Besicovitch, Sraffa, and the existence of the Standard commodity

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31 July 2010

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by

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1. **Introduction**.

The proof of the existence of the Standard commodity contained in Sraffa's book (section 37) has been debated recently. Lippi (2008) has argued that the algorithm in section 37 of Sraffa's book is not precisely stated and does not need to converge to the desired eigenvalue and eigenvector. The first part of the proposition was known since the proof-reading stage of Sraffa's book when it was sustained by Alister Watson (cf. Kurz and Salvadori, 2001, p. 272-3). But the second part has escaped the attention of all commentators before Lippi. Indeed examples can be found in which an algorithm corresponding to the description provided by Sraffa converges to a vector which is not an eigenvector and it is certainly Lippi's credit to have uncovered the problem. In Appendix A I report the example I provided in a paper in which I investigated what are the properties that an algorithm needs to have in order to converge to the desired eigenvalue and eigenvector (cf. Salvadori, 2008). In an appendix of his paper Lippi provided a complete proof of the existence of the Standard commodity by using a very special algorithm from among all the algorithms corresponding to the description of section 37 and another special algorithm has been provided, without a proof, by Kurz and Salvadori (2001, p. 284). The fact that Sraffa has not chosen a particular algorithm makes us think that he was convinced that any algorithm would do the job. This is wrong, but, as I proved in the mentioned paper (Salvadori, 2008), any algorithm, based on a continuous function, which can start from any feasible point does actually do the job.

In this paper I want to bring some more light on the issue from an historical perspective. A proof of the existence of the Standard commodity was provided to Sraffa by Besicovitch on 21 September, 1944. This proof has not yet been discussed In the literature. In Appendix B there is a transcription of the file D3/12/39: 42 including it. In this paper I will show that also the proof by Besicovitch is incomplete, but it can easily be completed. Once completed, also this proof concerns a family of algorithms, but all the algorithms of the family converges to the desired eigenvalue and eigenvector. Sraffa thought that the exposition of the proof could be simplified. Alas, in carrying out such a simplification he failed.

2. Sraffa' Section 37

Sraffa starts section 37 of his book with the following two paragraphs.

> That any actual economic system of the type we have been considering can always be transformed into a Standard system may be shown by an imaginary experiment. (The experiment involves two types of alternating steps. One type consists in changing the proportions of the industries; the other in reducing in the same ratio the quantities produced by all industries, while leaving unchanged the quantities used as means of production.)

What Sraffa calls "imaginary experiment" is clearly what mathematicians call an algorithm: given an initial state, a definite list of well-defined instructions are given to proceed through a well-defined sequence of successive states, eventually terminating in an end-state. In order to reconstruct formally Sraffa's argument, let us introduce the square nonnnegative matrix $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ and the positive vector $\mathbf{l} = \begin{bmatrix} l_1, l_2, ..., l_n \end{bmatrix}^T$ as the material input matrix and the labour input vector, on the assumption that the output matrix is the identity matrix $\mathbf{I}$. Matrix $\mathbf{A}$ is assumed to be also indecomposable, that is, all non basic commodities are explicitly not considered. Let us continue our reading of section 37.

> We start by adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement.
>
> Let us next imagine gradually to reduce by means of successive small proportionate cuts the product of all the industries, without interfering with the quantities of labour and means of production that they employ.
>
> As soon as the cuts reduce the production of any one commodity to the minimum level required for replacement, we readjust the proportions of the industries so that there should again be a surplus of each product (while keeping constant the quantity of labour employed in the aggregate).

The initial state of the algorithm is the "actual economic system". This is able to produce a surplus, but does not need to procuce a surplus consisting of all (basic) commodities, so the first step consists in determining $\mathbf{x}_0 \in \left\{ \mathbf{x} > \mathbf{0} \middle| \mathbf{x}^T \mathbf{l} = \beta, \mathbf{x}^T [\mathbf{I} - \mathbf{A}] > \mathbf{0}^T \right\}$ and then building up two sequences: $\{\mathbf{x}_t\}$ and $\{\lambda_t\}$, where

$$\lambda_t = \lambda(\mathbf{x}_{t-1}) = \max_j \frac{\mathbf{x}_{t-1}^T \mathbf{A} \mathbf{e}_j}{\mathbf{x}_{t-1}^T \mathbf{e}_j}$$

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so that $\mathbf{x}_{t-1}^T[\lambda_t\mathbf{I}-\mathbf{A}]\geqq\mathbf{0}^T$ and $\mathbf{x}_{t-1}^T[\lambda_t\mathbf{I}-\mathbf{A}]\not>\mathbf{0}^T$, and $\mathbf{x}_t$ $(t>0)$ is a vector such that $\mathbf{x}_t>\mathbf{0}$, $\mathbf{x}_t^T\mathbf{1}=\beta$ and $\mathbf{x}_t^T[\lambda_t\mathbf{I}-\mathbf{A}]>\mathbf{0}^T$. Sraffa comments "This is always feasible so long as there is a surplus of some commodities and a deficit of none." However he does not provide a proof of this sentence. As we will see, this proof is an immediate consequence of the first three Theorems provided by Besicovitch. Then Sraffa proceeds to the end-state of the algorithm.

> We continue with such an alternation of proportionate cuts with the re-establishment
> of a surplus for each product until we reach the point where the products have been
> reduced to such an extent that all-round replacement is just possible without leaving
> anything as surplus product.

The "imaginary experiment" concludes, in Sraffa's opinion, when $\mathbf{x}_\infty>\mathbf{0}$, $\mathbf{x}_\infty^T\mathbf{1}=\beta$ and $\mathbf{x}_\infty^T[\lambda_\infty\mathbf{I}-\mathbf{A}]=\mathbf{0}^T$. Sraffa never states that the algorithm may need an infinite number of steps, but we know indeed that this is so. Finally, we have the last paragraph of section 37.

> Since to reach this position the products of all the industries have been cut in the
> same proportion we are now able to restore the original conditions of production by
> increasing the quantity produced in each industry by a uniform rate; we do not, on
> the other hand, disturb the proportions to which the industries have been brought.
> The uniform rate which restores the original conditions of production is R and the
> proportions attained by the industries are the proportions of the Standard system.

Hence we arrive at the equation

$$\mathbf{x}_\infty^T[\mathbf{I}-(1+R)\mathbf{A}]=\mathbf{0}^T$$

where, obviously, $1+R=1/\lambda_\infty$. As Alister Watson, Kurz and Salvadori (2001) and Lippi (2008), among others, have remarked, the algorithm is not well defined since there are infinitely many ways to define $\mathbf{x}_t$. Completing the definition of the algorithm means to define a function $\phi(\mathbf{q})$ such that $\mathbf{x}_t=\phi(\mathbf{x}_{t-1})$, each t. In order to be more precise, we introduce the sets

$$\mathsf{R}=\left\{\mathbf{q}\in\Re^n\,\middle|\,\mathbf{q}\geq\mathbf{0},\,\mathbf{q}^T\mathbf{1}=\beta,\,\mathbf{q}^T[\mathbf{I}-\mathbf{A}]\geqq\mathbf{0}^T\right\}$$

$$\mathsf{R}^*=\left\{\mathbf{q}\in\Re^n\,\middle|\,\exists\rho\geq0\,:\,\mathbf{q}\geq\mathbf{0},\,\mathbf{q}^T\mathbf{1}=\beta,\,\mathbf{q}^T[\rho\mathbf{I}-\mathbf{A}]=\mathbf{0}^T\right\}$$

$$\mathsf{S}=\mathsf{R}-\mathsf{R}^*$$

and the set of functions

$$\mathsf{Z}(\mathsf{S}_0)=\left\{\phi\,:\,\mathsf{S}_0\to\mathsf{R}\,\middle|\,\forall\mathbf{q}\in\mathsf{S}_0\,:\,\phi(\mathbf{q})\in\mathsf{S}_0\cup\mathsf{R}^*,\lambda(\mathbf{q})\phi(\mathbf{q})-\mathbf{A}^T\phi(\mathbf{q})>\mathbf{0}\right\},$$

where S_0 is any subset of S. Each function of the set $\bigcup_{\mathsf{S}_0 \subseteq \mathsf{S}} \mathsf{Z}(\mathsf{S}_0)$ defines a different algorithm which corresponds to Sraffa's description.

If function $\phi(\mathbf{q})$ has a fixed point in S, then sequence $\{\mathbf{x}_t\}$ may converge on the fixed point of function $\phi(\mathbf{q})$. As a consequence, sequence $\{\lambda_t\}$ may converge to a number which does not even need to be close to the eigenvalue of matrix $\mathbf{A}$. This fact cannot hold if function $\phi(\mathbf{q})$ has the mentioned inequality properties in the whole S and, therefore, the set of functions to be considered $\bigcup_{\mathsf{S}_0 \subseteq \mathsf{S}} \mathsf{Z}(\mathsf{S}_0)$ equals

$$\mathsf{Z} = \mathsf{Z}(\mathsf{S}) = \left\{ \phi \ : \ \mathsf{S} \to \mathsf{R} \middle| \ \forall \mathbf{q} \in \mathsf{S} \ : \ \phi(\mathbf{q}) \geq \mathbf{0}, \ \lambda(\mathbf{q})\phi(\mathbf{q}) - \mathbf{A}^T \phi(\mathbf{q}) > \mathbf{0}, \ \mathbf{l}^T \phi(\mathbf{q}) = \beta \right\}$$

This is the extra assumption found by Salvadori (2008). The interpretation is close at hand: the function $\phi(\mathbf{q})$ is such that $\phi(\mathbf{q}) \geq \mathbf{0}$, $\lambda(\mathbf{q})\phi(\mathbf{q}) - \mathbf{A}^T \phi(\mathbf{q}) > \mathbf{0}$, $\mathbf{l}^T \phi(\mathbf{q}) = \beta$, whatever is point $\mathbf{q} \in \mathsf{R}$ and not just in the support of sequence $\{\mathbf{x}_t\}$, as Sraffa's description may be interpreted. In the following section I will show that Besicovitch proposed a better defined algorithm and proved that the algorithm converges to the desired solution (apart for a small point to be completed).

3. Towards Besicovitch proof
Besicovitch's proof is devided in four "Theorems". Only the last one is the required proof. The first three prepare the field. In this section we discuss the first three theorems. Besicovitch does not follow the matricial notation we have used above in order to have a more compact presentation.
The first Theorem of file D3/12/39: 42 reads in plain English: *With positive prices any distribution of the net outputs can be attained*. This Theorem starts from the assumption that there is a system with no profits and positive prices and a positive wage rate. The aim is to prove that industries can be operated in such a way that any proportion in which the surplus is distributed among industries is feasible. The no profit assumption is not necessary, but probably follows the exercise that Sraffa is performing. Obviously the rate of profit must be lower than the maximum one since the wage rate must be positive and this is really what is needed. On the other side, if the wage rate is nought, then the rate of profits equals *R* and the existence of the Standard commodity is immediately obtained. In modern notation the first Theorem states:

$$\exists \mathbf{p} > \mathbf{0}, w > 0 : \mathbf{A}\mathbf{p} + w\mathbf{l} = \mathbf{p} \Rightarrow \exists \mathbf{x} \geq \mathbf{0} : \mathbf{x}^T = \mathbf{x}^T \mathbf{A} + \mathbf{q}^T \ \forall \mathbf{q} \geq \mathbf{0},$$

where *w* is assumed to be a positive scalar. Obviously the semipositive vector $\mathbf{q}$ is the vector of what Besicovitch calls "the Surplus outputs" (net outputs in the ebove). In order to obtain this result it is enough to prove that matrix $\mathbf{I} - \mathbf{A}$ is invertible and its inverse is positive and we know that this

is the case when matrix $\mathbf{A}$ is indecomposable and there is a positive vector $\mathbf{p}$ such that $\left[\mathbf{I} - \mathbf{A}\right]\mathbf{p} \geq \mathbf{0}$, because of the Perron-Frobenius Theorem. However, Besicovitch does not refer to the Perron-Frobenius Theorem.[1]

The proof provided by Besicovitch is very ingenious, but may need some explanation. Like the Gauss-Jordan elimination way to solve a linear system of equations it is based on consecutive applications of two elementary steps: (i) multiplication of an equation by a non-zero scalar, and (ii) addition to an equation of non-zero scalar multiples of other equations. Besicovitch proves that since prices are positive the *non-zero* scalar multiplications involved in both steps are indeed *positive* scalar multiplications. Let us follow step by step this recursive proof. In the first step only the last industry, n, is considered. Since

$$a_{n1}p_1 + ... + a_{nn}p_n + l_n w = p_n$$

and since $a_{n1}p_1 + ... + a_{nn-1}p_{n-1} + l_n w > 0$, we have that $1 - a_{nn} > 0$. Hence it is possible to find a positive λ_n such that $\lambda_n\left(1 - a_{nn}\right)$ can take any positive value.

In the second step the last two industries are considered. Taking account of the equations

$$a_{n-1,1}p_1 + ... + a_{n-1,n-1}p_{n-1} + a_{n-1,n}p_n + l_{n-1}w = p_{n-1}$$
$$a_{n1}p_1 + ... + a_{nn-1}p_{n-1} + a_{nn}p_n + l_n w = p_n$$

and using the first step, we can multiply the latter by a λ_n such that $\lambda_n\left(1 - a_{nn}\right) = a_{n-1,n}$ in such a way as to obtain that the surplus of industry n equals the input of commodity n into industry $n-1$:

$$a_{n-1,1}p_1 + ... + a_{n-1,n-1}p_{n-1} + a_{n-1,n}p_n + l_{n-1}w = p_{n-1}$$
$$\frac{a_{n-1,n}}{1 - a_{nn}}a_{n1}p_1 + ... + \frac{a_{n-1,n}}{1 - a_{nn}}a_{nn-1}p_{n-1} + \frac{a_{n-1,n}}{1 - a_{nn}}a_{nn}p_n + \frac{a_{n-1,n}}{1 - a_{nn}}l_n w = \frac{a_{n-1,n}}{1 - a_{nn}}p_n$$

As a consequence, by summing up the two equations we obtain

$$\left(a_{n-1,1} + \frac{a_{n-1,n}}{1 - a_{nn}}a_{n1}\right)p_1 + ... + \left(a_{n-1,n-1} + \frac{a_{n-1,n}}{1 - a_{nn}}a_{nn-1}\right)p_{n-1} + \left(l_{n-1} + \frac{a_{n-1,n}}{1 - a_{nn}}l_n\right)w = p_{n-1}$$

since

$$a_{n-1,n} + \frac{a_{n-1,n}}{1 - a_{nn}}a_{nn} = \frac{a_{n-1,n}}{1 - a_{nn}}.$$

Once again, since

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[1]   The proof of the existence of the Standard commodity can be interpreted as a proof of the Perron-Frobenius Theorem (see Kurz and Salvadori, 1993).

$$\left(a_{n-1,1} + \frac{a_{n-1,n}}{1-a_{nn}} a_{n1} \right) p_1 + \ldots + \left(a_{n-1,n-2} + \frac{a_{n-1,n}}{1-a_{nn}} a_{nn-2} \right) p_{n-2} + \left(l_{n-1} + \frac{a_{n-1,n}}{1-a_{nn}} l_n \right) w > 0$$

we have that

$$1 - \left(a_{n-1,n-1} + \frac{a_{n-1,n}}{1-a_{nn}} a_{n,n-1} \right) = \frac{\det \begin{bmatrix} 1-a_{n-1,n-1} & -a_{n-1,n} \\ -a_{n,n-1} & 1-a_{nn} \end{bmatrix}}{1-a_{nn}} > 0$$

Hence we can find two positive scalars λ_n and λ_{n-1} such that $\lambda_{n-1} - \lambda_n a_{n,n-1} - \lambda_{n-1} a_{n-1,n-1}$ can take any positive value and $\lambda_n - \lambda_n a_{nn} - \lambda_{n-1} a_{n-1,n} = 0$, that is, we can proportion the two equations in such a way that the output of commodity n equals the sum of the inputs of commodity n in the last two industries and the output of commodity $n-1$ is as desired. In a similar way we can proportion the two equations in such a way that the output of commodity $n-1$ equals the sum of the inputs of commodity $n-1$ in the last two industries and the output of commodity n is as desired. Thus we can proportion the two equations in such a way that there is the desired surplus of the last two commodities.

The third step analyzes the last three industries. By using the second step we can proportion the last two equations in such a way that the outputs of the last two commodities equal the sum of their inputs in the last three industries.

$$a_{n-2,1} p_1 + \ldots + a_{n-2,n-1} p_{n-1} + a_{n-2,n} p_n + l_{n-2} w = p_{n-2}$$

$$\frac{\Delta_1}{\Delta} a_{n-1,1} p_1 + \ldots + \frac{\Delta_1}{\Delta} a_{n-1,n-1} p_{n-1} + \frac{\Delta_1}{\Delta} a_{n-1,n} p_n + \frac{\Delta_1}{\Delta} l_{n-1} w = \frac{\Delta_1}{\Delta} p_{n-1}$$

$$\frac{\Delta_2}{\Delta} a_{n1} p_1 + \ldots + \frac{\Delta_2}{\Delta} a_{nn-1} p_{n-1} + \frac{\Delta_2}{\Delta} a_{nn} p_n + \frac{\Delta_2}{\Delta} l_n w = \frac{\Delta_2}{\Delta} p_n$$

where

$$\Delta = \det \begin{bmatrix} 1-a_{n-1,n-1} & -a_{n-1,n} \\ -a_{n,n-1} & 1-a_{nn} \end{bmatrix}, \Delta_1 = \det \begin{bmatrix} a_{n-2,n-1} & -a_{n-1,n} \\ a_{n-2,n} & 1-a_{nn} \end{bmatrix}, \Delta_1 = \det \begin{bmatrix} 1-a_{n-1,n-1} & a_{n-2,n-1} \\ -a_{n,n-1} & a_{n-2,n} \end{bmatrix}.$$

By adding up, we obtain

$$\left(a_{n-2,1} + \frac{\Delta_1}{\Delta} a_{n-1,1} + \frac{\Delta_2}{\Delta} a_{n1} \right) p_1 + \ldots + \left(a_{n-2,n-2} + \frac{\Delta_1}{\Delta} a_{n-1,n-2} + \frac{\Delta_2}{\Delta} a_{nn-2} \right) p_{n-2} + l_{n-2} w = p_{n-2}$$

since

$$a_{n-2,n-1} + \frac{\Delta_1}{\Delta} a_{n-1,n-1} + \frac{\Delta_2}{\Delta} a_{n,n-1} = \frac{\Delta_1}{\Delta}, \quad a_{n-2,n} + \frac{\Delta_1}{\Delta} a_{n-1,n} + \frac{\Delta_2}{\Delta} a_{n,n} = \frac{\Delta_2}{\Delta}$$

Once again, since prices are positive, we obtain that there is a surplus of commodity $n-2$, that is

$$1 - \left(a_{n-2,n-2} + \frac{\Delta_1}{\Delta} a_{n-1,n-2} + \frac{\Delta_2}{\Delta} a_{n,n-2} \right) = \frac{\det \begin{bmatrix} 1 - a_{n-2,n-2} & -a_{n-2,n-1} & -a_{n-2,n} \\ -a_{n-1,n-2} & 1 - a_{n-1,n-1} & -a_{n-1,n} \\ -a_{n,n-2} & -a_{n,n-1} & 1 - a_{nn} \end{bmatrix}}{\Delta} > 0 \ .$$

and that multipliers can be found such that the surplus of commodity $n-2$ can take any positive value, whereas the outputs of the last two commodities equal the sum of their inputs in the last three industries. This is enough to find multipliers such that there is the desired surplus of commodity $n-2$, the desired surplus of commodity $n-1$, and the desired surplus of commodity n. And so on.

The second Theorem reads in plain English: *If the wage is posive and prices are positive, then net outputs cannot be all nought and, therefore, there is a surplus of at least one commodity*. In modern notation the second Theorem states:

$$\exists \mathbf{p} > \mathbf{0}, w > 0 : \mathbf{Ap} + w\mathbf{l} = \mathbf{p} \Rightarrow \mathbf{x}^T \neq \mathbf{x}^T \mathbf{A} \ \forall \mathbf{x} \geq \mathbf{0}$$

If not, we obtain $\mathbf{x}^T \mathbf{Ap} + w\mathbf{x}^T \mathbf{l} = \mathbf{x}^T \mathbf{p} = \mathbf{x}^T \mathbf{Ap}$, and therefore $w\mathbf{x}^T \mathbf{l} = 0$, which is not possible. The proof by Besicovitch does not need a *reductio ad absurdum*. If $\mathbf{x}^T \mathbf{Ae}_i = \mathbf{x}^T \mathbf{e}_i$ each $i \neq j$, where $\mathbf{e}_i$ is the i-th unit vector, then

$$\mathbf{x}^T \mathbf{Ap} + w\mathbf{x}^T \mathbf{l} = \mathbf{x}^T \mathbf{Ae}_j \left(\mathbf{e}_j^T \mathbf{p} \right) + w\mathbf{x}^T \mathbf{l} + M = \mathbf{e}_j^T \mathbf{p} + M$$

where $M = \sum_{i \neq j} \mathbf{x}^T \mathbf{Ae}_i \left(\mathbf{e}_i^T \mathbf{p} \right) = \sum_{i \neq j} \mathbf{e}_i^T \mathbf{p}$ and since $w\mathbf{x}^T \mathbf{l} > 0$, we have $\mathbf{x}^T \mathbf{Ae}_j < 1$ as required.

The third Theorem reads in plain English: *If the surplus of a commodity is positive and that of the others is nought then the prices are positive*. Note that it is always implicit that the wage rate is positive. The aim is to prove that if there is a positive surplus of at least one commodity (and a negative surplus of none), then the wage is posive and prices are positive. Also in this case it is enough to prove that matrix $\mathbf{I} - \mathbf{A}$ is invertible and its inverse is positive. In modern notation the third Theorem states:

$$\exists \mathbf{x} \geq \mathbf{0} : \mathbf{x}^T \geq \mathbf{x}^T \mathbf{A} \Rightarrow \exists \mathbf{p} > \mathbf{0} : \mathbf{Ap} + w\mathbf{l} = \mathbf{p}$$

However in the document D3/12/39: 42 of 21 September 1944 this Theorem is not proven. What is proven is that if there is a surplus in one commodity and no surplus in all the others, then the equations can be proportioned in such a way that a surplus is obtained in every commodity (even this proof is not complete; in particular it would be false, if the input matrix were decomposable; further it does not show why it works when the input matrix is indecomposable). However in the

document D3/12/39: 42 there is a note by Sraffa saying: "Refer to blue page 1". The reference is no doubt to D3/12/39: 7, which is written on a blue piece of paper and contains a proof by Besicovitch of the fact that if there is a surplus in every commodity, then prices are positive.[2] The transcriptions of this document is reported here as Appendix C.

Before arguing on the proof of the third theorem I will discuss the proof in D3/12/39: 7. The statement in modern notation is:

$$\mathbf{e}^T > \left(1+r\right)\mathbf{e}^T\mathbf{A}, \ \left(1+r\right)\mathbf{A}\mathbf{p} + w\mathbf{l} = \mathbf{p}, \ w > 0 \Rightarrow \mathbf{p} > \mathbf{0}$$

Note that the mentioned equation admits always a solution since it is homogeneous in $\left(\mathbf{p}, \ w\right)$, however, we are assuming here something more, i.e., that a solution with a positive w exists.[3] Suppose that in this solution some price (at least one) is negative or nought, and all the others (possibly none) are positive. With no loss of generality assume that the first h are negative or nought, $1 \le h \le n$, and the last $n-h$ are positive. Then, with obvious meanings of symbols,

$$\left(1+r\right)\mathbf{A}_{12}\mathbf{p}_2 + w\mathbf{l}_1 = \left[\mathbf{I} - \left(1+r\right)\mathbf{A}_{11}\right]\mathbf{p}_1$$

which is impossible since $\mathbf{e}_h^T\left[\mathbf{I} - \left(1+r\right)\mathbf{A}_{11}\right]\mathbf{p}_1 \le 0$ whereas $\left(1+r\right)\mathbf{e}_h^T\mathbf{A}_{12}\mathbf{p}_2 + w\mathbf{e}_h^T\mathbf{l}_1 > 0$, where $\mathbf{e}_h$ is the sum h-vector, that is, an h-vercor of 1's. Note that this proof holds even if matrix $\mathbf{A}$ is decomposable, and therefore some commodities are non-basic, provided that labour enters directly into the production of all commodities and, therefore, $\mathbf{l}_1 > \mathbf{0}$ (it still holds if labour enters directly or indirectly into the production of all commodities, but I will not deal with this issue here).

Now we can discuss the proof of the third theorem in the document D3/12/39: 42. With no loss of generality assume that the first h commodities have a positive surplus, $1 \le h \le n$, whereas the last $n-h$ have no surplus (and no loss). Therefore $\mathbf{e}_h^T > \mathbf{e}_h^T\mathbf{A}_{11} + \mathbf{e}_{n-h}^T\mathbf{A}_{12}$ and $\mathbf{e}_{n-h}^T = \mathbf{e}_h^T\mathbf{A}_{12} + \mathbf{e}_{n-h}^T\mathbf{A}_{22}$. Therefore, Besicovitch mantains, if u is a real number lower than 1, but so close to 1 that it is still true that $u\mathbf{e}_h^T > u\mathbf{e}_h^T\mathbf{A}_{11} + \mathbf{e}_{n-h}^T\mathbf{A}_{12}$, we must have that $\mathbf{e}_{n-h}^T > u\mathbf{e}_h^T\mathbf{A}_{12} + \mathbf{e}_{n-h}^T\mathbf{A}_{22}$. However, this is not necessarily true. Indeed if matrix $\mathbf{A}$ is decomponible and $\mathbf{A}_{12} = \mathbf{0}$, this is certainly false. It is

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[2]   D3/12/39: 8 is also written on a blue piece of paper and contains a proof by Besicovitch, but on a different issue.

[3]   As a matter of fact, a known theorem allows to state that if the mentioned strong inequality holds, then w is nought if and only if all prices are nought too (see Kurz and Salvadori, 1995, p. 510, Theorem A.3.1). The same theorem, however, proves also the theorem that Besicovitch wants to prove.

reasonable to assume that Besicovitch assumed that all commodities are basic and, therefore, matrix $\mathbf{A}$ is indecomponible. Even in this case, however, the proof is incomplete ($\mathbf{e}_h^T \mathbf{A}_{12}$ is semi-positive, but does not need to be positive) since we may need to iterate the process to bring home the result. In fact, if matrix $\mathbf{A}$ is indecomponible, we are sure that $\mathbf{e}_{n-h}^T \geq u\mathbf{e}_h^T \mathbf{A}_{12} + \mathbf{e}_{n-h}^T \mathbf{A}_{22}$ and therefore the number of commodities with a positive surplus is increased and still no commodity has a negative surplus. Further, since at any iteration of the process the number of products with a positive surplus increases, the number of iterations needed to obtain a surplus in all commodities is certainly finite since it is lower than $n - h$.

The first three theorems of file D3/12/39: 42 have the role to mantain two facts. First, if there is a surplus of any type, industries may be proportioned in such a way as to get the surplus everywhere the observer wants. Second, there is a surplus if and only if prices are positive and the wage rate is positive. The relationship with section 37 of the book by Sraffa (1960) is obvious. One of the two steps of the algorithm introduced there consists exactly in "adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement". The fourth theorem concerns the existence of the Standard commodity and will be analyzed in the next section.

4. Besicovitch's proof

The fourth theorem is quite cryptic and requires some explanation. In plain English it reads: *If prices are positive, then there exist positive multipliers q_a, ... ,q_k such that the net output is proportional to the total of every kind of raw material*. The proof is similar to that provided by Sraffa, but is more detailed and closer to the description of an algorithm. It starts by assuming that there is a surplus with regard to all commodities. If there were a surplus only in some industries, then we can find a starting point with a surplus in all industries, $\mathbf{x}_0 \in \left\{ \mathbf{x} > \mathbf{0} \big| \mathbf{x}^T \mathbf{l} = \beta, \mathbf{x}^T [\mathbf{I} - \mathbf{A}] > \mathbf{0}^T \right\}$, since the condition of Theorem 1 holds. Then the second step used by Sraffa is applied. That is, it is found

$$\lambda_1 = \lambda(\mathbf{x}_0) = \max_j \frac{\mathbf{x}_0^T \mathbf{A} \mathbf{e}_j}{\mathbf{x}_0^T \mathbf{e}_j}$$

so that $\mathbf{x}_0^T [\lambda_1 \mathbf{I} - \mathbf{A}] \geq \mathbf{0}^T$ and $\mathbf{x}_0^T [\lambda_1 \mathbf{I} - \mathbf{A}] \not> \mathbf{0}^T$. Then all the equations of commodities for which there is a surplus are multiplied for a common scalar lower than 1. Besicovitch thinks this is enough

to get that all commodities are in surplus, but this does not need to be true since input coefficients are not all positive. However, since all commodities are assumed to be basic, the input matrix $\mathbf{A}$ is indecomposable and therefore we can get the desired result by iterating the same procedure, as seen above, in the analysis of the third theorem by Besicovitch. Let us consider the point in a more formal way.

Let $\mu \in \mathfrak{R}$ and $\mathbf{x} \in \mathsf{S}$ be such that $\mu\mathbf{x}^T \geq \mathbf{x}^T\mathbf{A}$ and define the set of indices

$$\mathsf{I}_{\mu\mathbf{x}} = \left\{ i \in \{1,\, 2,\, \dots,\, n\} \,\middle|\, \mu x_i > \sum_{j=1}^{n} x_j a_{ji} \right\}$$

$$\hat{\mathsf{I}}_{\mu\mathbf{x}} = \left\{ i \in \{1,\, 2,\, \dots,\, n\} \,\middle|\, \mu x_i = \sum_{j=1}^{n} x_j a_{ji} \right\}$$

The aim of this step consists in finding an intensity vector $\phi(\mathbf{x})$ such that $\mathsf{I}_{\mu\phi(\mathbf{x})} = \{1,\, 2,\, \dots,\, n\}$ and, as a consequence, $\hat{\mathsf{I}}_{\mu\phi(\mathbf{x})} = \varnothing$. Besicovitch considers that this can be obtained if $\phi(\mathbf{x})$ is the function $\mathbf{g}(\mu,\mathbf{x})$, where

$$g_i(\mu,\mathbf{x}) = \begin{cases} x_i & \text{if } i \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \\ \eta x_i & \text{if } i \in \mathsf{I}_{\mu\mathbf{x}} \end{cases}$$

where η is a scalar lower than 1, but so close to 1 that

$$\mu(\eta x_i) > \sum_{j \in \hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j a_{ji} + \eta \sum_{j \in \mathsf{I}_{\mu\mathbf{x}}} x_j a_{ji} \qquad\qquad \text{each } i \in \mathsf{I}_{\mu\mathbf{x}}$$

That is,

$$\max_{i \in \mathsf{I}_{\mu\mathbf{x}}} \frac{\sum_{j \in \hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j a_{ji}}{\mu x_i - \sum_{j \in \mathsf{I}_{\mu\mathbf{x}}} x_j a_{ji}} < \eta < 1\,.$$

As mentioned above, this is not enough to obtain that $\mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})} = \{1,\, 2,\, \dots,\, n\}$ because some a_{ji} may be nought. However, by construction, $i \in \mathsf{I}_{\mu\mathbf{x}}$ implies that $i \in \mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})}$ and therefore $\mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})} \supseteq \mathsf{I}_{\mu\mathbf{x}}$. On the other side, $\mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})} = \mathsf{I}_{\mu\mathbf{x}}$ if and only if $a_{ji} = 0$ each $i \in \mathsf{I}_{\mu\mathbf{x}}$ and each $j \in \hat{\mathsf{I}}_{\mu\mathbf{x}}$. But then matrix $\mathbf{A}$ would be decomposable. This being impossible, we get that $\mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})} \supset \mathsf{I}_{\mu\mathbf{x}}$. This is enough to say that the procedure can be iterated for a number of times lower to the (finite) number of commodities (also because if $\mathsf{I}_{\mu\mathbf{x}} = \{1,\, 2,\, \dots,\, n\}$, then $\mathbf{g}(\mu,\mathbf{x})$ is proportional to $\mathbf{x}$). Hence we can define:

$$\mathbf{h}_1(\mathbf{x}) = \mathbf{g}(\lambda(\mathbf{x}),\mathbf{x})$$

$$\mathbf{h}_j(\mathbf{x}) = \mathbf{g}(\lambda(\mathbf{x}),\mathbf{h}_{j-1}(\mathbf{x})) \qquad\qquad j = 2,\, \dots,\, n-1$$

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$$\phi(\mathbf{x}) = \mathbf{h}_{n-1}(\mathbf{x})$$

There is one further aspect considered by Besicovitch. In a remark he argued that 'we may keep one of our industries intact' in order to avoid that all multipliers become nought. With no loss of generality, assume that such industry is industry 1. Therefore function $\mathbf{g}(\mu,\mathbf{x})$ must be redefined as

$$g_i(\mu,\mathbf{x}) = \begin{cases} x_i & \text{if } i \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \text{ and } 1 \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \\ \eta x_i & \text{if } i \in \mathsf{I}_{\mu\mathbf{x}} \text{ and } 1 \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \\ \dfrac{1}{\eta} x_i & \text{if } i \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \text{ and } 1 \notin \hat{\mathsf{I}}_{\mu\mathbf{x}} \\ x_i & \text{if } i \in \mathsf{I}_{\mu\mathbf{x}} \text{ and } 1 \notin \hat{\mathsf{I}}_{\mu\mathbf{x}} \end{cases}$$

This function has also the property that if $\mathsf{I}_{\mu\mathbf{x}} = \{1, 2, \dots, n\}$, then $\mathbf{g}(\mu,\mathbf{x}) = \mathbf{x}$. As seen in section 1, Sraffa followed a different, but equivalent strategy to avoid that all multipliers become nought. He kept fixed the amount of labour. If we follow this strategy we have that function $\mathbf{g}(\mu,\mathbf{x})$ must be redefined as

$$g_i(\mu,\mathbf{x}) = \begin{cases} \theta x_i & \text{if } i \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \\ \theta \eta x_i & \text{if } i \in \mathsf{I}_{\mu\mathbf{x}} \end{cases}$$

where

$$\theta = \frac{\beta}{\sum_{j \in \hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j l_j + \eta \sum_{j \in \mathsf{I}_{\mu\mathbf{x}}} x_j l_j}$$

Also this function has the property that if $\mathsf{I}_{\mu\mathbf{x}} = \{1, 2, \dots, n\}$, then $\mathbf{g}(\mu,\mathbf{x}) = \mathbf{x}$.

Also in Besicovitch's proof there is a family of potential algorithms involved. In order to have a proper algorithm we must have a way to define how η is chosen. For example if we chose η in the middle of the range in which it can vary we would have

$$\eta = \frac{1}{2} + \frac{1}{2} \max_{i \in \mathsf{I}_{\mu\mathbf{x}}} \frac{\sum_{j \in \hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j a_{ji}}{\mu x_i - \sum_{j \in \mathsf{I}_{\mu\mathbf{x}}} x_j a_{ji}}$$

and in general any possible choice could be defined as a choice of $0 < \alpha < 1$ in the expression

$$\eta = \alpha + (1-\alpha) \max_{i \in \mathsf{I}_{\mu\mathbf{x}}} \frac{\sum_{j \in \hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j a_{ji}}{\mu x_i - \sum_{j \in \mathsf{I}_{\mu\mathbf{x}}} x_j a_{ji}}$$

For each sequence $\{\alpha_i\}$, $0 < \alpha_i < 1$, we have a different algorithm; but whatever sequence $\{\alpha_i\}$ is chosen, it is easily proved that the conditions stated by Salvadori (2008) hold and therefore all the

potential algorithms considered by Besicovitch converge to the desired result. In fact, function $\phi(\mathbf{x})$ is continuous and can start from any point in S.

Sraffa found it too laborious to use the function $\phi(\mathbf{x})$ used by Besicovitch. He recognised that what is relevant is the "adjusting [of] the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement" and that at each step the desired result is closer, but he did not consider the fact that the "imaginary experiment" may go over an infinite number of steps without approaching the Standard commodity.

5. **Conclusion**

In this paper I have investigated the relationship between the proof of the existence of the Srandard commodity contained in section 37 of Sraffa's (1960) book and the proof supplied to Sraffa by Besicovitch on 21 September 1944 and I have investigated the completeness and consistence of such a proof.

Appendix A. An example

Let

$$\mathbf{A} = \begin{bmatrix} 0 & h \\ k & 0 \end{bmatrix} \qquad \mathbf{l} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \beta = 1 \qquad\qquad 0 < h < k < 1$$

It is easily calculated that the eigenvalue of maximum modulus of matrix $\mathbf{A}$ is $\sqrt{hk}$ and that the left eigenvector associated with this eigenvalue normalized by the condition $\mathbf{x}^T\mathbf{l} = \beta$ is

$$\left[\frac{2\left(k - \sqrt{hk}\right)}{k - h} \quad \frac{2\left(\sqrt{hk} - h\right)}{k - h} \right]$$

It is easily recognised that

$$\mathsf{R} = \left\{ \mathbf{q} \in \Re^2 \,\middle|\, \frac{2k}{1+k} \le q_1 \le \frac{2}{1+h}, \ q_2 = 2 - q_1 \right\}.$$

Finally, let us consider the function

$$\phi(\mathbf{q}) = \begin{bmatrix} 1 \\ 1 - 2\varepsilon \end{bmatrix} + \varepsilon\mathbf{q} \tag{6}$$

with

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$$0 < \varepsilon < \frac{1}{2} - \frac{1}{2}\sqrt{\frac{h}{k}} \tag{7}$$

From inequalities (7) we obtain the inequality

$$\frac{1}{1-\varepsilon} < \frac{2\left(k - \sqrt{hk}\right)}{k - h}$$

from which we obtain[4] that $\lambda(\mathbf{q})\phi(\mathbf{q}) - \mathbf{A}^T\phi(\mathbf{q}) > \mathbf{0}$ for each $\mathbf{q} \in \mathsf{S}_1$, whereas this property does not hold for $\mathbf{q} \in \mathsf{S}_2$, where

$$\mathsf{S}_1 = \left\{ \mathbf{q} \in \mathsf{S} \,\middle|\, q_1 < \frac{1}{1-\varepsilon} \right\}$$

$$\mathsf{S}_2 = \left\{ \mathbf{q} \in \mathsf{S} \,\middle|\, \frac{1}{1-\varepsilon} \le q_1 < \frac{2\left(k - \sqrt{hk}\right)}{k - h} \right\}$$

Further, it is easily verified that $\mathbf{q} \in \mathsf{S}_1 \Rightarrow \phi(\mathbf{q}) \in \mathsf{S}_1$. Therefore each element of any sequence defined by the conditions

$$\mathbf{q}_0 \in \mathsf{S}_1, \qquad \mathbf{q}_{t+1} = \begin{cases} \mathbf{q}_t & \text{if } \lambda(\mathbf{q}_t)\mathbf{q}_t - \mathbf{A}^T\mathbf{q}_t = \mathbf{0} \\ \phi(\mathbf{q}_t) & \text{if } \lambda(\mathbf{q}_t)\mathbf{q}_t - \mathbf{A}^T\mathbf{q}_t \ne \mathbf{0} \end{cases}$$

satisfies the conditions stated by Sraffa, but

$$\lim_{i \to \infty} \lambda(\mathbf{q}_i) = (1 - 2\varepsilon)k > \sqrt{hk}$$

$$\lim_{i \to \infty} \mathbf{q}_i = \begin{bmatrix} \dfrac{1}{1-\varepsilon} \\ \dfrac{1-2\varepsilon}{1-\varepsilon} \end{bmatrix} \ne \begin{bmatrix} \dfrac{2\left(k - \sqrt{hk}\right)}{k - h} \\ \dfrac{2\left(\sqrt{hk} - h\right)}{k - h} \end{bmatrix}$$

The last limit is the unique fixed point of funtion (6).

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Besic.: – **21.9.44**: (42) 1-4:

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[4]   If $q_1 \le 2\left(k - \sqrt{hk}\right)(k - h)^{-1}$, then $\lambda(\mathbf{q}) = k(2 - q_1)q_1^{-1}$. Further $\lambda(\mathbf{q})\mathbf{e}_1^T\phi(\mathbf{q}) - \mathbf{e}_1^T\mathbf{A}^T\phi(\mathbf{q}) > 0$ if and only if $q_1 < (1-\varepsilon)^{-1}$ whereas $\lambda(\mathbf{q})\mathbf{e}_2^T\phi(\mathbf{q}) - \mathbf{e}_2^T\mathbf{A}^T\phi(\mathbf{q}) > 0$ for $q_1 \le 2\left(k - \sqrt{hk}\right)(k - h)^{-1}$, provided that inequalities (7) hold.

<u>'Th 1</u> <u>If prices are +ve, any distribution of the Surplus outputs can be attained'</u>

<u>Proof</u>

(i) $A_k p_a + ... + K_k p_k + l_k w = K p_k$

obviously any desirable surplus of K can be produced since $K > K_k$ ($\because$ of +ve prices).

(ii) $\begin{cases} A_j p_a + ... + J_j p_j + K_j p_k + l_j w = J p_j \\ A_k p_a + . \qquad . \qquad . \qquad = K p_k \end{cases}$

Let the surplus of the *K*-industry be K_j. Then *J* industry produces some Surplus. Multiply{in}g both =ions {equations} by the Same factor the *J* surplus may take any assigned value. S{imilar}ly we can make *J* not to have any Surplus & *K* to have any assigned surplus. Then add{in}g the two firs{t} =ions {equations} & the two second ones we get an assigned <u>Surplus of *J* & f{o}r *K*</u> , a. s.o., q.e.d.

<u>'Th 2</u> <u>If the prices are +ve {positive} and the surplus of *B*, ..., *K* is 0 then the surplus of *A* is +ive {positive}.</u>

<u>Proof</u> For take the surplus of *B*, ..., *K* (wrt {with respect to} *B* ... *K*) to be B_a, ... , K_a. Then the surplus f{o}r *B*, ... *K* wrt {with respect to} *A*, ... , *K* is 0. Write

$A_a p_a + ... + l_a w = A p_a$

$\overline{\qquad\qquad}$

$A_k p_a + ... + l_k w = K p_k$

& add them & drop *B*, ... , *K* terms from both sides as they are = {equal}. The result is

$(A_a + ... + A_k) p_a + (l_a + ... + l_k) w = A p_a$.

i.e. $A_a + ... + A_k < A$, q.e.d.

<u>'Th 3</u> <u>If the Surplus of *A* is +ve {positive} & of *B*, ..., *K* is 0 then the prices are +ve {positive}.</u>

For multiplying *A*=ion {equation A} by *u* (< 1) sufficiently near 1 we shall still have the surplus of *A* +ve {positive} & we shall make surplus of *B*, ... , *K* +ve {positive}.

{Addition by Sraffa on bottom of page: (Refer to blue page 1)}

<u>'Th 4 If prices are +ve {positive} there exist +ve {positive} multipliers $q_a,...,q_k$ such that the</u> <u>Surplus output is proportional to the total of every kind of raw materials.</u>

<u>Proof</u>.

$$A_a p_a + ...+ l_a w = A p_a$$

(1)   ....

$$A_k p_a + ...= K p_k$$

assuming the surplus for each to be +ve {positive}, i.e.

$$A_a + ...+ A_k < A$$

(2)   ....

$$K_a + ...+ K_k < K$$

Consider

$$A_a + ...+ A_k < Au$$

(3)   ....

$$K_a + ...+ K_k < Ku$$

The ≠ties {inequalities} remain true as u decreases from 1 until it reaches a certain value $u_0 > 0$ for which some of the ≠ties {inequalities} become =ties {equalities}, f.i {for instance} the first two. Then we multiply the $C, ... , K$ =ions {equations} by $k < 1$ but near 1, so that the surplus of $C, ... , K$ still remain positive. This will release a surplus of A & B. Then (3) will be true wrt {with respect to} the reformed system for $u = u_0$. Now we decrease u beyond u_0 a.s.o. In this way we shall reach as System

$$q_a \left(A_k p_a + ...\right) = q_a A p_a$$

....

$$q_k \left(........................\right) = q_k K p_k$$

for which

$$q_a A_a + ... q_k A_k < q_a Au$$

....

$$q_a K_a + ... < q_k Ku$$

for $u_1 < u \le 1$, & when $u = u_1$ all the ≠ies {inequalities} become =ies {equalities}.

<u>Remark</u>. All $q_a,...,q_k$ cannot become 0 since in all our adjustments we may keep one of our industries intact, f. i A, & from this it follows that $u_1 \ge A_a/A$ ($\therefore$ 1st =ion {first equation} of (3))

Appendix C. D3/12/39: 7

$$(A_a p_a + ... + K_a p_k)(1+r) + L_a w = A p_a$$

$$...........$$

$$(A_k p_a + ... + K_k p_k)(1+r) + L_k w = A p_k$$

<u>If r is such that</u>

$$(A_a + ... + A_k)(1+r) < A$$
$$(K_a + ... + K_k)(1+r) < K$$

<u>then all prices are positive, assuming w > 0</u>

<u>Proof.</u> Suppose not. Let $p_a < 0$, $p_b < 0$, the rest > 0. Then adding the first two equations and taking to the right A and B terms we shall have

$$\{(C_a + C_b)p_c + ... + (K_a + K_b)p_k\}(1+r) + (L_a + L_b)w$$
$$= \{A - (A_a + A_b)(1+r)\}p_a + \{B - (B_a + B_b)(1+r)\}p_b$$

which is impossible, since the expression on the left hand side is > 0, and on r. h. side < 0.

ASB

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