Dynamic nonpoint-source pollution control policy: ambient transfers and uncertainty

Athanassoglou, Stergios

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Stergios Athanassoglou *

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Abstract

When a regulator cannot observe or infer individual emissions, corrective policy must rely on ambient pollution data. Assuming this kind of environment, we study a class of differential games of pollution control with profit functions that are polynomial in the global pollution stock. Given an open-loop emissions strategy satisfying mild regularity conditions, an ambient transfer scheme is exhibited that induces it in Markov-perfect equilibrium (MPE). Proposed transfers are a polynomial function of the difference between actual and desired pollution levels; moreover, they are designed so that in MPE no tax or subsidy is ever levied. Their applicability under stochastic pollution dynamics is studied for a symmetric game of polluting oligopolists with linear demand. We discuss a quadratic scheme that induces agents to adopt Markovian emissions strategies that are stationary and linearly decreasing in total pollution. Total expected ambient transfers are non-positive and their magnitude is linearly increasing in physical volatility, the size of the economy, and the absolute value of the slope of the inverse demand function. However, if the regulator is interested in inducing a constant emissions strategy then, in expectation, transfers vanish. The total expected ambient transfer is compared to its point-source equivalent.

Keywords: differential games, nonpoint source pollution, stochastic dynamics, policy design

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*Post-Doctoral Fellow, The Earth Institute at Columbia University, New York, NY; e-mail: sa2164@columbia.edu.

1 Introduction

When individual pollution discharges are not observable, a regulator may wish to impose corrective policy measures that are based on observed total (ambient) pollution levels. As a result, there is an extensive literature on ambient transfers as a means of nonpoint-source pollution control going back to the work of Segerson (1988), whose analysis builds on earlier theoretical work of Holmstrom (1982). Xepapadeas (1992) extends Segerson's contribution to a dynamic setting under both deterministic and random specifications on pollutant accumulation. Since then a significant and growing literature has developed, shedding light into the theoretical design and practical implementation of ambient transfer schemes.

From a practical standpoint, ambient policy has been employed in a variety of settings. Segerson (1999) describes a number of applications of the basic theoretical ideas: (i) The Everglades Forever Act, in which the government instituted a cropland tax based on aggregate phosphorus contamination from agricultural runoff; (ii) the Coastal Zone Management Reauthorization Amendments of 1990 that regulated nonpoint-source pollution in coastal areas of the United States; (iii) a policy in Lake Okeechobee, Florida, in which dairy farmers were compelled to adhere to ambient water quality standards, (iv) the Oregon Salmon Restoration Program in which salmon species were to be listed as endangered unless farmers ensured that agricultural runoff did not significantly deplete local fisheries.

A common criticism of ambient transfers rests on their dependence on total pollution levels and, in particular, the fact that they may result in excessive and inequitable penalties Karp (2005). In an environment with no uncertainty Karp (2005), drawing on earlier work of Karp and Livernois (1994), investigates these concerns by comparing the tax burdens of (a compelling type of) Pigouvian and ambient taxes. In his model, which deals with flow rather than stock pollutants, both tax schemes are linear and evolve over time in an intuitive fashion; moreover, they are designed to induce a common steady state level of pollution. Karp rigorously investigates the conditions under which the open-loop equilibrium steady-state tax burden of ambient policy is lower than the Pigouvian tax, mitigating some of the concerns regarding its potential inequity.

At the same time, it is possible to design ambient transfers so that, in steady-state equilibrium, no tax or subsidy is ever imposed. In particular, one can make the tax scheme a function of the observed difference between actual and desired pollution levels, ensuring that when that difference is zero transfers accordingly vanish (Xepapadeas, 1992). Indeed, Karp and Livernois' Karp and Livernois (1994) ambient scheme (which is revisited in Karp (2005)) can be modified in this way as well. It should be noted, however, that the equilibrium analysis in Karp and Livernois (1994) and Karp (2005) deals with necessary conditions for a MPE. Moreover, Xepapadeas (1992) relies on conjecture functions and examines non-degenerate Markovian Nash equilibria, which may or may not be Markov perfect.

One important point that the literature has largely left unaddressed is *how* desirable steady states are reached.[1] That is, researchers have generally not been interested in entire emissions trajectories, choosing instead to focus on the steady state. Issues of potential inter-temporal welfare loss (in relation to a social optimum) en route to the steady-state equilibrium are not explored. Such considerations can be important in instances when convergence to a steady-state is slow, especially if agents have reason to be disgruntled by the short-run implications of the instituted policy. In addition, equilibrium dynamics can be important if the regulator's goal is to ensure that pollution never exceeds a given level, for example, by enforcing a dynamic environmental standard. In the case of water pollution, such a standard could be to keep pollution levels low enough so that water bodies are "swimmable and fishable" at all times. By focusing on entire paths of emissions instead of just steady-state levels, this paper accommodates such concerns.

We initially focus on a class of deterministic infinite horizon differential games of pollution control in which agents' payoffs are polynomial in the total stock of pollution.[2] Moreover, we allow for potential irreversibility or hysteresis effects in the pollution accumulation process. Such phenomena are typically observed in many ecological processes, notably so in shallow lake systems (see, for e.g., (Maler et al., 2003; Kossioris et al., 2008)), and carry profound implications for pollution control policy. Given an open-loop emissions strategy satisfying a mild regularity condition, I exhibit an ambient transfer scheme that induces it in MPE.[3] The target open-loop strategy can

[1]Exceptions include papers by Benchekroun and Van Long (1998), Sorger (2005), and Akao (2008). But these authors allow for knowledge of individual agents' actions in the design of policy and thus do not focus on ambient transfers.

[2]Specific instances of this model can be found in many previous contributions including (Segerson, 1988; Tsutsui and Mino, 1990; Xepapadeas, 1992; Dockner and Van Long, 1993; Karp and Livernois, 1994; Dockner et al., 1996; Benchekroun and Van Long, 1998; Karp, 2005; Wirl, 2007; Dutta and Radner, 2009) .

[3]This result has certain parallels to the neoclassical-growth work of Boldrin and Montrucchio (1986) who, given a candidate policy, exhibit an optimal growth problem that produces it as an optimal solution.

be thought of as the solution to a suitably defined optimal control problem, which may focus on or reconcile such considerations as (i) social welfare maximization; (ii) meeting dynamic environmental standards at a minimum cost; or, (iii) maximizing the utility of the agent who is worst-off, among many others.

The proposed transfer scheme is a polynomial function of the observed difference between actual and desired total pollution and is designed so that, in MPE, no tax or subsidy is levied *at any point in time*, not just at the steady-state. Since equilibrium emissions strategies are open-loop and subgame-perfect, it is less likely that agents will find themselves off equilibrium. Thus, actual pollution levels will, at least in theory, plausibly match desired ones so that no transfers ever occur. (This paper employs the MPE criterion described in Definition 4.4 of Dockner et al. (2000), for which sufficient conditions are given in Theorem 4.4 of the same reference.) We illustrate the results by deriving the ambient transfer that induces welfare-maximizing emissions for a linear-quadratic oligopoly game introduced by Benchekroun and Van Long (1998).

Of course, deviations from the equilibrium can happen for a variety of reasons and are observed in experimental studies. A striking example can be found in Cochard et al. (2005) where ambient transfers perform quite poorly. At the same time, and in contrast to Cochard et al. (2005), Spraggon (2002) finds ambient transfers to be effective in inducing socially optimal behavior. These occasionally dramatic discrepancies between experimental studies are not thoroughly understood, though collusion seems to play a prominent role in the inefficiency observed in Cochard et al. (2005).

An additional implication of the deterministic analysis is that, with moderate monitoring, first-best outcomes can be achieved in settings in which they cannot be sustained as MPE without the use of policy. As an example, consider the linear quadratic game studied in Dockner and Van Long (1993), which draws on foundational work by Tsutsui and Mino (1990). The best one can hope for in this setting (assuming the discount rate is low enough) is a MPE in nonlinear strategies that leads to socially optimal *steady state* pollution levels. At the same time, Wirl (2007) shows that even this outcome depends crucially on the quadratic nature of the profit function, and does not hold in its absence. On a more abstract level, the analysis establishes that differential games with "bad" equilibrium properties can be, via the manipulation of the state-dependent component of agents' objective functions, transformed into ones possessing at least one MPE that is obvious and, where applicable, socially desirable.

This neat result breaks down when uncertainty is introduced into the pollutant accumulation process. From a purely technical point of view, the differential game becomes stochastic and its analysis is substantially complicated. Determining the temporal distribution of pollution as a result of agents' emissions rests on solving a stochastic differential equation, an exercise of considerable mathematical difficulty. Moreover, even when such an equation allows for analytical insight, the resulting process will typically fail to have a stationary distribution unless certain modeling assumptions are imposed. Such assumptions, while standard in the literature on stochastic models of economic growth (see Merton (1975)), are not natural in a pollution control context. Xepapadeas (1992) incorporates stochastic dynamics in his model but focuses on long-run asymptotics (once again relying on conjecture functions) and does not discuss the dynamic effect of policy implementation. He also does not quantify the magnitude of the transfers that are needed to induce the socially optimal steady state. I address some of these issues in this work.

In a stochastic environment, it is no longer reasonable for a regulator to solely focus on open-loop strategies. This is because such strategies do not make efficient use of available information and are likely to be suboptimal even in instances where there are no strategic interactions (see Example 3.1). Indeed, in stochastic control, optimal paths have a random feedback representation. Therefore, we widen the scope of the regulator's goals to include general Markovian strategies and go on to provide an analog of the results of the deterministic section. In the model's full generality, little can be said about the probabilistic properties of the global pollution stock trajectory and the resulting transfers. To make the analysis meaningful, we concentrate on the model by Benchekroun and Van Long (1998) that was discussed in the deterministic section. Assuming linear demand, we focus on schemes that induce emissions strategies that are symmetric, stationary, and linearly decreasing in total pollution. This class of target strategies is appealing for its simplicity. Moreover, when environmental damages are quadratic, its elements include the social optimum.

Under this specification on target strategies, the stochastic process of total pollution accumulation is a special case of the well-studied Cox-Ingersoll-Ross process (Cox et al., 1985), which is extensively used in finance and whose probabilistic and asymptotic properties are completely characterized. The underlying stochastic control problem is tractable and it is possible to gauge the effect of ambient transfers. In particular, given a target strategy, we exhibit a simple quadratic ambient transfer scheme that induces it in MPE and provide closed-form expressions for expected transfers at any point in time. These (expected) transfers are non-positive and their magnitude increases linearly with physical volatility, the size of the economy, and the absolute value of the slope of the inverse demand function. Moreover, we show that expected transfers vanish when the regulator wishes to induce a constant emissions strategy. To the best of our knowledge, this is the first paper that provides as precise a probabilistic analysis of dynamic nonpoint-source pollution control policy. This section ends in the spirit of Karp (2005) with a comparison of the expected transfers of ambient and point-source transfer schemes.

The paper is organized as follows. Section 2 discusses the deterministic model and its policy implications. Section 3 extends the analysis to stochastic environments. Section 4 provides concluding remarks. Technical proofs are collected in the Appendix.

2 The Deterministic Model

Suppose there are n agents who are involved in a pollution-generating economic activity. Agent i's emissions at time t are denoted by $e_i(t)$ and the global stock of pollution by $x(t) \in \Re_+$. Agent i's profit at time t is denoted by (vectors are indicated in bold)

$$\pi_i(\mathbf{e}(t), x(t)),$$

where the function $\pi_i(\cdot) : \Re^{n+1} \mapsto \Re$ is strictly concave in e_i, and is both *polynomial* and decreasing in x (note however that it is not necessarily separable in $\mathbf{e}$ and x). In the nonpoint source pollution literature π_i is typically only a function of e_i, with the damages from pollution entering only in the social welfare function. We relax this assumption to allow for agents directly affecting each others' profits as well as potentially incurring some of the costs of total pollution accumulation.

Agents' emissions are constrained by technology so that for all $i \in \{1, 2, .., n\}$ there exists $e_i^{max} \in \Re_+$ such that $e_i(t) \leq e_i^{max}$. We allow for the possibility of abatement, in the form of a negative emission rate. The time evolution of pollution is governed by the following differential equation

$$\dot{x}(t) = \sum_{i=1}^{n} e_i(t) - g(x(t)), \tag{1}$$

where $g(x)$ is a *polynomial* function that denotes the physical rate of natural purification, which satisfies

$$\lim_{x\to\infty} g(x) = \infty.$$

The function $g(\cdot)$ may have convex-concave nonlinearities in order to capture potential irreversibility or hysteresis effects in the pollution accumulation process.[4] To preclude the possibility of nonsensical trajectories, we impose the state non-negativity constraint $x(t) \geq 0$.

Suppose that initial pollution levels are universally bounded by a large constant K. Our assumptions imply that the global stock of pollution will be bounded so that, $x(t) \in \big[0, x^{max}\big]$, where

$$x^{max} = \max\Bigg\{K,\ \ \max\Bigg\{x \in \Re_+ : g(x) = \sum_{i=1}^{n} e_i^{max}\Bigg\}\Bigg\}.$$

Suppose that the regulator imposes an ambient transfer scheme ϕ that is a function of total pollution and calendar time. That is, at time 0, the regulator pre-commits to a policy, announcing to agents that they will be subject to a particular dynamic transfer of $\phi_i(x,t)$ if at time t the total pollution stock is equal to x. Focusing on agent i, and denoting other agents' emissions by $\tilde{\mathbf{e}}_{-\mathbf{i}}(x,t)$, ϕ gives rise to the following differential game:

$$\begin{aligned}
\max_{e_i(\cdot)} \quad & \int_0^{\infty} \mathrm{e}^{-\delta t}\big[\pi_i(e_i(t), \tilde{\mathbf{e}}_{-\mathbf{i}}(x(t),t), x(t)) + \phi_i(x(t),t)\big]\mathrm{d}t \\
\text{subject to:} \quad & \dot{x}(t) = e_i(t) + \sum_{j\neq i}^{n} \tilde{e}_j(x(t),t) - g(x(t)) \\
& e_i(t) \leq e_i^{max},\ \ x(t) \geq 0,\ \ x(0) = x_0,
\end{aligned} \tag{2}$$

where δ is the discount rate.

The regulator wishes to induce an open-loop emissions strategy $\hat{\mathbf{e}}$ where

$$\hat{\mathbf{e}} = \big\{\hat{e}_i(t) :\ \ t \geq 0,\ \ i \in \{1, 2, .., n\}\big\}.$$

This open-loop strategy $\hat{\mathbf{e}}$ may represent a control path that achieves or reconciles many social goals such as (a) maximizing social welfare; (b) ensuring that specific environmental standards are always met at a reasonable cost, or; (c) maximizing the utility of the worst-off agent, among others.

[4]For (a non-polynomial) example applicable to shallow lake dynamics, see Maler et al. (2003) and Kossioris et al. (2008).

Typically, we can think of $\hat{\mathbf{e}}$ as the solution to a particular optimal control problem. Given an initial condition $\hat{x}(0) = x_0$ on total pollution, $\hat{\mathbf{e}}$ gives rise to an associated pollution path $\hat{x}$, where

$$\hat{x} = \big\{\hat{x}(t), \ \ t \geq 0\big\}.$$

Theorem 1 shows that the regulator can induce the open-loop strategy $\hat{\mathbf{e}}$ in MPE with the use of a relatively simple ambient transfer scheme. First, we introduce some notation.

Definition 1 *An open-loop strategy $\hat{\mathbf{e}}(t)$ is admissible if it is continuously differentiable and satisfies the following inequalities*

$$\sum_{j=1}^{n} \hat{e}_j(t) \geq 0 \ \ \textit{and} \ \ \hat{e}_i(t) \leq e_i^{max}, \ \ \textit{for all} \ \ t \geq 0 \ \ \textit{and} \ \ i \in \{1, 2, ..., n\}.$$

Admissibility ensures feasibility along any subgame.[5] Given an admissible open-loop strategy $\hat{\mathbf{e}}$, we define the functions $f_i^{\hat{\mathbf{e}}} : \big[0, x^{max}\big] \times [0, \infty) \mapsto \Re$, where

$$\begin{aligned} f_i^{\hat{\mathbf{e}}}(x,t) \quad = \quad & -\delta \int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\mathrm{d}x + \frac{\partial}{\partial t}\Big[\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\mathrm{d}x\Big] \\ & + \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\Big[\sum_{j=1}^{n} \hat{e}_j(t) - g(x)\Big] - \pi_i(\hat{\mathbf{e}}(t), x), \end{aligned} \tag{3}$$

for $i \in \{1, 2, .., n\}$. The model assumptions imply that the functions $f_i^{\hat{\mathbf{e}}}$ are well-defined and polynomial in x. Let $\hat{m}_i \geq 1$ denote the polynomial degree of $f_i^{\hat{\mathbf{e}}}$. The following theorem summarizes the paper's first result.

Theorem 1 *Consider an admissible open-loop strategy $\hat{\mathbf{e}}$ and the functions $f_i^{\hat{\mathbf{e}}}$ given by Eq. (3). Suppose that the functions $V^i(x,t) : \big[0, x^{max}\big] \times [0, \infty) \mapsto \Re$, where*

$$V^i(x,t) = -\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\mathrm{d}x - \int_t^{\infty} f_i^{\hat{\mathbf{e}}}(\hat{x}(s), t)\mathrm{e}^{-\delta(s-t)}\mathrm{d}s,$$

are bounded from below and satisfy $\limsup_{t\to\infty} \mathrm{e}^{-\delta t} V^i(\hat{x}(t), t) \leq 0$*, for all initial conditions x_0 and $i \in \{1, 2, ..., n\}$. The ambient transfer*

$$\hat{\phi}_i(x,t) = \sum_{k=1}^{\hat{m}_i} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t), t)\frac{[x - \hat{x}(t)]^k}{k!}, \ \ i \in \{1, 2, ..., n\} \tag{4}$$

[5] Definition 2 generalizes this concept for arbitrary Markovian strategies.

induces $\hat{\mathbf{e}}$ *in Markov perfect equilibrium.*

Proof. Consider the Hamilton-Jacobi-Bellman (HJB) equation for agent i, assuming that other agents choose the open-loop emission strategies $\hat{\mathbf{e}}_{-\mathbf{i}}$,

$$\delta V^i(x,t) - V_t^i(x,t) = \max \left\{ \pi_i(e_i, \hat{\mathbf{e}}_{-\mathbf{i}}(t), x) + \hat{\phi}_i(x,t) + V_x^i(x,t)\left[e_i + \sum_{j\neq i} \hat{e}_j(t) - g(x)\right] \middle| \; x = 0 \Rightarrow e_i + \sum_{j\neq i} \hat{e}_j(t) \geq 0, \;\; e_i \leq e_i^{max} \right\}. \tag{5}$$

The fact that $\hat{\mathbf{e}}$ is admissible implies that $\sum_{j=1}^{n} \hat{e}_j(t) \geq 0$ and $\hat{e}_i(t) \leq e_i^{max}$. Thus, state non-negativity is not violated and upper bound constraints are respected. To ensure that agent i's best response is given by $\hat{e}_i(t)$, the right-hand-side of Eq. (5) must be maximized at that level of emissions. As the function π_i is strictly concave in e_i, it is sufficient to impose that the value function $V^i(x,t)$ satisfy

$$V_x^i(x,t) = -\frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x). \tag{6}$$

Eq. (6) in turn implies

$$V^i(x,t) = -\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\mathrm{d}x + \hat{A}_i(t), \tag{7}$$

where $\hat{A}_i(t)$ is a time-dependent function. Substituting the value function given by (7) into the HJB conditions obtains the following equation

$$\begin{aligned} &\delta\left[-\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\mathrm{d}x + \hat{A}_i(t)\right] - \frac{\partial}{\partial t}\left[-\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\mathrm{d}x + \hat{A}_i(t)\right] = \\ &= \pi_i(\hat{\mathbf{e}}(t), x) + \hat{\phi}_i(x,t) - \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)\left[\sum_{j=1}^{n} \hat{e}_j(t) - g(x)\right]. \end{aligned} \tag{8}$$

Recalling Eq. (3) and rearranging terms, Eq. (8) may be written in the following way

$$\begin{aligned} & \hat{\phi}_i(x,t) - \delta\hat{A}_i(t) + \frac{d}{\mathrm{d}t}\hat{A}_i(t) = f_i^{\hat{\mathbf{e}}}(x,t) \\ \overset{(4)}{\Rightarrow} \quad & \sum_{k=1}^{\hat{n}} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t),t)\frac{[x-\hat{x}(t)]^k}{k!} - \delta\hat{A}_i(t) + \frac{d}{\mathrm{d}t}\hat{A}_i(t) = f_i^{\hat{\mathbf{e}}}(x,t) \\ \Rightarrow \quad & \sum_{k=1}^{\hat{n}} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t),t)\frac{[x-\hat{x}(t)]^k}{k!} - f_i^{\hat{\mathbf{e}}}(x,t) - \delta\hat{A}_i(t) + \frac{d}{\mathrm{d}t}\hat{A}_i(t) = 0. \end{aligned} \tag{9}$$

Considering the Taylor expansion of $f_i^{\hat{\mathbf{e}}}(x,t)$ (recall that $f_i^{\hat{\mathbf{e}}}$ is polynomial in x) about $(\hat{x}(t),t)$, Eq. (9) obtains the following differential equation

$$\frac{d}{\mathrm{d}t}\hat{A}_i(t) - \delta\hat{A}_i(t) - f_i^{\hat{\mathbf{e}}}(\hat{x}(t),t) = 0. \tag{10}$$

Solving differential equation (10) yields

$$\hat{A}_i(t) = \mathrm{e}^{\delta t}\left[\hat{A}_i(0) + \int_0^t f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)\mathrm{e}^{-\delta s}\mathrm{d}s\right].$$

Setting

$$\hat{A}_i(0) = -\int_0^\infty f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)\mathrm{e}^{-\delta s}\mathrm{d}s$$

implies the particular solution

$$\hat{A}_i(t) = -\int_t^\infty f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)\mathrm{e}^{\delta(t-s)}\mathrm{d}s.$$

The theorem's assumptions imply that $V^i(x,t)$ satisfies sufficient conditions for optimality given by Theorem 4.4 in Dockner et al. (2000). ■

Economic Interpretation. Equations (3) and (4) may at first glance seem difficult to interpret. However, keeping in mind the theory of optimal control they can be explained in an intuitive way. Suppose that the open-loop strategy $\hat{\mathbf{e}}$ satisfies the HJB equations (and thus is a MPE) without the use of any transfer scheme. Denote by V_{NT}^i the optimal value functions, which would apply to this hypothetical no-transfer (NT) case. To be precise, the functions $V_{NT}^i(x,t)$ denote the maximized stream of discounted future profits at time t and current state x, given that the open-loop strategy $\hat{\mathbf{e}}$ satisfies the HJB conditions and is a MPE. Optimality at $\hat{\mathbf{e}}$ ensures that they are given by Eq. (7) for $\hat{A}_i(t) \equiv 0$ so that

$$V_{NT}^i(x,t) = -\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t),x)\mathrm{d}x.$$

A little bit of calculus then yields the following expression for $f_i^{\hat{\mathbf{e}}}$

$$f_i^{\hat{\mathbf{e}}}(x,t) = -\pi_i(\hat{\mathbf{e}}(t),x) - \frac{\mathrm{d}}{\mathrm{d}\Delta}\left[\mathrm{e}^{-\delta\Delta}V_{NT}^i\big(x(t+\Delta),t+\Delta\big)\right]_{\Delta=0,\ x(t)=x}. \tag{11}$$

(For a straightforward justification of Eq. (11), refer to the derivation of the HJB conditions on pages 41-43 of Dockner et al. (2000)) Viewed in this light, the function $f_i^{\hat{\mathbf{e}}}$ has a natural interpretation. It is the difference evaluated at (x,t) between (a) the negative of an agent's profit function applied to $\hat{\mathbf{e}}$ and; (b) the rate of change of his maximized stream of discounted future profits if $\hat{\mathbf{e}}$ satisfied the HJB conditions, and therefore constituted a MPE, without the use of transfers. [Note that if $\hat{\mathbf{e}}$ were a MPE without the use of policy this difference would be identically zero, i.e. $f_i^{\hat{\mathbf{e}}} \equiv 0$, negating the need for any transfers.]

As the open-loop strategy $\hat{\mathbf{e}}$ will not in general constitute a MPE, it is necessary to introduce a transfer scheme that will actually induce it. As previously discussed, policy makers are constrained to using instruments that only keep track of cumulative pollution levels. In addition, proposed schemes must be realistic; they cannot generate huge budget imbalances, nor can they rely on command-and-control measures. To this end, and similar to Xepapadeas (1992), a reasonable class of instruments is one which determines the size of the dynamic transfer as an explicit function of the deviation of the global pollution stock from its desired dynamic target $\{\hat{x}(t),\ t \geq 0\}$. To this effect, the following ambient transfer is proposed

$$\phi_i(x,t) = \sum_{k=1}^{\hat{m}_i} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t),t)\frac{[x-\hat{x}(t)]^k}{k!}.$$

The assumption that the functions π_i and g are polynomial in x implies that $f_i^{\hat{\mathbf{e}}}$ will also be polynomial in x. Thus, its Taylor expansion is valid on the whole domain (in other words, $f_i^{\hat{\mathbf{e}}}$ is an *entire* function) and we may write

$$\hat{\phi}_i(x,t) = \sum_{k=1}^{\hat{m}_i} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}(\hat{x}(t),t)\frac{[x-\hat{x}(t)]^k}{k!} = f_i^{\hat{\mathbf{e}}}(x,t) - f_i^{\hat{\mathbf{e}}}(\hat{x}(t),t), \ \ \forall (x,t) \in \big[0, x^{max}\big] \times [0,\infty). \tag{12}$$

When added to an agent's profit function, the transfer $\hat{\phi}_i(x,t)$ ensures that the right-hand-side of Eq. (11) will be zero, so long as the optimal value functions are appropriately altered. That is, the transfer will have the effect of equating the negative of the agent's profit function evaluated at $\hat{\mathbf{e}}$ to the rate of change of an appropriately defined maximized stream of discounted future profits. To this end, the functions $V^i(x,t)$ remain consistent to the form given by Eq. (7). However, in contrast to the hypothetical NT case, they will now also include a nontrivial function of t, $\hat{A}_i(t)$, so that

$$V^i(x,t) = V^i_{NT}(x,t) + \hat{A}_i(t).$$

We first reason intuitively as to what the function $\hat{A}_i(t)$ should look like. We see from the Taylor expansion of Eq. (12) that the transfer scheme ϕ introduces an instantaneous term which only depends on calendar time, namely $-f_i^{\hat{\mathbf{e}}}(\hat{x}(t),t)$. This transfer is incurred independently of agents' actions or current states. Relating this aspect of the scheme to the time-dependent component of the value function, we intuitively suspect that $\hat{A}_i(t)$ will equal the discounted stream of these time-dependent transfers so that

$$\hat{A}_i(t) = -\int_t^{\infty} \mathrm{e}^{-\delta(s-t)} f_i^{\hat{\mathbf{e}}}(\hat{x}(s),s)\mathrm{d}s. \tag{13}$$

Moving on to a formal argument, the function $\hat{A}_i(t)$ is most certainly not arbitrary. The HJB conditions imply that the negative of an agent's profit function must be equal to the rate of change of his maximized stream of discounted profits. Thus, $\hat{A}_i(t)$ must be a (non asymptotically divergent) solution of the following differential equation

$$-\big[\phi_i(x,t) \quad + \quad \pi_i(\hat{\mathbf{e}}(t),x)\big] - \frac{\mathrm{d}}{\mathrm{d}\Delta}\left[\mathrm{e}^{-\delta\Delta}\underbrace{\left(V^i_{NT}\big(x(t+\Delta),t+\Delta)\big) + \hat{A}_i(t+\Delta)\right)}_{V^i(x(t+\Delta),t+\Delta)}\right]_{\Delta=0,\ x(t)=x} = 0$$

$$\Rightarrow \qquad f_i^{\hat{\mathbf{e}}}(\hat{x}(t),t) + \delta\hat{A}_i(t) - \frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_i(t) = 0, \quad i \in \{1,2,...,n\}. \tag{14}$$

Does our intuitive guess meet this criterion? The answer is yes, as the expression of Eq. (13) satisfies differential equation (14) while ensuring that the value functions do not diverge as $t \to \infty$.

Note that the polynomial assumption on π_i and g is absolutely critical for the above scheme to work. This is because in its absence the Taylor approximation of $f_i^{\hat{\mathbf{e}}}$ would only be locally valid. In particular, Eq. (12) would only hold in a neighborhood around $(\hat{x}(t),t)$ and Markov perfection would be lost. Finally, it is important to explain why Theorem 1 does not extend to arbitrary, non open-loop, Markovian strategies. This is because, again, the polynomial nature of $f_i^{\hat{\mathbf{e}}}$ is potentially lost if $\hat{\mathbf{e}}(t)$ is substituted by a feedback strategy $\hat{\mathbf{e}}(x,t)$. That said, in some instances and for some target non-open loop strategies (one of which is extensively discussed in Example 3.1) polynomiality is preserved, and one can induce these feedback strategies with the proposed set of schemes.

Theorem 1 gives rise to two immediate corollaries.

Corollary 1 *All admissible open-loop strategies* $\hat{\mathbf{e}}$ *for which* $\frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)$ *and* $\frac{\partial^2}{\partial t \partial e_i}\pi_i(\hat{\mathbf{e}}(t), x)$ *are bounded for all* $i \in \{1, 2, .., n\}$ *satisfy the assumptions of Theorem 1.*

Proof. Recall that under our assumptions both $\mathbf{e}$ and x are bounded. The result follows. ■

As an example, Corollary 1 is satisfied for profit functions π_i that are separable in $\mathbf{e}$ and x, provided the target $\hat{\mathbf{e}}$ is bounded from below by a strictly positive number (an interior assumption that is typically true of first-best solutions) and does not change too rapidly. These modeling assumptions are present in many well-studied dynamic games of pollution control.

Corollary 2 *Eq. (4) implies that* $\hat{\phi}(\hat{x}(t), t) \equiv \mathbf{0}$*. Thus, in equilibrium, the mechanism prescribed by Theorem 1 ensures that no transfers are ever made.*

The practical relevance of Corollaries 1 and 2 hinges on the robustness of our equilibrium analysis. On this score, Theorem 1 implies that a desirable open-loop emissions strategy may be induced in Markov-perfect equilibrium. This finding suggests that agents are less likely to deviate from the equilibrium path as they do not condition their emissions on anything else but calendar time, knowing that their actions will constitute a best response regardless of perceived pollution levels. The predictive capacity of an equilibrium open-loop strategy that is subgame perfect is, at least in theory, quite robust. As a result, provided there is no uncertainty in the evolution of the global pollution stock, Corollary 2 indicates that it is unlikely for any tax or subsidy to ever be levied.

Example 2.1 Adopting the model of polluting oligopolists by Benchekroun and Van Long (1998), suppose there are n identical agents producing a homogeneous good. I take the output of each agent to equal his emissions and assume that each agent has a constant unit cost $c \geq 0$. Furthermore, assume that the underlying demand for the produced good is linear so that the inverse demand function $P(\cdot)$ is given by

$$P(\mathbf{e}) = B - b\sum_{j=1}^{n} e_j.$$

Hence, agent i's profit at time t is given by

$$\pi_i(\mathbf{e}(t), x(t)) \equiv \pi_i(\mathbf{e}(t)) = \left[B - b\sum_{j=1}^{n} e_j(t)\right] e_i(t) - ce_i(t).$$

The pollution stock's rate of natural purification is linear so that $g(x) = \beta x$, $\beta > 0$. We are interested in calculating a transfer scheme along the lines of Theorem 1 that will induce an open-loop strategy $\hat{\mathbf{e}}(t)$. As the profit function in this example is independent of the state, the transfer scheme is quite simple to calculate. The functions f_i are linear in x so that applying Eq. (3) obtains

$$\begin{aligned} f_i^{\hat{\mathbf{e}}}(x,t) &= -\delta\left[B - b\sum_{j\neq i}\hat{e}_j(t) - 2b\hat{e}_i(t) - c\right]x - b\left[\sum_{j\neq i}\frac{d}{dt}\hat{e}_j(t) + 2\frac{d}{dt}\hat{e}_i(t)\right]x \\ &\quad + \left[B - b\sum_{j\neq i}\hat{e}_j(t) - 2b\hat{e}_i(t) - c\right]\left[\sum_{j=1}^{n}\hat{e}_j(t) - \beta x\right] - \left[N - b\sum_{j=1}^{n}\hat{e}_j(t)\right]\hat{e}_i(t) + c\hat{e}_i(t). \end{aligned}$$

Hence, the sum that appears in Eq. (4) will consist of just one term so that

$$\hat{\phi}_i(x,t) = \left(-(\delta+\beta)\left[B - b\sum_{j\neq i}\hat{e}_j(t) - 2b\hat{e}_i(t) - c\right] - b\left[\sum_{j\neq i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_j(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_i(t)\right]\right)[x - \hat{x}(t)],$$
$$i \in \{1, 2, ..., n\}. \tag{15}$$

We go on to illustrate Eq. (15) with a specific open-loop strategy. Suppose, along the lines of Xepapadeas (1992) and Benchekroun and Van Long (1998), that pollution creates environmental damages, which are not internalized by polluting agents, denoted by a convex and increasing function $D(x(t))$. In this example, we take damages to be quadratic, so that

$$D(x(t)) = -\theta x(t)^2, \;\; \theta > 0.$$

The socially optimal emission schedule is defined to be the solution of the following optimal control problem $\big($by symmetry we may simply focus on $e(t) = \sum_{i=1}^{n} e_i(t)\big)$:

$$\begin{aligned} &\max_{e(\cdot)} && \int_0^\infty \mathrm{e}^{-\delta t}\Big(\big[B - be(t)\big]e(t) - ce(t) - \theta x(t)^2\Big)\mathrm{d}t \\ &\text{subject to:} && \dot{x}(t) = e(t) - \beta x(t) \\ & && e(t) \le e^{max}, \;\; x(t) \ge 0, \;\; x(0) = x_0. \end{aligned} \tag{16}$$

Let the constants α_1 and α_2 equal

$$\alpha_1 = \left[(\delta + 2\beta)b - \sqrt{(\delta + 2\beta)^2 b^2 + 4\theta b}\right] \Big/ 2 < 0$$

$$\alpha_2 = \alpha_1 \frac{(B-c)/b}{\delta + \beta - \frac{\alpha_1}{b}} < 0.$$

It is straightforward to show that an interior optimal control $e^*(t)$ will be equal to

$$e^*(t) = \frac{\alpha_1}{b} x^*(t) + \frac{B - c + \alpha_2}{2b}, \tag{17}$$

where $x^*(t)$ satisfies the differential equation

$$\frac{\mathrm{d}x^*(t)}{\mathrm{d}t} = \frac{B - c + \alpha_2}{2b} + \left[\frac{\alpha_1}{b} - \beta\right] x^*(t),$$

subject to initial condition $x^*(0) = x_0$. Thus,

$$e^*(t) = \frac{\alpha_1}{b} \underbrace{\left[-\frac{B - c + \alpha_2}{2b\left(\frac{\alpha_1}{b} - \beta\right)} + \left(x_0 + \frac{B - c + \alpha_2}{2b\left(\frac{\alpha_1}{b} - \beta\right)} \right) \mathrm{e}^{\left(\frac{\alpha_1}{b} - \beta\right)t} \right]}_{x^*(t)} + \frac{B - c + \alpha_2}{2b}. \tag{18}$$

Suppose for simplicity that model primitives are such that $e^*(t) \in \left[0, e^{max}\right]$ for all t. We proceed to calculate a transfer scheme along the lines of Theorem 1 that will induce the open-loop strategies

$$\frac{e^*(t)}{n}, \quad i \in \{1, 2, ..., n\}.$$

Applying Eq. (15) to the desired strategies yields the following ambient transfer

$$\phi_i^*(x,t) = -\left((\delta + \beta) \left[B - b\frac{n+1}{n} e^*(t) - c \right] + b\frac{n+1}{n} \frac{\mathrm{d}}{\mathrm{d}t} e^*(t) \right) [x - \hat{x}(t)], \quad i \in \{1, 2, ..., n\}. \tag{19}$$

3 The Stochastic Setting

In a physical environment in which pollutant accumulation evolves stochastically, state-dependent policy needs to be designed with caution. This is because deviations from any kind of dynamic target are inevitable and actual transfers will have to be made between agents and the regulating authority. Appropriate policy tools should arguably result in transfers that are moderate, or at the very least predictable (in a probabilistic sense). In this section, we attempt to address some of these issues in a systematic fashion.

Departing from a deterministic physical environment we assume that the evolution of the pollution stock is governed by a stochastic analogue of Eq. (1), i.e., the following stochastic differential equation

$$\mathrm{d}x(t) = \left[\sum_{j=1}^{n} e_j(t) - g(x(t))\right]\mathrm{d}t + \sqrt{h(x(t))}\mathrm{d}W_t, \quad x(0) = x_0, \tag{20}$$

where the functions $g(x)$ is defined as before, $h(x)$ is a non-negative polynomial, and W_t is a Wiener process. Thus, pollutant accumulation is a diffusion process with instantaneous drift $\sum_{j=1}^{n} e_i(t) - g(x(t))$, and variance $h(x(t))$.

In a stochastic environment, a regulator may justifiably wish to induce emissions strategies that are not necessarily open-loop. This is because the evolution of the state is not perfectly predictable like it is in the deterministic case and an open-loop policy does not make efficient use of available information. Indeed, in stochastic control, optimal paths have a (random) feedback representation (see Example 3.1). Therefore, it is reasonable for the regulator to wish to adapt his or her goals to the random trajectory of the total pollution stock.

Suppose the regulator wishes to induce a Markovian emissions strategy $\hat{\mathbf{e}}$ such that

$$\hat{\mathbf{e}} = \left\{\hat{e}_i(x,t) : \quad (x,t) \in \Re_+ \times [0,\infty), \quad i \in \{1,2,..,n\}\right\}.$$

This strategy $\hat{\mathbf{e}}$ may represent a control path that achieves or reconciles many social goals such as (a) maximizing expected social welfare; (b) ensuring that specific environmental standards are always met at a reasonable cost, or; (c) maximizing the utility of the worst-off agent, among others. Equivalently to the deterministic settting, we can think of $\hat{\mathbf{e}}$ as the solution to a particular stochastic optimal control problem. Given an initial condition $\hat{x}(0) = x_0$ on total pollution, $\hat{\mathbf{e}}$ has an associated random pollution path $\hat{x}$, where

$$\hat{x} = \left\{\hat{x}(t), \quad t \geq 0\right\},$$

that is governed by stochastic differential equation (23).

We again provide necessary notation. First, the concept of admissibility is extended to Markovian strategies.

Definition 2 *A Markovian strategy* $\hat{\mathbf{e}}(x,t)$ *is admissible if it is continuously differentiable and satisfies the following inequalities*

$$\sum_{j=1}^{n} \hat{e}_j(0,t) \geq 0 \ \text{ for all } t \in [0,\infty) \text{ and } \ \hat{e}_i(x,t) \leq e_i^{max} \ \text{ for all } \ (x,t) \in \Re_+ \times [0,\infty), \ \ i \in \{1,2,...,n\}.$$

Consequently, Eq. (3) is altered in the following way to accommodate randomness and the strategy's feedback representation,

$$\begin{aligned} f_i^{\hat{\mathbf{e}}}(x,t) \quad = \quad & -\delta \int \frac{\partial}{\partial e_i} \pi_i(\hat{\mathbf{e}}(x,t),x)\mathrm{d}x + \frac{\partial}{\partial t}\left[\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(x,t),x)\mathrm{d}x\right] \\ & + \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(x,t),x)\left[\sum_{j=1}^{n}\hat{e}_j(x,t) - g(x)\right] + \frac{\partial}{\partial x \partial e_i}\pi_i(\hat{\mathbf{e}}(x,t),x)\frac{h(x)}{2} \\ & -\pi_i(\hat{\mathbf{e}}(x,t),x), \quad i \in \{1,2,...,n\}. \end{aligned} \tag{21}$$

This function admits an equivalent economic interpretation as the one discussed after the proof of Theorem 1. We now proceed to provide a stochastic equivalent to Theorem 1. In what follows, $\mathbf{E}$ denotes the expected value operator.

Theorem 2 *Consider an admissible Markovian strategy* $\hat{\mathbf{e}}(x,t)$ *and suppose that the functions* $f_i^{\hat{\mathbf{e}}}$ *given by Eq. (21) are polynomial in* x. *Furthermore, suppose that the functions* $V^i(x,t)$: $\Re_+ \times [0,\infty) \mapsto \Re$, *where*

$$V^i(x,t) = -\int \frac{\partial}{\partial e_i}\pi_i(\hat{\mathbf{e}}(x,t),x)\mathrm{d}x - \int_t^{\infty} f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)],t\big)\mathrm{e}^{-\delta(s-t)}\mathrm{d}s,$$

are bounded from below and satisfy $\limsup_{t\to\infty} \mathrm{e}^{-\delta t}\mathbf{E}\big[V^i(\hat{x}(t),t)\big] \leq 0$, *for all initial conditions* x_0 *and* $i \in \{1,2,...,n\}$. *The ambient transfer*

$$\hat{\phi}_i(x,t) = \sum_{k=1}^{\hat{m}_i} \frac{\partial^k f_i^{\hat{\mathbf{e}}}}{\partial x^k}\big(\mathbf{E}[\hat{x}(t)],t\big)\frac{\big[x - \mathbf{E}[\hat{x}(t)]\big]^k}{k!}, \quad i \in \{1,2,...,n\} \tag{22}$$

induces $\hat{\mathbf{e}}(x,t)$ *in Markov perfect equilibrium.*

Proof. Identical to Theorem 1, except that we invoke Theorem 8.5 in Dockner et al. (2000). ■

Theorem 2 is primarily presented to make the analysis consistent with that of Section 2. Its boundedness conditions seem to be quite strong (especially since the state space is now unbounded from above), which implies that its reach may not be wide as that of Theorem 1. In the example that follows we do not apply Theorem 2, relying instead on a different proof technique that does not require boundedness of the value functions.

Example 3.1 Consider the stochastic equivalent of Example 2.1. We follow Xepapadeas (1992) and assume that $h(x) = \sigma^2 x$. Thus, the evolution of the pollution stock is governed by the following stochastic differential equation

$$\mathrm{d}x(t) = \left[\sum_{j=1}^{n} e_j(t) - \beta x(t)\right]\mathrm{d}t + \sigma\sqrt{x(t)}\mathrm{d}W_t, \ \ x(0) = x_0. \tag{23}$$

Suppose that the regulator is interested in inducing a symmetric and stationary Markovian emissions strategy that is linearly decreasing in total pollution levels. The target emissions strategy, $\hat{\mathbf{e}}$, therefore satisfies

$$\hat{e}_i(x,t) = \hat{e}_i(x) = \frac{1}{n}\big[E - \gamma x\big], \ \ i \in \{1, 2, .., n\} \tag{24}$$

where $\gamma \geq 0$ and $0 < E \leq e^{max}$. Observe that $\hat{\mathbf{e}}$ is an admissible Markovian strategy, ensuring that state non-negativity is always maintained.

With this target strategy specification, the pollutant dynamics (23) can be rewritten in the following way:

$$\mathrm{d}x(t) = (\beta + \gamma)\left[\frac{E}{\beta+\gamma} - x(t)\right]\mathrm{d}t + \sigma\sqrt{x(t)}\mathrm{d}W_t, \ \ x(0) = x_0. \tag{25}$$

Eq. (25) is an instance of the celebrated Cox-Ingersoll-Ross (Cox et al., 1985) process, which is extensively used in finance. Fortunately, its evolution and steady-state properties are completely characterized. The following proposition summarizes.

Proposition 1 *(Cox et al., 1985) Stochastic differential equation (25) has a unique solution given by the diffusion process* $\big\{\hat{x}(t) : t \geq 0\big\}$ *where*

(a) $\hat{x}(t)$ *has a noncentral chi-square distribution with expectation*

$$\mathbf{E}[\hat{x}(t)] = \hat{x}_0 \mathrm{e}^{-(\beta+\gamma)t} + \frac{E}{\beta+\gamma}\big[1 - \mathrm{e}^{-(\beta+\gamma)t}\big],$$

and variance

$$\mathbf{Var}[\hat{x}(t)] = \hat{x}_0 \frac{\sigma^2}{\beta+\gamma}\big[\mathrm{e}^{-(\beta+\gamma)t} - \mathrm{e}^{-2(\beta+\gamma)t}\big] + \frac{E\sigma^2}{2(\beta+\gamma)^2}\big[1 - \mathrm{e}^{-(\beta+\gamma)t}\big]^2.$$

(b) If $2E > \sigma^2$ then $\{\hat{x}(t) : t \geq 0\}$ has a stationary distribution that is Gamma$\left(\frac{2E}{\sigma^2}, \frac{\sigma^2}{2(\beta+\gamma)}\right)$.

Due to its relevance for financial applications, much numerical analysis has been undertaken to describe the precise nature of this noncentral chi-square distribution.[6] For our purposes, knowledge of its mean and variance will suffice.

In view of Proposition 1, the class of target strategies introduced in Eq. (24) holds considerable appeal. This is because, if somehow induced, its elements lead to an equilibrium pollutant accumulation process that can be described in precise probabilistic terms. Moreover, this class of target strategies is insightful because it lends itself to simple policy prescriptions. In particular, a quadratic ambient transfer scheme is presented that induces, in MPE, a strategy satisfying Eq. (24), which meets the stability condition $2E > \sigma^2$.

Adapting Eq. (21) to fit our example leads to the following expression (as the target strategy is stationary we suppress the time argument from $f_i^{\hat{\mathbf{e}}}$)

$$\begin{aligned} f_i^{\hat{\mathbf{e}}}(x) &= -\delta\left[\gamma b\frac{n+1}{2n}x^2 - \left(B - c - \frac{b(n+1)}{n}E\right)x\right] - \Big[B - b(E-\gamma x)\Big]\frac{E-\gamma x}{n} + \frac{c}{n}\big[E-\gamma x\big] \\ &\quad + \left[\gamma b\frac{n+1}{n}x + \left(B - c - \frac{b(n+1)}{n}E\right)\right]\big[E - (\beta+\gamma)x\big] + \frac{\sigma^2}{2}\gamma b\frac{n+1}{2n}x, \end{aligned} \tag{26}$$

for $i \in \{1, 2, .., n\}$. Similar to the proof of Theorem 1, this function will appear in the HJB equation of the stochastic control problem faced by the agents. The third result of the paper is summarized in the following Proposition. Its proof does not rely on the sufficient conditions of Theorem 2, and is presented separately in the paper's Appendix.

Proposition 2 *Consider the problem setting of Example 3.1. Let $\hat{\mathbf{e}}$ denote a target Markovian strategy given by Eq. (24) such that $2E > \sigma^2$. The ambient transfer scheme $\hat{\phi}$ such that*

$$\hat{\phi}_i(x,t) = \frac{\mathrm{d}^2 f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x^2}\Big(\mathbf{E}[\hat{x}(t)]\Big)\frac{\big[x - \mathbf{E}[\hat{x}(t)]\big]^2}{2} + \frac{\mathrm{d}f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x}\Big(\mathbf{E}[\hat{x}(t)]\Big)\big[x - \mathbf{E}[\hat{x}(t)]\big], \quad i \in \{1, 2, ..., n\}, \tag{27}$$

where all relevant quantities are defined in Proposition 1 and Eq. (26), induces $\hat{\mathbf{e}}$ in Markov-perfect equilibrium.

[6] See Dyrting (2004) for a discussion of efficient numerical methods to determine its probability distribution function.

Proof. See Appendix. ■

Note that the function $\frac{\mathrm{d}^2 f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x^2}$ is a constant such that

$$\frac{\mathrm{d}^2 f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x^2} = -\frac{b\gamma\big[\delta(n+1) + 2n(\beta+\gamma) + 2\beta\big]}{n}. \tag{28}$$

Thus, in equilibrium, at time t an agent incurs an expected ambient transfer that is equal to

$$\mathbf{E}\big[\phi_i(\hat{x}(t), t)\big] = -\frac{b\gamma\big[\delta(n+1) + 2n(\beta+\gamma) + 2\beta\big]}{n}\frac{\mathbf{Var}[\hat{x}(t)]}{2}, \tag{29}$$

where $\mathbf{Var}[\hat{x}(t)]$ is given by Proposition 1. This transfer is non-positive and can be clearly seen to be zero in the deterministic $\sigma = 0$ case. Interestingly, it is also zero when γ vanishes (i.e., when the regulator wishes to induce a constant emissions strategy). The next proposition gives a precise description of the total discounted cost of policy implementation.

Proposition 3 *Expected total ambient transfers (across time and agents) are equal to*

$$-\sigma^2 b\gamma\left[\delta(n+1) + 2n(\beta+\gamma) + 2n\beta\right]\frac{E + \delta\hat{x}_0}{\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))} \tag{30}$$

Proof. We proceed to calculate

$$\mathbf{E}\left[\int_0^\infty \mathrm{e}^{-\delta t}\hat{\phi}_i(\hat{x}(t), t)\mathrm{d}t\right] = \mathbf{E}\left[\int_0^\infty \mathrm{e}^{-\delta t}\left(\frac{\mathrm{d}^2 f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x^2}\frac{\big[\hat{x}(t) - \mathbf{E}[\hat{x}(t)]\big]^2}{2} + \frac{\mathrm{d}f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x}\Big(\mathbf{E}[\hat{x}(t)]\Big)\big[\hat{x}(t) - \mathbf{E}[\hat{x}(t)]\big]\right)\mathrm{d}t\right].$$

By Proposition 1 and Fubini's Theorem, the expectation and integral operators can be interchanged so that

$$\begin{aligned}
\mathbf{E}\left[\int_0^\infty \mathrm{e}^{-\delta t}\hat{\phi}_i(\hat{x}(t), t)\mathrm{d}t\right] &= \int_0^\infty \mathrm{e}^{-\delta t}\frac{\mathrm{d}^2 f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x^2}\frac{\mathbf{Var}[\hat{x}(t)]}{2}\mathrm{d}t = \frac{\mathrm{d}^2 f_i^{\hat{\mathbf{e}}}}{\mathrm{d}x^2}\frac{\sigma^2(E + \delta\hat{x}_0)}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))} \\
&= -\frac{b\gamma\big[\delta(n+1) + 2n(\beta+\gamma) + 2\beta\big]}{n}\frac{\sigma^2(E + \delta\hat{x}_0)}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))} \\
&= -\sigma^2 b\gamma\left[\frac{\delta(n+1)}{n} + 2(\beta+\gamma) + \frac{2\beta}{n}\right]\frac{E + \delta\hat{x}_0}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))}
\end{aligned}$$

Adding all of the above terms for $i \in \{1, 2, ..., n\}$ establishes the result. ■

Proposition 4 *Expected total ambient transfers are non-positive. They are equal to zero when γ is equal to zero. Their absolute value is decreasing in β and δ, increasing in γ, and linearly increasing in E, n, b, and σ^2.*

Proof. The monotonicity results regarding β and γ can be established by taking the appropriate derivatives of Eq. (30) and observing their signs. All other statements are obvious by inspection. ■

Propositions 3 and 4 provide a precise account of the expected tax burden agents will bear. That their magnitude is linearly increasing in σ^2 and E is because of the linear effect these parameters have on the global pollution stock's volatility, and subsequent deviation from expected value levels (see Proposition 1). Equivalently, the linear effect of b can be traced to the coefficient of the ambient transfer's quadratic term exhibited in Eq. (28). Moreover, this coefficient's sign is responsible for the fact that expected transfers are non-positive.

We observe that ambient policy becomes more expensive as n, the size of the economy, grows. This point makes intuitive sense: the greater the number of polluting agents, the less an individual one will feel that his or her actions affect the global pollution stock. Therefore, it becomes more and more expensive to influence agent behavior. We revisit this point when we briefly address point-source policy. Monotonicity results regarding β, γ and δ can be understood by noting the countervailing effects these parameters have on (a) the transfer coefficient of Eq. (28) and (b) the discounted stream of pollution volatility, i.e.,

$$\frac{\sigma^2(E + \delta\hat{x}_0)}{2\delta(\delta + \beta + \gamma)(\delta + 2(\beta + \gamma))}$$

which was calculated in the proof of Proposition 3.

Equivalently to Example 2.1, we illustrate Proposition 2 and Eq. (27) by applying it to the social optimum. (We once again posit a quadratic damage function of $\theta x(t)^2$). Recalling the constants α_1 and α_2 from Example 2.1, let

$$\tilde{\alpha}_1 = \alpha_1$$
$$\tilde{\alpha}_2 = \alpha_2 + \frac{\alpha_1\sigma^2}{\delta + \beta - \frac{\alpha_1}{b}}.$$

Assuming the regularity condition

$$\frac{B - c + \tilde{\alpha}_2}{b} > \sigma^2$$

it is straightforward to show that the following symmetric and stationary Markovian strategy maximizes total expected welfare

$$e_i^*(x) = \frac{1}{n}\Bigg[\underbrace{\frac{B - c + \tilde{\alpha}_2}{2b}}_{E^*} - \Bigg(\underbrace{-\frac{\tilde{\alpha}_1}{b}}_{\gamma^*}\Bigg)x\Bigg]. \tag{31}$$

Thus, the socially optimal policy is consistent with our class of target strategies given by Eq. (24). Applying Eq. (31) to Eqs. (26) and (27) obtains the following transfer scheme

$$\begin{aligned}\phi_i^*(x,t) \quad = \quad & -\frac{b\gamma^*\big[\delta(n+1) + 2n(\beta+\gamma^*) + 2\beta\big]}{n}\frac{\big[x - \mathbf{E}[x^*(t)]\big]^2}{2} + \frac{1}{2n}\Big[-2(B-c)\big[(n-1)\gamma^* + \\ & n(\beta+\delta)\big] + 2b\beta(n+1)\big(E^* - 2\gamma^*\mathbf{E}[x^*(t)]\big) + 2b\delta(n+1)\big(E^* - \gamma^*\mathbf{E}[x^*(t)]\big) + 4b\gamma^* nE^* \\ & + b\gamma^*(n+1)\sigma^2 - bn(\gamma^*)^2\mathbf{E}[x^*(t)]\Big]\big[x - \mathbf{E}[x^*(t)]\big], \quad i \in \{1, 2, ..., n\}. \end{aligned} \tag{32}$$

where, by Proposition 1,

$$\mathbf{E}[x^*(t)] = \hat{x}_0 \mathrm{e}^{-(\beta+\gamma^*)t} + \frac{E^*}{\beta+\gamma^*}\big[1 - \mathrm{e}^{-(\beta+\gamma^*)t}\big].$$

But, while we focus on the social optimum, an important implication of Proposition 2 is that it is possible to induce strategies that reconcile many different considerations. One may wish, for instance, to induce a linear strategy that maximizes steady-state payoffs while ensuring that the mean and variance of steady-state pollution levels be below certain exogenously determined levels. In view of Proposition 1, determining such a target strategy (i.e., solving for the relevant E and γ) would amount to solving a two-variable nonlinear optimization problem with quadratic constraints.

Static setting. We see that, in this example, the main effect of adding uncertainty to the problem is that in contrast to the deterministic case agents will, in expectation, incur a tax burden. A reasonable question to ask is whether this insight translates to a stochastic, but static environment. It is not difficult to observe that it does not. In this sense, the assumption of a dynamic pollution stock is crucial. The argument is a simple stochastic analog of the analysis appearing in the beginning of section 3.1 in Xepapadeas (1992).

Suppose the regulator wishes to induce an emissions profile $\hat{\mathbf{e}}$. This results in total pollution $\hat{x}$, where

$$\mathbf{E}[\hat{x}] = \sum_{j=1}^{n} \hat{e}_i.$$

Suppose, further, that the regulator institutes a linear ambient transfer $\hat{\phi}$ such that

$$\hat{\phi}_i(x) = k_i \Big[x - \mathbf{E}[\hat{x}] \Big], \quad k_i \in \Re, \quad i \in \{1, 2, ..., n\}.$$

First order conditions establish that setting

$$k_i = -\frac{\partial \mathbf{E}[\pi_i]}{\partial e_i}(\hat{\mathbf{e}}), \quad k_i \in \{1, 2, .., n\}$$

induces $\hat{\mathbf{e}}$ in equilibrium. Expected transfers will clearly be zero as the transfer is linear in x.

We briefly note how this result is consistent to the dynamic analysis. This is because, while in a dynamic framework expected transfers may be strictly negative, the relevant comparison involves a setting in which the regulator wishes to induce a *constant* (i.e., static) emissions strategy. In this case, Proposition 4 establishes that, similar to a static framework, expected transfers will once again vanish.

Comparison to generic point source transfers. The prudence of instituting ambient policy will, in large part, depend on whether the transfer it imposes is excessive. To this end, it is useful to compare the expected tax burden of Proposition 3 to one that a generic point-source tax/subsidy would generate. Karp (2005) pursues this goal for the case of a flow pollutant and rigorously compares the two tax burdens in the open-loop equilibrium case, providing conditions under which one dominates the other.

Suppose that the regulator is able to observe individual emissions and charge a tax/subsidy per unit of individual emissions. That is, the regulator can impose a transfer of (similar to Benchekroun and Van Long (1998) and Karp (2005))

$$\phi_i(x)e_i$$

to an agent i. The goal is to once again induce a target strategy that satisfies Eq. (24). Using cumbersome but straightforward analysis[7] one can show that a transfer scheme such that

$$\phi_i(x) = \beta_1 x + \beta_2, \;\; i \in \{1, 2, .., n\}$$

where

$$\beta_1 = \frac{\frac{-b\gamma}{n}\Big[\frac{\delta}{2n} + (n+1)\beta + n\gamma\Big]}{\frac{\delta}{2} + \beta + \gamma\frac{n-1}{n}} \leq 0$$

$$\beta_2 = \frac{\gamma\Big[-(B-c)(n-1) + (n+1)\beta\frac{\sigma^2}{2} + 2bEn\Big] + \beta_1\Big[E(n-1) + n\frac{\sigma^2}{2}\Big] - (B-c)(\beta+\delta)n + bE(\beta+\delta)(n+1)}{(\beta+\delta)n + \gamma(n-1)}, \tag{33}$$

induces a strategy satisfying Eq. (24) that meets the stability condition $2E > \sigma^2$. Consequently, the total expected discounted transfer across agents and time is equal to

$$\begin{aligned}
&\mathbf{E}\int_0^\infty \mathrm{e}^{-\delta t}\big[\beta_1\hat{x}(t) + \beta_2\big]\big[E - \gamma\hat{x}(t)\big]\mathrm{d}t \\
= \quad & \int_0^\infty \mathrm{e}^{-\delta t}\Big[-\beta_1\gamma\mathbf{E}\big[\hat{x}(t)^2\big] + \big[E\beta_1 - \beta_2\gamma\big]\mathbf{E}\big[\hat{x}(t)\big] + \beta_2 E\Big]\mathrm{d}t \\
= \quad & -\beta_1\gamma\Big[\frac{\sigma^2(E+\delta\hat{x}_0)}{2\delta(\delta+\beta+\gamma)(\delta+2(\beta+\gamma))} + \frac{2E^2 + 2\delta E x_0 + \delta(\beta+\delta+\gamma)x_0^2}{\delta(\beta+\delta+\gamma)(2\beta+\delta+2\gamma)}\Big] \\
& + \big[E\beta_1 - \beta_2\gamma\big]\frac{E+\delta x_0}{\delta(\delta+\beta+\gamma)} + \frac{\beta_2 E}{\delta}.
\end{aligned} \tag{34}$$

The complexity of Eq. (34) does not permit easy comparisons between the magnitudes of the two tax/subsidy burdens. Still, observing Eqs. (30) and (34) there are a few key points worth making.

First, under nonpoint-source policy the expected tax burden diverges as the size of the economy grows. This does not occur under a point source tax/subsidy where, as Eq. (34) shows, the tax/subsidy converges. Thus, sustaining a given policy as the economy grows becomes harder under the former compared to the latter. Then again, this should not be especially surprising. As Karp (2005) discusses in his paper, when the size of the economy becomes large ambient policy becomes intuitively untenable. An agent has to feel that his actions have a nontrivial impact on the total pollution stock in order to be affected by ambient standards, and alter his behavior accordingly. For

[7] Derivation and Mathematica output available upon request.

example, an ambient standard for automobile emissions in say, New York City, would be patently unrealistic.

Second, there exists a threshold value of volatility for which the nonpoint source tax burden becomes smaller than its equivalent point source one. This is because as volatility decreases, the probabilistic evolution of the pollution stock more closely resembles the evolution of its absolute value. For low values of volatility the actual process is very close to its expected value and therefore ambient transfers along the lines of Proposition 2 become accordingly small.

Third, recall that as γ becomes small the expected tax burden in the nonpoint-source policy gets driven down to zero. This does not occur in the point-source case. Here, as γ tends to 0 the transfer converges to

$$\frac{-(B-c)(\beta+\delta)n+bE(\beta+\delta)(n+1)}{n(\beta+\delta)}\cdot\frac{E}{\delta}.$$

Thus, if the regulator is interested in inducing a strategy that is relatively insensitive to changes in total pollution he or she may be well served to focus on ambient transfers.

We end this section by briefly noting that point-source policy can, in the spirit of Akao (2008), be designed so that in MPE no tax/subsidy is ever levied. This can be accomplished by modifying the standard policy in the following way. Consider introducing a per-unit tax/subsidy that depends on the *deviation* of an agent from the target Markovian strategy, so that the transfer is given by the following expression

$$\phi_i(x)\cdot\left[e_i-\frac{E-\gamma x}{n}\right].$$

In this case, setting ϕ_i such that

$$\phi_i(x)=\gamma_1 x+\gamma_2,\quad i\in\{1,2,..,n\}$$

where

$$\begin{aligned}\gamma_1&=\frac{\frac{-b\gamma}{n}\left[\frac{\delta}{2n}+(n+1)\beta+n\gamma\right]}{\frac{\delta}{2}+\beta+\gamma}\leq\beta_1\leq 0\\ \gamma_2&=\frac{\gamma\left[-(B-c)(n-1)+(n+1)\beta\frac{\sigma^2}{2}+2bEn\right]+\gamma_1 n\left[E+\frac{\sigma^2}{2}\right]-(B-c)(\beta+\delta)n+bE(\beta+\delta)(n+1)}{n(\beta+\delta+\gamma)},\end{aligned} \tag{35}$$

induces the desired Markovian strategy with zero actual transfers.

4 Conclusion

This paper sheds light on the ability of ambient transfers to influence MPE behavior for a large class of differential games of pollution control. The analysis suggests that, under deterministic pollution accumulation, these policy tools are able to induce a wide set of open-loop strategies in MPE. Moreover, proposed schemes are designed so that, in equilibrium, no tax or subsidy is ever levied. The fact that we have a MPE in open loop strategies indicates that deviations from the equilibrium path are, at least in theory, relatively unlikely.

The applicability of these results is explored under a stochastic framework for pollutant accumulation. When physical dynamics are uncertain, it is no longer possible to guarantee zero transfers in equilibrium and it becomes important to gauge the scale of potential taxes or subsidies. This exercise is undertaken for a simple linear oligopoly model and a regulating authority that is interested in inducing emissions that are symmetric, stationary, and linearly decreasing in total pollution. We derive closed-form expressions for expected ambient transfers at any point in time and find that they are non-positive, with their magnitude increasing linearly with volatility, the size of the economy, and the slope of the inverse demand function. In addition, these expected transfers vanish if the regulating authority wishes to induce a constant emission strategy. The simplicity of the stochastic analysis implies that one may solve for the target strategy that maximizes profits subject to the constraint that the mean and variance of steady-state levels of pollution be below certain exogenously determined levels. Finally, expected transfers are compared to those generated by a generic point source policy.

Appendix

Proof of Proposition 2 Consider the Hamilton-Jacobi-Bellman equation for agent i,

$$\delta V^i(x,t) - V_t^i(x,t) = \max_{e_i \le e_i^{max}} \Bigg\{ \big[B - b\sum_{j\neq i} e_j(x,t) - be_i\big]e_i - ce_i + \hat{\phi}_i(x,t)$$
$$+V_x^i(x,t)\Bigg[e_i + \sum_{j\neq i} e_j(x,t) - \beta x\Bigg] + V_{xx}^i(x,t)\frac{\sigma^2 x}{2} \;\Bigg|\; x = 0 \Rightarrow e_i + \sum_{j\neq i}\hat{e}_j(x,t) \geq 0\Bigg\}. \tag{36}$$

Assuming that other agents choose the stationary Markovian strategies $\hat{e}_j(x) = \frac{E-\gamma x}{n}$ and dropping superscripts, Eq. (36) obtains

$$\delta V(x,t) - V_t(x,t) = \max_{e_i \leq e_i^{max}} \left\{ \left[B - \frac{b(n-1)}{n}(E-\gamma x) - be_i \right] e_i - ce_i + \hat{\phi}(x,t) \right.$$
$$\left. + V_x(x,t)\left[e_i + \frac{(n-1)}{n}(E-\gamma x) - \beta x \right] + V_{xx}(x,t)\frac{\sigma^2 x}{2} \;\middle|\; x = 0 \Rightarrow e_i + (n-1)\frac{E-\gamma x}{n} \geq 0 \right\}. \tag{37}$$

To ensure that agent i's best response is given by $\hat{e}_i(x) = \frac{E-\gamma x}{n}$ (note also that this response is clearly feasible), the right-hand-side of Eq. (5) must be maximized at that level of emissions. Thus, it is sufficient to impose that the value function $V(x,t)$ satisfy

$$V_x(x,t) = -\left[B - c - \frac{b(n+1)}{n}(E-\gamma x) \right]. \tag{38}$$

Following identical reasoning as in the proof of Theorem 1 the specification of $\hat{\phi}$ ensures that the value function

$$V(x,t) = -\gamma \frac{b(n+1)}{2n}x^2 - \left[B - c - \frac{b(n+1)}{n}E \right] x - \int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big) \mathrm{e}^{-\delta(s-t)} \mathrm{d}s. \tag{39}$$

solves the HJB equation (37) for the desired maximizing control $\hat{e}_i(x) = \frac{E-\gamma x}{n}$. That is, $V(x,t)$ satisfies the partial differential equation

$$\begin{aligned} \delta V(x,t) - V_t(x,t) &= \left[B - b(E-\gamma x) \right] \frac{E-\gamma x}{n} + \frac{c}{n}\big[E - \gamma x\big] + \hat{\phi}(x,t) \\ &+ V_x(x)\big[E - (\beta+\gamma)x\big] + V_{xx}(x)\frac{\sigma^2 x}{2}. \end{aligned} \tag{40}$$

But, while this choice of $V(x,t)$ solves the HJB equation, it is not possible to invoke standard sufficiency theorems to establish optimality. This is because the state space is no longer bounded; hence, the candidate value function will not be bounded or even bounded from below. For this reason, it is necessary to use an alternative sufficiency theorem given by Theorem 3.4 Dockner et al. (2000) that relies on finite horizon approximations of the value function. A similar approach was recently used in the context of a fishery game by Wang and Ewald (2010).

To this end, consider a finite-horizon version of our problem over $t \in [0,T]$ with no salvage function and postulate that a value function of the form

$$V(x,t;T) = A^1(t;T)x^2 + A^2(t;T)x + A^3(t;T) \tag{41}$$

solves the Hamilton-Jacobi-Bellman equation (37) for a maximizing control of $e_i(x) = \big[E - \gamma x\big]/n$, with the added terminal time constraint $V(x, T; T) = 0$. In particular,

$$\begin{aligned}
& \left[\delta A^1(t;T) - \frac{d}{\mathrm{d}t}A^1(t;T)\right]x^2 + \left[\delta A^2(t;T) - \frac{d}{\mathrm{d}t}A^2(t;T)\right]x + \delta A^3(t;T) - \frac{d}{\mathrm{d}t}A^3(t;T) \\
= \quad & \Big[B - b(E - \gamma x)\Big]\frac{E - \gamma x}{n} + \frac{c}{n}\big[E - \gamma x\big] + \hat{\phi}(x,t) \\
& + \big[2A^1(t;T)x + A^2(t;T)\big]\big[E - (\beta+\gamma)x\big] + A^1(t;T)\sigma^2 x.
\end{aligned} \tag{42}$$

and

$$A^1(T;T) = A^2(T;T) = A^3(T;T) = 0.$$

Using Eq. (40), it is possible to cancel out $\hat{\phi}(x,t)$ and to rewrite Eq. (42) in the following way

$$\begin{aligned}
& \left[\delta A^1(t;T) - \frac{d}{\mathrm{d}t}A^1(t;T) + \frac{\delta\gamma b(n+1)}{2n}\right]x^2 + \left[\delta A^2(t;T) - \frac{d}{\mathrm{d}t}A^2(t;T) + \delta\left(B - c - \frac{Eb(n+1)}{n}\right)\right]x \\
& + \delta A^3(t;T) - \frac{d}{\mathrm{d}t}A^3(t;T) + \underbrace{\delta\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)\mathrm{e}^{-\delta(s-t)}\mathrm{d}s - \frac{d}{\mathrm{d}t}\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)\mathrm{e}^{-\delta(s-t)}\mathrm{d}s}_{\text{equivalently to Eq. (10), this expression equals } -f_i^{\hat{\mathbf{e}}}\left(\mathbf{E}[\hat{x}(t)]\right)}. \\
= \quad & \left[2A^1(t;T)x + A^2(t;T) + B - c - \frac{b(n+1)}{n}(E - \gamma x)\right]\big[E - (\beta+\gamma)x\big] + \left[A^1(t;T) + \frac{\gamma b(n+1)}{2n}\right]\sigma^2.
\end{aligned} \tag{43}$$

Collecting the terms involving x^2, $A^1(t;T)$ must satisfy the following differential equation

$$-\frac{d}{\mathrm{d}t}A^1(t;T) + \big[\delta + 2(\beta+\gamma)\big]A^1(t;T) = -\gamma\frac{b(n+1)}{2n}(\delta + 2(\beta+\gamma)). \tag{44}$$

The solution of (44) satisfying $A^1(T;T) = 0$ is given by

$$A^1(t;T) = -\gamma\frac{b(n+1)}{2n}\mathrm{e}^{(\delta+2(\beta+\gamma))t}\int_t^T (\delta+2(\beta+\gamma))\mathrm{e}^{-(\delta+2(\beta+\gamma))s}\mathrm{d}s = -\gamma\frac{b(n+1)}{2n}\left[1 - \mathrm{e}^{-(\delta+2(\beta+\gamma))(T-t)}\right],$$

so that

$$\lim_{T\to\infty} A^1(t;T) = -\gamma b\frac{n+1}{2n}. \tag{45}$$

Similarly, collecting the terms involving x, $A^2(t;T)$ must satisfy

$$
\begin{aligned}
&-\frac{d}{\mathrm{d}t}A^2(t;T)+\big[\delta+\beta+\gamma\big]A^2(t;T)=-\big[\delta+\beta+\gamma\big]\left[B-c-\frac{b(n+1)}{n}E\right],\\
&+\Bigg[\underbrace{A^1(t;T)+\frac{\gamma b(n+1)}{2n}}_{\frac{\gamma b(n+1)}{2n}\mathrm{e}^{-(\delta+2(\beta+\gamma))(T-t)}}\Bigg]\big[2E+\sigma^2\big]. \qquad (46)
\end{aligned}
$$

The solution of (46) satisfying $A^2(T;T)=0$ is given by

$$
A^2(t;T)=-\left[B-c-\frac{b(n+1)}{n}E\right]\left[1-\mathrm{e}^{-(\delta+\beta+\gamma)(T-t)}\right]+K\mathrm{e}^{(\delta+\beta+\gamma)t}\int_t^T \mathrm{e}^{-(\delta+2(\beta+\gamma))(T-s)}\mathrm{e}^{-(\delta+\beta+\gamma)s}\mathrm{d}s,
$$

where $K=\big[2E+\sigma^2\big]\frac{\gamma b(n+1)}{2n}$. It is easy to see that $A^2(t;T)$ will satisfy

$$
\lim_{T\to\infty}A^2(t;T)=-\left[B-c-\frac{b(n+1)}{n}E\right] \qquad (47)
$$

Finally, $A^3(t;T)$ will need to satisfy

$$
\delta A^3(t;T)-\frac{d}{\mathrm{d}t}A^3(t;T)=f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(t)]\big)+\left[A_2(t;T)+B-c-\frac{b(n+1)}{n}E\right]E. \qquad (48)
$$

Using identical reasoning as before it is easy to show that

$$
\lim_{T\to\infty}A^3(t;T)=-\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)\mathrm{e}^{-\delta(s-t)}\mathrm{d}s, \qquad (49)
$$

so that collecting Eqs. (45), (47), and (49) obtains

$$
\lim_{T\to\infty}V(x,t;T)=V(x,t). \qquad (50)
$$

Finally, it is necessary to examine the limiting properties of $\mathbf{E}\big[V(\hat{x}(t),t)\big]$:

$$
\begin{aligned}
&\limsup_{t\to\infty}\mathrm{e}^{-\delta t}\mathbf{E}\big[V(\hat{x}(t),t)\big]=\limsup_{t\to\infty}\mathrm{e}^{-\delta t}\Bigg[-\gamma\frac{b(n+1)}{2n}\mathbf{E}[\hat{x}(t)^2]-\left[B-c-\frac{b(n+1)}{n}E\right]\mathbf{E}[\hat{x}(t)]\\
&\quad-\int_t^\infty f_i^{\hat{\mathbf{e}}}\big(\mathbf{E}[\hat{x}(s)]\big)\mathrm{e}^{-\delta(s-t)}\mathrm{d}s\Bigg]\\
&=\limsup_{t\to\infty}\mathrm{e}^{-\delta t}\Bigg[-\gamma\frac{b(n+1)}{2n}\Big[\mathbf{Var}[\hat{x}(t)]+\big[\mathbf{E}[\hat{x}(t)]\big]^2\Big]-\left[B-c-\frac{b(n+1)}{n}E\right]\mathbf{E}[\hat{x}(t)]\Bigg]. \qquad (51)
\end{aligned}
$$

Given Proposition 1, it is easy to see that the process $\{\hat{x}(t):t\geq 0\}$ converges to the relevant Gamma distribution in L_2 so that

$$
\limsup_{t\to\infty}\mathrm{e}^{-\delta t}\mathbf{E}\big[V(\hat{x}(t),t)\big]=\lim_{t\to\infty}\mathrm{e}^{-\delta t}\mathbf{E}\big[V(\hat{x}(t),t)\big]=0. \qquad (52)
$$

Given Eqs. (50) and (52), applying the stochastic equivalent of Theorem 3.4 in Dockner et al. (2000) completes the proof. ■

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