# Systemic Stability of Housing and Mortgage Market: A state-dependent four-phase model

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# Systemic Stability of Housing and Mortgage Market

## — A state-dependent four-phase model[1]

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## *Abstract*

Motivated by the revealed preference approach to consumer theory, this study constructs a dynamic theoretical model which infers the unobservable household behavior from the observable patterns of housing and mortgage market activities. The model emphasizes the role of sellers and their asymmetric behavior in different phases of a housing market cycle in generating certain price-volume patterns. Such role has so far largely been ignored in both theoretical and empirical studies of housing markets. The model also

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establishes, theoretically, multiple channels via which housing and mortgage markets interact and via which speculative forces are propagated. In addition, it generates a testable result regarding the stability of the system formed by the two markets, which may be extended by endogenizing some important policy instruments.

***Key words:*** systemic stability, speculation, asymmetric seller response, state-dependency

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## I. Introduction

The objective of the current study is to capture some important characteristics of a speculative community and the implications of such characteristics on the stability of housing and mortgage markets, via the construction of a hypothetic community. The members of this community are entirely driven by their desire to profit from buying and selling assets, rather than from producing goods and services. Although producing nothing and being small in size, this community is highly influential on the much larger production community within which it resides. We will henceforth call the former the financial economy and the latter the real economy.

The activities in the financial economy are vital parts of a process which determines the allocation of scarce resources in the real one. When in moderation, these activities will ensure that resources are directed to the most productive sectors in the real economy; when in excess, they do quite the contrary, as has been witnessed by many countries around the world in the past century (C. P. Kindleberger, 2005, Q. Xiao, 2005, Qin Xiao, 2010).

Housing asset, for its sheer size, perhaps exerts the most influence on the real economy within this financial economy.[i] Nonetheless, our understanding of this asset market at a macro-level is so far mostly based on households utility maximization, branched out from the standard theory of consumer choices (see for example (J. Y. Campbell and J. F. Cocco, 2007, J. F. Cocco, 2005, M. Flavin and T. Yamashita, 2002, M. Iacoviello, 2004)). Alternatively, it is treated like stocks and bonds using the present-value approach. Either way, the supply of housing is usually assumed to be fixed hence left out of the picture (except in a few papers, see for instance (Edward L. Glaeser et al., 2008, James M. Poterba, 1984) whose merits and drawbacks will be discussed in the next section).Yet, housing does not bear close comparison with food, or clothes, or refrigerator, or even cars, nor is it close in characteristics to stocks and bonds. On the one hand, it does deliver consumables which generate pleasure the way food and clothes do; on the other hand, it stores value in a way no ordinary[ii] food or clothes can possibly manage. It is, in most cases, the only significant asset a household will ever acquire in its lifetime (Stephanie Curcuru, John Heaton, Deborah Lucas and Damien Moore, 2004). Furthermore, this asset is distinctively different from stocks and bonds: it is heterogeneous; it is lumpy; and more importantly, its acquisition would not be possible in most cases without the empowerment of a mortgage loan.

## II. Stylized Facts

The housing market is cyclical with irregularities so far deemed by the real estate community as unpredictable. However, there are some facts about such cycle which are fairly predictable. It has been observed that, in a market in which confidence is crumbling, to-be house owners often postpone previously planned purchases in anticipation of future price decline; on the contrary, when optimism is running high, anxious to-be owners bring forward purchase plans in expectation of future price increases. These expectations accelerate price changes both on the price's rising and falling phases, but in an asymmetric manner, partly because of sellers' behaviors. House-owners are loss-averse (D. Genesove and C. Mayer, 2001). In a falling market, sellers' resistance to sell at a loss helps to slow down the speed of price sliding. On the other hand, when the market is rising, sellers' resistance to sell in anticipation of even higher future prices simply adds fuel to the flame. In a recent study, Haurin et al. (Donald R. Haurin et al., 2010) shows that households sentiment on good-time-to-buy (GTTB) moved more or less in line with that on good-time-to-sell (GTTS) before the most recent housing market downturn in the United States. Thereafter, GTTB has changed little while GTTS has dropped dramatically. Such swings in sentiments are manifested by the large decrease in housing transaction volumes in both UK and US in the recent downturn, especially in UK where house price declines have been modest compared to the experiences of US during the same period (figure 7 and 8) [iii].

The discussion above implies that housing supply plays a key role — which has so far largely been ignored by the literature. In many cases, housing supply has been implicitly assumed to be fixed. The fixity of housing supply (or at least the growth rate of housing supply) is not entirely an absurd assumption if housing supply is identical to housing stock. Historical data show that, in countries like UK or US where the urbanization process has long reached a stable equilibrium state, the growth in housing stock is one per cent per annum or less[iv]. Given the depreciation rate of a similar figure (Brent C Smith, 2004), treating the value of total housing stock as fixed would not be too far from the truth. The fact is, however, at any time only a small fraction (typically less than 10 per cent[v]) of the housing stock is being actively traded, and that fraction varies greatly at different phases of a market cycle (see figure 1). As far as the researcher is aware, the existing literature on housing market either does not make this distinction (James M. Poterba, 1984) or, if it does, treats the fraction of existing stock on-sale as a constant number (Edward L. Glaeser, Joseph Gyourko and Albert Saiz, 2008). The author will show in section C.2. that variations in this fraction are important reasons behind some observed patterns in the housing market, for instance the varying growth in house price at different stages of a market cycle.

The previously described market phases are usually associated with very high volatilities in price and transaction volume. Such high volatility is a result of great uncertainty, market anxiety as well as controversial beliefs held by market participants: has the price risen or fallen enough? Is the market due for correction? Is an observed market correction a temporary confusion, or a signal that the tide is now turning? As neither pessimists nor optimists dominate the market in these phases, we would observe scores of participants buying as well as scores of participants selling. The price therefore swings violently as these heterogeneous participants trade actively on their widely-parted beliefs.

There are two other phases of a market cycle which should not be left out of the picture. On approaching the peak of a cycle, a housing market is characterized by stagnation in trading volumes, coupled sometimes by continued price hikes. Part of the explanation of this phenomenon lies in information uncertainty. At the peak, the buyer with the highest willingness and ability to pay has already paid the price and is in possession of the housing asset he/she desired. But this information is unknown to the sellers. On observing the high selling price, the next seller simply wishes to snatch an even higher one. So he pitches the price of his house at a higher or similar level and wait for the buyer to arrive. But that buyer will not come, not now. On approaching the trough, both price and volume tend to be stagnant. The seller who has the lowest reservation price has already sold his/her house. But the buyers have no way of observing this information. They wait and wait, for the next seller with an even lower reservation price, until their patience runs out. The market tide turns around! In these two phases, low volatility is characteristic of the market because of greater unanimity among market participants. At near the peak, even the above average optimists are gradually converted to believe that the price is due for a downward correction; at near the trough, even the above average pessimists are gradually persuaded that the price is due for an upward correction. The price quiets down as market beliefs converge.

Therefore, a housing market cycle has four phases: a booming phase featured by fast price growths and high market volatility; an over-heating phase by stagnant or falling trading volume (accompanies by either stagnant or rising price); a collapsing phase where both price and volume are falling; and a recovering phase where volume picks up gradually with price to follow. These descriptions are visualized in figure 2 and 3, with figure 2 the stylized and 3 the observed price-volume co-movement for England and Wales, and the four different regions in the US. The four phases together paint out the major trend of a housing market cycle. The movement along this trend is, however, not linear but cyclical itself (see figure 3).

How do we then understand these observations using the established economic framework? One possibility is to explain the market as in disequilibrium or in adjustment towards a single equilibrium. However, this disequilibrium market can definitely not explain the existence of an overheating market (refer to figure 4). Another possibility is to describe the market as in continuous equilibrium, guaranteed by price adjustment, or as in adjustment from multiple short-run equilibria to a single long-run equilibrium. This scenario is depicted in figure 5, which looks explaining the four-phase cycle fairly comfortably. The intuitions behind the existence of a continuous equilibrium market lies in differentiating between potential and active traders, or future and current demand/supply. The current price is driven by the current active traders and reflects the current demand and supply. Furthermore, it reveals the preferences of the current buyer and seller, not of those who are looking to trade but have found the price unattractive — least of all of those who have not even thought about participating in the market. An analogy can be drawn from the labor market, where the current wage reveals the preferences of those who are employed, but not of those who are in the labor force but unemployed, and not of those who have no intention of joining the labor force in any foreseeable future. The implied time paths of price and volume of this continuous equilibrium scenario is shown in figure 6, which does not seem to contradict the picture shown in figure 7 and 8 which paint the observed time paths for house price and sales volume in US, England and Wales.

Another empirical observation is that mortgage debt often moves in tandem with the housing market (see figure 7 and 8). Such observations have to do with the fact that mortgage debt is a secured debt: it is secured by the housing asset associated with it. Therefore, when the housing market is on the rise, the value of the security increases which improves the balance sheet of a bank and increases the capacity and willingness of the bank to lend (Q. Xiao, 2005, Qin Xiao, 2010). The in-tandem movement of housing market and mortgage market can be partly understood with the help of figure 9 (adopted from (Robert M. Buckley, 1982)), which connect the housing market to the mortgage market via the interest rate. This adopted analytic framework can be used for comparative static analysis, for instance, the equilibrium-to-equilibrium implication of a more elastic mortgage and housing supply on house price and sales volume.

However, the graphical analytical frameworks described above cannot easily be adapted to explain the cyclicality of the housing (or mortgage) market along the trend (figure 3, 7, and 8), least can it be easily twisted to capture the other channels via which the housing asset and the mortgage market connect and interact. A mathematical model in this case will prove handy. The discussions above suggest that the growth of the housing and the mortgage markets depend on the level of the house price, the speed of change of this price, as well as their expected future values. Therefore, a dynamic higher-order differential equation model looks fitting the bill (refer to (Alpha C. Chiang, 1984) p.529-30). The theoretical model to be built below will capture the four-phase market trend and the cyclicality along this trend. The differences in market volatility at different phases will be left out of the current theoretical construction, as it can be easily picked up in an empirical estimation by allowing for GARCH disturbances.

## III. A Dynamic Model of Housing and Mortgage Markets

Following the spirit of ceteris paribus analysis, the current model will adopt a partial dynamic analysis. The analysis is partially, as opposed to fully, dynamic because that every variable will be treated as exogenously given except for house price, house transaction volume and mortgage debt. The purpose of this stylization is to decipher households' price-expectation-formation and speculation in driving housing and mortgage markets, and the implications of these on the stability of the system formed by the two markets.

The approach taken in this model is bottom-up[vi] in the sense it follows the line of thoughts in the theory of revealed preferences. Economics 101 teaches us that demand and supply in a market determines the price and quantity of the good traded in that market. Neither demand nor supply is observable. For instance, in a housing market, we observe prices and volumes of housing transactions, not the demand for and the supply of houses; in a mortgage market, we observe mortgage rates and households secured debts, but not the demand for and the supply of housing debt. The existing literature, in general, takes a top-down approach. For example, in the consumption approach to housing market, a consumer utility function (an unobservable) is first specified; a demand function (another un-observables) is then deduced from that utility function; the implication of these unobservable on prices and quantities (the observables) are derived thereafter. The current study will take the reverse route and ask the question: what, if any, do the observables tell us about the un-observables? To be more specific, what can we say about the underlying parameters governing households' behavior, by observing price and quantity? If the observations imply that the system can remain on an explosive path for a significant period of time, what market forces are holding up and prolonging this unsustainable development?

## A. Demographic and Housing Tenure Distribution

Consider a hypothetic economy. In this economy, the only consumption taking place is housing consumption, and the only production is housing production. Assume that at any point in time there are T generations of N households in total, with N and T exogenously given as the demographics of this economy is in a stationary state[vii]. Denote the new-born generation as generation 0 and the oldest as generation T-1. Each generation has n households with n = N/T, who lives for exactly T periods. A constant fraction $\lambda$[viii], $\lambda \in (0,1)$, of the n households in each generation will own housing asset once in its lifetime. We call these $\lambda n$ households in each generation owners and the remaining ones renters. When living in a rental house, that house is rented from a single absentee landlord; when buying a housing asset, the purchase is financed at least partly with a loan from an absentee financier. In the spirit of comparative analysis, we assume that owners and renters of the same generation are homogenous in every way except for their tastes for ownership of housing assets; that renters are homogenous but owners heterogeneous across generations. This owner-cross-generation difference, as will be shown later, is what generates certain observed patterns in the markets of concern.

At any time, owners can be classified as buyers, holders, or sellers of housing assets. In a non-speculative community, owners buy one unit of house at the point of entry and sell it at the point of exit (refer to figure 10 for a graphical display of the demographic and tenure pattern of a non-speculative economy). In a general case where speculation about future prices is allowed, a new-born household may choose to own housing asset now, in the future, or never; an owner may choose to sell its housing asset at the exiting point or earlier; and a household may own more than one unit of housing asset at any point in time. Figure 11 illustrates the decision tree of a generation born at time t and lives for three periods in a speculative community. In either case, buyers and sellers of housing assets are small in number compared with the size of the population at any time. Nonetheless *it is precisely this small group of households who generate the current volume of transactions and determine the current price of housing assets or mortgage debt*.

## B. Financing Home Ownership

When exit, the wealth (or debt) of each household in a given generation is collected by an absentee government and redistributed evenly as endowment among the new-borne households. Let $W_{i,t}$ denotes the real endowment of time $t$'s generation $i$, with $i = 0,1,\ldots,T-1$, and $w_{i,t}$ the real endowment of each household in this generation, inherited at time t-i hence known. By assumption $w_{i,t} = W_{i,t} / n$.

When a household inherits a debt, that debt is financed by borrowing from a single absentee financier at a constant risk-free real interest rate $r^f$; when inherits a wealth, the wealth is either deposited with this financier to earn a string of payments at the real rate $r^f$, or used as a down-payment for the purchase of a housing asset, or invested in another risky asset. The other risky asset generates a real return $\tilde{r}_t = r + \sigma\varepsilon_t$, with σ a constant, and $\varepsilon_t$ a random disturbance of zero mean and unit standard deviation, i.e. $E[\varepsilon_t] = 0$ and $s.d.[\varepsilon_t] = 1$.

Assume, without loss of reality in general, that $w_{i,t} < P_t$. This would force each intended owner to take out a mortgage loan from the financier at the rate $\tilde{r}^M = r^M + \sigma^M \varepsilon_t$, with $r^f < r^M < r$ and $\sigma^M < \sigma$, which implies that mortgage debt earns a lower rate of return but is less risky than the other risky asset. This assumption is in line with the general observation that stocks are riskier but have the potential to generate higher returns than housing assets ((David M. Geltner et al., 2007), Exhibit 11-4, p.252).The mortgage debt, principle plus interest payments, is fully repaid at the time the house is sold if the proceeds from selling the house is large enough; otherwise, it is rolled over and passed onto the next new-born generation when the current one exits[ix].

By nature, the demand for mortgage debt is a derived one: it is derived from the demand for housing asset. Therefore both price and volume of housing asset transaction will have bearings on the volume of mortgage debt, as will do the borrowing constraint embodied in the loan-to-value ratio set by the financier. Furthermore the level of activities in the mortgage market will be affected by the opportunity costs of buying a housing asset, represented by the real interest rate of mortgage loan and the real rate of return on the alternative risky asset. Their respective standard deviations will also impact the volume of mortgage debt. Hence $M_t = M\left(P_t, Q_t, r^f, r^M, r, \sigma, \sigma^M, LTV_t\right)$, where *M( )* is a well behaved implicit function that is continuously differentiable. The subscript *t* denotes the time the measurement is taken, $M$ the new real mortgage issuance, $Q$ the housing transaction volume. By differentiate M with respect to time we obtain an explicit function in its dynamic form:

$$\begin{aligned}&\dot{M}_t\\&= M_P\dot{P}_t + M_Q\dot{Q}_t + M_{r^f}\dot{r}^f + M_{r^M}\dot{r}^M + M_r\dot{r}\\&\quad + M_\sigma\dot{\sigma} + M_{\sigma^M}\dot{\sigma}^M + M_{LTV}L\dot{T}V_t\end{aligned} \qquad \mathbf{1}$$

where $\dot{M} = dM/dt, \dot{P} = dP/dt, etc.$; $M_Q = \partial M/\partial Q > 0$ by intuition, i.e. the volume of mortgage debt increases with the volume of housing asset transactions. When the riskiness of the alternative investment asset rises, investing in housing asset becomes more attractive to households and the financier is more willing to lend to house-buyers, hence $M_\sigma = \partial M/\partial\sigma > 0$. By the same token, $M_{\sigma^M} = \partial M/\partial\sigma^M < 0$. The marginal impact of *LTV (or P)* on mortgage debt is not as clear-cut as the ones above. If the borrowing constraint *is binding*, an increase in *LTV* or *P* will relax the constraint hence raise the volume of mortgage debt; when it is not binding, the marginal impact of *LTV* or *P* will be zero, i.e. $M_{LTV} = \partial M/\partial LTV \geq 0$ (and $M_P = \partial M/\partial P \geq 0$ ). The marginal impact of r and $r^f$ are negative, as an increase in the alternative investment return depresses the supply of mortgage loans; but the marginal impact of $r^M$, $M_r = \partial M/\partial r$, is ambiguous. On the one hand, a higher interest rate raises mortgage supply, other things equal; on the other hand, it depresses demand. The dynamic equation implies that, other things equal, *mortgage debt moves in tandem with house price and transaction volume*; *lower return and greater uncertainty in stock market directs speculative money to housing market*; *Furthermore, a sudden downward jump in house price is likely to trigger a credit crunch.*

By construction, the loan-to-value ratio, the means and standard deviations of the other investment returns are exogenous to housing and mortgage markets. To single out the mechanisms which transmit the forces of speculation backwards and forwards between housing and mortgage markets, define an exogenous variable X, with

$$\dot{X}_t = M_{r^f}\dot{r}^f + M_{r^M}\dot{r}^M + M_r\dot{r} + M_\sigma\dot{\sigma} + M_{\sigma^M}\dot{\sigma}^M + M_{LTV}L\dot{T}V_t \qquad 2$$

Hence

$$\dot{M}_t = M_P\dot{P}_t + M_Q\dot{Q}_t + \dot{X}_t \qquad \mathbf{3}$$

## C. *The Investment Market for Housing Assets*

In a world with perfect information hence the absence of speculation, the expected capital gain is zero at all time. The new-born households purchase their housing assets at the point of entry or never and all house owners sell their housing asset at the point of exit. In such a world, the volume of housing asset transactions, $Q_t$, will be constant over time, with $Q_t = \lambda n$; the price, $P_t$, will fully reflect the value of the housing services and grow in line with the rental price, $R_t$ (see subsection 1 below).

When speculation is introduced with opaque information into this world, the new-born to-be-owners may delay buying if they believe the price will fall in the future, and the existing owners may sell earlier if they believe doing so is more profitable. In the current model, these beliefs in future price changes have no economic ground, and are results of sheer information uncertainty[x]. If a large enough number of owners expect a reduction in future prices, their collective attempt to sell now will depress the current price, which may or may not induce an increase in the current transaction volume. If a large enough number of future owners believe that buying now is more profitable than in the future, their collective attempt to bring forward buying plans will drive up the current price, with or without an accompanied increase in volume.

*The volume will go up if home-owners and to-be-owners hold opposite beliefs*. In that event, the number of buyers will increase simultaneously with the number of sellers. When both groups believe in future price falls, buyers withhold buying while sellers try to rush out of the market, resulting in a crash in both volume and price (with varying impact on their respective magnitude, as discussed in section II). When near the trough of a market cycle, both groups believe in future price rises. In this phase, buyers want to buy now but sellers want to sell in the future. Hence an initial price rise is accompanied with a low transaction volume.

## 1. The house price dynamics

A housing asset is somewhat different from the other risky asset mentioned in section B: like the latter, it is an investment vehicle; unlike the latter, it generates consumption values in the form of a stream of housing services. As a consumption good, the value of a house should be reflected by the per-period real rental price (or simply rent), $R_t$, associated with it; as an investment asset, it should be reflected by the real capital value (price), $P_t$, of the house. Barring a prohibitively high transaction cost (financial and psychological), in a world with perfect information, cross-section and cross-time arbitrage would ensure the present-value of the expected cost of owning equals to the present-value of the expected cost of renting at anytime. With imperfect information, the course of future events is unknown to households. This uncertainty is the fundamental source of speculation which has a great potential of destabilizing the economy at large.

Let $\zeta_t^e$ denotes the time-t *expected* per-period owner cost associated with each pound worth of housing capital in the holding period. This owner cost can be decomposed as follows:

$$\zeta_t^e = \delta_t + \kappa_t + op_t^e - \pi H_t^e \qquad \textbf{4}$$

where the superscript *e* denotes the expected value of the variable, and the subscript *t* denotes the time when these costs occur or the expectation is formed; $\pi H^e$ is the expected nominal house price inflation rate; δ the rate of depreciation of housing structures; κ the operation costs and $op_t^e$ the expected opportunity cost, both associated with this unit of housing asset. The operation cost includes property tax and maintenance cost. The opportunity cost comes in the form of rate of borrowing if the purchase is financed with a mortgage debt, or the return foregone on the best alternative investment opportunity if with own funds, or a combination of the two if the sources of funding is of mixed origin. For neat notation, assume the mean expected opportunity cost equals $\bar{r}$. Further assume δ and κ are constant overtime, we can write the above equation as

$$\zeta_t^e = \delta + \kappa + \bar{r} - \pi H_t^e \qquad \textbf{5}$$

Denote the general inflation rate by π (assume time-invariant), and the expected real house price inflation rate by $\dot{p}_t^e$, with $\dot{p}_t^e = \pi H_t^e - \pi$, then

$$\begin{aligned} &\zeta_t^e \\ &= \delta + \kappa + \bar{r} - \pi - \dot{p}_t^e \\ &\equiv \nu - \dot{p}_t^e \end{aligned} \qquad 6$$

where $\nu \equiv \delta + \kappa + \bar{r} - \pi$. By definition,

$$\dot{p}_t^e = \frac{\dot{P}_t^e}{P_t} \qquad \mathbf{7}$$

with $\dot{P}_t \equiv dP_t/dt$ which is the expected instantaneous real price change at time t. Hence

$$\begin{aligned} &\dot{P}_t^e \\ &= \dot{p}_t^e P_t \\ &= \nu P_t - \zeta_t^e P_t \end{aligned} \qquad \mathbf{8}$$

Households optimization would result, in equilibrium (which we assume being continuous given the argument in section II), the expected per-period real cost of owning one housing structure, $\zeta_t^e P_t$, being equal to the per-period real value of housing services delivered by that housing structure, the latter measured by its associated rent, $R_t$. That is

$$\zeta_t^e P_t = R_t \qquad \mathbf{9}$$

Intuitively, the rental price of housing services is a function of the size of the population, the level of housing stock and the income levels of all households involved. In the current model, the population is fixed in size, and the income of a household is manifested by its endowment. Hence, we can write

$$R_t = R(H_t, W_t) \qquad \mathbf{10}$$

where *R( )* is a well-behaved, continuously differentiable function; $W_t$ the aggregate endowment at time *t*. The partial derivatives $R_H \equiv \partial R_t/\partial H_t < 0$ , and $R_W \equiv \partial R_t/\partial W_t \in (0.1)$. The partial derivatives stipulate that, if *W* rises by one unit across all generations, the rental price of housing services will increase but by less than one-for-one. On the other hand, an increase in housing stock will always drive down the rent. Hence

$$\dot{P}_t^e = \nu P_t - R(H_t, W_t) \quad \textbf{11}$$

In a perfect foresight world, $P_t = \dot{P}_t^e$, hence actual price will grow in line with the fundamentals embodied in ν and R. With imperfect information, actual price will bear the fingerprints of speculation as well. This imperfection, in the current model, is captured by asymmetric seller behaviours in different market phases (see next subsection).

To transform the implicit function of $R_t$ into an estimable form, take derivatives with respect to time, we have

$$\ddot{P}_t = \nu \dot{P}_t - R_H \dot{H}_t - R_W \dot{W}_t \quad \textbf{12}$$

Intuitively, wealth changes with the accumulation of housing stock; furthermore, a change in house price or the availability of mortgage debt also has a real wealth effect.

$$\dot{W}_t = W_P \dot{P}_t + W_M \dot{M}_t + W_Y \dot{Y}_t \quad \textbf{13}$$

where Y is a vector of exogenous variables, e.g. inheritance from outside, which affect the endowment of households in this economy; $W_Y = dW/dY > 0$, $W_P = dW/dP > 0$, and $W_M = dW/dM > 0$. so

$$\begin{aligned}\ddot{P}_t &= \nu \dot{P}_t - R_H \dot{H}_t - R_W \left(W_P \dot{P}_t + W_M \dot{M}_t + W_Y \dot{Y}_t\right) \\ &= \nu \dot{P}_t - R_H \dot{H}_t - (R_W W_P)\dot{P}_t - (R_W W_M)\dot{M}_t - (R_W W_Y)\dot{Y}_t\end{aligned} \quad \textbf{14}$$

## 2. The dynamics of transaction volumes

With continuous equilibrium, at anytime the quantity demanded is equal to the quantity supplied, both reflected by the transaction volume, $Q_t$. We distinguish housing supply,

$Q_t$, from housing stock, $H_t$, as the majority of housing stock is *not* for trading at any point in time. Furthermore, the stock can be altered only through new construction while the supply may, in addition, vary with the fraction of existing stock on market. For instance, higher than expected price may induce households to put their houses onto the market, an action they would not have undertaken had the price behaved more in line with their expectations. Conversely, they may withdraw housing unit from the asset market when prices are too low by their standards.

The supply therefore consists of housing units from the existing stock as well as from the newly completed ones:

$$Q_t = I_{t-l}\left(C_{t-l}, E_{t-l}P_t\right) + h\left(P_t, E_t P_{t+f}; H_{t-1}\right) \quad f = 1,2,...,T \qquad \textbf{15}$$

where $I(\ )$ and h( ) are well behaved, continuously differentiable function; $I_{t-l}$ is the new construction started at time *t-l*, with *l* the average period required for the completion of a construction project which we take as given. The new construction is a function of the exogenously determined time *t-l* construction cost, $C_{t-l}$,[xi] and the expected time t price with expectation formed at time *t-l*, $E_{t-l}P_t$. The second term in the equation, $h(\ )$, is the housing units supplied from the existing stock, which depends on the current price, $P_t$, the expected price in the future T periods, $P_{t+f}$ (f = 1,2,…,T), as well as the existing stock (excluding the newly added ones), $H_{t-1}$.

What about the time $t - l$ expectation of time *t* price, $E_{t-l}P_t$? Suppose this expected price is higher than the current price plus a transaction cost, i.e. $E_{t-l}P_t > (1+\tau)P_{t-l}$ where $\tau$ is transaction cost as a fraction of the price, then the expected higher future price will immediately push up $P_{t-l}$ until $E_{t-l}P_t = (1+\tau)P_{t-l}$; suppose $E_{t-l}P_t < (1+\tau)P_{t-l}$, this expected future price fall will depress the current price immediately until $E_{t-l}P_t = (1+\tau)P_{t-l}$. By the same argument, $E_t P_{t+f} = (1+\tau)P_t$. Hence, we can rewrite the above equation as

$$Q_t = I_{t-l}(C_{t-l}, P_{t-l}) + h(P_t, H_{t-1}) \qquad \mathbf{16}$$

By intuition, $I_C \equiv \partial I_{t-l} / \partial C_{t-l} < 0 \forall t, l$ and $h_H \equiv \partial h / \partial H_t > 0 \forall t$, i.e. higher construction costs reduces new housing supply whereas higher housing stock increases existing housing supply. In a perfect foresight world, the partial derivatives $I_P \equiv \partial I_{t-l} / \partial P_{t-l}$ and $h_P \equiv \partial h / \partial P_t$ will be constant with $I_P, h_p > 0$, as higher price implies higher profit, other things equal. With imperfect information hence price expectation-formation becomes necessary, however, their respective sign and magnitude will be state dependent: positive when the price is low and rising, as households take these an indication of the coming or arrival of a bull market (Phase IV and I); negative when price is very high and still rising, as households grow in unanimity in their expectation for a downward price correction (Phase II). In Phase III when the market is collapsing, the falling price prompts household loss-aversion, a further reduction in supply therefore a positive value in these partial derivatives. The above descriptions of the partial derivatives are summarized in table 1. *These state-depend partial derivatives capture the behaviors of the sellers at different phases of a market cycle*, *which hold the key to explaining the observed price-volume patterns described in section II.*

In a dynamic world

$$\dot{Q}_t = I_C \dot{C}_{t-l} + I_P \dot{P}_{t-l} + h_P \dot{P}_t + h_H \dot{H}_{t-1} \quad \textbf{17}$$

With $\dot{H}_{t-1} = I_{t-l-1}$, we can rewrite this equation as

$$\dot{Q}_t = I_C \dot{C}_{t-l} + I_P \dot{P}_{t-l} + h_P \dot{P}_t + h_H I_{t-l-1} \quad \textbf{18}$$

This can be expressed as a second-order differential equation in price (an endogenous variable) and construction cost (an exogenous variable):

$$\ddot{Q}_t = I_P \ddot{P}_{t-l} + h_P \ddot{P}_t + h_H I_P \dot{P}_{t-l-1} + h_H I_C \dot{C}_{t-l-1} \quad \textbf{19}$$

# IV. Analysis of the Dynamic System

Before looking at the system of three dynamic equations as a whole, recall that $\dot{H}_t = I_{t-l} = I(C_{t-l}, P_{t-l})$. The second-order price differential equation can therefore be expressed as a third-order differential equation in price alone.

$$\begin{aligned} &\dddot{P}_t \\ &= (\nu - R_W W_P)\ddot{P}_t - R_H \dot{I}_t - (R_W W_M)\ddot{M}_t - (R_W W_Y)\ddot{Y}_t \\ &= (\nu - R_W W_P)\ddot{P}_t - (R_H I_P)\dot{P}_{t-l} - (R_W W_M)\ddot{M}_t - \left[(R_H I_C)\dot{C}_{t-l} + (R_W W_Y)\ddot{Y}\right] \end{aligned} \quad \textbf{20}$$

Hence, we have a dynamics system of mortgage debt, house price, and housing transaction volume of the following form:

$$\dot{M}_t - M_P \dot{P}_t - M_Q \dot{Q}_t = \dot{X}$$

$$\begin{aligned} &(R_W W_M)\ddot{M}_t + \dddot{P}_t - (\nu - R_W W_P)\ddot{P}_t + (R_H I_P)\dot{P}_{t-l} \\ &= -\left[(R_H I_C)\dot{C}_{t-l} + (R_W W_Y)\ddot{Y}\right] \end{aligned}$$

$$-I_P \ddot{P}_{t-l} - h_P \ddot{P}_t - h_H I_P \dot{P}_{t-l-1} + \ddot{Q}_t = h_H I_C \dot{C}_{t-l-1}$$

The solution to this system consists of a column vector of particular integrals, $\begin{bmatrix} M_P & P_P & Q_P \end{bmatrix}'$, which represents the intertemporal equilibrium values of the three variables in question, and a vector of complementary functions, $\begin{bmatrix} M_C & P_C & Q_C \end{bmatrix}'$, which map out the deviation of these variables from the vector of equilibrium values. The intertemporal equilibrium of a variable is the path the variable has a tendency to return to, hence representing the "gravity pull" of the fundamental forces. However, in the short run, the actual path of a variable is also driven by forces other than fundamentals, exemplified by the complementary functions.

It can be shown (see Appendix) that if

$$\left\{(\nu - R_W W_P) - (R_W W_M)\left[M_P + M_Q(I_P + h_P)\right]\right\}^2 < 4\left[(R_W W_M) M_Q (h_H I_P) + (R_H I_P)\right]$$

the variables will behave in a cyclical manner on the way to their respective intertemporal equilibrium . The general solution then takes the form

$$M_t = e^{\alpha t}\left(A_3 \cos\theta t + A_4 \sin\theta t\right) + \overline{M}$$

$$P_t = e^{\alpha t}\left(B_3 \cos\theta t + B_4 \sin\theta t\right) + \overline{P}$$

$$Q_t = e^{\alpha t}\left(C_3 \cos\theta t + C_4 \sin\theta t\right) + \overline{Q}$$

the second term on the right hand side of each equation is the intertemporal equilibrium with

$$\overline{P} = -\left\{\frac{\left[(R_H I_C)\dot{C}_{t-l} + (R_W W_Y)\dddot{Y}\right]}{\left[2 - 2(\nu - R_W W_P)t + (R_H I_P)t^2\right]}\right\} - \left\{\frac{(R_W W_M)}{\left[2 - 2(\nu - R_W W_P)t + (R_H I_P)t^2\right]}\right\}\overline{M}$$

$$\overline{Q} = \left[2(I_P + h_P)t + h_H I_P t^2\right]\overline{P} + h_H I_C \dot{C}_{t-l-1}$$

$$\overline{M} = M_P t\overline{P} + M_Q\overline{Q} + \frac{\dot{X}}{t}$$

(recall that C, Y and X are exogenous to the system). Hence the value of $\overline{Q}$ and $\overline{M}$ are all pinned down by the value of $\overline{P}$ . A sudden drop in price will thus reduce the equilibrium values of both M and Q. There is also a feed back effect of $\overline{M}$ on $\overline{P}$ , a direct result of the wealth effect of mortgage availability on households budget.[xii] The first term is the cyclical deviation from its corresponding equilibrium. The behavior of this is governed by the rate of amplification α and the angular velocity $\theta$ radians per unit of time, with

$$\alpha = \frac{1}{2}\left\{\left(\nu - R_W W_P\right) - \left(R_W W_M\right)\left[M_P + M_Q\left(I_P + h_P\right)\right]\right\}$$
$$\theta = \frac{1}{2}\sqrt{4\left[\left(R_W W_M\right)M_Q\left(h_H I_P\right) + \left(R_H I_P\right)\right] - \left\{\left(\nu - R_W W_P\right) - \left(R_W W_M\right)\left[M_P + M_Q\left(I_P + h_P\right)\right]\right\}^2}$$

(interested reader may refer to Nelson et al. (E. W. Nelson et al., 1998) p.243 and p.463 for an explanation of the velocity concept). As $I_P$ and $h_p$ changes at different phases of a market cycle, both market trend and the cyclical movements along a given trend are state-dependant

It has been shown that the real root α < 0 is the necessary and sufficient condition for the system to be non-explosive (Alpha C. Chiang, 1984). In the current situation, it means we require $\left\{\left(\nu - R_W W_P\right) - \left(R_W W_M\right)\left[M_P + M_Q\left(I_P + h_P\right)\right]\right\} < 0$.

Recall

$$\dot{X}_t \equiv M_{r^f}\dot{r}^f + M_{r^M}\dot{r}^M + M_r\dot{r} + M_\sigma\dot{\sigma} + M_{\sigma^M}\dot{\sigma}^M + M_{LTV}L\dot{T}V_t$$

which is exogenous to this system. From equation 2 and 3, if the return on the alternative assets are higher and less risky, $\dot{X}$ hence $\overline{M}$ decreases. Alternatively, if the return on mortgage loan is higher and less risky, $\dot{X}_t$ hence $\overline{M}$ increases. Similarly, an increase in the loan-to-value ratio will raise the intertemporal equilibrium value of the mortgage debt. Similarly, an increase in the rate of change in the growth rate of Y will reduce $\overline{P}$, and an increase in the growth rate of construction cost will raise $\overline{P}$ but depress $\overline{Q}$.

## V. Conclusion

The most recent boom and bust in housing and mortgage markets in many developed countries have ignited renewed interests, among academics and policy makers, in forces behind this periodic phenomenon. Such interests are manifested by a call for "Study on housing markets and intra-euro area macroeconomic imbalances – identifying policy instruments" by European Commission, Directorate-General for Economic and Financial Affairs, and by the number of sessions and panel discussion devoted to this topic in 17th Annual ERES Conference (Milan, 2010).

Despite of the great efforts made by academics and policy makers, the boom and bust cycles in housing and mortgage markets seem to grow more spectacular and more destructive to the general economic well-being. The duration and magnitude of these booms and busts are beyond the explanations of economic fundamentals, such as income, interest rate and population growths.

The objective of the current study is to argue, facilitated with a system of dynamic models, that certain characteristics of household behavior in an uncertain environment can greatly affect the forces of demand and supply, and especially supply, hence prices and activities in housing and mortgage markets. The construction of the model follows the line of thoughts in the theory of revealed preferences. Unlike the majority of studies in this area which derives the observables (price and quantity) from the un-observables (households behavior), this paper takes a reverse route. It infers, from the characteristics of the growths in the observable prices and transaction volumes, the unobservable parameters of households' behavior. It is motivated by the most fundamental economic equilibrium concept, first in a static sense. This equilibrium concept is then extended to a dynamic world where both fundamentals and speculative forces prevail. It concludes that the observed boom and bust cycles are results of a continued battle between the two forces, which take turn to dominate. This battle is manifested by a tendency of the variables in concern to deviate, in a cyclical manner, from their respective intertemporal equilibrium; but that tendency is countered by a mean reverting force which exerts its influence both continuously and in jumps.

The model constructed captured a number of stylized facts of the markets: the non-linearity and cyclicality of growths in house price, volume and mortgage debt; the co-movements among these variables and the characteristics of such co-movements; the fact that lower return and greater uncertainty in stock market directs speculative money to housing market; and the observation that credit crunch is typically associated with a sudden drop in house price. The non-linear relationship is modeled with state-dependent coefficients. This model also generates some testable implications, for instance, the stability of the system. With the wealth effect of price and mortgage debt taken into account , this model allows not only house prices to determine the level of mortgage debt, but also a feedback effect of mortgage availability on house prices.

This theoretical framework can be adopted for use by market participants and policy makers alike. For the former group of economic agents whose objective is profit maximization, the model is usable for market forecasts — not only in housing but also in commercial real estate markets, as similar economic forces rule both worlds. For the second group of economic agents whose objective is to avoid short-term growth from damaging long-term economic stability, a straightforward application of this model is to use the stability condition to test the sustainability of a market boom. This condition should be refined when expanding the dynamic system to endogenize the policy instruments, such as interest rate and loan-to-value ratio. By endogenizing these policy instruments, it enables the design of an optimal mechanism for smooth government policy responses. A continuous smooth response to market moves, the author argue, is less likely to generate disproportionate jumps in the market hence likely to be destructive to markets (refer to (Ahmet Duran, 2006, Ahmet Duran and Gunduz Caginalp, 2005, 2007, Qin Xiao and Weihong Huang, 2010) on discussions of market over-reactions to shocks). A lesson of this kind of smooth policy response might be learnt from the foreign exchange rate management by the government of Singapore which, partly because of that, survived almost unscathed one of the most severe financial crisis of Southeast Asian in the late 1990s (see (Sheng-Yi1 Lee, 1984) for more discussion on this exchange management).

This model may also be extended by replacing the deterministic with stochastic differential equations. This may be of greater value for assets which are traded much more frequently, such as REITs. Whichever the avenue of expansion, the central message delivered by this model should not be fundamentally altered: there are three-way dynamic interactions, via fundamental and speculative forces, among house price, transaction volume and mortgage debt; and their mutual impacts vary with the phase of a housing market cycle.

(6298 words excluding endnotes, bibliography, graphs and table, and Appendix)

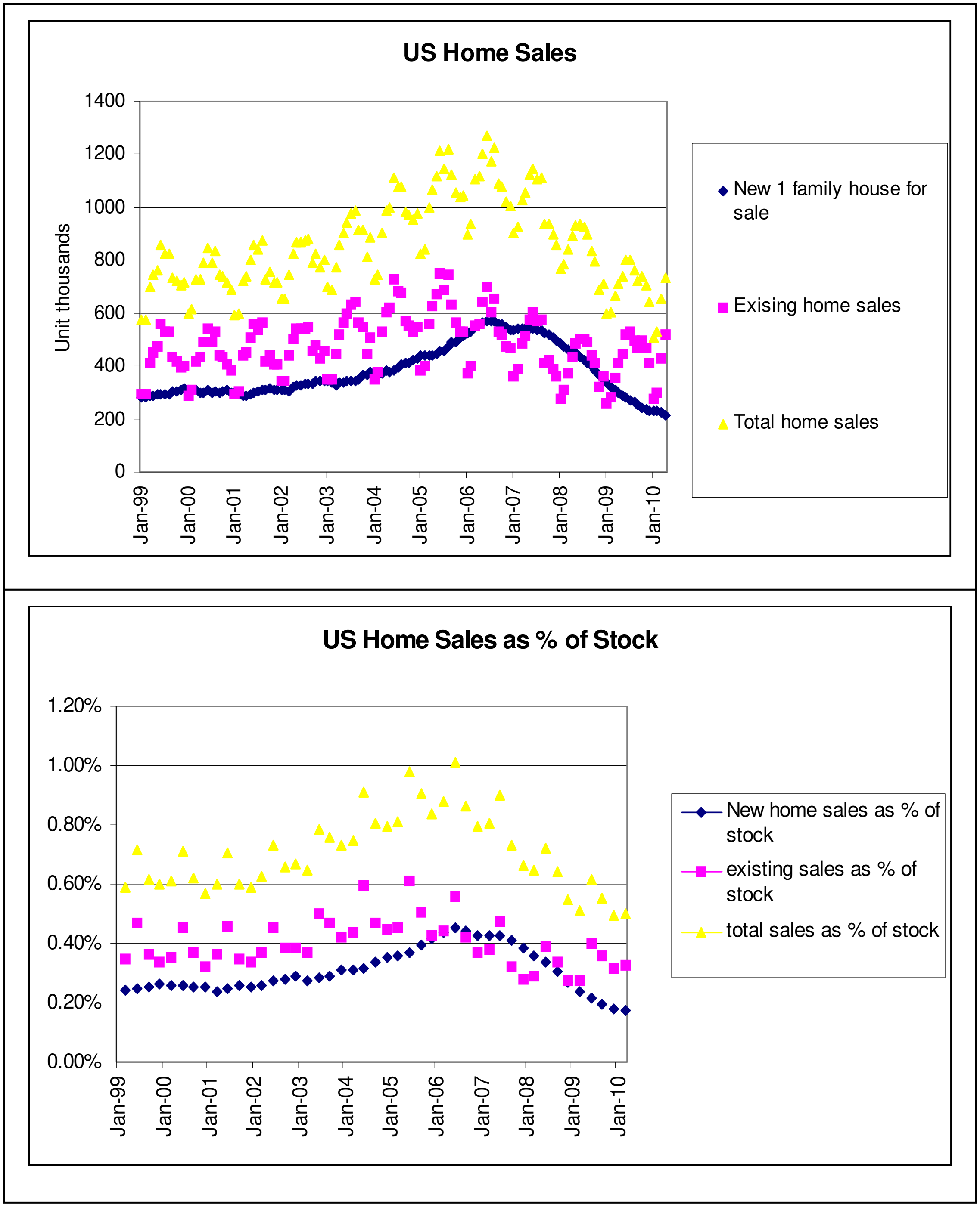



**Figure 1 US home sales in unit thousands and as percentage of housing stock. Source:** (CEIC Database)

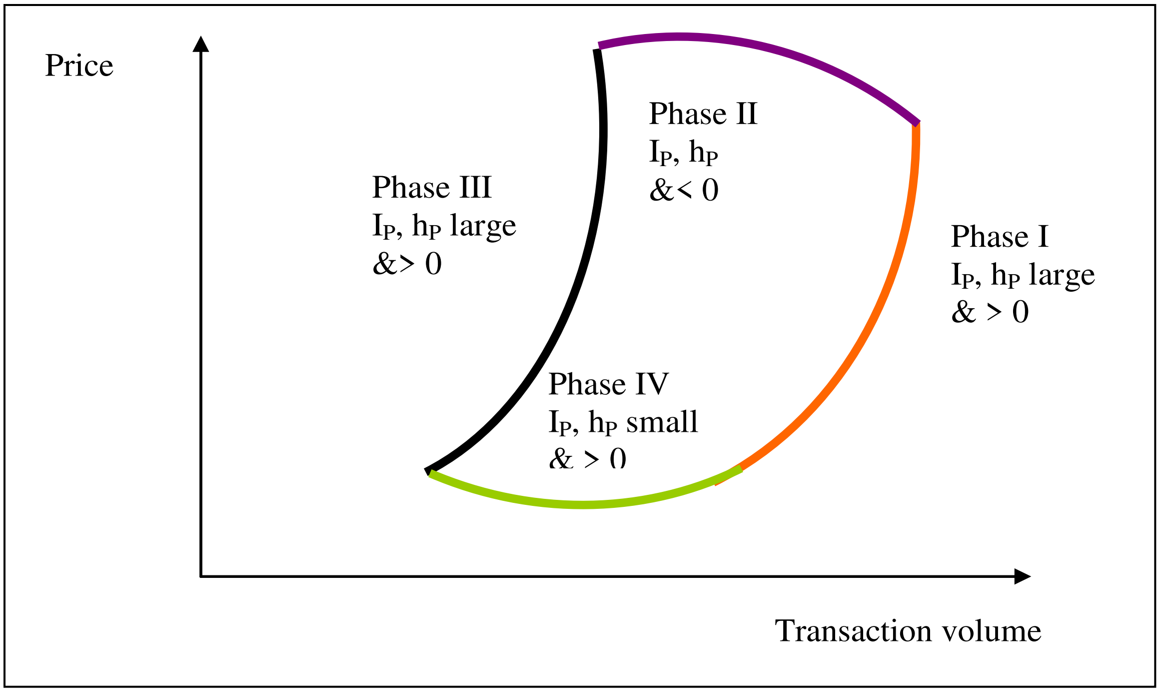
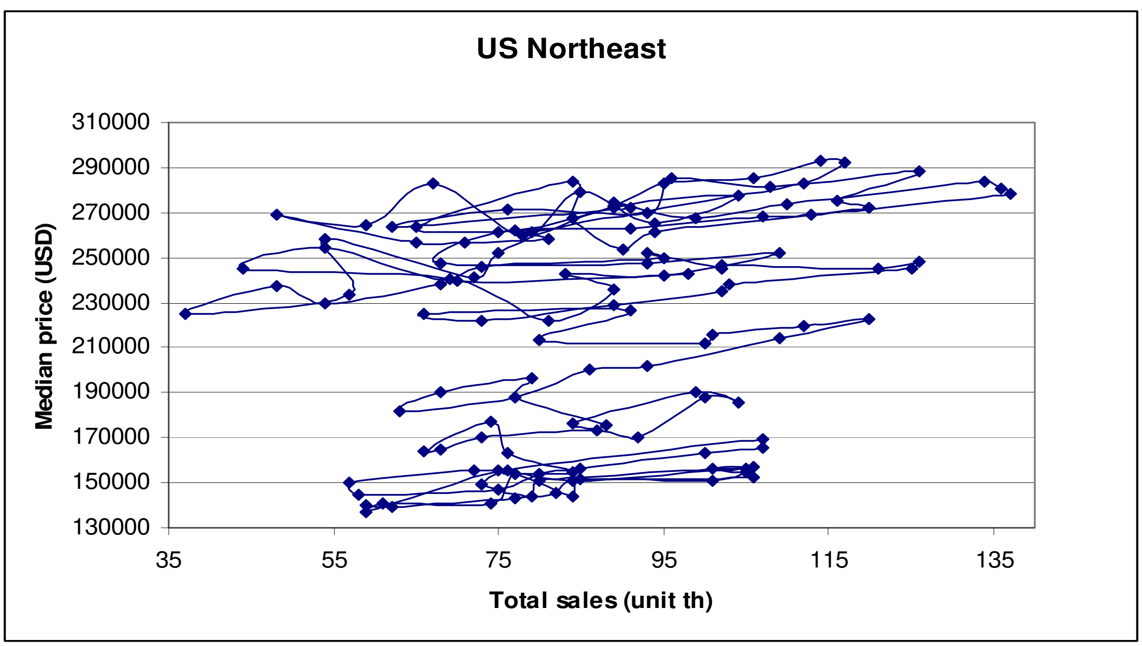
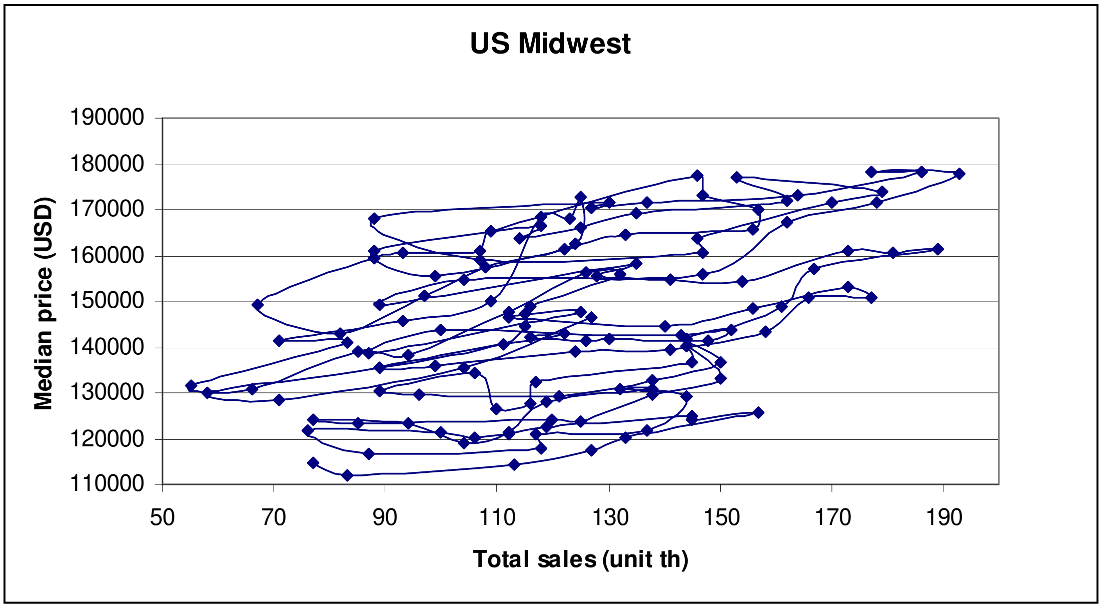



**Figure 2 Stylized facts on the co-movement of price and transaction volume in housing asset market.** The colors depicts different phases of a market cycle: in the green (recovery) phase, price and volume pick up after being stagnant; in the red phase, the increase in price and volume are gathering speed; in the purple (over-heating) phase, price continue to rise while volume dries out; in the black (crash) phase, price drops sharply which is coupled with a selling frenzy.




US Midwest
190000
180000
170000
160000
150000
140000
130000
120000
110000
Median price (USD)
50
70
90
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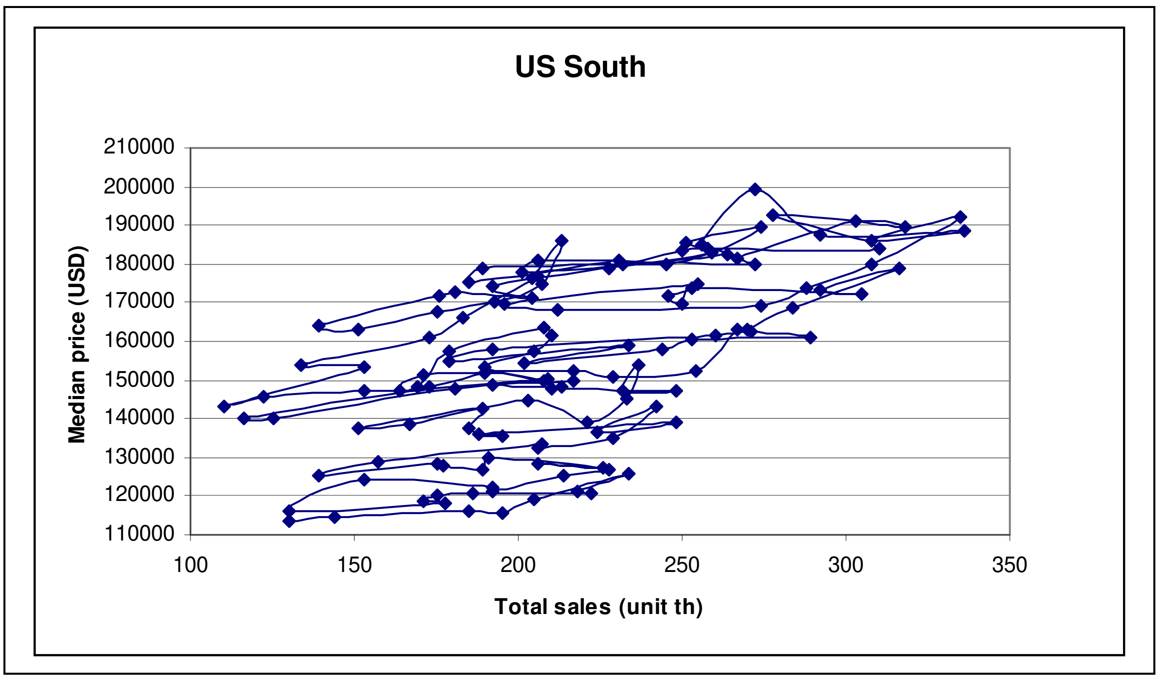
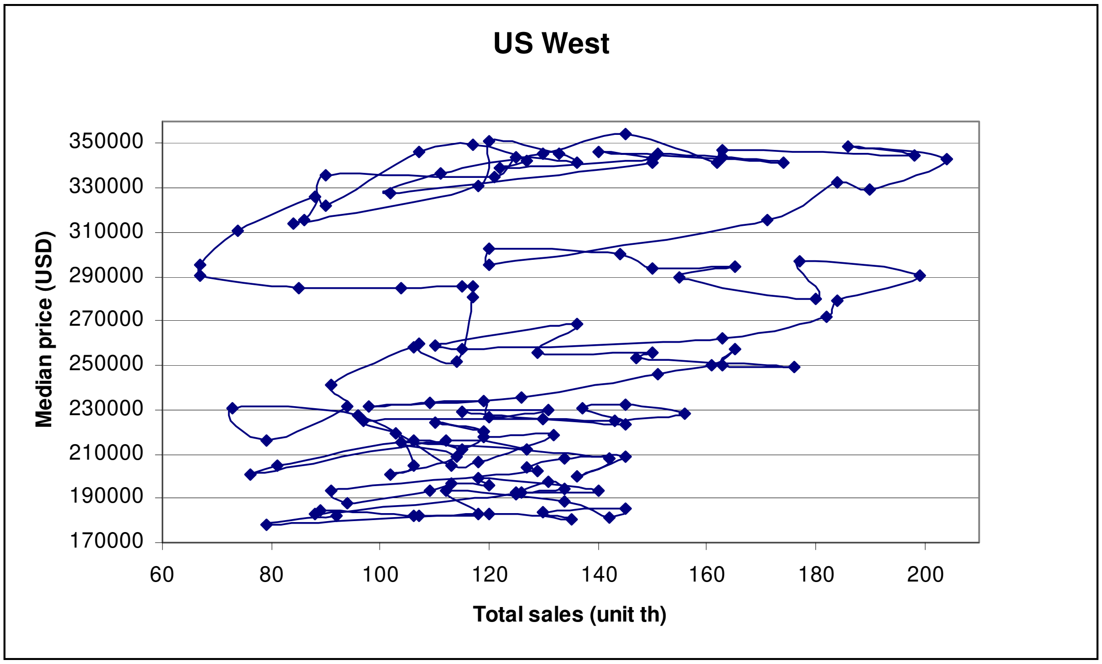
130
150
170
190
Total sales (unit th)


US South
210000
200000
190000
180000
170000
160000
150000
140000
130000
120000
110000
Median price (USD)
100
150
200
250
300
350
Total sales (unit th)



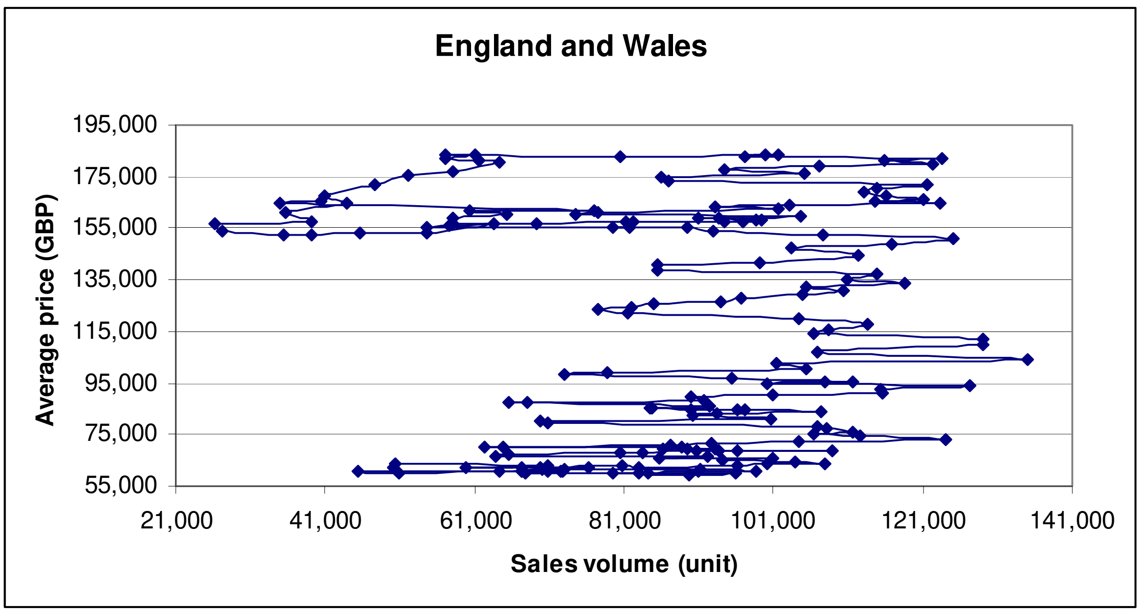




**Figure 3 Observed price and transaction volume co-movement in US (Jan 99 – Apr 10), England and Wales (Jan 95 – Feb 10).** The observations above seem to confirm the stylized fact: low transaction volume is associated with both high and low prices (Source: CEIC Database).

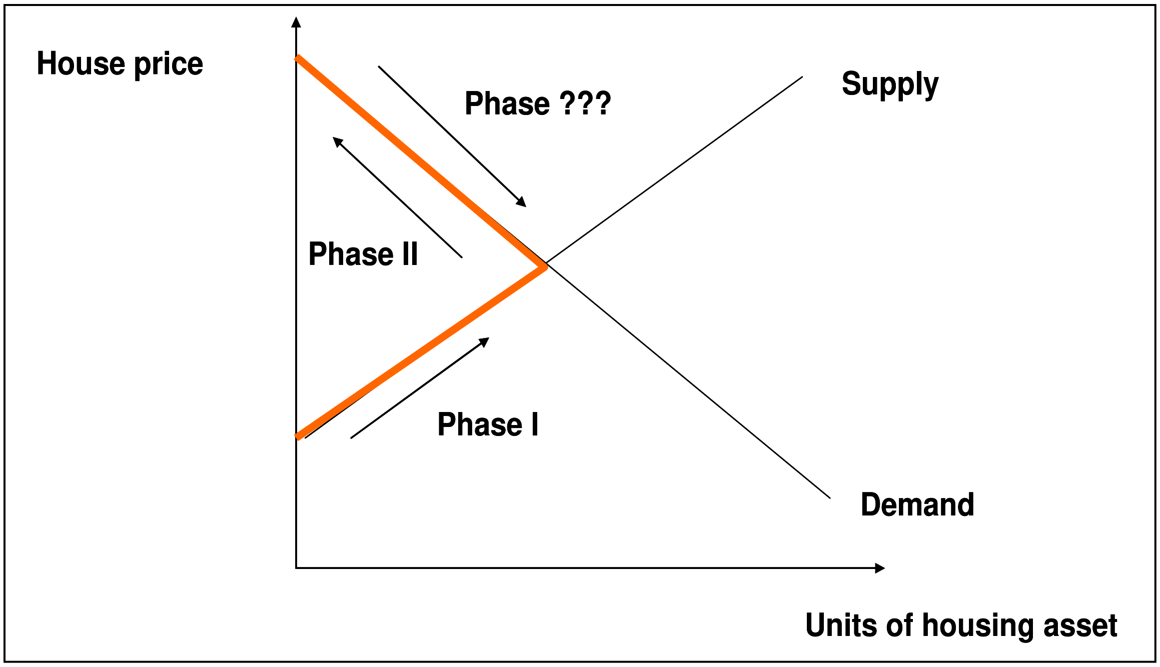



**Figure 4 A disequilibrium explanation of the housing market cycle.**

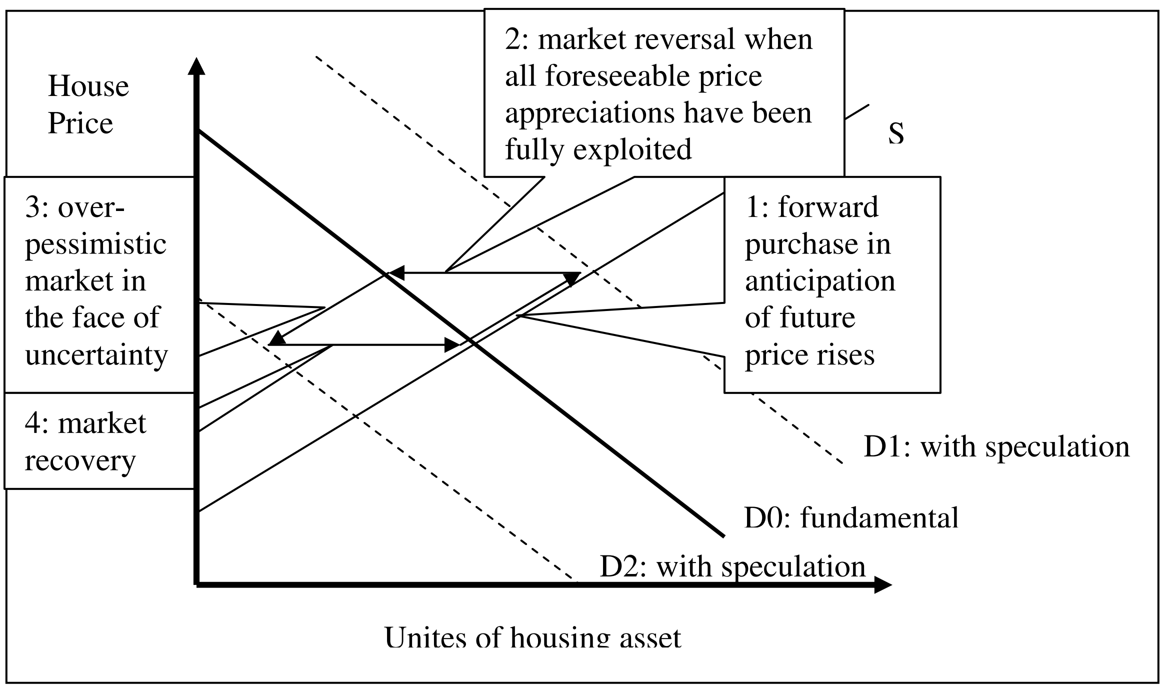



**Figure 5 A continuous equilibrium explanation of the housing market cycle.**

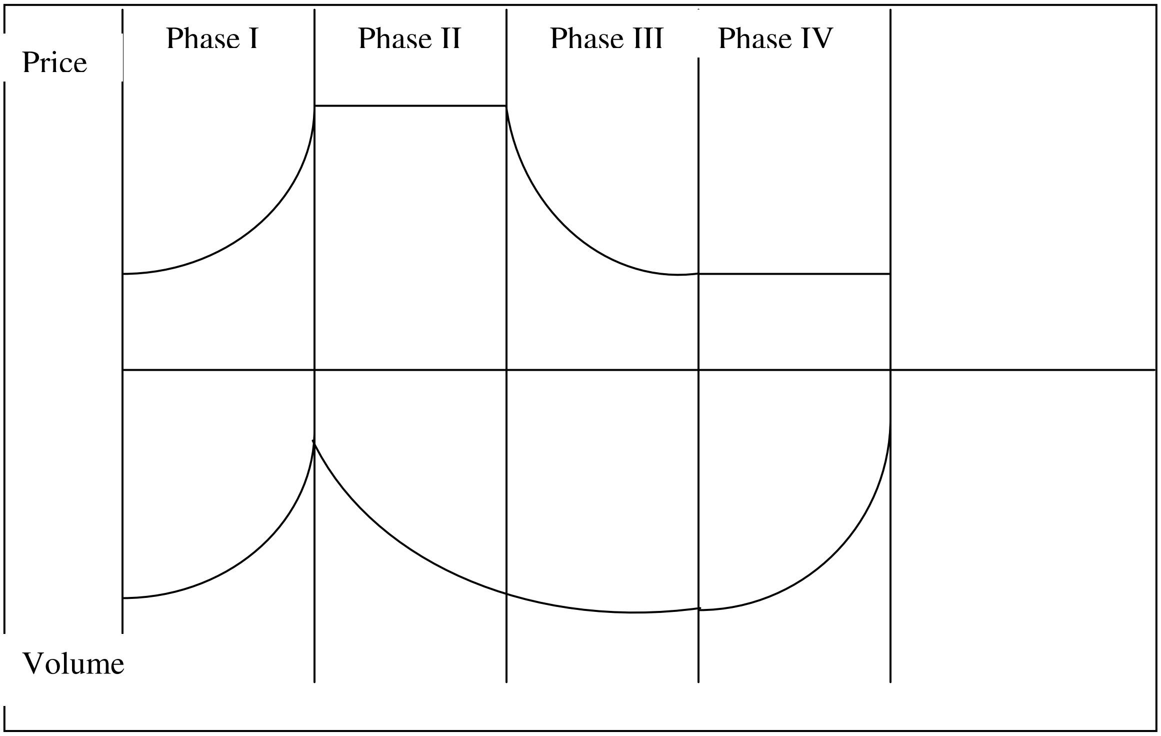



**Figure 6 House price and transaction volume in the four phases of a housing market cycle**

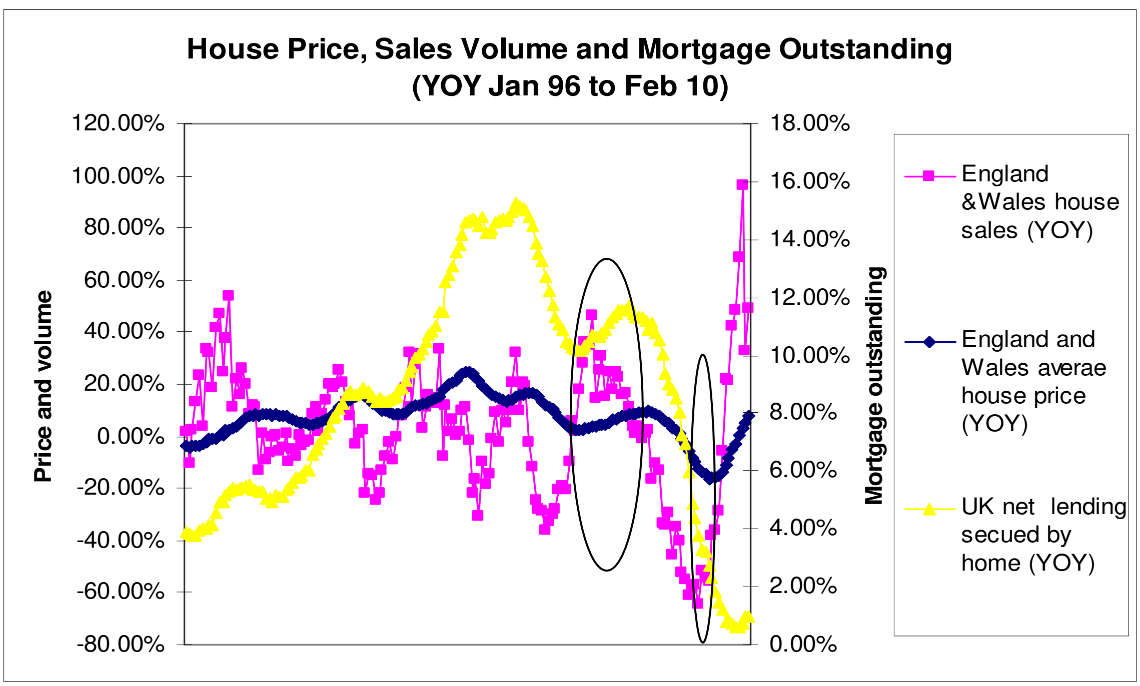



**Figure 7 House price, sales volume and mortgage outstanding** (Jan 96 – Feb 10). Source: Land Registry website (house price and sales) and Bank of England (net lending secured by home), both accessed on 18th June 2010.

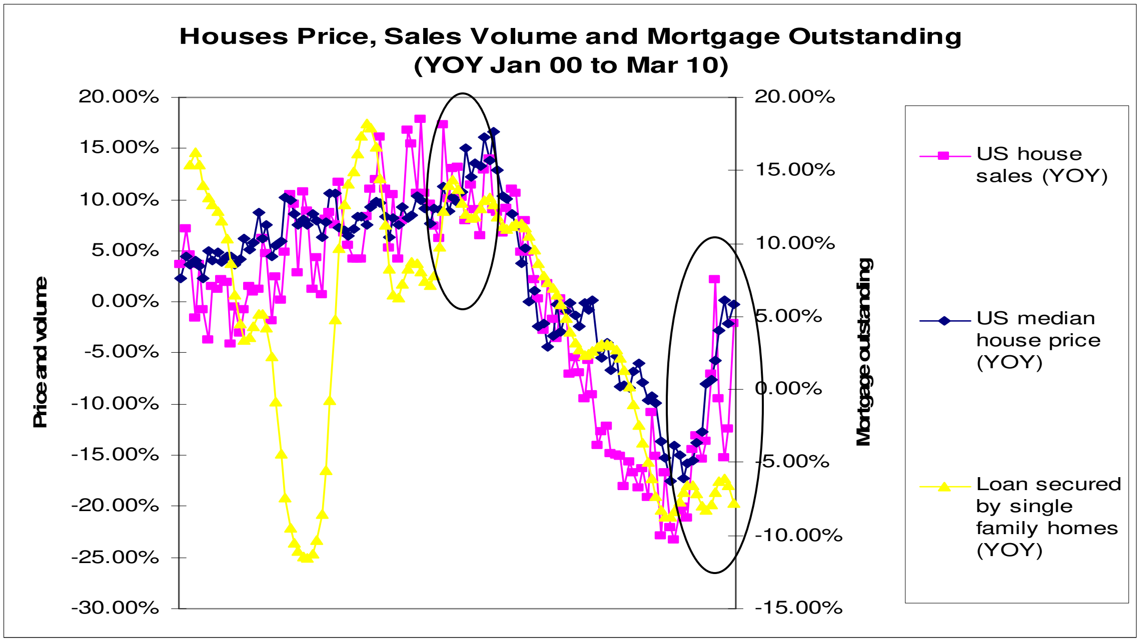



**Figure 8 US house price, sales volume and mortgage outstanding (Jan 00 – Mar 10).** Source: CEIC Database.

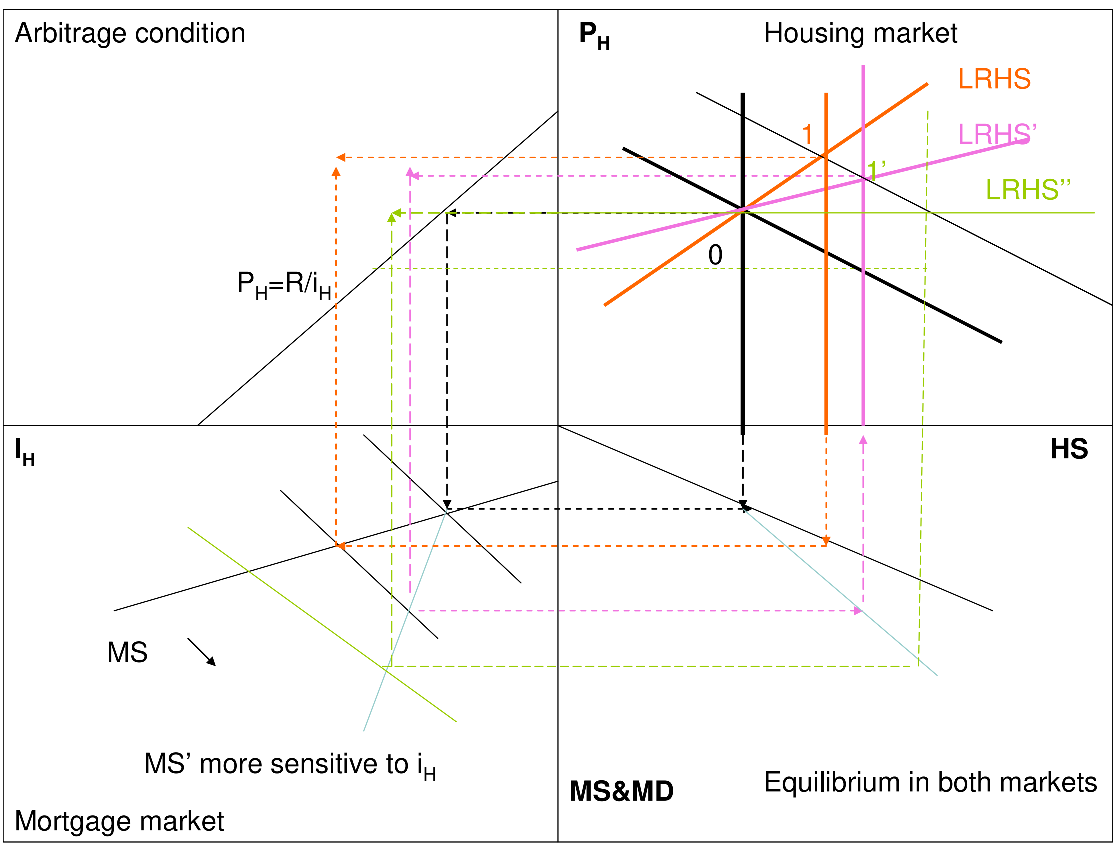



**Figure 9 Housing Market and Mortgage Market** (adopted from Buckley (1982)).

| | Generation 0 | Generation 1,2,…T-2 | Generation T-1 |
|---|---|---|---|
| Owners λN | Buyer λn | Holder (T-2)λn | Seller λn |
| Renters (1-λ)N | Renters (1-λ)n | Renters (T-2)(1-λ)n | Renters (1-λ)n |

**Figure 10 An example of stationary state demographic and housing tenure distribution at time t when speculation is absent.**

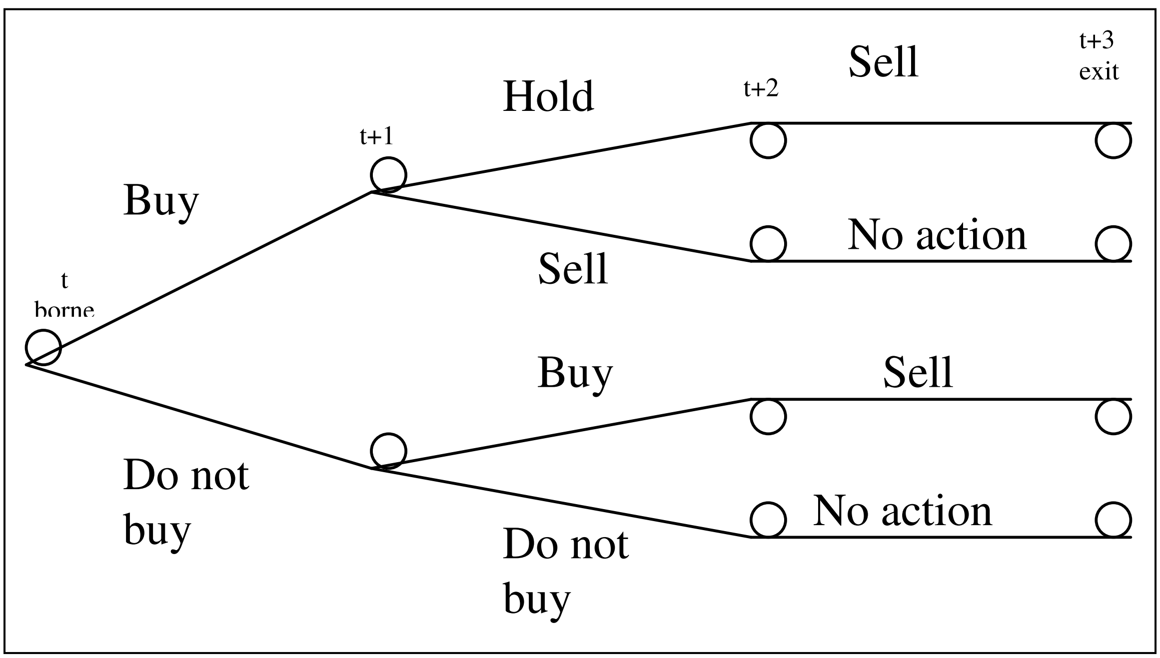



**Figure 11 The decision tree of a household borne at time t and lived for three periods when speculation about future price was allowed.**

**Table 1 Marginal Impact of Price on New and Existing Supply**

<table>
<tr><td></td><td>$\dot{P} > 0$ & high</td><td>$\dot{P} > 0$ & low</td><td>$\dot{P} < 0$ & low</td></tr>
<tr><td>P high</td><td colspan="2">Phase II<br>$I_P^{II}, h_P^{II} < 0$<br><br>(growing unanimity in expecting downward price correction)</td><td rowspan="2">Phase III<br>$I_P^{III}, h_P^{III} > 0$<br><br>(loss aversion and a collapsing supply in a falling market)</td></tr>
<tr><td>P low</td><td>Phase I<br>$I_P^{I}, h_P^{I} > 0$<br><br>(burgeoning optimism)</td><td>Phase IV<br>$I_P^{IV}, h_P^{IV} > 0$<br><br>(Growing unanimity in expecting upward price correction)</td></tr>
</table>

# Appendix

# Solving the System of Differential Equations

Given the dynamic system

$$\dot{M}_t - M_P \dot{P}_t - M_Q \dot{Q}_t = \dot{X}$$

$$\begin{aligned}&(R_W W_M)\ddot{M}_t + \dddot{P}_t - (\nu - R_W W_P)\ddot{P}_t + (R_H I_P)\dot{P}_{t-l} \\ &= -\left[(R_H I_C)\dot{C}_{t-l} + (R_W W_Y)\ddot{Y}\right]\end{aligned}$$

$$\ddot{Q}_t - I_P \ddot{P}_{t-l} - h_P \ddot{P}_t - h_H I_P \dot{P}_{t-l-1} = h_H I_C \dot{C}_{t-l-1}$$

The solution to which consists of two parts: a vector of particular integrals and a vector of complementary functions.

## VI. Finding the vector of particular integrals

Following the method described by Chiang (Alpha C. Chiang, 1984), try $M_P = \frac{\overline{M}}{2}t^2$, $P_P = \frac{\overline{P}}{3}t^3$, $Q_P = \frac{\overline{Q}}{2}t^2$, then

$$\overline{M}t - M_P \overline{P}t^2 - M_Q \overline{Q}t = \dot{X}$$

$$\begin{aligned}&(R_W W_M)\overline{M} + \left[2 - 2(\nu - R_W W_P)t + (R_H I_P)t^2\right]\overline{P} \\ &= -\left[(R_H I_C)\dot{C}_{t-l} + (R_W W_Y)\ddot{Y}\right]\end{aligned}$$

$$\overline{Q} - \left[2(I_P + h_P)t + h_H I_P t^2\right]\overline{P} = h_H I_C \dot{C}_{t-l-1}$$

We therefore have the intertemporal equilibrium values:

$$\overline{P} = -\left\{\frac{\left[(R_H I_C)\dot{C}_{t-l} + (R_W W_Y)\dot{\dot{Y}}\right]}{\left[2 - 2(\nu - R_W W_P)t + (R_H I_P)t^2\right]}\right\} - \left\{\frac{(R_W W_M)}{\left[2 - 2(\nu - R_W W_P)t + (R_H I_P)t^2\right]}\right\}\overline{M}$$

$$\overline{Q} = \left[2(I_P + h_P)t + h_H I_P t^2\right]\overline{P} + h_H I_C \dot{C}_{t-l-1}$$

$$\overline{M} = M_P t\overline{P} + M_Q \overline{Q} + \frac{\dot{X}}{t}$$

## VII. Finding the vector of complementary functions

Following Chiang (Alpha C. Chiang, 1984), try $M_C = Ae^{\eta t}$, $P_C = Be^{\eta t}$, and $Q_C = Ce^{\eta t}$, where A, B and C are arbitrary constants which can be definitized with boundary conditions. Substitutes these trial solutions and their respective derivatives into the reduced (homogenous) equations:

$$\eta e^{\eta t} A - \eta M_P e^{\eta t} B - \eta M_Q e^{\eta t} C = 0$$

$$(R_W W_M)\eta^2 e^{\eta t} A + \left[\eta^2 - (\nu - R_W W_P)\eta + (R_H I_P)\right]\eta e^{\eta t} B = 0$$

$$-\left[\eta(I_P + h_P) + (h_H I_P)\right]\eta e^{\eta t} B + \eta^2 e^{\eta t} C = 0$$

Simplify

$$A - M_P B - M_Q C = 0$$

$$(R_W W_M \eta)A + \left[\eta^2 - (\nu - R_W W_P)\eta + (R_H I_P)\right]B = 0$$

$$-\left[(I_P + h_P)\eta + (h_H I_P)\right]B + \eta C = 0$$

Write in matrix form

$$\begin{bmatrix} 1 & -M_P & -M_Q \\ [R_W W_M \eta] & [\eta^2 - (\nu - R_W W_P)\eta + (R_H I_P)] & 0 \\ 0 & -[(I_P + h_P)\eta + (h_H I_P)] & \eta \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For non-trivial solutions of A, B and C, we require

$$\begin{vmatrix} 1 & -M_P & -M_Q \\ [R_W W_M \eta] & [\eta^2 - (\nu - R_W W_P)\eta + (R_H I_P)] & 0 \\ 0 & -[(I_P + h_P)\eta + (h_H I_P)] & \eta \end{vmatrix}$$

$$= \begin{vmatrix} [\eta^2 - (\nu - R_W W_P)\eta + (R_H I_P)] & 0 \\ -[(I_P + h_P)\eta + (h_H I_P)] & \eta \end{vmatrix}$$

$$-(R_W W_M \eta)\begin{vmatrix} -M_P & -M_Q \\ -[(I_P + h_P)\eta + (h_H I_P)] & \eta \end{vmatrix}$$

$$\begin{aligned} &= [\eta^2 - (\nu - R_W W_P)\eta + (R_H I_P)]\eta \\ &-(R_W W_M \eta)\{-M_P\eta - [(I_P + h_P)\eta + (h_H I_P)]M_Q\} \\ &= 0 \end{aligned}$$

which is called the characteristic equation of the system of dynamic equations. Simplify to

$$\begin{aligned} &\eta^2 - \{(\nu - R_W W_P) - (R_W W_M)[M_P + M_Q(I_P + h_P)]\}\eta \\ &+ [(R_W W_M)M_Q(h_H I_P) + (R_H I_P)] = 0 \end{aligned}$$

and solve for the pair of characteristic roots $\eta_1, \eta_2$

$$\begin{aligned} &\eta_1, \eta_2 \\ &= \frac{1}{2}\{(\nu - R_W W_P) - (R_W W_M)[M_P + M_Q(I_P + h_P)]\} \\ &\pm \frac{1}{2}\sqrt{\{(\nu - R_W W_P) - (R_W W_M)[M_P + M_Q(I_P + h_P)]\}^2 - 4[(R_W W_M)M_Q(h_H I_P) + (R_H I_P)]} \end{aligned}$$

To have the observed cyclical patterns of time path, we require

$$\{(\nu - R_W W_P) - (R_W W_M)[M_P + M_Q(I_P + h_P)]\}^2 < 4[(R_W W_M)M_Q(h_H I_P) + (R_H I_P)]$$

i.e. the characteristic roots are a pair of conjugate complex numbers. In that case, we can write the above solution as

$$\eta_1, \eta_2 = \alpha + \theta i$$

where $i = \sqrt{-1}$, which is an imaginary number;

$\alpha = \frac{1}{2}\{(\nu - R_W W_P) - (R_W W_M)[M_P + M_Q(I_P + h_P)]\}$, the real root; and

$$\theta i = \frac{1}{2}\sqrt{\{(\nu - R_W W_P) - (R_W W_M)[M_P + M_Q(I_P + h_P)]\}^2 - 4[(R_W W_M)M_Q(h_H I_P) + (R_H I_P)]}i$$

Both real and imaginary roots are state-dependent because of $h_p$ and $I_p$. Hence the complementary functions:

$$M_C = A_1 e^{\alpha t + \theta i} + A_2 e^{\alpha t - \theta i} = e^{\alpha t}(A_3 \cos\theta t + A_4 \sin\theta t)$$

$$P_C = B_1 e^{\alpha t + \theta i} + B_2 e^{\alpha t - \theta i} = e^{\alpha t}(B_3 \cos\theta t + B_4 \sin\theta t)$$

$$Q_C = C_1 e^{\alpha t + \theta i} + C_2 e^{\alpha t - \theta i} = e^{\alpha t}(C_3 \cos\theta t + C_4 \sin\theta t)$$

where $A_j, B_j, C_j, j = 1,2$ are arbitrary constants to be definitized with boundary conditions; $X_3 \equiv X_1 + X_2$ and $X_4 \equiv (X_1 - X_2)i$ with $X_j = A_j, B_j, C_j$ and *j = 1, 2, 3,4.*

## VIII. The general solution

Combine the particular integral with the complex functions we have the general solution:

$$M_t = e^{\alpha t}(A_3 \cos\theta t + A_4 \sin\theta t) + \overline{M}$$

$$P_t = e^{\alpha t}(B_3 \cos\theta t + B_4 \sin\theta t) + \overline{P}$$

$$Q_t = e^{\alpha t}\left(C_3 \cos\theta t + C_4 \sin\theta t\right) + \overline{Q}$$

For these variables to converge to their respective intertemporal equilibrium, we require the real root α < 0, i.e. $\left\{\left(\nu - R_W W_P\right) - \left(R_W W_M\right)\left[M_P + M_Q\left(I_P + h_P\right)\right]\right\} < 0$

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[i] Housing is the single most important asset for most households in countries where middle-class forms the backbone of the economy. For instance, in the US, more than 40% of household assets are in the form of housing, c.f. **Curcuru, Stephanie; Heaton, John; Lucas, Deborah and Moore, Damien** eds**.** *Heterogeneity and Portfolio Choice: Theory and Evidence*. 2004.

[ii] The adjective "ordinary" is attached to exclude things such as saffron or articles worn by some influential historical figures.

[iii] In England and Wales, the year-on-year growth rate in sales dropped from 46.45% in March 06 to -66.16% in November 08 (a 112.61% change), while the growth rate in price dropped from 9.54% in August 07 to -15.91% in March 09 (a 25.45% change); In USA, the year-on-year growth in sales dropped from 17.82% in June 04 to -23.25% in February 09 ( a 41.07% change) and the growth in price dropped from 16.68% in October 05 to -17.53% in January 09 (a 34.21% change).

[iv] refer to Communities and Local Government website Accessible at http://www.communities.gov.uk. for UK housing statistics, or U.S. Census Bureau's American Housing Survey for US housing statistics, or CEIC Database for both

[v] Total housing sales as a percentage of the housing stock has been consistently below 1% in the USA and below 6% in UK between 1999 and 2010 (see US Census Bureau website for data on USA and Communities and Local Government web site on UK housing statistics).

[vi] A bottom-up approach is the piecing together of systems to give rise to grander systems. On the other hand, in a top-down approach, an overview of the system is first formulated.

[vii] In dynamic analysis, a *stationary state* is a state where each relevant variable grows at a zero rate, i.e. each is at its intertemporal equilibrium. A related concept is steady state where all relevant variables grow at an identical rate.

[viii] In the most general case, $\lambda$ will be a function of the relative cost of owning versus renting. In this model, we put the tenure choice problem at the background by assuming an exogenously given constant-value $\lambda$. This will allow us to single out the price impact of timing house buying or selling, a result of speculation through price expectation formation. Those who are interested in the specific treatment of a tenure choice problem are referred to **Sinai, Todd and Souleles, Nicholas.** "Owner-Occupied Housing as a Hedge against Rent Risk." *The Quarterly Journal of Economics*, 2005, *May*, pp. 763-89. and **Ortalo-Magné, Francois and Rady, Sven.** "Tenure Choice and the Riskiness of Non-Housing Consumption." *Journal of Housing Economics*, 2002, *11*(3), pp. 266-79.

[ix] The sustainability of borrowing into infinite future is taken for granted in this model.

[x] In the real world, such beliefs sometimes do have sound economic ground but are often greatly distorted as a result of opaque information, and as a result of uncertainty regarding the implication of a given piece of information. For instance, over-heating in housing market often triggers government intervention — that many people know. What form that intervention will take is unknown to most beforehand. The exact

impact of a given intervening policy will be unknown even to the policy maker himself, as that will depend on the complex interaction of all economic agents affected. A famous military analogue is described by Tolstoy in War and Peace regarding the role of Napoleon in the Battle of Borodino.

[xi] More general, $C_{t-l}, C_{t-l+1}, \ldots, C_{t-1}$ should all enter as arguments in the $I(\ )$ function as these past costs may affect the expected costs between the time the construction starts, *t-l*, and the time it completes, *t*. The choice of *l* will be a bit involved in empirical studies. The actual construction time for a residential housing varies greatly and can take from as little as two months to as much as two years (see http://www.b4ubuild.com/resources/schedule/6kproj.shtml and http://nwjoinery.com/budget.htm. Both accessed on 4th July 2010. Also refer to **Skitmorea, R. Martin and Ng, S. Thomas.** "Forecast Models for Actual Construction Time and Cost " *Building and Environment*, 2003, *38*(8), pp. 1075-83.).

[xii] Xiao and Sornette (Qin Xiao and Didier Sornette, 2009) show, using UK data (1975Q1 to 2007Q3) and a two-state Markov switching model, that while house price has a large impact on the growth of mortgage debt (roughly 2% growth in mortgage debt for every 1% growth in house price over five quarters), the feedback effect of mortgage debt on house price is small (about 0.1% in house price for every 1% in mortgage debt over four quarters), but not statistically insignificant.