been implemented in the meantime in other papers to model various behavioral biases, such as non-bayesian updating, cognitive dissonance, etc. For examples, see Epstein and Kopylov (2007) or Kopylov and Noor (2009). In another direction, the preference for commitment has been interpreted in Sarver(2008) not as being driven by the presence of temptation but by the anticipation of regret.

The rest of the paper is organized as follows. In section 2 we present the basic assumptions common to most of the literature on preferences over menus. Also, in Section 2 we define the representation in our model and compare it with other representations that built on the DLR (2001) framework. In section 3 we present our additional axioms and the main results which state the equivalence between the axioms and the representations, while section 4 concludes the paper. Most proofs are relegated to the Appendix.

2 The Framework and the Representations

Let Z be a finite space of *outcomes* or prizes and let $\Delta(Z)$ denote the set of probability measures on Z endowed with the topology of convergence in distribution. Let $K(\Delta(Z))$ denote the collection of all nonempty closed subsets of $\Delta(Z)$. Endowing $K(\Delta(Z))$ with the Hausdorff topology we make it a compact metric space. Elements of $\Delta(Z)$ will be called *lotteries* and will be denoted by x, y, z, etc., while the typical elements $K(\Delta(Z))$ will be called *menus* and will be denoted by A, B, C, etc. The decision maker is assumed to have a *revealed* preference relation $\succeq$ over the elements in $K(\Delta(Z))$. For any two menus $A, B \in K(\Delta(Z))$ and any $\alpha \in [0, 1]$, define their convex combination as $\alpha A + (1-\alpha)B \equiv \{z \in \Delta(Z) : z = \alpha x + (1-\alpha)y$, for some $x \in A$ and $y \in B\}$.

We will impose throughout the paper the following standard axioms on the preference. For a detailed interpretation and motivation of these axioms, see DLR(2001).

Axiom 1 ***(Weak Order).*** *$\succeq$ is a complete and transitive binary relation.*

Axiom 2 ***(Continuity).*** *For any $A \in K(\Delta(Z))$, the upper and lower contour sets, $\{B \in K(\Delta(Z)) : B \succeq A\}$ and $\{B \in K(\Delta(Z)) : B \preceq A\}$, are closed in the Hausdorff metric topology.*

Axiom 3 ***(Independence).*** *For all $A, B, C \in K(\Delta(Z))$ and any $\alpha \in (0,1)$, $A \succ B$ implies $\alpha A + (1-\alpha)C \succ \alpha B + (1-\alpha)C$.*

The representation in (1) is a particular case of the *additive expected utility representation* as defined and axiomatized for the first time by DLR (2001). We will also frequently refer to it throughout as the *DLR representation.*

Definition 4 *An* ***additive expected utility representation*** *of $\succeq$ is a nonempty possibly infinite set S, a state dependent utility function $U : \Delta(Z) \times S \to \mathbb{R}$ and a finitely additive signed Borel measure μ on S, such that $V : K(\Delta(Z)) \to \mathbb{R}$, defined for all $A \in K(\Delta(Z))$ by*

$$V(A) = \int_S \left[\max_{z \in A} U(z,s) \right] \mu(ds) \tag{3}$$

is continuous and represents $\succeq$ and each $U(\cdot, s)$ is an expected utility function in that for each $s \in S$ there exists $u_s : Z \to \mathbb{R}$ such that $U(z,s) = z \cdot u_s$.

Following DLR (2001), we allow for the uncertainty to be *completely subjective.* Thus, besides allowing for a subjective distribution over the ex post contingencies as in standard Savage type models, we also allow for the actual space of ex post contingencies to be subjective. See DLR (2001) for further details. Therefore, a *state* in the above representation can be uniquely identified by the corresponding ex post state utility.

For the case when the state space S is finite the representation can be equivalently written as:

$$V(A) = \sum\nolimits_{s \in S} \left[\max_{z \in A} U(z,s) \right] \mu(s) = \sum\nolimits_{s \in S^+} \left[\max_{z \in A} u_s(z) \right] - \sum\nolimits_{s \in S^-} \left[\max_{z \in A} u_s(z) \right] \tag{4}$$

where $S^+ \equiv \{s \in S : \mu(s) > 0\}$ and $S^- \equiv \{s \in S : \mu(s) < 0\}$ and $u_s(\cdot) \equiv |\mu(s)| U(\cdot, s)$. In writing the above we used the fact that the measure over states and the state utility are not separately identified in models of state-dependent utility, so they can be combined together.

Note that the above definition allows the measure μ over the states to be signed. DLR (2001) call *positive states* and *negative states*, the states in the support of the positively signed and respectively negatively signed components of μ. Intuitively, as stated in DLR (2001), the positive states would reveal the agent's desire for flexibility, while the negative states would reveal his desire for commitment. In our paper, unlike the other papers building on the DLR (2001) framework, the agent is assumed to not have any kind of commitment issues. Therefore we assume throughout an additional axiom on preferences called Monotonicity, which imposes that weakly larger sets in the partial order given by inclusion be weakly preferred by the decision maker.[5] This is a condition consistent with the assumption of the agent not experiencing commitment problems.

Axiom 5 ***(Monotonicity).*** *For all $A, B \in K(\Delta(Z))$ with $A \subseteq B$, we have $B \succeq A$.*

DLR (2001) and Dekel, Lipman, Rustichini and Sarver(2007) prove the following result.

Theorem 6 *When the set Z is finite, the preference $\succeq$ has an additive expected utility representation with a measure μ which is always positive if and only if it satisfies Weak Order, Independence, Continuity, and Monotonicity.*

The effects of imposing Monotonicity on preferences are the following. Firstly, the axiom insures that the measure over the states from the representation is everywhere positive. Secondly, it allows us to obtain a stronger property of the measure, that is σ-additivity instead of finite additivity as in DLR (2001). Finally, Dekel, Lipman, Rustichini and Sarver (2007) show that if Monotonicity is not imposed, the Continuity axiom as presented above needs to be strenghtened to an axiom which they call Strong Continuity in order to get the additive expected utility representation with a signed measure. The additional condition on preferences that is needed delivers the Lipschitz continuity of the representation. Here, since we do assume Monotonicity, we may impose the weaker continuity condition given by the Continuity axiom presented above.

We will also consider the case when the state space from the DLR representation is finite. A necessary and sufficient condition to obtain a finite state space was found in Dekel, Lipman and Rustichini (2009). The authors call this additional axiom Finiteness.[6]

Axiom 7 ***(Finiteness)*** *Every menu $A \in K(\Delta(Z))$ has a finite critical set, where a critical set of a menu A is a any set A' such that for all B with $A' \subseteq hull(B) \subseteq hull(A)$ we have $B \sim A$.*

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[5]The Monotonicity axiom is part of the axiomatization of the preference for flexibility in Kreps (1979).

[6]$hull(A) = \{z \in Z : z = \sum_{i=1}^{k} \lambda_i z_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$ and $z_i \in A\}$ denotes the convex hull of a set A.

Dekel, Lipman and Rustichini (2009) prove the following result.

Theorem 8 *When the set Z is finite, the preference $\succeq$ has an additive expected utility representation with a measure μ which is always positive and with a finite state space S if and only if it satisfies Weak Order, Independence, Continuity, Monotonicity and Finiteness.*

The Reference Dependent Representation under Uncertainty

The specifics of the representation from DLR (2001) require a number of normalizations. First, as explained in DLR (2001) the state space is just an index set that allows reference to different ex post preferences. Moreover, the ex post state utilities which are assumed to be of the expected utility form are identified only up to affine transformations, so we follow the approach in DLR (2001) and restrict the state space S to the set of normalized utilities

$$S^N \equiv \left\{ s \in \mathbb{R}^N : \sum\nolimits_{k=1}^{N} s^k = 0 \text{ and} \sum\nolimits_{k=1}^{N} (s^k)^2 = 1 \right\}. \tag{5}$$

Throughout the rest of the paper we use $s \in S^N$ to refer both to a second period contingency as well as to the normalized expected utility function representing the preferences in that state. Thus, the utility of $x \in \Delta(Z)$ in state s will be $U(x, s) = x \cdot s = \sum\nolimits_{k=1}^{N} x^k s^k$, where $s = (s^1, ..., s^N) \in S^N$ is the normalized expected utility function that represents the state s preferences. Note now that the restrictions of the Weak Order, Continuity and Independence axioms to $\Delta(Z)$ imply by standard results the existence of an expected utility function $v(\cdot)$ that represents the restriction of $\succeq$ to $\Delta(Z)$. Sarver(2008) shows that since S^N contains the normalization of any affine function on $\Delta(Z)$, there exists $s_* \in S^N$ and $\lambda \geq 0$ such that $v(x) = \lambda \sum\nolimits_{k=1}^{N} x^k s_*^k$ for all $x \in \Delta(Z)$.

We define now formally a normalized representation of the preferences exhibiting the reference point bias. As mentioned in the Introduction, we assume that in the second period, after the uncertainty is resolved the decision maker cannot reverse *or almost* reverse his ex ante tastes. Denote the ball of radius ε around s where $\varepsilon > 0$ and $s \in S$ by $N_\varepsilon(s) \equiv \{s' \in S : d(s', s) < \varepsilon\}$ where $d(\cdot, \cdot)$ is the usual Euclidean metric in $\mathbb{R}^N$.

Definition 9 *Let Z be any finite set. A* ***normalized reference-dependent representation under uncertainty*** *of $\succeq$ consists of a nonempty possibly infinite measurable set $S \subset S^N$, a Borel measure μ on S^N, with S being the unique support*[7] *of μ and a constant $\theta \in (0, 1)$, such that*

[7] *The support of a Borel σ-additive measure μ, if it exists, is a closed set, denoted S, satisfying: (1) $\mu(S^c) = 0$;*

(i) $V : K(\Delta(Z)) \to \mathbb{R}$, defined for all $A \in K(\Delta(Z))$ by

$$V(A) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu(ds) - \theta \min_{x\in A} \left[\int_S (x \cdot s)\mu(ds)\right] \tag{6}$$

represents the preference $\succeq$;

(ii) the utility of a lottery $x \in \Delta(Z)$ in state $s \in S$ is $x \cdot s$;

(iii) if $s_* \in S^N$ is the normalized utility that represents the restriction of $\succeq$ to $\Delta(Z)$ then there exists $\varepsilon > 0$ such that $S \subset S^N \backslash N_\varepsilon(-s_*)$.[8]

We emphasize that besides S and μ which are the usual elements in a normalized DLR representation, the parameter θ and the restriction *(iii)* on the set of ex post utility functions will also be deduced from preference as a part of the representation.

Note that the functional form in (6) for $V(\cdot)$ can be rewritten as:

$$V(A) = \int_S \left[\max_{z\in A}(z \cdot s)\right] \mu(ds) + \max_{x\in A} \theta \left[-\int_S (x \cdot s)\mu(ds)\right], \tag{7}$$

and thus our representation is indeed a particular form of an additive expected utility representation with all states having associated a positive measure. The condition *(iii)* on the set of ex post utilities allows identification of the behavioral bias modeled by our representation with the reference point bias. Note that (6) implies that the ex ante preferences over singletons are represented by the utility function $v(x) \equiv (1-\theta)\int_S (x \cdot s)\mu(ds)$. By inspecting (7), it is clear that the preferences represented by $v^-(x) \equiv -\int_S (x \cdot s)\mu(ds)$ could constitute just another ex post state in a DLR(2001) framework with the property that these ex post preferences are exactly the reverse of the ex ante preferences over singletons. We rule out this possibility by making the arguably reasonable assumption that in the second period, after the uncertainty is resolved, the decision maker cannot reverse *or almost* reverse his ex ante tastes. This is done by the identification of the term $\max_{x\in A} \theta \left[-\int_S (x \cdot s)\mu(ds)\right]$ from the equivalent representation in (7) with the impact of a reference-point bias and by the condition *(iii)* from Definition 9.

In the remaining of this Section we present the particular structure imposed on the ex post states of the DLR representation by various papers that built on that framework to underline

and (2) If G is open and $G \cap S \neq \varnothing$, then $\mu(G \cap S) > 0$. Theorem 10.13 in Aliprantis and Border (1999) shows that if the underlying topological space on which μ is defined is second countable or if μ is tight, then μ has a (unique) support. In our case, S^N is clearly second countable, so the definition is correct.

[8] *When the state space is finite, condition (iii) can be written as $-s_* \notin S$. Also, note that $-s_* \in S^N$.*

the differences between these representations and ours.[9] Note firstly from (7) that in our model there exist no negative states and there exists one positive state of strictly positive measure having the corresponding utility a negative affine transformation of the utilities of the rest of the states.

In Gul and Pesendorfer (2001) the equivalent DLR representation is the following:

$$V(A) = \max_{x \in A} u_1(x) - \max_{y \in A} u_2(y) \tag{8}$$

where u_1 is the utility that represents the second period preference relation and u_2 is the temptation component of these second period preferences. Therefore, in this representation there is one positive state and one negative state with no particular mathematical relation between them.

In Sarver (2008) the equivalent DLR representation of his *regret representation* is:

$$V(A) = \max_{z \in A} \left[(1+K) \int_S U(z,s)\mu(ds) \right] - \int_S \left[\max_{x \in A} KU(x,s) \right] \mu(ds) \tag{9}$$

where $K \geq 0$. Thus, in this case there is a number, possibly infinite of negative states and one positive state whose corresponding state utility is a positive affine transformation of the utilities corresponding to the negative states.[10]

The equivalent representation from Dekel, Lipman and Rustichini (2009) of what they call the *temptation representation* is:

$$V(A) = \sum\nolimits_{s \in S} \left[\max_{z \in A} U(z,s) \right] \mu(s) - \sum\nolimits_{s \in S} \left\{ \sum\nolimits_{j \in J_s} \left[\max_{y \in A} \overline{U}(y,j) \right] \right\} \mu(s) \tag{10}$$

which is a generalization of the one from Gul and Pesendorfer (2001) in the sense that it assumes multiple ex post states and for each ex post state multiple ex post temptations. This representation has a number of positive states and for each positive state a number of corresponding negative states with some underlying structure among them. Unlike the other representations, in Dekel, Lipman and Rustichini (2009) the state space is assumed to be finite.

[9] We emphasize that the actual representation from each of the papers is different from that presented here as being its equivalent DLR representation in order to capture the corresponding behavioral trait that is analyzed in each paper.

[10] The regret representation as defined in Sarver (2008) has the Borel measure μ positive. However, as mentioned in that paper as well, the equivalent DLR representation is signed and has a negative component.

3 The Axioms and the Main Results

As noted before, the representation in (6) is a special type of an additive expected utility representation with a positive measure. Thus, it will be necessary that the preference satisfy Weak Order, Continuity, Independence and Monotonicity. Two additional axioms will be sufficient for the preference $\succeq$ to have a reference-dependent representation when combined with the above four axioms. The first additional axiom captures the departure from the standard model of rationality that we study in this paper.

Axiom 10 ***(CEB: Context-Effects Bias) :*** *For any pair* $(A, x) \in K(\Delta(Z)) \times \Delta(Z)$*, such that* $\{y\} \succ \{x\}$ *for all* $y \in A$*, we have* $A \cup \{x\} \succ A$*.*

Axiom CEB states that if the decision maker has the set of possible choices A expanded by adding an option, say a singleton $\{x\}$, which from an ex ante point of view is strictly worse that the rest of the elements in the menu, then the agent will strictly prefer the new expanded set $A \cup \{x\}$ to the initial one A. The motivation for this preference is given by the fact that the inferior lottery x will be chosen as the new reference point and thus the overall attractiveness of the better options from the menu will increase. Note that since the preferences in the text of the axiom are strict, Axiom CEB imposes that the agent has a *strict* preference for menus having additional strictly inferior outcomes. This corresponds to the restriction that $\theta > 0$ in the representation in (6).

Axiom CEB provides the departure from the standard model of rationality as suggested by the presence of a behavioral bias. However, when allowing for an *infinite* state space this departure is identified only when combined with the Axiom CEB-2 presented below. This is because, when allowing for the presence of uncertainty, it may happen that an ex ante inferior option still provides some ex post flexibility to the elements of a set, and thus the pattern of choice suggested by Axiom CEB is valid without assuming any reference point bias. In order to have a departure from the standard rational preferences, there must exist at least one set A and at least one lottery y strictly worse from an ex ante point of view to all elements of A such that in any ex post state there exists an element in A that is at least as preferred as y. Then, imposing Axiom CEB to the sets A and $A \cup \{y\}$ would provide the departure. Now, in the case of a finite state space, the pair (A, y) with the desired properties always exists when we maintain the assumption that the second period preferences cannot be exactly the reversed ex ante preferences.[11] In the case of an infinite state space, the existence of such a pair (A, y) will be imposed by Axiom CEB-2 below. As the ex post preferences are not observable,

[11] See the necessity part of the proof of Theorem 14 below for a formal argument.

this is done by imposing a natural implication of the existence of the pair (A, y) on the ex ante preferences. Before presenting Axiom CEB-2 we will make a remark that suggests that imposing axiomatically the existence of such a pair is correct when the preferences that we are studying are represented by a utility function as in (6).

Remark 11 *When the preferences $\succeq$ admit a normalized reference-dependent representation as in (6), there exist a set $A \in K(\Delta(Z))$ with $A \subset int(\Delta(Z))$ and a lottery $y \in \Delta(Z)$ such that: (i) for any $x \in A$ we have $x \cdot s_* > y \cdot s_*$ and (ii) for any $s \in S$ there exists $x \in A$ such that $x \cdot s > y \cdot s$.*

Proof. See Appendix A1 for some notation on support functionals and then Lemma 29.

The second non standard axiom for the case of an infinite subjective state space is the following.

Axiom 12 ***(CEB-2):*** *There exists a set $A \in K(\Delta(Z))$ with $A \subset int(\Delta(Z))$ and a set $B \in K(\Delta(Z))$ with $A \subset B$ and $\{x\} \succ \{y\}$ for all $x \in A$ and some $y \in B$, such that for all lotteries $z \in \Delta(Z)$ with $\{y\} \succeq \{z\}$ for all $y \in B$, we have $B \cup \{z\} \sim A \cup \{z\}$.*

To see the motivation for this axiom, consider a pair $\{A, y\}$ such that y does not provide any ex post flexibility to A. Then, on the one hand, by choice of y, it provides no ex post flexibility to the set $A \cup \{z\}$. On the other, since y is weakly preferred to z from an ex ante point of view, y will not be the reference point chosen from $A \cup \{y, z\}$. Since under no circumstances the agent would choose y over the elements in $A \cup \{z\}$, he is as well off having at hand the menu $A \cup \{z\}$ as he is having the larger menu $A \cup \{y, z\}$. Therefore, we impose the required indifference. The condition that B is some superset of A instead of $A \cup \{y\}$ is a nonessential weakening of the axiom meant only to simplify the notation in the proof of the main theorem when we characterize a menu by the corresponding support functionals.

We mention that while the condition from the text of the axiom is valid for an infinite number of sets in $K(\Delta(Z))$, we do not impose this condition to hold for *all* these sets simply because there may exist $A \in K(\Delta(Z))$ such that there exists no lottery $y \notin A$ that does not provide any ex post flexibility to A. Also, we do not impose the indifference of the sets constructed as in the text of Axiom CEB-2, but for *all* B that contain a lottery y such that y is strictly less preferred to all elements of A from the ex ante point of view. This is because, as mentioned earlier, a lottery which is ex ante inferior to all $x \in A$ could turn out ex post to be better to all elements of A and then $B \cup \{z\} \succ A \cup \{z\}$. The weak restriction imposed in the text of the Axiom that the condition is valid for at least one pair (A, B) is sufficient to obtain the desired representation for *all* sets due to the additional structure provided by the EU form of the ex post utilities. Note also that we impose that an element of B be strictly worse than all elements of A. Without this condition, it is clear that the axiom would not have any bite since we could always let B be exactly the set A.

Now we are ready to state the main result of the paper:

Theorem 13 *The preference $\succeq$ has a representation as in (6) if and only if it satisfies Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Axiom CEB-2.*

Proof. See Appendix A1 and Appendix A2.

While the complete proof of Theorem 13 can be found in the Appendix, we present here for intuition a sketch of this proof. We start by showing how the argument goes if the state space S were *finite*. This reveals how Axiom CEB and Axiom CEB-2 work to give us the representation.

Weak Order, Continuity, Independence and Monotonicity imply that the preference over menus has the following representation, with μ a positive measure:

$$V(A) = \sum\nolimits_{s \in S} \left[\max_{z \in A} (z \cdot s) \right] \mu(s), \text{ for all } A \in K(\Delta(Z)) \tag{11}$$

Denote by $v(\cdot)$ the restriction of $V(\cdot)$ to singletons. Thus, v represents the ex ante preference over lotteries and by (11) we have: $v(z) = \sum\nolimits_{s \in S} (z \cdot s)\mu(s)$, for all $z \in \Delta(Z)$. Moreover, as claimed in Section 2, since v is an affine function, there exists $s_* \in S^N$, such that $v(z) = \lambda(z \cdot s_*)$ for all $z \in \Delta(Z)$. Now, recall that the representation in (6) is a particular case of a DLR representation in which the utility associated with one of the states is a negative affine transformation of the utilities associated with the rest of the states. We will prove here that under Axiom CEB and Axiom CEB-2, the representation in (11) must have exactly that structure on the ex post states, which comes down to showing that $-s_* \in S$. The rest of the proof consists of showing that given that structure, the representation in (11) can be written as in (6). This second part of the proof is just simple algebra manipulations and its presentation is relegated to the Appendix.

Let $A \in int(K(\Delta(Z)))$ and $B \in K(\Delta(Z))$ be as in the definition of Axiom CEB-2, that is $A \subset B$ and $\{x\} \succ \{y\}$ for all $x \in A$ and some $y \in B$. Then by Axiom CEB and Monotonicity we have $B \succ A$; using the representation in (11) it follows that:

$$\sum\nolimits_{s \in S} \left[\max_{x \in B} (x \cdot s) \right] \mu(s) > \sum\nolimits_{s \in S} \left[\max_{x \in A} (x \cdot s) \right] \mu(s). \tag{12}$$

Since the measure μ is positive, (12) implies that there must exist $s' \in S$ such that $\max\limits_{x \in B}(x \cdot s') > \max\limits_{x \in A}(x \cdot s')$. Denote the strict lower contour sets associated with an expected utility function s and a lottery $y \in \Delta(Z)$ by $L_s(y) \equiv \{x \in \Delta(Z) : x \cdot s < y \cdot s\}$. Then, given the linearity of the utility functions, a state utility s will be a negative affine transformation of s_* if and only if $L_s(y) \cap L_{s_*}(y) = \varnothing$ for all $y \in \Delta(Z)$. Assume by contradiction that there is no such state utility as the one that we are looking for, that is $L_s(y) \cap L_{s_*}(y) \neq \varnothing$ for all $s \in S$. We show that in this case, if Axiom CEB holds then Axiom CEB-2 must be violated.

Take some $y \in B$ such that $x \cdot s_* \geq y \cdot s_*$ for all $x \in B$ and then some $z \in L_{s'}(y) \cap L_{s_*}(y)$ which is nonempty by the contradiction assumption. Then, since $y \in B$ we will have $\max\limits_{x \in B}(x \cdot s') > z \cdot s'$ so $\max\limits_{x \in B \cup \{z\}}(x \cdot s') = \max\limits_{x \in B}(x \cdot s') > \max\left(\max\limits_{x \in A}(x \cdot s'), z \cdot s'\right) = \max\limits_{x \in A \cup \{z\}}(x \cdot s')$. Therefore:

$$V(B \cup \{z\}) = \sum\nolimits_{s \in S \setminus \{s'\}} \left[\max_{x \in B \cup \{z\}}(x \cdot s)\right]\mu(s) + \mu(s') \max_{x \in B \cup \{z\}}(x \cdot s') >$$

$$\sum\nolimits_{s \in S \setminus \{s'\}} \left[\max_{x \in B \cup \{z\}}(x \cdot s)\right]\mu(s) + \mu(s') \max_{x \in A \cup \{z\}}(x \cdot s') \geq$$

$$\sum\nolimits_{s \in S \setminus \{s'\}} \left[\max_{x \in A \cup \{z\}}(x \cdot s)\right]\mu(s) + \mu(s') \max_{x \in A \cup \{z\}}(x \cdot s') = V(A \cup \{z\}).$$

Therefore, $V(B \cup \{z\}) > V(A \cup \{z\})$ so there exists z with $x \cdot s_* > z \cdot s_*$ for all $x \in B$ such that $B \cup \{z\} \succ A \cup \{z\}$ which violates Axiom CEB-2 as claimed. In conclusion, there must exist a state $s \in S$ that is a negative affine transformation of s_*.

Proving the necessity of Axiom CEB is straightforward. To see that Axiom CEB-2 must also be satisfied when the preferences can be represented as in (6) with a finite state space, take some lottery $y \in int(\Delta(Z))$ and for each $s \in S$, take $x_s \in H_s(y) \cap L_{-s_*}(y) \cap \Delta(Z)$, where $H_s(y) \equiv \{x \in \Delta(Z) : x \cdot s = y \cdot s\}$. Let $A \equiv \cup_{s \in S} x_s$ and $B \equiv A \cup \{y\}$. Then, by the choice of the set A, we will have $\{x\} \succ \{y\}$ for all $x \in A$. On the other hand, for any $z \in L_{s_*}(y) \cup H_{s_*}(y)$ we have

$$V(A \cup \{z\}) = \sum\nolimits_{s \in S} \left[\max_{w \in A \cup \{z\}}(w \cdot s)\right]\mu(ds) - \theta \min_{w \in A \cup \{z\}} \left[\sum\nolimits_{s \in S}(w \cdot s)\mu(ds)\right] =$$

$$\sum\nolimits_{s \in S} \left[\max\left(\max_{w \in A}(w \cdot s), z \cdot s\right)\right]\mu(ds) - \theta\left[\sum\nolimits_{s \in S}(z \cdot s)\mu(ds)\right] =$$

$$\sum\nolimits_{s \in S} \left[\max\left(\max_{w \in B}(w \cdot s), z \cdot s\right)\right]\mu(ds) - \theta \min_{w \in B \cup \{z\}} \left[\sum\nolimits_{s \in S}(w \cdot s)\mu(ds)\right] = V(B \cup \{z\}).$$

In the above we used the fact that the restriction of the representation to singletons implies $\{x\} \succ \{y\} \Leftrightarrow (1-\theta)\sum_{s\in S}(x\cdot s) > (1-\theta)\sum_{s\in S}(x\cdot s)$ and the fact that since for each $s \in S$, there exists $x_s \in A \cap H_s(y)$ we have $\max_{w\in A\cup\{y\}}(w\cdot s) = \max_{w\in A}(w\cdot s)$.

While in the case of a finite state space, it is sufficient to show that $-s_*$ must be one of the states from the DLR representation, for the case of an infinite state space this is not enough. This is because the state $-s_*$ can always be added to the state space and assign a measure zero. Thus, for the proof of the sufficiency of the axioms for the infinite state space case, the main challenges are to show that the DLR measure of $-s_*$ is strictly positive and to show the existence of the empty neighborhood of $-s_*$. The first straightforward step in the general proof is to use Axiom CEB to assert the existence of a set $\widehat{S}_1 \subset S$ with $\mu(\widehat{S}_1) > 0$, such that $\max_{x\in B}(x\cdot s') > \max_{x\in A}(x\cdot s')$ for all $s' \in \widehat{S}_1$ (see Lemma 18). The main goal of the rest of the proof of sufficiency is to construct a set $\widehat{S}_5 \subset \widehat{S}_1$ with $\mu(\widehat{S}_5) > 0$ and find a lottery $z \in \Delta(Z)$, such that $\max_{x\in B}(x\cdot s') > z\cdot s'$ for all $s' \in \widehat{S}_5$ and $\min_{x\in B}(x\cdot s_*) > z\cdot s_*$. Then, an argument similar to the one from the case of a finite state space will complete the proof by showing that Axiom CEB-2 must be violated.

Now, note the following fact.

Fact: If $\cap_{s\in\{s_1,\dots,s_n\}}L_s(y) \neq \varnothing$ and $\left(\cap_{s\in\{s_1,\dots,s_n\}}L_s(y)\right) \cap L_{s_{n+1}}(y) = \varnothing$ for some $y \in \Delta(Z)$, then $s_{n+1} \in hull(-s_1, \dots, -s_n)$.

Thus, if we could find a set of $N-1$ linearly independent ex post states in $\widehat{S}_1$with:

$$\mu(int\,(hull(\{s_1, \dots, s_{N-1}\}))) > 0 \tag{13}$$

we could then first argue inductively using the above Fact that:

$$\cap_{s\in\{s_1,\dots,s_{N-1}\}}L_s(y) \neq \varnothing \tag{14}$$

(see Lemma 25) and then also argue that $\cap_{s\in hull(\{s_1,\dots,s_{N-1}\})}L_s(y) \neq \varnothing$ (see Lemma 26). In addition, if we ensure that:

$$s_* \notin hull(\{-s_1, \dots, -s_{N-1}\}) \tag{15}$$

then again by the above Fact, we would also have that $\left(\cap_{s\in\{s_1,\dots,s_{N-1}\}}L_s(y)\right) \cap L_{s_*}(y) \neq \varnothing$ and thus be able to take:

$$z \in \left(\cap_{s\in hull(\{s_1,\dots,s_{N-1}\})}L_s(y)\right) \cap L_{s_*}(y) \tag{16}$$

Finally, by letting:

$$\widehat{S}_5 \equiv hull(\{s_1, \dots, s_{N-1}\}) \tag{17}$$

and selecting z as above, we would achieve the desired result. However, there are two problems with this approach that do not allow the argument to go through as stated. Firstly, since Axiom CEB-2 does not state the existence of a *lottery* y, but of a *set* B, the above mentioned Fact is not true if we replace y with the set B. Secondly, since S^N is the subset of *normalized* utilities, finding $N-1$ linearly independent values with the desired properties is not immediately obvious, if possible at all.

In order to solve the first issue, we will expand the set $\Delta(Z)$ to the smallest affine set that contains it, that is to $\Omega \equiv \{z \in \mathbb{R}^N : \sum_{i=1}^N z_i = 1\}$. Then, by defining the expanded lower contour sets $L_s(B) \equiv \{y \in \Omega : y \cdot s < z \cdot s \text{ for all } z \in B\}$, we will be able to prove a counterpart of the above Fact (see Lemma 24). While in the end, we will obtain an element z' that belongs to the intersection of these *expanded* sets $\left(\cap_{s \in \widehat{S}_5} L_s(B)\right) \cap L_{s_*}(B)$ (see Lemma 26), since $\Delta(Z)$ has a non empty algebraic interior in Ω, we will be able to select a lottery $z \in \Delta(Z)$ that will have the desired properties (see first Lemma 17 to see why we may consider without loss of generality that $B \subset int(\Delta(Z))$ in the text of Axiom CEB-2 and then Lemma 27).[12] To solve the second problem, we will expand the set of ex post utilities from the normalized set S^N to the set $P^N = \left\{s \in \mathbb{R}^N : \sum_{k=1}^N s^k = 0\right\}$. A first effect of this expansion is that the counterpart of the above Fact will have now to be written in terms of *convex cones* instead of convex hulls (see Lemma 24). But since $z' \in \left(\cap_{s \in \{s_1,\dots,s_{N-1}\}} L_s(B)\right) \cap L_{s_*}(B)$ still implies $z' \in \left(\cap_{s \in cone(\{s_1,\dots,s_{N-1}\})} L_s(B)\right) \cap L_{s_*}(B)$ the argument will continue to go through (see Lemma 26). Therefore, we need to find $N-1$ linearly independent utilities in P^N such that $\mu(int\,(cone(\{s_1,...,s_{N-1}\})) \cap \widehat{S}_1) > 0$ and $s_* \notin cone(\{-s_1,...,-s_{N-1}\})$. Then, by letting $\widehat{S}_5 \equiv int\,(cone(\{s_1,...,s_{N-1}\})) \cap \widehat{S}_1$ and z be selected as explained above we would complete the argument. We mention here that considering $N-1$ linearly independent states that would include $-s_*$ would not solve the problem because then the interior of the cone generated by less than $N-1$ states in the $N-1$ dimensional space P^N would be empty and thus of zero measure. Therefore, the need for the more elaborate construction.

Now, using the contradiction assumption $\mu(\{-s_*\}) = 0$, we show in Lemma 19 and Lemma 20 that there exists a set $\widehat{S}_2 \subset \widehat{S}_1$ such that $\mu(\widehat{S}_2) > 0$ and $-s_* \notin cone\left(\widehat{S}_2\right)$. Next, Lemma 21 and Lemma 22 show that there exists a set $\widehat{S}_3 \subset \widehat{S}_2$ and $\epsilon > 0$ such that $-s_* \notin cone(\cup_{s \in \widehat{S}_3} \overline{N}_\epsilon(s))$, where $\overline{N}_\epsilon(s)$ is the closed ball of radius ϵ around s. In Lemma 23 we construct the set of linearly independent utilities $\{s_1,...,s_N\}$ with the properties presented above. This is done by covering the compact set $\widehat{S}_3$ with a finite partition extracted from a cover of $\widehat{S}_3$ whose elements are the intersections of this set $\widehat{S}_3$ with the convex cones generated for each $s \in \widehat{S}_3$ by some set of $N-1$ linearly independent utilities $\{s_1^s,...,s_{N-1}^s\}$ with the property that

[12] Note that while $z' \in \left(\cap_{s \in \widehat{S}_5} L_s(B)\right) \cap L_{s_*}(B)$, z will satisfy only the *weaker* set of properties $\max_{x \in B}(x \cdot s') > z \cdot s'$ for all $s' \in \widehat{S}_5$ and $z \in L_{s_*}(B)$. As argued above, this is sufficient to obtain the desired result.

$s \in int\left(cone\left(\{s_1^s, ..., s_{N-1}^s\}\right)\right)$. In addition, since $-s_*$ is sufficiently far away from the set $cone(\cup_{s\in\widehat{S}_3}\overline{N}_\epsilon(s))$, as ensured in the Lemma 22, we will have $s_* \notin cone(\{-s_1, ..., -s_{N-1}\})$. The proof of the existence of the empty neighborhood around $-s_*$ is presented in Theorem 28. It differs from the proof of the fact that $\mu(\{-s_*\}) > 0$ only in the way in which it uses the contradiction assumption in the proof of Lemma 19 as a first step in the construction of the set $\widehat{S}_2$. We will present in the Appendix only the part of the argument in which the two proofs are different.

The necessity part of the proof of Theorem 13 also needs a rather elaborate approach. This is because the infinite set $\cup_{s\in S}x_s$, with x_s chosen as in the intuitive argument presented above, is not necessarily closed and thus not necessarily compact. Thus, we need to take $A = cl(\cup_{s\in S}x_s)$. But then the fact that we select $x_s \in L_{-s_*}(y)$ for each $s \in S$ does not necessarily imply $\{x\} \succ \{y\}$ for all $x \in A$ and this invalidates the required conclusion. Part *(iii)* of the Definition 9 will help overcome this problem but the construction is still not straightforward. The proof of the necessity will share some steps which are similar to steps from the proof of Theorem 16 and those steps will be presented without proof in the Appendix. However, we emphasize here that while in both proofs the initial steps consist of partitioning some compact set of ex post states into a finite number of subsets such that the states contained in each subset share some common properties, there is an important difference in terms of the ultimate goal of these arguments. Thus, in the proof of Theorem 16, which is the result that shows that $\mu(\{-s_*\}) > 0$, the partitioning of $\widehat{S}_1$ was done so that we could in the end claim that one of these subsets, namely $\widehat{S}_5$, must be of strictly positive measure since the measure of the set $\widehat{S}_1$ was strictly positive. In the proof of the necessity of Axiom CEB-2, the goal is to partition the set $S^N \backslash N_\varepsilon(-s_*)$ into a *finite* number of subsets, each sharing some common relevant properties, so that we can resolve the problems raised by the infiniteness of the state space. We defer the presentation of the argument to the Appendix.

Next, we present an additional result that constitutes the representation theorem for the case of a *finite* ex post state space of uncertainty. As argued above, in this case the restriction on preferences given by Axiom CEB-2 is not necessary. To prove the theorem below, it is enough to show that by replacing Axiom CEB-2 with Finiteness, the resulting set of axioms imply Axiom CEB-2. Then the argument from the sketch of the proof of Theorem 13 would complete the proof. Showing that Axiom CEB-2 must be satisfied in this case can be done by following an argument close to the one used above as the intuitive proof of Theorem 13 for the finite case.

Theorem 14 *The preference $\succeq$ has a representation as in (6) with a finite state space if and only if it satisfies Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Finiteness.*

We close this section with a result that states the uniqueness of the representation for the uncertainty model. This result is important because it allows the interpretation the objects of the representation as intended. Thus, the fact that the parameters of the representation are identified ensures that when observing choice, it is feasible to disentangle the impact on behavior of the context effects from the impact of the presence of subjective uncertainty.

Our representation in (6) is identified by the elements of the set (μ, θ) where μ is a probability over the S^N and θ measures the strength of the behavioral bias. The following theorem shows that both μ and θ are identified from preferences.

Theorem 15 *Suppose that (μ_1, θ_1) is a normalized representation of some preferences $\succeq$ satisfying Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Axiom CEB-2. Then, if (μ_2, θ_2) is also a normalized representation of $\succeq$ we must have $\theta_1 = \theta_2$ and $\mu_1 = \mu_2$.*

Proof. See Appendix A3.

4 Conclusion

This paper studies a model of reference-dependent preferences over menus of lotteries. We extend the model of choices from categories in Barbos (2009) to allow for the presence of some underlying subjective uncertainty between the moment of the choice of a menu and the time a specific option within the menu is selected. The axiomatic foundations of this model allow for the disentangling of the context effects bias from the rational desire from flexibility that is usually captured by preferences over menus. We identify a weak condition on the set of ex post preferences under which the two effects are distinguishable from each other. We find the behavioral implications of both a finite and an infinite space of uncertainty.

Appendix

A1. Construction of the state space for the uncertainty model

We present here briefly the construction of the state space from Dekel, Lipman and Rustichini (2001) as we will utilize the concepts introduced there extensively in the rest of the proof.

Firstly, as shown in DLR (2001) under Weak Order, Continuity and Independence any set of lotteries in $\Delta(Z)$ is indifferent to its convex hull. Thus, we can restrict attention to the set of *convex* sets[13] in $K(\Delta(Z))$, which we denote from now on with $\widetilde{K}(\Delta(Z))$. Recall that the number of outcomes in Z is denoted by N and that S^N is the set of normalized expected utility functions on $\Delta(Z)$. Define by $C(S^N)$ the set of real-valued continuous functions on S^N and endow it with the topology given by the sup-norm metric. Embed $\widetilde{K}(\Delta(Z))$ into $C(S^N)$ by identifying each menu with its support function: $A \to \sigma_A$, with $\sigma_A(s) = \max\limits_{x \in A} \sum_{k=1}^N x^k s^k$. It is a standard result that the above mapping is an embedding, one-to-one and monotonic. Thus, for all $A, B \in \widetilde{K}(\Delta(Z))$, $\sigma_A(\cdot) = \sigma_B(\cdot)$ implies $A = B$ and $A \subset B$ implies $\sigma_A \leq \sigma_B$. The order used on $C(S^N)$ is the usual pointwise partial order. Also the support functional is affine, that is: $\sigma_{\beta A + (1-\beta)B} = \beta\sigma_A + (1-\beta)\sigma_B$.

Let C denote the subset of $C(S^N)$ that σ maps $\widetilde{K}(\Delta(Z))$ into, that is $C \equiv \{\sigma_A \in C(S^N) : A \in \widetilde{K}(\Delta(Z))\}$. Using this mapping and the Weak Order and Continuity axioms, DLR (2001) construct the continuous linear functional $W : C \to \mathbb{R}$ that represents the preference $\succeq$ over $\widetilde{K}(\Delta(Z))$:

$$W(\sigma_A) \geq W(\sigma_B) \text{ if and only if } A \succeq B. \tag{18}$$

As in the main text, define $v : \Delta(Z) \to \mathbb{R}$ to be the restriction of W to the set of support functions of the singleton sets: $v(x) \equiv W(\sigma_{\{x\}})$. It can be shown using the linearity of the support functions that v is affine, that is $v(\beta x + (1-\beta)y) = \beta v(x) + (1-\beta)v(y)$. In addition, as mentioned in Section 2, there exists $s_* \in S^N$ and $\lambda \geq 0$ such that $v(x) = \lambda \sum_{k=1}^{N} x^k s_*^k$ for all $x \in \Delta(Z)$.

Dekel, Lipman, Rustichini and Sarver (2007) show in the proof of their Theorem 2 that under Monotonicity, the functional W is increasing on the space $H^* = \{r_1\sigma_1 - r_2\sigma_2 : \sigma_1, \sigma_2 \in C$ and $\sigma_1, \sigma_2 \geq 0\}$ which is dense in $C(S^N)$. Since $f \leq ||f|| \cdot \mathbf{1}$ for any $f \in H^*$, where $\mathbf{1}$ is the function identically equal to 1, by the monotonicity of W we will have $W(f) \leq ||f|| W(1)$ so W is bounded on H^*. Therefore, as in DLR (2001), W can be extended uniquely from C to the whole $C(S^N)$ preserving continuity and linearity. Also, since H^* is dense in $C(S^N)$, it follows immediately that W will be monotone on the whole $C(S^N)$. As in Royden (1988, page 355), W can be decomposed as $W = W^+ - W^-$ where W^+ and W^- are two positive linear functional forms. Using again the monotonicity of W and the definition of W^+ from Royden (1988) it is straightforward to show that $W(\cdot) = W^+(\cdot)$ and $W^-(\cdot) = 0$ on $C(S^N)$.

Then, W is a positive linear functional on $C(S^N)$ so since S^N is compact, the functions in $C(S^N)$ have compact support since closed subsets of compact spaces are compact, so the

[13] Note that $\int_S U(x,s)\mu(ds)$ is a linear function in x so even when A is not convex, the minimum of $\int_S U(x,s)\mu(ds)$ over A will be attained at an element of A. Thus, the reference point will always belong to A.

Riesz-Markov Theorem from Royden (1988, page 352) can be used to write $W(f)$ as an integral of f against a σ-additive positive measure μ over S^N for any $f \in C(S^N)$. In particular,

$$W(\sigma_A) = \int_{S^N} \sigma_A(s)\mu(ds) \text{ for any } A \in \widetilde{K}(\Delta(Z)) \tag{19}$$

This last step delivers the DLR representation of the preference $\succeq$. However, note that we use here a different version of the Riesz Representation Theorem than the one used in DLR (2001). This is because the Monotonicity Axiom makes the functional W positive and thus we can obtain a σ-additive and positive Borel measure as opposed to a finitely additive and signed measure as in DLR (2001). As it will be seen below, the σ-additivity of the measure is necessary both for obtaining our reference-dependent representation as well as for proving the uniqueness of this representation. Next, we will impose the additional restrictions on preferences given by Axiom CEB and Axiom CEB-2 to obtain our specific representation from (6).

A2. Proof of Theorem 13

As a first step in the proof, we will rewrite Axiom CEB and Axiom CEB-2 by using the support functionals and the functional W instead of the preference relation. Note that since $v(x)$ represents the preference over lotteries in $\Delta(Z)$, using the results from Appendix B1 we have:

$$\{x\} \succ \{y\} \Leftrightarrow \lambda \sum_{k=1}^{N} y^k(-s_*^k) \geq \lambda \sum_{k=1}^{N} x^k(-s_*^k) \Leftrightarrow \lambda\sigma_{\{y\}}(-s_*) \geq \lambda\sigma_{\{x\}}(-s_*) \tag{20}$$

Given two sets $A, B \in \widetilde{K}(\Delta(Z))$, if there exists $y \in B$ such that $\{x\} \succ \{y\}$ for all $x \in A$ we will have that $\lambda\sigma_{\{y\}}(-s_*) > \lambda\sigma_{\{x\}}(-s_*)$ for all $x \in A$ so $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$. Thus, in general if there exists a lottery in B that is strictly worse than all lotteries in A we can write this in a compact way as $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$. Similarly, if y is *weakly* worse than all elements in A, we have $\lambda\sigma_B(-s_*) \geq \lambda\sigma_A(-s_*)$. Also note that in order for Axiom CEB-2 to hold, more exactly for a lottery $y \in \Delta(Z)$ to exist such that $\{x\} \succ \{y\}$ for some other $x \in \Delta(Z)$, we need $\lambda > 0$ since otherwise all elements in $\Delta(Z)$ are indifferent to each other. Therefore, under Axiom CEB-2 we have $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$ if and only if $\sigma_B(-s_*) > \sigma_A(-s_*)$. Finally, for any two support functionals $\sigma_A, \sigma_B \in C$, denote their join by $\sigma_A \vee \sigma_B$, that is $(\sigma_A \vee \sigma_B)(\cdot) \equiv \max(\sigma_A(\cdot), \sigma_B(\cdot))$ and note that $\sigma_{A\cup B} = \sigma_A \vee \sigma_B$.

Using these results, the fact that $A \sim hull(A)$ for any $A \in K(\Delta(Z))$ and the fact that $A \subset B$ iff $\sigma_A \leq \sigma_B$, we can write Axiom CEB and Axiom CEB-2 in the following equivalent forms:

Axiom CEB: *For any* $A, B \in \widetilde{K}(\Delta(Z))$, *with* $\sigma_A \leq \sigma_B$ *such that* $\lambda\sigma_B(-s_*) > \lambda\sigma_A(-s_*)$, *we have* $W(\sigma_B) > W(\sigma_A)$.

Axiom CEB-2: *There exists a set* $A \in \widetilde{K}(\Delta(Z))$ *such that* $A \subset int(\Delta(Z))$ *and a set* $B' \in \widetilde{K}(\Delta(Z))$ *with* $\sigma_A \leq \sigma_{B'}$ *and* $\lambda\sigma_{B'}(-s_*) > \lambda\sigma_A(-s_*)$, *such that for all lotteries* $z \in \Delta(Z)$ *with* $\lambda\sigma_{\{z\}}(-s_*) \geq \lambda\sigma_{B'}(-s_*)$, *we have* $W(\sigma_{hull(B' \cup \{z\})}) = W(\sigma_{hull(A \cup \{z\})})$.

We introduce now some new notation in addition to the one already presented in Section 3. For any set $B \in \widetilde{K}(\Delta(Z))$, denote its expanded weak lower and upper contour sets corresponding to $s_i \in P^N$ by: $L_{s_i}(B) \equiv \{y \in \Omega : y \cdot s_i \leq z \cdot s_i$ for all $z \in B\}$ and $U_{s_i}(B) = \{y \in \Omega : y \cdot s_i > z \cdot s_i$ for all $z \in B\}$. For $q \in \Delta(Z)$ denote the hyperplane generated by s_i as: $H_{s_i}(q) = \{z \in \Omega : z \cdot s_i = q \cdot s_i\}$. For any set S of points in $P^N \backslash \{0\}$, denote the convex cone and convex hull generated by S with: $cone(S) \equiv \{s \in P^N : s = \sum_{i=1}^{k} \lambda_i s_i$ with $\lambda_i \geq 0$ and $s_i \in S$ for all $i \in \{1, ..., k\}$ and $k \in \mathbb{N}\}$ and $hull(S) \equiv \{s \in P^N : s = \sum_{i=1}^{k} \lambda_i s_i$ with $\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1$ and $s_i \in S$ for all $i \in \{1, ..., k\}$ and $k \in \mathbb{N}\}$.

Proof of Sufficiency in Theorem 13:

We will prove two results, Theorem 16 and Theorem 28, which together will bring us one step away from obtaining the structure on the state space from the DLR representation necessary for writing the representation of $W(\cdot)$ as in (6).

Theorem 16 *Under axioms CEB and Identification we must have* $\mu(\{-s_*\}) > 0$.

Proof. Assume by contradiction that $\mu(\{-s_*\}) = 0$ and take the set set $A \in \widetilde{K}(\Delta(Z))$ given by the Axiom CEB-2 with $A \subset int(\Delta(Z))$ and the superset $B' \in \widetilde{K}(\Delta(Z))$ of A such that there exists $x \in B' \backslash A$ with $\lambda\sigma_{\{x\}}(-s_*) > \lambda\sigma_A(-s_*)$. Note that in order for such a set B' to exist it must be that $\lambda > 0$ so we must also have $\sigma_{\{x\}}(-s_*) > \sigma_A(-s_*)$. We will break up most of the rest of the proof of Theorem 16 into a series of lemmas.

Lemma 17 *There exists a set* $B \in \widetilde{K}(\Delta(Z))$ *with* $A \subset B \subset B' \cap int(\Delta(Z))$ *and* $\sigma_B(-s_*) > \sigma_A(-s_*)$. *Moreover, under Axiom CEB we have* $\int_{S^N} \sigma_B(s)\mu(ds) > \int_{S^N} \sigma_A(s)\mu(ds)$.

Proof. If $x \in int(\Delta(Z))$, then let $B \equiv hull(A \cup \{x\})$. Since B' and $int(\Delta(Z))$ are convex and $A \cup \{x\} \subset B' \cap int(\Delta(Z))$, we have $hull(A \cup \{x\}) \subset B' \cap int(\Delta(Z))$. On the other hand, $\sigma_B(-s_*) = \sigma_{A \cup \{x\}}(-s_*) = \max(\sigma_{\{x\}}(-s_*), \sigma_A(-s_*)) = \sigma_{\{x\}}(-s_*) > \sigma_A(-s_*)$. If $x \notin int(\Delta(Z))$ we will find some $x' \in int(\Delta(Z)) \cap (B' \backslash A)$ with $\sigma_{\{x'\}}(-s_*) > \sigma_A(-s_*)$ and then define $B \equiv$

$hull(A \cup \{x'\})$ and repeat the argument above to prove the first part of the claim. Note that since A is closed it must be that $\sigma_A(-s_*) = \sigma_{\{y\}}(-s_*)$ for some $y \in A$. Also, since $-s_* \in S^N$, we have $-s_* \neq 0$ so $y \notin int(A)$. Take $x' = \frac{1}{2}(x+y)$ and note that $x' \in int(\Delta(Z)) \cap (B' \backslash A)$. On the other hand, by the affine property of $\sigma_{(\cdot)}$ we have $\sigma_{\{x'\}}(-s_*) = \frac{1}{2}\sigma_{\{x\}}(-s_*) + \frac{1}{2}\sigma_{\{y\}}(-s_*) > \sigma_A(-s_*)$. For the second part of the claim, note that $A \subset B$ implies $\sigma_A \leq \sigma_B$ and since $\sigma_B(-s_*) > \sigma_A(-s_*)$ we can appeal to Axiom CEB to conclude that $W(\sigma_B) > W(\sigma_A)$. Using the DLR(2001) representation from (19), this can be rewritten as $\int_{S^N} \sigma_B(s)\mu(ds) > \int_{S^N} \sigma_A(s)\mu(ds)$. ■

Lemma 18 *There exists an open set $\widehat{S}_1 \subset S^N$ with $\mu(\widehat{S}_1) > 0$ and $-s_* \in \widehat{S}_1$ such that $\sigma_B(s) > \sigma_A(s)$ for any $s \in \widehat{S}_1$.*

Proof. The result follows from Theorem 5 in Royden(1988, pp. 82) and Lemma 17. ■

Lemma 19 *There exists $\varepsilon > 0$ such that $\mu(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))) > 0$ where $\overline{N}_\varepsilon(-s_*)$ is the closed ball of radius ε around $-s_*$ in $\mathbb{R}^N$.*

Proof. If this were not true we would then have $\mu\left(\widehat{S}_1 \backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = 0$ for all $n \geq 1$ so: $\mu\left(cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right) = \mu\left(\widehat{S}_1\right) - \mu\left(\widehat{S}_1 \backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) > 0$ for any $n \geq 1$. Note that $\left\{cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right\}$ is a decreasing sequence of sets with $\cap_{n=1}^{\infty}\left(cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right) = cone(\{-s_*\}) \cap \widehat{S}_1$. But $cone(\{-s_*\}) = \{\lambda(-s_*) : \lambda \geq 0\}$ and since $\widehat{S}_1 \subset S^N$ in which the utilities are normalized so that $\sum_{k=1}^N (s^k)^2 = 1$, we have $cone(\{-s_*\}) \cap \widehat{S}_1 = \{-s_*\}$. So, since $\mu\left(cone\left(\overline{N}_1(-s_*)\right) \cap \widehat{S}_1\right) \leq \mu\left(\widehat{S}_1\right) < \infty$ and μ is σ-additive we can use for instance Theorem 9.8(ii) in Aliprantis and Border(1999, pp. 337) to conclude that: $\mu(\{-s_*\}) = \lim_{n\to\infty} \mu\left(cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right) \cap \widehat{S}_1\right) = \lim_{n\to\infty} \mu\left(\widehat{S}_1\right) > 0$ which contradicts the assumption that $\mu(\{-s_*\}) = 0$. Thus, the set $\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$ will be of strictly positive measure. ■

Lemma 20 *There exists a set $\widehat{S}_2 \subset \widehat{S}_1$ such that $\mu(\widehat{S}_2) > 0$ and $-s_* \notin cone\left(\widehat{S}_2\right)$.*

Proof. Even though $-s_* \notin \widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$, we cannot yet claim that $-s_* \notin cone(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*)))$. To obtain a set with this property, we will partition $\widehat{S}_2$ into 2^{N-2} elements constructed as follows. Firstly, note that by the normalization $\sum_{k=1}^N s^k = 0$ for all $s \in P^N$, we must have $-s_* \cdot v_1 = 0$, where $v_1 \equiv (1, ..., 1) \in \mathbb{R}^N$. Select next some other $N-3$ vectors such that $\{v_1, v_2, ..., v_{N-2}\}$ is a linearly independent set and $-s_* \cdot v_i = 0$, for all $i \in \{1, ..., N-2\}$. Note on the one hand that choosing $N-2$ such vectors is possible because the dimension of the underlying space P^N

is $N-1$. On the other hand, since $\{v_1, v_2, ..., v_{N-2}\}$ are linearly independent, the dimension of the set: $R \equiv \{s \in P^N : s \cdot v_i = 0, \text{ for } i \in \{1, ..., N-2\}\}$ is 1. Thus, since $-s_* \in R$, we have that $s \in R$ implies $s = \kappa(-s_*)$ for some $\kappa \in \mathbb{R}$.

Let $H_1 \equiv \{s \in P^N : s \cdot v_1 \geq 0\}$ and $H_2 \equiv \{s \in P^N : s \cdot v_1 \leq 0\}$ and then construct iteratively the following sets: $H_{i_1,...,i_n,1} \equiv \{s \in H_{i_1,...,i_n} : s \cdot v_{n+1} \geq 0\}$, $H_{i_1,...,i_n,2} \equiv \{s \in H_{i_1,...,i_n} : s \cdot v_{n+1} \leq 0\}$ for $n = \{1, ..., N-3\}$. Let: $S_{i_1,...,i_{N-2}} \equiv H_{i_1,...,i_{N-2}} \cap \left(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))\right)$ for all $\{i_1, ..., i_{N-2}\} \in \{1,2\}^{N-2}$. Note that the 2^{N-2} elements $S_{i_1,...,i_{N-2}}$ thus constructed form a finite partition of $\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$, so since $\mu(\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))) > 0$, one of the elements of the partition which we denote $\widehat{S}_2$ must be of strict positive measure. Without loss of generality we may assume that $s \cdot v_i \geq 0$ for all $s \in \widehat{S}_2$ and $i \in \{1, ..., N-2\}$. This is because when $s \cdot v_i \leq 0$ for some i we may take $v_i' = -v_i$ instead of v_i and then, except for some notation, the elements of the partition of $\widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$ will be the same.

We will show now that $-s_* \notin cone(\widehat{S}_2)$. Assume by contradiction that this is not true, that is there exist $\{s_1, ..., s_m\} \subset \widehat{S}_2$ and $\{\phi_1, ..., \phi_m\} \in \mathbb{R}_+^m$ such that $-s_* = \sum_{i=1}^m \phi_j s_j$. We may assume without loss of generality that $s_j \neq s_*$ for any j, because when this is not true we must still be able to write $-s_*$ as a positive combination of the remaining elements from $\{s_1, ..., s_m\}$. Now, for any $i \in \{1, ..., N-2\}$ we have $-s_* \cdot v_i = 0$, $s_j \cdot v_i \geq 0$ and $-s_* = \sum_{i=1}^m \phi_j s_j$ imply $s_j \cdot v_i = 0$ for all $j \in \{1, ..., m\}$. Therefore, for any $j \in \{1, ..., m\}$, we have $s_j \cdot v_i = 0$ for all $i \in \{1, ..., N-2\}$ which implies $s_j \in R \cap S^N$. But $R \cap S^N = \{-s_*, s_*\}$ because of the normalization $\sum_{k=1}^N \left(s^k\right)^2 = 1$ for the elements in S^N and of the fact that $s \in R$ implies $s = \kappa(-s_*)$ for some $\kappa \in \mathbb{R}$. Since $s_j \neq s_*$ we must therefore have $s_j = -s_*$ for all j which is impossible because $-s_* \notin \widehat{S}_1 \backslash cone(\overline{N}_\varepsilon(-s_*))$. This completes the proof of Lemma 20. ■

We denote by $diam(S) \equiv \sup\{d(s, s') : s, s' \in S\}$ the diameter of a nonempty set.

Lemma 21 *There exists a closed set $\widehat{S}_3 \subset \widehat{S}_2$ such that $\mu(\widehat{S}_3) > 0$ and $diam(\widehat{S}_3) \leq \delta$ for some $\delta < \frac{1}{2}$.*

Proof. We will use Theorem 15 from Royden(1988, pp. 63) which states that if E is a measurable set and $\varepsilon > 0$, then there exists a closed set $F \subset E$ such that $\mu(E \backslash F) < \varepsilon$. Since by Lemma 20 we have $\mu(\widehat{S}_2) > 0$, there exists ε such that $\mu(\widehat{S}_2) > \varepsilon > 0$. Applying the result from Royden, we conclude that there exists a closed set $\widehat{S}_4 \subset \widehat{S}_2$ such that $\mu(\widehat{S}_2 \backslash \widehat{S}_4) < \varepsilon$. But since $\widehat{S}_4 \subset \widehat{S}_2$, we have: $\mu(\widehat{S}_2 \backslash \widehat{S}_4) = \mu(\widehat{S}_2) - \mu(\widehat{S}_4) > \varepsilon - \mu(\widehat{S}_4)$ from which it follows that $\mu(\widehat{S}_4) > 0$. Now, take some $\delta < \frac{1}{2}$ and consider the open cover of $\widehat{S}_4$ consisting of the sets $\{N_{\frac{\delta}{2}}(s) \cap \widehat{S}_4\}_{s \in \widehat{S}_4}$. Since $\widehat{S}_4$ is a closed subset of the compact set S^N, it is compact so there exists a finite subcover of $\widehat{S}_4$. Since $\mu(\widehat{S}_4) > 0$, one of the elements of the subcover, let's say $N_{\frac{\delta}{2}}(s) \cap \widehat{S}_4$, must be of strict positive measure. Applying again the result from Royden(1988) to the set $N_{\frac{\delta}{2}}(s) \cap \widehat{S}_4$, we conclude that there exists a closed set $\widehat{S}_3 \subset \widehat{S}_2$ such that $\mu(\widehat{S}_3) > 0$ and $\widehat{S}_3 \subset N_{\frac{\delta}{2}}(s)$ so that $diam(\widehat{S}_3) \leq diam\left(N_{\frac{\delta}{2}}(s)\right) \leq \delta$. ■

Lemma 22 *There exists $\epsilon > 0$ such that $-s_* \notin cone(\cup_{s\in\widehat{S}_3}\overline{N}_\epsilon(s))$.*

Proof. Assume by contradiction that the claim is not true so that for any $n > 1$, we have $-s_* \in cone(\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s))$. Thus, for any $n \geq 2$, there exist $\{\lambda_i^n, r_i^n\}_{i\in\{1,\ldots,p(n)\}}$ with $\lambda_i^n > 0$ and $r_i^n \in \cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)$ such that $-s_* = \sum_{i=1}^{p(n)} \lambda_i^n r_i^n$. We firstly claim that it is without loss of generality to take $p(n) = N$ for all n. To see this, note that: $-s_* = \beta\left(\sum_{i=1}^{p(n)} \alpha_i r_i^n\right)$, where $\beta \equiv \left(\sum_{i=1}^{p(n)} \lambda_i^n\right)$ and $\alpha_i \equiv \frac{\lambda_i^n}{\beta}$. Since $\sum_{i=1}^{p(n)} \alpha_i = 1$, we have: $r^n \equiv \sum_{i=1}^{p(n)} \alpha_i r_i^n \in hull\left(\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)\right)$. By Carathéodory's Convexity Theorem (see for instance Theorem 5.17 from Aliprantis and Border(1999, pp. 173)) in an $(N-1)$-dimensional vector space, every vector in the convex hull of a nonempty set can be written as a convex combination of at most N vectors from that set. Thus, in our case there exist $\{\alpha_i^n, \overline{r}_i^n\}_{i\in\{1,\ldots,\ N\}}$ with $\alpha_i^n > 0$ and $\overline{r}_i^n \in \cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)$ such that $r^n = \sum_{i=1}^N \alpha_i^n \overline{r}_i^n$. Therefore, $-s_* = \sum_{i=1}^N (\beta\alpha_i^n)\overline{r}_i^n$ as desired.

Now, since $\widehat{S}_3$ is closed it follows that $\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{n}}(s)$ is also closed. Moreover, for any $r_i^n \in \cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s) \subset \cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{4}}(s)$ we have: $||r_i^n|| \leq ||r_i^n - s|| + ||s|| \leq \frac{5}{4}$ because $||s|| = 1$ when $s \in S^N$. Therefore, $\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2}}(s)$ is a closed subset of the compact set $\left\{s \in \mathbb{R}^N : ||s|| \leq \frac{5}{4}\right\}$ so it is compact. Since $\{r_1^n\}$ is a sequence in a compact set, it has a convergent subsequence $r_1^{n_i} \to r_1^0$. Thus, it is without loss of generality to assume that $r_1^n \to r_1^0$ and then repeating the argument iteratively we can take $r_i^n \to r_i^0$ for all $i \in \{1, ..., N\}$. We claim that $r_i^0 \in \widehat{S}_3$ for all i. To see this, note that if $r_i^0 \notin \widehat{S}_3$ for some i, since $\widehat{S}_3$ is closed we will have $d(r_i^0, \widehat{S}_3) = \chi > 0$. But then, take M' such that for any $n \geq M'$ we have $r_i^n \in N_{\frac{\chi}{2}}(r_i^0)$ and let $M \equiv \max(M', \frac{1}{\chi}) + 1$. Then: $\chi = d(r_i^0, \widehat{S}_3) \leq d(r_i^0, r_i^M) + d(r_i^M, \widehat{S}_3) < \frac{\chi}{2} + \frac{\chi}{2}$ which is impossible so it must be that $r_i^0 \in \widehat{S}_3$ for all i. Next, we show that for any i the real sequence $\{\lambda_i^n\}$ is bounded so that we can extract some convergent subsequence. Thus, we have $-s_* = \sum_{i=1}^N \lambda_i^n r_i^n = \left(\sum_{i=1}^N \lambda_i^n\right) r^n$, for some $r^n \in hull\left(\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)\right)$. Now, note that since $diam(\widehat{S}_3) \leq \delta$ we will have: $diam\left(hull\left(\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)\right)\right) = diam\left(\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)\right) \leq \delta + \frac{1}{n}$. Thus, for any $r \in hull\left(\cup_{s\in\widehat{S}_3}\overline{N}_{\frac{1}{2n}}(s)\right)$ we have: $||r|| = d(r, 0) \geq d(s, 0) - d(r, s) \geq ||s|| - (\delta + \frac{1}{n}) = (1 - \frac{1}{n} - \delta)$ for any $s \in \widehat{S}_3$ and $|| - s_*|| = \left(\sum_{i=1}^N \lambda_i^n\right) ||r^n|| \geq \left(\sum_{i=1}^N \lambda_i^n\right)\left(1 - \frac{1}{n} - \delta\right)$ which since $|| - s_*|| = 1$ and $\lambda_i^n \geq 0$ implies $\lambda_i^n \leq \frac{1}{\frac{1}{2} - \delta}$. Therefore, repeating the argument from above, we may assume without loss of generality that $\lambda_i^n \to \lambda_i^0 \geq 0$ for each $i \in \{1, ..., N\}$. But then, the sequence $\sum_{i=1}^N \lambda_i^n r_i^n \to \sum_{i=1}^N \lambda_i^0 r_i^0$ as $n \to \infty$. Therefore, $-s_* = \sum_{i=1}^N \lambda_i^0 r_i^0$ with $r_i^0 \in \widehat{S}_3$ and $\lambda_i^0 \geq 0$ so

$-s_* \in cone(\widehat{S}_3)$ which is a contradiction and thus the proof of Lemma 22 is complete.[14] ■

Lemma 23 *There exists a set of* $N-1$ *linearly independent utilities*[15] $\{s_1, ..., s_{N-1}\} \subset P^N \backslash \{0\}$ *such that* $\mu(int(cone(\{s_1, ..., s_{N-1}\})) \cap \widehat{S}_3) > 0$ *and* $-s_* \notin cone(\{s_1, ..., s_{N-1}\})$.

Proof. For any $s \in P^N$, denote by $\overline{s} \in \mathbb{R}^{N-1}$ the vector consisting of the first $N-1$ coordinates of s. Then, note that $\sum_{i=1}^{N-1} \lambda_i s_i = 0$ if and only if $\sum_{i=1}^{N-1} \lambda_i \overline{s}_i = 0$, so finding $N-1$ linearly independent elements in P^N is equivalent to finding $N-1$ linearly independent states in $\mathbb{R}^{N-1}$. For each $i \in \{1, ..., N-1\}$, let $f_i \equiv (0, ..., 1, ..., 0) \in \mathbb{R}^{N-1}$ with 1 on the i^{th} position and $e_i \equiv f_i - \frac{1}{N-1}(1, ..., 1)$. It is straightforward to show that $\{e_1, ..., e_{N-1}\}$ is a linearly independent set in $\mathbb{R}^{N-1}$ so it constitutes a basis for $\mathbb{R}^{N-1}$. For for each $i \in \{1, ..., N-1\}$, let $\overline{s}_i^s \equiv \overline{s} + \eta^s e_i$ for some $0 < \eta^s < \min(\epsilon, 1)$, where ϵ is given by Lemma 22 and note that $\overline{s} = \sum_{i=1}^{N-1} \frac{1}{N-1} \overline{s}_i^s$.

We claim that for any $s \in \widehat{S}_3$ we can choose η^s such that the set $\{\overline{s}_1^s, ..., \overline{s}_{N-1}^s\}$ is linearly independent. For this, we will show that $\sum_{i=1}^{N-1} \lambda_i \overline{s}_i^s = 0$ must imply $\lambda_i = 0$ for all i. Since $\{e_1, ..., e_{N-1}\}$ is a basis in $\mathbb{R}^{N-1}$, $\overline{s} = \sum_{i=1}^{N-1} \gamma_i^s e_i$ for some $\gamma_i \in \mathbb{R}$. Let $\lambda \equiv \sum_{i=1}^{N-1} \lambda_i$ and $\gamma^s \equiv \sum_{i=1}^{N-1} \gamma_i^s$ and note that

$$\sum_{i=1}^{N-1} \lambda_i \overline{s}_i^s = \sum_{i=1}^{N-1} (\lambda \gamma_i^s + \eta^s \lambda_i) e_i = \left(\lambda \gamma_1^s + \eta^s \lambda_1 - \frac{\lambda \gamma^s + \eta^s \lambda}{N-1}, ..., \lambda \gamma_{N-1}^s + \eta^s \lambda_{N-1} - \frac{\lambda \gamma^s + \eta^s \lambda}{N-1} \right) \tag{21}$$

Setting this equal to 0, we obtain a system of $N-1$ equations with $N-1$ unknowns $\{\lambda_1, ..., \lambda_{N-1}\}$, where the i^{th} equation is:

$$\lambda_1(\gamma_i^s - \frac{\gamma^s + \eta^s}{N-1}) + ... + \lambda_i(\gamma_i^s + \eta^s - \frac{\gamma^s + \eta^s}{N-1}) + ... + \lambda_{N-1}(\gamma_i^s - \frac{\gamma + \eta^s}{N-1}) = 0 \tag{22}$$

[14] Note here that unless we bound $p(n)$ above with N, the argument as presented here does not go through because it may well be that $p(n) \to \infty$ as $n \to \infty$.

[15] *We emphasize here that the set* $\{s_1, ..., s_{N-1}\}$ *is not required to belong to* S^N, *but to* $P^N \backslash \{0\}$. *While we could adapt Lemma 24 below to conclude that* $s_{n+1} \in hull(\{-s_1, ..., -s_n\})$ *and then also adapt the rest of the proof of Theorem 16 to avoid using cones and work only with states in* S^N *it is not immediately obvious, if possible at all, in the proof of this Lemma to choose* $\{s_1, ..., s_{N-1}\}$ *in* S^N *to satisfy the desired properties. Therefore, the choice to work in the extended state space* $P^N \backslash \{0\}$ *and use cones instead of convex hulls.*

We will show now that the $(N-1)\times(N-1)$ coefficient matrix of this system has a non-zero determinant D. Thus:

$$
\begin{aligned}
D &= \begin{vmatrix}
\gamma_1^s+\eta^s-\frac{\gamma^s+\eta^s}{N-1} & \gamma_1-\frac{\gamma^s+\eta^s}{N-1} & \dots & \gamma_1^s-\frac{\gamma^s+\eta^s}{N-1} \\
\gamma_2^s-\frac{\gamma^s+\eta^s}{N-1} & \gamma_2^s+\eta^s-\frac{\gamma^s+\eta^s}{N-1} & \dots & \gamma_2^s-\frac{\gamma^s+\eta^s}{N-1} \\
\dots & \dots & \dots & \dots \\
\gamma_{N-1}^s-\frac{\gamma^s+\eta^s}{N-1} & \gamma_{N-1}^s-\frac{\gamma^s+\eta^s}{N-1} & \dots & \gamma_{N-1}^s+\eta^s-\frac{\gamma^s+\eta^s}{N-1}
\end{vmatrix} \\
&= \begin{vmatrix}
1 & 1 & \dots & 1 \\
\gamma_2^s-\frac{\gamma^s+\eta^s}{N-1} & \gamma_2^s+\eta^s-\frac{\gamma^s+\eta^s}{N-1} & \dots & \gamma_2^s-\frac{\gamma^s+\eta^s}{N-1} \\
\dots & \dots & \dots & \dots \\
\gamma_{N-1}^s-\frac{\gamma^s+\eta^s}{N-1} & \gamma_{N-1}^s-\frac{\gamma^s+\eta^s}{N-1} & \dots & \gamma_{N-1}^s+\eta^s-\frac{\gamma^s+\eta^s}{N-1}
\end{vmatrix} \frac{N-2}{N-1}(\gamma^s+\eta^s) = \\
&= \frac{N-2}{N-1}(\gamma^s+\eta^s)\,(\eta^s)^{N-2}
\end{aligned}
$$

For the first equation, we added rows 2 through $N-1$ to the first row and then factored out the term $\frac{N-2}{N-1}(\gamma^s+\eta^s)$. For the second equation, we subtracted from each row $i\in\{2,...,N-1\}$, the first row multiplied with $\gamma_i^s-\frac{\gamma^s+\eta^s}{N-1}$. Now note that since the only restriction on η^s is $0<\eta^s<\min(\epsilon,1)$ we can always select η^s such that $\eta^s\neq-\gamma^s$ so $D\neq 0$. Therefore, the system has a unique solution and since $\lambda_i=0$ for $i\in\{1,...,N-1\}$ solves the system, we obtain that $\{\overline{s}_1^s,...,\overline{s}_{N-1}^s\}$ is a linearly independent set in $\mathbb{R}^{N-1}$. Then, for each $\overline{s}_i^s$ construct: $s_i^s\equiv(\overline{s}_{i,1}^s,...,\overline{s}_{i,N-1}^s,-(\overline{s}_{i,1}^s+...+\overline{s}_{i,N-1}^s))\in P^N$ and we obtained the $N-1$ linearly independent states in P^N.

We will show now that $s\in int(cone(\{s_1^s,...,s_{N-1}^s\}))$, for which since $cone(\{s_1^s,...,s_{N-1}^s\})$ is a convex set in an Euclidean space it suffices to show that $s\in al-int(cone(\{s_1^s,...,s_{N-1}^s\}))$, the algebraic interior of the set $cone(\{s_1^s,...,s_{N-1}^s\})$ in P^N. Thus, we will show that for any $p\in P^N$, there exists some $\overline{\alpha}^s>0$ such that for all $\alpha\in[0,\overline{\alpha}^s)$, we have $(1-\alpha)s+\alpha p\in cone(\{s_1^s,...,s_{N-1}^s\})$. Since $\{s_1^s,...,s_{N-1}^s\}$ are linearly independent, they form a basis in the $N-1$ dimensional space P^N so $p=\sum_{i=1}^N\delta_i s_i^s$ with $\delta_i\in\mathbb{R}$. On the other hand, by construction $s=\sum_{i=1}^{N-1}\frac{1}{N-1}s_i^s$ so $(1-\alpha)s+\alpha p=\sum_{i=1}^{N-1}\left((1-\alpha)\frac{1}{N-1}+\alpha\delta_i\right)s_i^s$. Now, by denoting $\beta_i^\alpha\equiv(1-\alpha)\frac{1}{N-1}+\alpha\delta_i$ we will have $(1-\alpha)s+\alpha p=\sum_{i=1}^{N-1}\beta_i^\alpha s_i^s$ and noting that for α sufficiently small $\beta_i^\alpha\geq 0$ for all i, the argument is complete. Employing the procedure presented above and using the Axiom of Choice construct the family of sets: $\mathcal{F}=\{int(cone(\{s_1^s,...,s_N^s\}))\cap\widehat{S}_3 : s\in\widehat{S}_3\}$. Since $s\subset int(cone(\{s_1^s,...,s_N^s\}))\cap\widehat{S}_3$ for any s, the elements of $\mathcal{F}$ are nonempty and open relative to $\widehat{S}_3$. Thus, $\mathcal{F}$ is an open cover of $\widehat{S}_3$ which is compact as a closed subset of the compact set S^N so there exists a finite family $\mathcal{F}'\subset\mathcal{F}$ such that $\widehat{S}_3\subset\bigcup_{F\in\mathcal{F}'}F$. Since $\mu(\widehat{S}_3)>0$, one of the elements of $\mathcal{F}'$ must be of strictly positive measure so $\mu(int(cone(\{s_1^s,...,s_{N-1}^s\}))\cap\widehat{S}_3)>0$ for some $s\in\widehat{S}_3$. Now, since $\sum_{k=1}^{N-1}e_{i,k}=0$ for all i, where $e_{i,k}$ denotes the k^{th} coordinate of the vector e_i, note that $d(s_i^s, s) = d(\overline{s}_i^s, \overline{s})$ so: $d(s_i^s, s) = ||\eta^s e_i|| = \eta^s \frac{N-2}{N-1} < \epsilon$ so we have $int(cone(\{s_1^s, ..., s_{N-1}^s\})) \subset int(cone(\overline{N}_\epsilon(s)))$. Therefore by Lemma 22 it follows that: $-s_* \notin cone(\{s_1^s, ..., s_{N-1}^s\})$. Finally, for any i, $||s_i^s|| = ||s + \eta^s e_i|| \geq ||s|| - ||\eta^s e_i|| = 1 - \eta^s \frac{N-2}{N-1} > \frac{1}{N-1}$ implies $s_i^s \neq 0$. ■

Lemma 24 *Let $\{s_1, ..., s_n\} \subset P^N \backslash \{0\}$ with $n \geq 1$ be such that $\cap_{i=1}^n L_{s_i}(B) \neq \varnothing$. Then, if $s_{n+1} \in P^N \backslash \{0\}$ is such that $(\cap_{i=1}^n L_{s_i}(B)) \cap L_{s_{n+1}}(B) = \varnothing$, we must have $s_{n+1} \in cone(\{-s_1, ..., -s_n\})$.*

Proof. Note that since B is compact, $L_{s_i}(B) = L_{s_i}(z_{s_i})$ for some lottery $z_{s_i} \in B$ for all $i \in \{1, ..., n+1\}$. Moreover, $\cap_{i=1}^n L_{s_i}(B) \supset \cap_{i=1}^n L_{s_i}(q) \neq \varnothing$ for some $q \in \cap_{i=1}^n L_{s_i}(B)$ because for any $x \in \cap_{i=1}^n L_{s_i}(q)$ and any $i \in \{1, ..., n\}$ we will have $x \cdot s_i < q \cdot s_i \leq z_{s_i} \cdot s_i$. Therefore, the condition that $(\cap_{i=1}^n L_{s_i}(B)) \cap L_{s_{n+1}}(B) = \varnothing$ implies that $(\cap_{i=1}^n L_{s_i}(q)) \cap L_{s_{n+1}}(z_{n+1}) = \varnothing$. Also, $q \cdot s_{n+1} > z_{n+1} \cdot s_{n+1}$ because otherwise $(\cap_{i=1}^n L_{s_i}(B)) \cap L_{s_{n+1}}(B) \neq \varnothing$ since all elements in $\cap_{i=1}^n L_{s_i}(q)$ would be also in $L_{s_{n+1}}(B)$. We will show now that: $L_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q)) = \varnothing$ and to this end, assume by contradiction that there exists some $y \in L_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q))$. Consider the set: $V \equiv \{x \in \Omega : q + \tau(y - q)$ for some $\tau > 0\}$ and note for any $x \in V$ and $i \in \{1, ..., n\}$, we have $x \cdot s_i < q \cdot s_i$ because $y \cdot s_i < q \cdot s_i$. Therefore, $V \subset \cap_{i=1}^n L_{s_i}(q)$ so to prove our claim it is enough to show that $V \cap L_{s_{n+1}}(z_{n+1}) \neq \varnothing$. For this we need to find some $\tau > 0$ such that $(q + \tau(y - q)) \cdot s_{n+1} < z_{n+1} \cdot s_{n+1}$. Since $q \cdot s_{n+1} > z_{n+1} \cdot s_{n+1}$ as stated above and $y \in L_{s_{n+1}}(q)$ by the contradiction assumption, any $\tau > \frac{(q-z) \cdot s_{n+1}}{(q-y) \cdot s_{n+1}}$ would satisfy this requirement.

Consider now the following sets:

$$H_{s_{n+1}}(q) \equiv \{z \in \Omega : z \cdot s_{n+1} = q \cdot s_{n+1}\} \tag{23}$$

$$Y \equiv \{w \in \mathbb{R}^n : w = ((z - q) \cdot s_1, ..., (z - q) \cdot s_n) \text{ or } w = (-(z - q) \cdot s_1, ..., -(z - q) \cdot s_n) \tag{24}$$

$$\text{for some } z \in H_{s_{n+1}}(q)\} \tag{25}$$

$$Y' \equiv \{w \in \mathbb{R}^n : w \leq 0\} \tag{26}$$

Clearly, Y and Y' are closed and convex. We will show next that $Y \cap int(Y') = \varnothing$. Thus, we want to show that if $z \cdot s_{n+1} = q \cdot s_{n+1}$ then it cannot be that $z \cdot s_i < q \cdot s_i$ for all $i \in \{1, ..., n\}$ or $z \cdot s_i > q \cdot s_i$ for all $i \in \{1, ..., n\}$. We can assume that $s_i \neq -s_{n+1}$ because otherwise we would be done with the proof of the lemma, so what remains to prove is that $H_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q)) = \varnothing$ and $H_{s_{n+1}}(q) \cap (\cap_{i=1}^n U_{s_i}(q)) = \varnothing$. The first claim follows from the results we obtained above. Thus, note that if this were not true, that is if there exists $x \in H_{s_{n+1}}(q) \cap (\cap_{i=1}^n L_{s_i}(q))$, since $\cap_{i=1}^n L_{s_i}(q)$ is open, we could take a sufficiently small $\delta > 0$ such that $N_\delta(x) \subset \cap_{i=1}^n L_{s_i}(q)$. Since $x \in H_{s_{n+1}}(q)$, we have that $\beta x + (1-\beta) y \in N_\delta(x) \cap L_{s_{n+1}}(q)$ for some $y \in L_{s_{n+1}}(q)$ and some β sufficiently small and we would thus obtain a contradiction with the fact that $L_{s_{n+1}}(q) \cap (\cap_{i=1}^{n} L_{s_i}(q)) = \varnothing$. As for the second part of the claim, note that if there exists $x \in H_{s_{n+1}}(q) \cap (\cap_{i=1}^{n} U_{s_i}(q))$ we would have $x \cdot s_i > q \cdot s_i$ for all $i \in \{1, ..., n\}$ and $x \cdot s_{n+1} = q \cdot s_{n+1}$. Consider then the element $x' = q + \alpha(q - x)$ for some $\alpha > 0$. We will then have $x' \cdot s_i < q \cdot s_i$ for all $i \leq n$ and $x' \cdot s_{n+1} = q \cdot s_{n+1}$ so $x' \in H_{s_{n+1}}(q) \cap (\cap_{i=1}^{n} L_{s_i}(q))$ which we know that cannot hold by the first part of the claim and thus we are done.

Given that Y and Y' are closed and convex and $Y \cap int(Y') = \varnothing$ we can use the Separating Hyperplane Theorem to obtain that there exists a vector $\phi \in \mathbb{R}^n \backslash \{0\}$ and a number $k \in \mathbb{R}$ such that such that $\phi \cdot w \geq k$ for all $w \in Y$ and $\phi \cdot w \leq k$ for all $w \in Y'$. But since $((q-q) \cdot s_1, ..., (q-q) \cdot s_n) \in Y \cap Y'$ we must have $k = \phi \cdot 0 = 0$. Also, note that for any $w \in Y$ we have $-w \in Y$ so $\phi \cdot w \geq 0$ and $\phi \cdot (-w) \geq 0$ so $\phi \cdot w = 0$. Moreover, note that since $\phi \cdot w \leq k = 0$ for all $w \in Y'$ we must have $\phi \geq 0$. Therefore, we obtained that for any $z \in H_{s_{n+1}}(q)$, that is for any $z \in \Omega$ with $(z - q) \cdot s_{n+1} = 0$ we must have: $(z - q) \cdot (\phi_1 s_1 + ... + \phi_n s_n) = 0$. Then, denoting as above by $\overline{s}_i$ the elements of $\mathbb{R}^{N-1}$ consisting of the first $N-1$ coordinates of s_i and using for instance Theorem 5.81 from Aliprantis and Border (1999, pp. 207), we have that $\overline{s}_{n+1} = \psi(\phi_1 \overline{s}_1 + ... + \phi_n \overline{s}_n)$ for some $\psi \in \mathbb{R}$ and then $s_{n+1} = \psi(\phi_1 s_1 + ... + \phi_n s_n)$. From the fact that $s_{n+1} \in P^N \backslash \{0\}$ it follows that $\psi \neq 0$. Since $(\cap_{i=1}^{n} L_{s_i}(q)) \cap L_{s_{n+1}}(q) = \varnothing$ we must also have $\psi < 0$ so $s_{n+1} = \sum_{i=1}^{n} \alpha_i s_i$ with $\alpha_i \equiv \psi \phi_i \leq 0$ and the proof of the Lemma 24 is complete. ■

Lemma 25 $\cap_{i=1}^{N-1} L_{s_i}(B) \neq \varnothing$.

Proof. We will prove the lemma by induction. Clearly, we have $L_{s_1}(B) \neq \varnothing$ so assume that $\cap_{i=1}^{n} L_{s_i}(B) \neq \varnothing$ and by contradiction that $\cap_{i=1}^{n+1} L_{s_i}(B) = \varnothing$. By Lemma 24, it would follow that $s_{n+1} \in cone(\{-s_1, ..., -s_n\})$ so $s_{n+1} = \sum_{i=1}^{n} \alpha_i (-s_i)$ with $\alpha_i \geq 0$. But then, $s_{n+1} + \sum_{i=1}^{n} \alpha_i s_i = 0$ which contradicts the fact that $\{s_1, ..., s_{n+1}\}$ are linearly independent. Therefore, we must have $\cap_{i=1}^{n+1} L_{s_i}(B) \neq \varnothing$ and this completes the induction proof. ■

Lemma 26 *There exists a set $\widehat{S}_5 \subset \widehat{S}_3$ with $\mu(\widehat{S}_5) > 0$ and $z' \in \Omega$ such that $z' \in U_{-s_*}(B) \cap \left(\cap_{s \in \widehat{S}_5} L_s(B)\right)$.*

Proof. Since $\cap_{i=1}^{N-1} L_{s_i}(B) \neq \varnothing$ by Lemma 25 and $-s_* \notin cone(\{s_1, ..., s_{N-1}\})$ which implies immediately that $s_* \notin cone(\{-s_1, ..., -s_{N-1}\})$ we can use Lemma 24 to conclude that: $L_{s_*}(B) \cap \left(\cap_{i=1}^{N-1} L_{s_i}(B)\right) \neq \varnothing$But since $L_{s_*}(B) = \{y \in \Omega : y \cdot s_* < z \cdot s_* \text{ for all } z \in B\} = \{y \in \Omega : y \cdot (-s_*) > z \cdot (-s_*) \text{ for all } z \in B\} = U_{-s_*}(B)$ it follows that $U_{-s_*}(B) \cap (\cap_{i=1}^{N-1} L_{s_i}(B)) \neq \varnothing$. So we can take some: $z' \in U_{-s_*}(B) \cap (\cap_{i=1}^{N-1} L_{s_i}(B))$. Moreover, since $z' \in \cap_{i=1}^{N-1} L_{s_i}(B)$ it follows that: $z' \in \cap_{s \in cone(\{s_1, ..., s_{N-1}\})} L_s(B)$. To see this, note firstly that $z' \cdot s_i < x \cdot s_i$ for all $x \in B$ and for each i. Take some $s = \sum_{i=1}^{N-1} \alpha_i s_i$ with $\alpha_i \geq 0$ for all i. Then for any $x \in B$

we will have $z' \cdot s = \sum_{i=1}^{N-1} \alpha_i(z' \cdot s_i) < \sum_{i=1}^{N-1} \alpha_i(x \cdot s_i) = x \cdot s$ so that $z' \in L_s(B)$. Now, denote by: $\widehat{S}_5 \equiv cone(\{s_1, ..., s_{N-1}\}) \cap \widehat{S}_3$. Since $z' \in U_{-s_*}(B) \cap \left(\cap_{s \in cone(\{s_1,...,s_{N-1}\})} L_s(B)\right) \subset U_{-s_*}(B) \cap \left(\cap_{s \in \widehat{S}_5} L_s(B)\right)$ and $\widehat{S}_5 \subset \widehat{S}_3$ with $\mu(\widehat{S}_5) > 0$ by Lemma 23, the proof of the Lemma 26 is complete. ■

Lemma 27 *There exists a lottery $z \in \Delta(Z)$ such that for all $s \in \widehat{S}_5$ we have $\sigma_B(s) > \sigma_{\{z\}}(s)$ and $\sigma_B(-s_*) < \sigma_{\{z\}}(-s_*)$.*

Proof. Since B is compact we have $U_{-s_*}(B) = U_{-s_*}(\{z''\})$ for some $z'' \in B$. Since $B \subset int(\Delta(Z))$ we have $z'' \in int(\Delta(Z))$ so there exists: $z \equiv \alpha z'' + (1-\alpha)z' \in int(\Delta(Z)$ for some sufficiently high $\alpha < 1$. Since $z' \in U_{-s_*}(B)$ we will have $\sigma_B(-s_*) < z' \cdot (-s_*)$. On the other hand, by choice of z'' we have $z'' \cdot (-s_*) \geq x \cdot (-s_*)$ for all $x \in B$ so $z'' \cdot (-s_*) \geq \sigma_B(-s_*)$. Therefore: $\sigma_{\{z\}}(-s_*) = \alpha z'' \cdot (-s_*) + (1-\alpha)z' \cdot (-s_*) > \sigma_B(-s_*)$. Finally, for any $s \in \widehat{S}_5$ we have $\sigma_B(s) \geq z'' \cdot s$ while $z' \in L_s(B)$ implies $\sigma_B(s) > z' \cdot s$. Thus, $\sigma_B(s) > \alpha z'' \cdot s + (1-\alpha)z' \cdot s = z \cdot s = \sigma_{\{z\}}(s)$ and the proof of the lemma is complete. ■

We will complete now the proof of Theorem 16. Thus, consider the sets $A \cup \{z\}$ and $B \cup \{z\}$ and we want to show that we must have: $W(\sigma_{hull(B\cup\{z\})}) > W(\sigma_{hull(A\cup\{z\})})$ which would be sufficient to exclude the case when $\mu(\{-s_*\}) = 0$. To see this, note that $B \subset B'$ implies $W(\sigma_{hull(B'\cup\{z\})}) \geq W(\sigma_{hull(B\cup\{z\})})$ so we found $z \in \Delta(Z)$ with $\lambda\sigma_{\{z\}}(-s_*) \geq \lambda\sigma_B(-s_*)$ and $W(\sigma_{hull(B'\cup\{z\})}) > W(\sigma_{hull(A\cup\{z\})})$. This contradicts Axiom CEB-2 because B' was chosen arbitrarily from those sets satisfying the requirements of the axiom. Thus, we have:

$$W(\sigma_{hull(B\cup\{z\})}) = \int_{S^N\setminus\widehat{S}_5} (\sigma_B \vee \sigma_{\{z\}})(s)\mu(ds) + \int_{\widehat{S}_5} (\sigma_B \vee \sigma_{\{z\}})(s)\mu(ds) \geq$$

$$\geq \int_{S^N\setminus\widehat{S}_5} (\sigma_A \vee \sigma_{\{z\}})(s)\mu(ds) + \int_{\widehat{S}_5} (\sigma_B \vee \sigma_{\{z\}})(s)\mu(ds) >$$

$$> \int_{S^N\setminus\widehat{S}_5} (\sigma_A \vee \sigma_{\{z\}})(s)\mu(ds) + \int_{\widehat{S}_5} (\sigma_A \vee \sigma_{\{z\}})(s)\mu(ds) = W(\sigma_{hull(A\cup\{z\})})$$

where the weak inequality comes from the fact that $A \subset B$ so $\sigma_B(s) \geq \sigma_A(s)$ for all s. The strict inequality comes from the fact that for any $s \in \widehat{S}_5 \subset \widehat{S}_3$ with $\mu(\widehat{S}_5) > 0$ we have $\sigma_B(s) > \sigma_{\{z\}}(s)$ and $\sigma_B(s) > \sigma_A(s)$ so that $(\sigma_B \vee \sigma_{\{z\}})(s) > (\sigma_A \vee \sigma_{\{z\}})(s)$. Therefore, we must have: $\mu(\{-s_*\}) > 0$ and thus the proof of Theorem 16 is complete. ■

Theorem 28 *Under axioms CEB and Identification there exists $\varepsilon > 0$ such that $\mu(N_\varepsilon(-s_*)\setminus\{-s_*\}) = 0$.*

Proof. Most steps in the proof of this theorem are identical to steps from the proof of the previous theorem so we will present in detail only the step at which the two proofs differ.

Assume Axiom CEB is satisfied and by contradiction that the statement of Theorem 28 is false. Thus, for any $\varepsilon > 0$ we have $\mu(N_\varepsilon(-s_*)\backslash\{-s_*\}) > 0$. Repeat the steps from Lemmas 17-18 in the proof of Theorem 16 to construct the open set $\widehat{S}_1 \subset S^N$ with $\mu(\widehat{S}_1) > 0$ and $-s_* \in \widehat{S}_1$ such that $\sigma_B(s) > \sigma_A(s)$ for any $s \in \widehat{S}_1$. We will next show that the result from Lemma 19 is true in this case as well.[16] Then, the rest of the proof will go through as above and thus we would conclude that Axiom CEB-2 is violated which would constitute the contradiction. We have $-s_* \in \widehat{S}_1$ and we claim that there exists $\varepsilon > 0$ such that $\mu\left(\widehat{S}_1\backslash cone(\overline{N}_\varepsilon(-s_*))\right) > 0$ where $\overline{N}_\varepsilon(-s_*)$ is the *closed* ball of radius ε around $-s_*$ in $\mathbb{R}^N$. If this were not true we would then have $\mu\left(\widehat{S}_1\backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = 0$ for all $n \geq 1$. Note that $\left\{\widehat{S}_1\backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right\}$ is an increasing sequence of sets with $\cup_{n=1}^{\infty}\left(\widehat{S}_1\backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = \widehat{S}_1\backslash cone(\{-s_*\})$. But $\widehat{S}_1\backslash cone(\{-s_*\}) = \widehat{S}_1\backslash \{-s_*\}$ because $cone(\{-s_*\}) \cap \widehat{S}_1 = \{-s_*\}$ as argued in the proof of the Lemma 19 from Theorem 16. So, we can use Theorem 9.8(i) in Aliprantis and Border (1999, pp. 337) to conclude that: $\mu(\widehat{S}_1\backslash\{-s_*\}) = \lim_{n\to\infty} \mu\left(\widehat{S}_1\backslash cone\left(\overline{N}_{\frac{1}{n}}(-s_*)\right)\right) = 0$. Since $-s_* \in \widehat{S}_1$ and $\widehat{S}_1$ is open, there must exist an open neighborhood $N_\delta(-s_*)$ of $-s_*$ included in $\widehat{S}_1$ such that $\mu(N_\delta(-s_*)\backslash\{-s_*\}) = 0$ which would contradict our assumption. Therefore, there must exist some $\varepsilon > 0$ such that $\mu\left(\widehat{S}_1\backslash cone(\overline{N}_\varepsilon(-s_*))\right) > 0$ which completes the proof of the Theorem 28. ■

We complete now the proof of the sufficiency of the Axioms. Using (19) for any $A \in \widetilde{K}(\Delta(Z))$, we can write:

$$W(\sigma_A) = \int_{S^N\backslash\{-s_*\}} \left[\max_{x\in A}(x \cdot s)\right] \mu(ds) + \max_{x\in A}(x \cdot (-s_*))\mu(\{-s_*\}). \tag{27}$$

In particular, for $A = \{z\}$ we will have:

$$W(\sigma_{\{z\}}) = \int_{S^N\backslash\{-s_*\}} (z \cdot s)\mu(ds) + (z \cdot (-s_*))\mu(\{-s_*\}). \tag{28}$$

But as shown above, $W(\sigma_{\{z\}}) = v(z) = \lambda(z\cdot s_*)$ so we have: $z\cdot s_* = \frac{1}{\lambda+\mu(\{-s_*\})}\int_{S^N\backslash\{-s_*\}}(z\cdot s)\mu(ds)$ so using (27) we get:

$$W(\sigma_A) = \int_{S^N\backslash\{-s_*\}} \left[\max_{x\in A}(x \cdot s)\right] \mu(ds) + \max_{z\in A}\left[-\frac{\mu(\{-s_*\})}{\lambda + \mu(\{-s_*\})}\int_{S^N\backslash\{-s_*\}}(z \cdot s)\mu(ds)\right] \tag{29}$$

[16] Note that it is the proof of Lemma 19 where we used the contradiction assumption that $\mu(\{-s_*\}) = 0$ in the proof of Theorem 16.

In conclusion, since $W(\sigma_A) = V(A)$ and $A \sim hull(A)$, for any $A \in K(\Delta(Z))$ we get the desired normalized reference-dependent representation:

$$V(A) = \int_S \left[\max_{x \in A} (x \cdot s)\right] \widetilde{\mu}(ds) - \theta \min_{z \in A} \left[\int_S (z \cdot s) \widetilde{\mu}(ds)\right] \tag{30}$$

where $\theta \equiv \frac{\mu(\{-s_*\})}{\lambda + \mu(\{-s_*\})}$, $S \equiv S^N \backslash N_\varepsilon(-s_*)$ and $\widetilde{\mu}(ds) \equiv \frac{\mu(ds)}{\mu(S^N) - \mu(\{-s_*\})}$ for $s \neq -s_*$ and $\widetilde{\mu}(\{-s_*\}) \equiv 0$. Note that since $\mu(\{-s_*\}) > 0$ by Theorem 16 and $\lambda > 0$ we will have $\theta \in (0, 1)$. Also, since we have $\mu(N_\varepsilon(-s_*) \backslash \{-s_*\}) = 0$ by Lemma 28 it follows that: $\widetilde{\mu}(N_\varepsilon(-s_*)) = \widetilde{\mu}(N_\varepsilon(-s_*) \backslash \{-s_*\}) + \widetilde{\mu}(\{-s_*\}) = \frac{\mu(N_\varepsilon(-s_*) \backslash \{-s_*\})}{\mu(S^N) - \mu(\{-s_*\})} = 0$ and thus condition *(iii)* from Definition 9 is also satisfied. This completes the sufficiency part of the proof of Theorem 13. ■

Proof of Necessity in Theorem 13:

We show next that the a preference relation which can be represented by a utility function as in (6) must satisfy Weak Order, Continuity, Independence, Monotonicity, Axiom CEB and Axiom CEB-2. The fact that the preference will satisfy the first three of the axioms is true because the representation in (6) is just a particular form of a DLR representation which implies those axioms. Also, given the equivalent representation in **(2)** it is clear that the preference must also satisfy Monotonicity and it is straightforward to show the necessity of Axiom CEB. Therefore, it remains to show that Axiom CEB-2 must also be satisfied.

The following lemma will constitute the main step of the argument. Note firstly that by part *(iii)* of the representation in (6), there exists $\varepsilon > 0$ such that $\mu(N_\varepsilon(-s_*)) = 0$.

Lemma 29 *When the preferences $\succeq$ admit a normalized reference-dependent representation as in (6), there exist a compact set $A' \subset int(\Delta(Z))$ and a lottery $y \in int(\Delta(Z))$ such that $\sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*)$ and $\sigma_{A'}(s) > \sigma_{\{y\}}(s)$ for all $s \in S^N \backslash N_\varepsilon(-s_*)$.*

Proof. Firstly, since $-s_* \notin S^N \backslash cone(N_\varepsilon(-s_*))$ we can use an argument similar to the one from the proof of Lemma 20 from Theorem 16 to cover $S^N \backslash N_\varepsilon(-s_*)$ with 2^{N-2} elements $\{S_1, ..., S_{2^{N-2}}\}$ such that $-s_* \notin cone(S_j)$ for any j. By taking their closures, we can assume that the elements are all closed sets. Then, using the approach from Lemma 21, we can partition each S_j to obtain a cover of $S^N \backslash N_\varepsilon(-s_*)$ with elements indexed by a finite set J, such that $diam(S_j) \leq \delta$ for some $\delta < \frac{1}{2}$ and all $j \in J$. Again, by taking closures we can assume that S_j are closed for all j. Next, as in Lemma 22 we can show that for each $j \in J$ there exists $\epsilon_j > 0$ such that for each j we have $-s_* \notin cone(\cup_{s \in S_j} \overline{N}_{\epsilon_j}(s))$. Thus, as in Lemma 23 we can find a cover of S_j with a finite family of sets of the form $\{int(cone(\{s_{1,i}, ..., s_{N-1,i}\})) \cap S_j\}_{i \in I_j}$ such that for any $i \in I_j$, $\{s_{1,i}, ..., s_{N-1,i}\}$ are linearly independent and $-s_* \notin int(cone(\{s_{1,i}, ..., s_{N-1,i}\}))$. Let $j(i)$ be the index j such that $i \in I_j$ and let $I \equiv \cup_{j \in J} I_j$. Note that by construction, I is a finite set. Take some arbitrary lottery $y \in int(\Delta(Z))$.

For each $i \in I$, since $\{s_{1,i}, ..., s_{N,i}\}$ are linearly independent we can employ Lemma 24 as in the proof of Lemma 25 from Theorem 16 to conclude that $\cap_{k=1}^{N-1} L_{s_{k,i}}(y) \neq \varnothing$ and then immediately that $\cap_{j=1}^{N-1} L_{-s_{k,i}}(y) \neq \varnothing$. Therefore, using again Lemma 24 for the set $\{-s_{1,i}, ..., -s_{N-1,i}\}$ and $-s_*$ we will have that $\left(\cap_{k=1}^{N-1} L_{-s_{k,i}}(y)\right) \cap L_{-s_*}(y) \neq \varnothing$ so $\left(\cap_{k=1}^{N-1} U_{s_{k,i}}(y)\right) \cap L_{-s_*}(y) \neq \varnothing$ because $L_{-s_{k,i}}(y) = U_{s_{k,i}}(y)$. Now, for each $i \in I$, take $x'_i \in \left(\cap_{k=1}^{N-1} U_{s_{k,i}}(y)\right) \cap L_{-s_*}(y)$ and note that by an argument similar to the one from Lemma 26 we will have: $x'_i \in \left(\cap_{s \in int(cone(\{s_{1,i},...,s_{N-1,i}\})) \cap S_{j(i)} \cap (S^N \setminus N_\varepsilon(-s_*))} U_s(y)\right) \cap L_{-s_*}(y)$. Since x'_i as chosen above is not necessarily in $\Delta(Z)$ consider the set $\{\lambda x'_i + (1-\lambda) y : \lambda \in (0,1)\}$ and note that since $y \in int(\Delta(Z))$ which is algebraically open, for a small enough λ we will have $x_i \equiv \lambda x'_i + (1-\lambda) y \in int(\Delta(Z))$. In addition, since $(\lambda x'_i + (1-\lambda) y) \cdot s \geq y \cdot s$ when $x'_i \cdot s \geq y \cdot s$ and $(\lambda x'_i + (1-\lambda) y) \cdot (-s_*) < y \cdot (-s_*)$ when $x'_i \cdot (-s_*) < y \cdot (-s_*)$ it follows that $x_i \in \left(\cap_{s \in int(cone(\{s_{1,i},...,s_{N-1,i}\})) \cap S_{j(i)} \cap (S^N \setminus N_\varepsilon(-s_*))} U_s(y)\right) \cap L_{-s_*}(y) \cap int(\Delta(Z))$. Let $A' \equiv \cup_{i \in I} x_i$. Firstly, note that since $x_i \in L_{-s_*}(y)$ for each i we have $x_i \cdot (-s_*) < y \cdot (-s_*) \Rightarrow \sup_{i \in I} (x_i \cdot (-s_*)) = \max_{i \in I} (x_i \cdot (-s_*)) < y \cdot (-s_*)$ so $\sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*)$. On the other hand, the family $\{int(\{cone(\{s_{1,i}, ..., s_{N-1,i}\})) \cap S_{j(i)} \cap \left(S^N \setminus N_\varepsilon(-s_*)\right)\}_{i \in I}$ being a cover of $S^N \setminus N_\varepsilon(-s_*)$, for any $s \in S^N \setminus N_\varepsilon(-s_*)$ we will have $s \in int(cone(\{s_{1,i}, ..., s_{N-1,i}\})) \cap S_{j(i)}$ for some $i \in I$. Therefore, $\sigma_{A'}(s) \geq \sigma_{\{x_i\}}(s) > \sigma_{\{y\}}(s)$ which completes the proof of the Lemma 29. ■

Let $A \equiv hull(A')$ and $B \equiv hull(A \cup\, y)$, where A' and y are given by the Lemma 29 and we will show that A and B thus defined will satisfy the conditions of Axiom CEB-2 from Appendix B which we already proved that is equivalent to Axiom CEB-2. Firstly, since $\sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*)$ it follows that:

$$\sigma_B(-s_*) = \sigma_{A \cup \{y\}}(-s_*) = \sigma_A(-s_*) \vee \sigma_{\{y\}}(-s_*) = \sigma_{A'}(-s_*) \vee \sigma_{\{y\}}(-s_*) > \sigma_{A'}(-s_*) = \sigma_A(-s_*) \tag{31}$$

where we employed repeatedly the fact that $\sigma_C(\cdot) = \sigma_{hull(C)}(\cdot)$. Using similar steps and the monotonicity of $\sigma_{(\cdot)}(s)$, it can be shown that $\sigma_{A'}(s) > \sigma_{\{y\}}(s)$ implies $\sigma_A(s) = \sigma_B(s)$ for $s \in S^N \setminus N_\varepsilon(-s_*)$. Secondly, we want to show that for any lottery $z \in \Delta(Z)$ with $\lambda \sigma_{\{z\}}(-s_*) \geq \lambda \sigma_B(-s_*)$, we have $W(\sigma_{hull(B \cup \{z\})}) = W(\sigma_{hull(A \cup \{z\})})$.

For $C \in \{A, B\}$ we have

$$W(\sigma_{hull(C \cup \{z\})}) = \int_{S^N \setminus N_\varepsilon(-s_*)} \sigma_{hull(C \cup \{z\})}(s) \mu(ds) + \int_{N_\varepsilon(-s_*)} \sigma_{hull(C \cup \{z\})}(s) \mu(ds) \tag{32}$$

Since $\mu(N_\varepsilon(-s_*)) = 0$ and $|\sigma_{hull(C \cup \{z\})}(s)| \leq |\sigma_{\Delta(Z)}(s)| < \infty$ because $\Delta(Z)$ is compact, we have

$W(\sigma_{hull(C\cup\{z\})}) = \int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(C\cup\{z\})}(s)\mu(ds)$. On the other hand,

$$\int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(A\cup\{z\})}(s)\mu(ds) = \int_{S^N\setminus N_\varepsilon(-s_*)} \sigma_{hull(B\cup\{z\})}(s)\mu(ds) \tag{33}$$

because $\sigma_A(s) = \sigma_B(s)$ for $s \in S^N\setminus N_\varepsilon(-s_*)$. Thus, we have $W(\sigma_{hull(B\cup\{z\})}) = W(\sigma_{hull(A\cup\{z\})})$ as desired. Finally, note that standard results guarantee that A and B are compact sets since A' is finite and $A\cup\, y$ is compact.

This completes the proof of the necessity of the axioms for the representation. We mention here that this slightly elaborate construction of the set A is necessary. Thus, note that it would have not been enough to select a lottery $x_s \in U_s(y) \cap L_{-s_*}(y)$ for each $s \in S^N\setminus N_\varepsilon(-s_*)$ appealing to the Axiom of Choice and then define $A \equiv cl\left(\cup_{s\in S^N\setminus N_\varepsilon(-s_*)} x_s\right)$. This is because $x_s \in L_{-s_*}(y)$ for all $s \in S^N\setminus N_\varepsilon(-s_*)$ would not necessarily imply $\sup_{x\in A}(x\cdot(-s_*)) < y\cdot(-s_*)$ as needed in order to show the required condition that $\sigma_{\{y\}}(-s_*) > \sigma_A(-s_*)$. On the other hand, $\cap_{s\in S^N\setminus N_\varepsilon(-s_*)} U_s(y)$ is in general not necessarily nonempty so we cannot just take an element in the intersection of this set with $L_{-s_*}(y)$ and let A be that element. An alternative approach would be to take some element $y' \in L_{-s_*}(y) \cap int(\Delta(Z))$ and then to try take elements $x_s \in U_s(y) \cap L_{-s_*}(y')$ with the aim of obtaining the strict condition $\sigma_{\{y\}}(-s_*) > \sigma_{\{y'\}}(-s_*) \geq \sigma_A(-s_*)$. However, this approach also runs into problems because even though $U_s(y)\cap L_{-s_*}(y') \neq \varnothing$ we cannot insure in general that $U_s(y)\cap L_{-s_*}(y')\cap\Delta(Z) \neq \varnothing$ as necessary to obtain $A \subset \Delta(Z)$. ■

A3. Proof of Theorem 15

Since the preferences $\succeq$ satisfy Weak Order, Independence and Continuity, Theorem 2 in DLR (2001) shows that the function that represents these preferences must be unique up to an affine transformation. Thus, if $V_i(A) = \int_S \left[\max_{z\in A}(z\cdot s)\right]\mu_i(ds) - \theta_i \min_{x\in A}\left[\int_S (x\cdot s)\mu_i(ds)\right]$ are two normalized reference-dependent representations of $\succeq$, then $V_1 = \alpha V_2 + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. If $v_i(z) \equiv V_i(\{z\})$ are the corresponding restrictions to the singletons, we must have $v_1 = \alpha v_2 + \beta$. As argued in Appendix B1, for each $i \in \{1,2\}$ there exists $s_*^i \in S^N$ and $\lambda_i \geq 0$ such that $v_i(z) = \lambda_i(z\cdot s_*^i)$ for all $z \in \Delta(Z)$. Moreover, as argued in Appendix B2, we must have $\lambda_i > 0$. Therefore, for any $z \in \Delta(Z)$ we have $\lambda_1(z\cdot s_*^1) = \alpha\lambda_2(z\cdot s_*^2) + \beta$. Because of the normalization $\sum_{k=1}^N s^k = 0$ in S^N, if we take $z^* = (\frac{1}{N},...,\frac{1}{N}) \in \Delta(Z)$ we have $z^*\cdot\, s_*^i = 0$. Thus, $\lambda_1(z^*\cdot s_*^1) = \alpha\lambda_2(z^*\cdot s_*^2) + \beta$ implies $\beta = 0$.

Therefore, $z\cdot(\lambda_1 s_*^1) = z\cdot(\alpha\lambda_2 s_*^2)$ for any $z \in \Delta(Z)$ which in turn implies that $\lambda_1 s_*^1 = \alpha\lambda_2 s_*^2$. To see this, for each $k \in \{1,...,N\}$ take $z_k = (0,...,0,1,0,...,0) \in \Delta(Z)$ with the 1 on k^{th}

position and note that $z_k \cdot (\lambda_1 s^1_*) = z_k \cdot (\alpha\lambda_2 s^2_*)$ implies $(\lambda_1 s^1_*)_k = (\alpha\lambda_2 s^2_*)_k$ where by $(w)_k$ we denote the k^{th} coordinate of a finite dimensional vector w. Thus, s^1_* is an affine transformation of s^2_* which immediately implies that $s^1_* = s^2_*$, because $s^i_* \in S^N$ for $i \in \{1,2\}$ and we know that S^N contains the *unique* normalization of any affine function. On the other hand, as shown in Appendix A for any $A \in \widetilde{K}(\Delta(Z))$ we have $V_i(A) = W_i(\sigma_A) = \int_{S^N} \sigma_A(s)\mu_i(ds)$ where μ_i is the measure from the DLR representation. Since $V_1(A) = \alpha V_2(A)$ we have $\int_{S^N} \sigma_A(s)\mu_1(ds) = \int_{S^N} \sigma_A(s)(\alpha\mu_2)(ds)$ and then Lemma 18 in Sarver (2008) shows that this implies that $\mu_1 = \alpha\mu_2$. But μ_1 and μ_2 are both normalized to be probability measures so it must be that $\alpha = 1$ and then $\mu_1 = \mu_2$. Finally, $\alpha = 1$ together with $\lambda_1 s^1_* = \alpha\lambda_2 s^2_*$ and $s^1_* = s^2_*$ imply $\lambda_1 = \lambda_2$.

Now, recall that at the end of the sufficiency part of the proof of Theorem 13 we used the elements of the DLR representation to define the elements of our normalized reference-dependent representation. More specifically, with a slight abuse of notation we have $\theta_i \equiv \frac{\mu_i(\{-s^i_*\})}{\lambda_i + \mu_i(\{-s^i_*\})}$ and $\widetilde{\mu}_i(ds) \equiv \frac{\mu_i(ds)}{\mu_i(S^N) - \mu_i(\{-s^i_*\})}$ for $i \in \{1,2\}$. Since $s^1_* = s^2_*$, $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$ it follows that $\theta_1 = \theta_2$ and $\widetilde{\mu}_1 = \widetilde{\mu}_2$ which completes the proof of Theorem 15. ■

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