Controllability, Observability, Regularity, and the so-called Problem of Transforming Values into Prices of Production

Theodore Mariolis

Department of Public Administration, Panteion University

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THEODORE MARIOLIS**

ABSTRACT

This paper specifies and interprets those direct relations which exist between the dual concepts of complete controllability/observability (Kalman, 1960), on the one hand, and the concept of the regular technique of production (Schefold, 1971), on the other. Specifically, it shows, first, that there is a certain dynamic system for determining labour values, which is connected with the usual system for determining prices of production (à la Sraffa, 1960, Part I) via the z-(Laplace) transform, and, second, that the said system of values is completely controllable when and only when the system of production prices is regular. In view of the above, it could be considered that the z-(Laplace) transform constitutes the solution to the – suitably reformulated – Marxian 'problem of transforming values into prices of production'. However, on the basis of an economic interpretation of the z-(Laplace) transform, this paper shows, ultimately, not only that such a consideration is erroneous, but also that the supposed 'problem' is devoid of economic meaning.

Key words: Control theory; controllability; observability; regular production technique; Sraffian theory; transformation problem

JEL classifications: B51, C61, C67, D46, D57, E11

1. Introduction

As is well known, the dual concepts of *controllability* and *observability*, which were first introduced by Kalman, 1960, are fundamental to the so-called 'Modern Theory of Control Systems'.[1] On the other hand however, as is also known, the concept of the regular system/technique of production, which was first introduced by Schefold, 1971, pp. 19-20, 1976, is fundamental to

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** Department of Public Administration, Panteion University, Athens, Greece; E-mail: mariolis@hotmail.gr

[1] Regarding the application of these concepts to the theory of economic policy, see, for example, Aoki, 1976, chs 3, 5 and Wohltmann, 1981, while regarding the relationships between the Traditional and Modern Theory of Control Systems, see Gilbert, 1963, Kalman, 1963 and, for example, D'Azzo and Houpis, 1988, chs 1, 13, 18-20 (for a brief discussion, see § 2.2 of the present paper).

the modern 'Theory of Production', and in particular for investigating the relations between long-period relative prices of commodities, the distribution of income and the technical conditions of production, within the framework of production systems *à la* Sraffa, 1960.[2] In this paper, first, we shall show that there is a one-to-one correspondence between a completely controllable and completely observable linear dynamic system and a regular linear system/technique of single production, and, second, on the basis of this relationship, we shall present a critique of the so-called 'problem of transforming values into prices of production'. Although this critique is connected with what has already been pointed out regarding the aforementioned 'problem' (see, primarily, Steedman, 1977, 1985, and consider Steedman and Tomkins, 1998, Mariolis, 1999, pp. 45-53, 2002), we believe that it makes a further contribution to highlighting those *insurmountable* problems which are associated with any attempt to find an economically meaningful relationship between values and prices of production.

Part 2 of the paper presents the main – for the investigation which follows – definitions and theorems.[3] Part 3 specifies the relationship which exists between a completely controllable and completely observable linear dynamic system and a regular system/technique of single production. Part 4 presents a critique of the 'transformation problem'. Part 5 summarises the conclusion of this investigation.

2. Main Definitions and Theorems

2.1. As is well known, an important class of linear, time-invariant, dynamic systems (e.g., physical or economic) can be *described* by the following system of difference equations (regarding the axiomatic foundation of this description: Kalman, 1963, pp. 154-7 and 189-90):

[2] Regarding these issues, see Bidard and Salvadori, 1995, Kurz and Salvadori, 1995, ch. 6. It should be noted that the definition of the regular production system, which was initially provided by B. Schefold, contains a superfluous condition (see Bidard and Salvadori, 1995, p. 389).

[3] Those who are familiar with the concepts of controllability and observability need not dwell on this Part of the paper.

$$v(t+1) = v(t)A + u(t)\ell, \quad t = 0,1,... \qquad (1a)$$

$$y(t) = v(t)b \qquad (1b)$$

where $v(t) \equiv [v_j(t)]$, j = 1,2,…, n, is the real 1×n *state vector* of the system, *A* is the real, constant, n×n *system matrix* (also known as 'plant coefficient matrix'), *u*(*t*) is the *input* of the system, which constitutes a scalar function of time (also known as 'one-dimensional control vector'), ℓ is a real, constant, 1×n vector, b is a real, constant, n×1 vector and *y*(*t*) is the *output* of the system.[4]

The *state* of a dynamic system is a mathematical structure containing the *n* variables $v_j(t)$, that is, the so-called *state variables*. The *initial values* $v_j(0)$ of these variables and the input *u*(*t*) of the system are sufficient for uniquely determining the system behaviour for any *t* ≥ 0. The state variables need *not* be observable and measurable quantities (e.g., from a physical or, respectively, economic viewpoint). They may be purely mathematical, abstract, quantities. On the contrary, the input and output of the system are concrete, observable and measurable quantities, that is, quantities which have a concrete meaning (e.g., physical or, respectively, economic). *State space* is the *n*-dimensional space, in which the components of the state vector represent its coordinate axes. Lastly, it should be noted that the choice (by the theorist) of state variables, that is, the choice of the *smallest* possible set of variables for uniquely determining the future behaviour of the system, is *not* unique. Of course, to each specific choice of the said variables corresponds a uniquely determined state of the system (regarding all these issues, see also Luenberger, 1979, ch. 4, D'Azzo and

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[4] The equations (1a, b) express a first-order, single-input single-output, time-invariant and linear discrete-time system. *Ceteris paribus*, the analog of (1a) in continuous time is the following system of differential equations

$$\dot{v}(t) = v(t)\left[A - I\right] + u(t)\ell \qquad (1aa)$$

where $\dot{v}(t)$ is the derivative of *v*(*t*) and I is the n×n identity matrix. In what follows, we shall be dealing only with systems which are sufficiently described by the equations (1a, b) *or* by (1aa, b). However, for brevity's sake, we shall refer to the latter equations only when we consider this to be completely necessary.

Houpis, 1988, chs 2 and 5).[5]

As far as the issues of interest to us here are concerned, only the following definitions and theorems are of particular importance (Gilbert, 1963, Kalman, 1963, Johnson, 1966, Ford and Johnson, 1968, Luenberger, 1979, chs 5 and 8, Kuo, 1995, ch. 5):

Definition 2.1.1. The two descriptions $[A,\ell,b]$ και $[A^*,\ell^*,b^*]$ of the dynamic system under consideration are said to be *strictly equivalent* when their state vectors $v(t)$ and, respectively, $v^*(t)$ are related for all t as follows:

$$v^*(t) = v(t)T \qquad (2)$$

where T is a nonsingular, constant, n×n matrix.

Remark: As can easily be shown (on the basis of (1a,b) and (2)), the existence of strict equivalence between the aforementioned descriptions entails the validity of the following relations (and vice versa):

$$A^* = T^{-1}AT \qquad (3a)$$

$$\ell^* = \ell T \qquad (3b)$$

$$b^* = T^{-1}b \qquad (3c)$$

which also define, evidently, a 'similarity transformation by the similarity matrix T' or, which is the same thing, a change of the coordinate system in the state space.

Definition 2.1.2. The system (1a) (that is, the description $[A,\ell]$) is said to be completely controllable if the initial state $v(0) = 0$ can be transferred, by application of inputs $u(t)$, to any state, in some finite time. In the opposite case, it is said to be uncontrollable.

Theorem 2.1.1. The system (1a) is completely controllable iff

[5] As underlined by Kalman, 1963, p. 154, the present axiomatic description of a dynamic system is based on the 'Newtonian Principle of Causality', according to which "the motion of a system of particles is fully determined for all future time by the present positions and momenta of the particles and by the present and future forces acting on the system. How the particles actually attained their present positions and momenta is immaterial. Future forces can have no effect on what happens at present". Therefore, the present description is entirely unsuitable for those quite complex systems (mental, economic, social), in the framework of which the future (to be precise: the expectations, hopes, fears, etc. of the subjects vis-à-vis the future) affects the present and which, consequently, cannot possibly be described without the category of 'futurity' (see e.g., Willke, 1993, § 4.3). As is well known, the models for determining exchange rates (see e.g., Mussa, 1979) constitute typical examples

the n×n matrix *L* (*controllability matrix*), whose rows are the vectors $\ell, \ell A, \ell A^2, ..., \ell A^{n-1}$, has rank n.

Remark: It emerges from this theorem that if the system is completely controllable (according to the aforementioned definition), then each initial state can be transferred to each final state, within n steps (regarding this delicate issue: Luenberger, 1979, pp. 277 (footnote) and 279).

Definition 2.1.3. The system (1a, b) (that is, the description $[A, \ell, b]$) is said to be completely observable if the knowledge of the values of the input and of the output, over a finite interval of time, is sufficient to determine the value of the initial state *v*(0). In the opposite case, it is said to be unobservable.

Theorem 2.1.2. The system (1a, b) is completely observable iff the n×n matrix *B* (*observability matrix*), whose columns are the vectors $b, Ab, A^2b, ..., A^{n-1}b$, has rank n.

Remark: It emerges from this theorem that when the system is completely observable, the knowledge of *n* values of input and of output is sufficient for determining *v*(0) (Luenberger, 1979, p. 287). In addition, it may be easily deduced from the above theorems that the properties of complete controllability and observability are independent of the choice of basis in the state space, i.e., that all the strictly equivalent descriptions of a dynamic system are also equivalent from the viewpoint of complete controllability/observability. Furthermore, in order to give a clear representation of the said dual properties, let us assume that the matrix *A* has *n* linearly independent eigenvectors, and consequently that it is diagonalisable (regarding the opposite case, the investigation of which is based on the reduction of matrix *A* to an upper or lower triangular matrix, see, for example, Chen and Desoer, 1968). So, if we transform *A* into matrix $T^{-1}AT$, where the columns of *T* are the right eigenvectors of *A*, then we may easily conclude (consider (2), (3a-c)) that the system is completely controllable (observable) iff vector ℓ (vector *b*) is not orthogonal to any right (left) eigenvector of

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of such systems.

A. For, when and only when this condition is violated, is matrix $LT(T^{-1}B)$, and therefore also matrix *L*(*B*) non-singular.[6] This implies, of course, that there is at least one component of v*(t), which is *not* affected by (does *not* affect) the input (the output) of the system. Thus, those components of *v**(*t*), which correspond to the zero components of $\ell*(b*)$, constitute the uncontrollable (unobservable) state variables of the system.

Theorem 2.1.3. If and only if the system (1a, b) is not completely controllable and completely observable, is it strictly equivalent to a system of the type

$$v*(t+1)=v*(t)\begin{bmatrix} A_{11}^* & 0 & 0 & 0 \\ A_{21}^* & A_{22}^* & 0 & 0 \\ A_{31}^* & 0 & A_{33}^* & 0 \\ A_{41}^* & A_{42}^* & A_{43}^* & A_{44}^* \end{bmatrix}+u(t)\left[\ell_1^*,\ell_2^*,0,0\right] \quad (4a)$$

$$y(t)=v*(t)\begin{bmatrix} 0 \\ b_2^* \\ 0 \\ b_4^* \end{bmatrix} \quad (4b)$$

where $v^* \equiv \left[v_1^*, v_2^*, v_3^*, v_4^*\right]$, v_k^* is an $1\times n_k$ vector (*k* = 1,…, 4 and $n=n_1+\ldots+n_4$) and $A_{mk}^*, \ell_k^*, b_m^*$ (m = 1,…, 4) are submatrices and subvectors of suitable dimensions. Thus, the overall system is split into four, mutually exclusive, subsystems (which is why the present theorem of R. E. Kalman is called the 'decomposition theorem'): subsystem 1 is completely controllable and unobservable, subsystem 2 is completely controllable and completely observable, subsystem 3 is uncontrollable and unobservable, and subsystem 4 is uncontrollable and completely observable.

Remark: It is shown that the aforementioned four subsystems are

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[6] If there is an eigenvalue $\tilde{\lambda}$ of matrix *A*, to which two linearly independent eigenvectors correspond, then there is no vector ℓ(b) for which the system is completely controllable (observable). For, in this case, *each* linear combination of eigenvectors of $\tilde{\lambda}$ is an eigenvector of *A* and, consequently, there is always an eigenvector of *A*, which is orthogonal to each given ℓ(b). Thus, the rank of *L* (of matrix *B*) is less than *n*. Furthermore, it is possible to show that the system is always uncontrollable and unobservable, irrespective of the choice of ℓ and b, iff the matrix *A* satisfies a polynomial equation of degree less than *n* (regarding all this: Johnson, 1966, Ford and Johnson, 1968).

not uniquely determined, that is, the numerical values of A^*, ℓ^*, b^* are not uniquely determined. In fact, depending on the choice of matrix *T*, it is possible for certain submatrices of *A** to be made equal to zero (see Kalman, 1963, pp. 165-7 and 172-5).

2.2. Apart from the previous method of describing a dynamic system, i.e., by means of state variables, there is also, as is well known, *another* method of description, in the context of which the relations between the input and the output of the system are represented by means of the so-called *transfer function* of the system. This latter method of description constituted the basis of the traditional theory of control, while the former constituted the basis of the modern theory. The relations existing between these two descriptions (or 'languages', as Kalman refers to them, 1963, p. 152), which interest us here, can be rendered in brief as follows (Aseltine, 1958, chs 1-3, 8-9 and 16, Gilbert, 1963, Kalman, 1963, D'Azzo and Houpis, 1988, chs 4, 13 and 22, Luenberger, 1979, ch. 8, Kuo, 1995, pp. 281-99):

Definition 2.2.1. The *z-transform* of a given sequence of numbers $y(0)$, $y(1)$, $y(2)$, …,is the series

$$Y(z) \equiv \sum_{t=0}^{\infty} \left[y(t)/z^t \right] \tag{5}$$

where *z* is a complex variable.

Remark: On the condition that the series (5) converges, the z-transform thus converts a sequence of numbers defined in the real domain into an expression defined in the complex *z*-domain.

Definition 2.2.2. The transfer function of the system (1a, b) is said to be the ratio of the *z*-transform of the output to the *z*-transform of the input, when the initial conditions are all zero ($v(0)=0$).

If, therefore, we apply the *z*-transform to (1a, b), then we get[7]

$$zV(z) - zv(0) = V(z)A + U(z)\ell \tag{6a}$$

$$Y(z) = V(z)b \tag{6b}$$

[7] As can easily be shown (on the basis of (5)), given a sequence $y(t)$, the sequence $y(t+1)$ has the *z*-transform z $z[Y(z) - y(0)]$.

where *V*(*z*), *U*(*z*), *Y*(*z*) are, respectively, the *z*-transforms of *v*(*t*), *u*(*t*) and *y*(*t*). Thus, setting *v*(0) = 0 results in the following:

$$V(z) = U(z)\ell\left[zI - A\right]^{-1} = U(z)\ell(1/z)\left[I - A(1/z)\right]^{-1} \tag{7a}$$

$$H(z) \equiv \left[Y(z)/U(z)\right] = \ell\left[zI - A\right]^{-1} b \tag{7b}$$

where *H*(*z*) is the transfer function of the system.[8] So, *H*(*z*) is a rational function which, it appears, completely represents the components A, ℓ, b of the system description (this perception of completeness which is fundamental to the traditional theory of control, see e.g., Aseltine, 1958, ch. 9, is in reality, as shown within the context of the modern theory and as we shall see below, erroneous). The values of *z* that make the denominator of the transfer function equal to zero (which denominator is, evidently, a polynomial of degree no greater than *n*) are called *poles* of the system (and are none other than the eigenvalues of matrix *A*), while the values of *z* which make the numerator equal to zero (the numerator is, evidently, a polynomial of degree no greater than *n*-1) are called *zeros*. The location of the poles and the zeros in the *z*-complex plane determines, as is well known, the transient response of the system, which can be specified by finding the inverse *z*-transform (or the inverse Laplace transform, in the continuous-time case) of functions *H*(*z*), *Y*(*z*) (e.g. see D'Azzo and Houpis, 1988, chs 4 and 6-7).

Theorem 2.2.1. If and only if the system (1a, b) is not completely controllable and completely observable, does the trans-

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[8] Respectively, the transfer function of the system (1aa, b) is said to be the ratio of the Laplace transform of the output to the Laplace transform of the input, when the initial values of the state variables are all zero. As is well known, the Laplace transform of a function y(t), t ≥ 0, is (by definition)

$$Y(s) \equiv \int_0^\infty y(t) e^{-st} dt \tag{5a}$$

where s is a complex variable. Thus, setting v(0) = 0 in (1aa, b) results in the following:

$$V(s) = U(s)\ell\left[(1+s)I - A\right]^{-1} \tag{7aa}$$

$$H(s) \equiv \left[Y(s)/U(s)\right] = \ell\left[(1+s)I - A\right]^{-1} b \tag{7ba}$$

where *V*(*s*), *U*(*s*), *Y*(*s*) are, respectively, the Laplace transforms of *v*(*t*), *u*(*t*), *y*(*t*) and *H*(*s*) is the transfer function of the system. Lastly, regarding the general relations between the *z*-transform and the Laplace transform, see e.g., Aseltine, 1958, Appendix A, D'Azzo and Houpis, 1988, ch. 22.

fer function represent *only* the completely controllable and completely observable subsystem of the system under consideration. That is, the following hold (consider relations (4a, b) and (7a, b)):

$$\ell\left[zI - A\right]^{-1} = \ell^{*}\left[zI - A^{*}\right]^{-1}T^{-1} \tag{8a}$$

$$\left[zI - A\right]^{-1}b = T\left[zI - A^{*}\right]^{-1}b^{*} \tag{8b}$$

$$H(z) = \ell^{*}\left[zI - A^{*}\right]^{-1}b^{*} \Rightarrow$$

$$H(z) = \ell_{2}^{*}\left[zI_{2} - A_{22}^{*}\right]^{-1}b_{2}^{*} \tag{8c}$$

where I_2 is the $n_2 \times n_2$ identity matrix, while, as can easily be seen from (8a, b), the vector $\ell[zI - A]^{-1}$ (and respectively the vector $[zI - A]^{-1}b$) represents only the completely controllable (the completely observable) subsystems, i.e., subsystems 1 and 2 (2 and 4).

Theorem 2.2.2. If the transfer function has pole-zero cancellation, the system will be either uncontrollable or unobservable, or both, depending on how state variables are defined. On the other hand, if the transfer function does not have pole-zero cancellation, the system can always be represented by dynamic equations as a completely controllable and observable system.

It emerges, therefore, from these two theorems that only if the system (1a, b) is completely controllable and completely observable is the description of the system in the state space completely equivalent to its description by means of the transfer function. In other words, only in this case does the resulting transfer function contain *all* the information which characterises the dynamic behaviour of the system or, which is the same thing, the knowledge of the transfer function is sufficient for uniquely determining the dynamic equations (1a, b) of the system.[9]

[9] It is possible to show (Kalman, 1963, Luenberger, 1979, ch. 8) that precisely what was said in this Part of the paper about the discrete-time case applies also to the continuous-time case (with the exception of one modification, which concerns the formulation of the definitions of controllability and observability but which, however, does not alter the meaning of these concepts).

3. Controllability, Observability, and Regularity of a Production Technique

We assume a linear, profitable and indecomposable single production technique *à la* Sraffa, 1960, Part I, which is defined by the pair $[A,\ell]$, where A (≥ 0) is the semi-positive n×n matrix of material inputs and $\ell(>0)$ the positive 1×n vector of labour inputs.[10] The profitability of the technique means that the Perron-Frobenius eigenvalue λ^* of A is smaller than unity, while the indecomposability of the technique means that all the commodities produced are basic *à la* Sraffa, 1960, §§ 6-7 (regarding these issues, see, for example, Kurz and Salvadori, 1995, ch. 4).

If we assume, furthermore, that wages are paid at the beginning of the production period, then for a given uniform rate of profit r the 1×n vector of prices of production $p(r)$ and the uniform nominal wage rate $w(r)$ are determined up to a factor by the system

$$p(r) = \left(p(r)A + w(r)\ell\right)(1+r) \tag{9}$$

The economically meaningful interval of the rate of profit is $0 \leq r \leq R$, where $R \equiv (1-\lambda^*)/\lambda^*$ is known as 'the maximum rate of profit of the technique $[\mathrm{A},\ell]$', because only within this interval is $p > 0$ *and* $w \geq 0$. Lastly, prices are normalised by setting

$$p(r)b = c \tag{10}$$

where b is a given semi-positive n×1 vector[11] and c a positive constant, the dimension of which is units of money per unit of commodity b, while (9) and (10) give a continuous and strictly decreasing (for $-1 < r \leq R$) function between nominal wage rate and rate of profit, that is, the so-called '*w*-*r* relationship or trade-off' (regarding all the above, see, for example, Kurz and Salvadori, 1995, ch. 4):

$$-1 < r < R \Rightarrow p(r) = w(r)\ell(1+r)\left[I - A(1+r)\right]^{-1}$$

[10] The reader will ascertain later on (Part 4) why we introduce exactly the same symbols, (A, ℓ), as those in Part 2.

[11] Also regarding the symbol *b*, the comment made in the previous footnote (10) applies.

$$\Rightarrow w(r) = c \Big/ \left\{ \ell(1+r)\left[I - A(1+r)\right]^{-1} b \right\} \qquad (11a)$$

$$r = R \Rightarrow w = 0 \qquad (11b)$$

Regarding the issues of interest here, suffice it for us to refer only to the following (Schefold, 1971, pp. 19-20, 1976, Miyao, 1977, Bidard and Salvadori, 1995):

Definition 3.1. If the n×n matrix *L* (*labour profile matrix*), whose rows are the vectors $\ell,\ \ell A,\ \ell A^2, \dots,\ \ell A^{n-1}$, has rank *n*, the technique $[A, \ell]$ is said to be *regular*. In the opposite case it is said to be *irregular*.

Theorem 3.1. We assume that the matrix *L* has rank *m* ($1 \le m < n$). Then, and only then, are there *m* commodities, the prices of which are linearly independent (that is, there is no nonzero vector e, such that $p_m e = 0$, where p_m is the vector of prices of these *m* commodities) and *n-m* commodities, the prices of which can be expressed as constant linear combinations of those m prices for $-1 < r \le R$.

Theorem 3.2. Let $[A, \ell]$ and $[A', \ell']$ be two n×n techniques and let $m = \min\{rankL, rankL'\}$, where *L′* is the labour profile matrix of the latter technique. The following statements are equivalent:

(i) the prices and wage rates of the two techniques coincide at *m+1* different profit rates, $-1 < r < \min(R, R')$ (where *R′* is the maximum rate of profit of the technique $[A', \ell']$.

(ii) the prices and wage rates of the two techniques coincide at each profit rate, $-1 < r < R = R'$, and $m = rankL = rankL'$.

(iii) The two techniques are connected through the relations $A' = A + C$ and $\ell = \ell'$, where *C* is a matrix, for which *LC* = 0 holds. Moreover, if *m* = *n*, then[12] $A' = A$.

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[12] As is well known, it is entirely possible to consider that the following quantity system (see, e.g. Kurz and Salvadori, 1995, ch. 4) corresponds to the price system (9), (10):

$$x = (Ax + db)(1+g) \qquad (9a)$$

$$\ell x \equiv 1 \qquad (10a)$$

where *x* is the n×1 vector of gross outputs per unit of labour employed, *b* is a given semi-positive n×1 vector, which represents the composition of consumption, *d* is a scalar, which represents the level of consumption per unit of labour employed, and *g* is the uniform (by assumption) rate of growth of the sys-

If we assume that there is a *dynamic* economic system, which can be described by the equations (1a, b) (or, respectively, by the equations (1aa, b)), where A, ℓ express the technical data of production, then we may, taking into consideration that which has already been set out, conclude the following (the tenability and content of the aforementioned assumption will be discussed in Part 4 of this paper):

a) Through the *z*-transform (or the Laplace transform, in the continuous-time case), the relations between $v(t)$, $u(t)$, $y(t)$, which are defined in the *time* domain, are converted (compare (7a, b) or, respectively, (7ab, ba) with (11a)) into relations $p(r)/w(r)$, $p(r)b/w(r)$, which are defined in the *profit rate* domain[13] $r \equiv (1-z)/z(r \equiv -s/(1+s))$ in the continuous-time case).

b) Although complete controllability and regularity relate to different systems, which are however interconnected via the *z*-transform (or the Laplace transform), their coexistence is always given. Just as uncontrollability, firstly, means that there are state variables, which are not affected (either directly or indirectly) by the input of the system, and secondly, that knowledge of the vector $V(z)/U(z)(=\ell[zI-A]^{-1})$ is not sufficient for uniquely determining the system (1a) (or, respectively, (1aa)), irregularity means, first, that there are commodities, the *relative* prices of which are not affected by changes in the distribution of income[14], and, second, knowledge of the vector

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tem. Thus, whatever holds for the relations between *p*, *w*, *r* and the rank of matrix *L*, holds analogously also for the relations between *x*, *d*, *g* and the rank of the matrix, the columns of which are the vectors $b, Ab, A^2b, ..., A^{n-1}b$ (compare, therefore, with *Theorem 2.1.2*). Lastly, for dynamic versions of the systems (9), (10) and (9a), (10a), which can be analysed on the basis of the concepts of controllability and observability, see e.g., Aoki, 1976, pp. 31-3 and 86-7, Takayama, 1987, pp. 503-21. Thus, for example, it holds that the technique is regular when and only when the following dynamic version of the system of prices (in which it is assumed that the rate of profit is *exogenously* given and constant)

$$p(t+1) = (p(t)A + w(t)\ell)(1+\bar{r}) \qquad (12)$$

is completely controllable.

[13] It should be stressed that there is *no* transformation (contrary to what one might for a moment believe) of $u(t)$ into $w(r)$ and $v(t)$ into $p(r)$. We shall return to this issue later (Part 4).

[14] In the special (and extreme) case in which ℓ is the left eigenvector of A ($\Leftrightarrow rankL = 1$), $p(r)$ and ℓ are linearly dependent (this is the familiar case of the existence of a 'uniform organic composition of capital'). If, in addition,

$p(r)/w(r)(=\ell(1+r)[I-A(1+r)]^{-1})$ at $m+1$ different values of the rate of profit, $-1<r<R$, is not sufficient for uniquely determining the production technique.[15]

c) The reciprocal of the wage rate measured in terms of commodity *b* (or, equivalently, the 'labour commanded' price of commodity *b*) constitutes the transfer function of the system. Consequently, the w-r relationship expresses *only* the completely controllable and completely observable subsystem of the system under consideration.[16]

4. On the Transformation Problem

We assume a system, which uses the production technique $[A,\ell]$ and let *v* be the 1×n vector of quantities of labour 'embodied' in the different commodities (or labour values). As is well known, the vector *v* is determined (by definition) by the following system:

$$v \equiv vA+\ell \Rightarrow v \equiv \ell\left[I-A\right]^{-1} \tag{13}$$

If we assume, now, that, first, a technical change takes place, which is characterised by the relations $A(t)=A$ and $\ell(t)=\ell u(t)$, where $u(t)$ is a positive scalar function of time, and, second, inputs occur at time *t*, while gross output appears at *t*+1, then the labour values are determined by the following system: [17]

$$v(t+1)=v(t)A+\ell(t)=v(t)A+u(t)\ell \tag{14}$$

which is none other than the system (1a). Consequently, by com-

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$v(0) = 0$ holds, then $v(t)$ $(t>0),\ \ell$ are also linearly dependent.

[15] Put differently, "exactly $n^2 + n$ magnitudes are needed to describe a regular technique with *n* commodities" (Kurz and Salvadori, 1995, p. 175).

[16] It easily emerges from the application of *Theorems 2.1.3 and 2.2.1* to the system of prices of production (9), (10) and from the Perron-Frobenius theorems for semi-positive matrices that $p_3^*=0,\ p_4^*=0$ and that the Perron-Frobenius eigenvalue of *A* constitutes an eigenvalue of the submatrix A_{22}^* (where the subsystems 1, 2, 3, 4 are defined in agreement with *Theorem 2.1.3*). It is, lastly, evident that with the application of *Theorem 2.1.3* the system of prices under consideration may be partitioned into basic and non-basic subsystems *à la* Sraffa, 1960 (of course, these subsystems are not always economically meaningful precisely because A_{mk}^*, ℓ_k^* can also contain negative components).

[17] For a version of this system of values, as well as for the system of production prices which corresponds to it, see e.g., Wolfstetter, 1973, pp. 804-7, Okishio, 1993, pp. 90-4.

bining what was expounded in Parts 2 and 3, we are in a position to conclude the following:

a) Traditionally, the so-called 'problem of transforming values into prices of production' consists in investigating the quantitative relations between systems (13) and (9) (for a systematic presentation of the issue, see e.g., Pasinetti, 1977, ch. 5, Appendix). This investigation shows that the said 'problem' has absolutely no economic meaning (see mainly Steedman, 1977, 1985 and, further, Mariolis, 1999, 2002). The present analysis showed that there is a specific quantitative relation between systems (14) and (9): if we assume that $v(0) = 0$ holds, the vector of 'labour commanded' prices is equal to the ratio of the z-transform (where $z \equiv 1/(1+r)$) of the time sequence of the vector of values $(0, \ell u(0), \ell A u(0) + \ell u(1), ...)$ to the z-transform of the index of technical change sequence $\left(u(0), u(1), u(2), ...\right)$, that is, the following holds

$$p(r)/w(r) = V(z)/U(z) \Rightarrow p(r) = V(z)\left[w(r)/U(z)\right] \qquad (15)$$

Thus, the content and economic meaning of this transformation should be examined.

b) If the definition of the z-transform (see *Definition 2.2.1*) is *interpreted* from an economic viewpoint, it is possible to say that the magnitudes $V(z)$, $U(z)$ represent the 'present value' – computed on the basis of the rate of interest z-1 – of the streams of the vector of values $v(t)$ and, respectively, of the index of technical change $u(t)$ (when period 0 represents the 'present'). It follows, therefore, that if these 'present values' are computed on the basis of an interest rate equal to $-r/(1+r)$, then their ratio is equal to the vector of 'labour commanded' prices[18].

c) However, this result should not be overestimated. Because, first, just as (9) is written (for $-1 < r < R$)

[18] It may be deduced from the economic interpretation of the Laplace transform (see (5a)) that the magnitudes $V(s)$, $U(s)$ represent the 'present value' – computed on the basis of interest rate s – of the streams of $v(t)$ and, respectively, of $u(t)$. Thus, whatever holds for the discrete-time case also holds, analogously, for the continuous-time case.

$$p = w\ell(1+r)\left[I - A(1+r)\right]^{-1} = w\sum_{k=0}^{\infty}\ell A^k (1+r)^{k+1} \qquad \text{(9aa)}$$

and expresses (can be said to express) the making equal to zero of the 'present values' of the following *n* 'flow input – point output' production processes (see Sraffa, 1960, ch. 6, Howard, 1980, Ahmad, 1991, Part II)

$$\left\{..., w\ell A^k, w\ell A^{k-1}, ..., w\ell A, w\ell, 0\right\} \to \left\{..., 0, 0, ..., 0, 0, p\right\} \qquad \text{(16)}$$

so too (15) is written

$$p(r)U(z) = w(r)V(z) \qquad \text{(15a)}$$

and expresses (can be said to express) the making equal to zero of the 'present values' of the following *n* 'flow input – flow output' production processes[19]

$$\left\{0, wv(1), wv(2), wv(3), ...\right\} \to \left\{pu(0), pu(1), pu(2), pu(3), ...\right\} \qquad \text{(17)}$$

Second, because (15) does not express a transformation of $v(t)$ into $p(r)$. In reality, it expresses a 'transformation' of *prices into prices*, on the one hand because *r* enters into $V(z)$ and $U(z)$ and, on the other, because through $w(r)$, $p(r)$ enters into the 'transformation operator' $(w(r)/U(z))$ of $V(z)$ into $p(r)$. Besides, in order for there to indeed be a transformation of $v(t)$ into $p(r)$ there should be a transformation of the 'labour' dimension into the 'money' dimension, which, however, is both non-existent and impossible (as has been shown in the context of the critique of the 'transformation problem' which already appears in the literature). For precisely this reason, the 'transformation' of the *j* component of $V(z)$ (the dimension of which is units of labour per unit of commodity *j*) into the component $p_j(r)$ necessarily (see (15)) takes place through $w(r)$, that is, through a magnitude whose dimension is units of *money* per unit of labour.[20] In

[19] In the specific case, in which $u(t)$ is the so-called 'unit impulse function', that is $u(t) = 1$, at $t = 0$, and $u(t) = 0$, for $t \geq 0$, holds, $U(z) = 1$ results and, thus, (9aa) and (15a) (or, equivalently, (16) and (17)) coincide. For the definition of the unit impulse function in the continuous-time case, the application of which to the system leads to an equivalent result (precisely because $U(s) = 1$ holds), see e.g., Aseltine, chs 3 and 9.

[20] It could be considered that this issue does not arise when $u(t)$ is such that $U(z) = w(r)(\Leftrightarrow V(z)b = 1)$ results or when $w(r)$ is a non-dimensional magnitude (e.g., because prices can be normalised by setting $p(r)b = vb$). However, the former of these two cases is non-existent (as may easily be deduced from the

other words, whatever holds for the relation between $p(r)$ and v, which is the following (deduced from (9) and (13)):

$$p(r) = vT(r) \qquad (18)$$

where $T(r) \equiv w(r)\left[I - Dr\right]^{-1}(1+r)$ is the 'transformation operator' and $D \equiv A[I - A]^{-1}$, also holds (analogously) for the relation between $p(r)$ and $v(t)$. Third, the validity of (15) is not always given, because it not only presupposes the invariability of A but also the economically meaningful convergence of the series $\sum_{t=0}^{\infty}[u(t)/z^t]$.

For example, if $u(t) = 1$ holds (in which case no technological change takes place[21]), the said series does not converge for $0 \leq r < R$, while if $u(t) = u^t$ holds, the interval (circle) of convergence of the said series includes the entire economically meaningful interval of r if and only if $u < \lambda^*$. Lastly, even if we disregard all the previous points of the critique, it should be noted that (15) does not express a 'transformation' of $v(t)$ into $p(r)$, but rather a 'transformation' (see (8a), (8c) and Part 3, point c) of the completely controllable and completely observable variables of the system into $p(r)$ (the same holds analogously also for the 'transformation' of v into $p(r)$).

5. Conclusions

On the basis of the principles of the Theory of Control, it was shown that the application of the z-(Laplace) transform to a suitably defined dynamic system for determining labour values leads to a system, which is *analogous* to the usual system for determining prices of production (and *vice versa*). Thus, those direct relations were identified and interpreted, which exist between the dual concepts of complete controllability/observability and the concept of regularity.[22]

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definition of the z-transform and from the equation of $w(r)$), while the existence of the latter case does not constitute, evidently, a *solution* to the said issue. So, for all these reasons, that which was stated in footnote 13 is correct.

[21] In this case the system (14) is asymptotically stable, precisely because the production technique is (by assumption) profitable ($\Leftrightarrow \lambda^* < 1$).

[22] It is clear that the said relationship could be analysed (also) on the basis

If the said transformation indeed constituted a 'transformation of values into prices of production' (with the economic content and meaning of the term), then it should be said that the descriptions of capitalist economic systems in terms of values and in terms of production prices are not always equivalent and, above all, that the former description is the most reliable (because the transition from the former description to the latter entails a loss of information when the system under consideration is not completely controllable/observable). However, because, first, in the context of the capitalist mode of production (as a concept) by definition only the latter description holds,[23] and, second, the *z*-(Laplace) transform (although a specific economic interpretation can be attributed to it) does not constitute a 'transformation of values into production prices', it follows that only the latter description is economically meaningful.

In other words, while in the context of the Theory of Control the loss of information, which possibly takes place in the transition from the description of a system in the state space to its description in terms of the transfer function constitutes a crucial point of the critique of the latter description, in the context of the economic theory of the capitalist mode of production the loss of information, which possibly takes place in the transition from the description of the economic system in terms of labour values to its description in terms of prices of production (and therefore of the '*w*-*r* relationship') is devoid of any meaning whatsoever. To be precise, in fact, the description in terms of labour values is itself devoid of any meaning.

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of systems (12) and (9). In this case, we would have $r \equiv (1+\bar{r}-z)/z$, and thus the rate of interest for computing the 'present values' (see Part 4) would be $(\bar{r}-r)/(1+r)$ (that is, $\bar{r}$ could be considered as the nominal rate of interest and *r* as the rate of inflation). The present discussion, consequently, was conducted on the basis of (14) $(\bar{r}=0, p(t)=v(t))$ and (9), precisely because it aimed at a critique of the 'transformation problem'.

[23] Within the context of this mode of production, the commodities are not exchanged as products of labour but as products of *capital*. Thus, the prices systematically deviate from the labour values.

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