The Value of Information in Auctions with Default Risk

Lamping, Jennifer

University of Colorado at Boulder

16 November 2007

The Value of Information in Auctions with Default Risk*

Jennifer Lamping†

November 16, 2007

Abstract

After the close of an auction, the winning bidder may find that he is unable to carry out his bid offer. This paper seeks to determine what measures the seller should take to maximize his share of the surplus when bidders are privately informed about their risk of default. Special attention is paid to the effect of imposing a default penalty, the value of gathering information about each bidder's default risk, and the role of commitment.

It is shown that the value of gathering information is negligible when the seller has commitment power and negative when the seller lacks commitment power. When the seller observes each bidder's risk, the seller benefits from the ability to commit. However, when the seller does not observe each bidder's risk, he is able to capture nearly all the surplus independent of whether or not he has commitment power.

JEL classification: C72; D44; D82. Keywords: Asymmetries; Auctions; Auction Theory; Bidding; Information Revelation; Signaling

*We thank Faruk Gul, Alessandro Lizzeri, and Bilge Yilmaz for valuable input in the early stages of this work. We also thank Yongmin Chen, Craig Kerr, Josepa M. Miquel-Florensa, Matthew Rhodes-Kropf, Michael Riordan, Anna Rubinchik, and Josep M. Vilarrubia for insightful comments. All remaining errors are ours.

†Department of Economics, University of Colorado at Boulder. Correspondence: 256 UCB, Boulder, CO 80309-0256. Phone: (303) 492 3827. Email: lamping@colorado.edu.

1 Introduction

Most of the auction theory literature assumes the winning bidder fulfills his bid immediately after the auction closes. However, some auctions – particularly those involving extended contracts – are not designed around such a "cash and carry" approach. It is in these cases that default is of particular concern. Information revealed after the submission of bids may make compliance with the bid offer undesirable, or even impossible. For instance, shortly after the conclusion of the 1996 "C-block" radio frequency spectrum auction, it was determined that the market value of the licenses was far less than the \$10 billion bid for them. As a result, several bidders were unable to make their payments to the United States Federal Communications Commission (Congressional Budget Office, 1997).

Given the potential cost of default, sellers in high value auctions often take measures to reduce the risk of a winning bidder defaulting on his offer. Online auction kingpin eBay ensures that users who submit bids in excess of \$15,000 are "serious about completing the transaction" by requiring a valid credit card (eBay, 2005a). Petròleos de Venezuela, S.A. (PDVSA) took a different approach in the 1990s when it auctioned operations contracts for marginal oil fields: it used its assessment of each bidder's "experience, technical capability, and financial resources" as a factor in selecting the winning bidders (Chalot, 1996).

The purpose of this paper is to determine what measures the seller should take to maximize his share of the surplus when bidders are privately informed about their risk of default. Special attention is paid to the value of gathering information about default risk from an outside source and the effects of implementing a default penalty.

We also investigate how the situation changes when the seller lacks the power to commit to the optimal mechanism. A seller is said to have *commitment power* if he can allocate the contract according to some predetermined rule even when he prefers a different selection ex post. In the absence of commitment power, the seller is unable to reject the offer whose combination of bid payment and expected risk maximizes the seller's expected utility.

Our results indicate that the value of gathering information about default risk is negligible when the seller has commitment power and negative when the seller lacks commitment power. We also show that there is a cost to lacking commitment power when the seller is informed about each bidder's risk but that when the seller is uninformed, he is able to capture nearly all the surplus independent of whether or not he has commitment power.

We develop a model in which the seller offers a single contract whose value is common to all bidders and commonly known. Each bidder is characterized by his exogenously determined probability of default with each bidder knowing his own default risk but not the risk levels of his competitors. If the winner delivers his bid offer, he collects the value of the contract, but if he defaults, the seller reappropriates the contract and imposes a penalty.

There are a number of papers that address the problem of default risk in an auction setting. Spulber (1990) shows that when the penalty is sufficiently low, bidders pool by bidding the highest possible value for the contract and defaulting unless the highest value is realized. Waehrer (1995) also demonstrates that a low default penalty results in more aggressive bidding and a greater likelihood of default but goes on to show that the associated loss to the seller can be reduced via renegotiation. In contrast, Harstad and Rothkopf (1995) argue that when opportunities for renegotiation are available, a low default penalty leads to less aggressive behavior as bidders' attention moves away from winning the contract in the initial round and toward obtaining the contract at a lower price in the renegotiation stage. Hansen and Lott (1991) and Zheng (2001) allow the penalty to vary across bidders and show that bidders who face a lower penalty bid more aggressively and are more likely to default.

In all of these papers, default is endogenously determined: bidders weigh the loss associated with delivering their bids against the loss associated with defaulting and paying the penalty. In some cases, however, the bidder may find himself in a situation where he is unable, rather than unwilling, to carry out his bid. One may imagine a case where a sudden shock to the supply chain affects his ability to deliver on time. The probability of such a shock is independent of the firm's bid and varies across bidding firms. While the previous models can be applied more easily to cases like the C-block auctions, our model is better suited to examining cases in which default is directly determined by sources outside the immediate transaction that are bidder specific.

We begin by assuming the seller can commit to any mechanism. If the seller knows each bidder's risk of default, he can extract the entire surplus by simply making a take-it-or-leave-it offer to the bidder with the lowest risk of default.

If the seller is not informed about each bidder's default risk, he can still capture nearly all the surplus. Suppose the seller holds a standard first-price auction. In the absence of a default penalty, price competition drives bids up to the common value of the contract. Since the seller cannot distinguish high risk bidders from low risk bidders, he selects a winner at random, and the outcome is likely to be inefficient. However, if the seller introduces a default penalty, low risk bidders will have a higher expected value for the contract because they are less likely to have to pay the penalty. As a result, low risk bidders can afford to outbid their high risk counterparts, and the seller obtains the desired combination of high bid and low risk.

Although the allocation is efficient, the bidder does enjoy some information rents due to the spread in valuations generated by the penalty. As the size of the penalty increases, do does the spread of the bidders' expected values for the contract. The wider the spread, the weaker the incentive to raise bids, and the larger the information rents. Thus, the seller is well advised to choose a penalty that is arbitrarily small so as to minimize the spread of valuations, stimulate price competition, and capture a greater share of the surplus.

In sum, by imposing a default penalty that is positive but arbitrarily the small, the seller can capture nearly the entire surplus, and the loss associated with being uninformed is negligible. Hence, if the cost of gathering information about each bidder's default risk exceeds the cost of holding an auction, the seller does better by choosing to remain uninformed and hold an auction.

If the seller lacks commitment power, he may be unable to implement either the take-it-or-leave-it offer or the first-price auction. In fact, the only allocation rule to which he can credibly commit is to award the contract to the bidder who offers the bid-risk combination that suits him best. In order to determine the effect of commitment on the seller's share of the surplus, we examine a *first-score* auction, in which each bidder bids on price alone but the contract goes to the bidder whose combination of both bid and expected risk maximizes the seller's expected utility.[1]

Suppose the seller does not observe each bidder's risk of default. We find that even in the absence of commitment, the seller can extract nearly all the surplus. As was the case with the first-price auction, introducing a positive penalty generates heterogeneity in the valuations such that low risk bidders have a higher expected value for the contract than high risk bidders do. However, unlike the first-price auction, the first-score auction allocates the contract on the basis of risk as well as bid. Low risk bidders would prefer to identify themselves as such, and since they can afford to bid more than their high risk counterparts, they can credibly signal their low risk status via higher bids.[2] As a result, the seller is able to obtain the desired combination of high bid and low risk.

Moreover, all that is required to generate this combination is that the penalty be a positive amount. To minimize the rents that accrue to the winning bidder, the seller selects a penalty that is arbitrarily small. As the penalty approaches zero, the bidder's information rents also approach zero, and the seller extracts nearly all the surplus – which is precisely the outcome of the first-price auction. Hence, when the seller is uninformed, he gains nothing from the ability to commit.

[1] To the extent that the seller allocates the contract on the basis of more than just price, this mechanism is reminiscent of the literature on multidimensional auctions (e.g., Che 1993, Branco 1997, and Zheng 2000), in which firms bid on price as well as other factors, such as quality. Our model is more similar to the work of Rezende (2006) and Lamping (2007) in that bidders bid only on price but the allocation rule account for factors not directly bid on.

[2] Goeree (2003) and Katzman and Rhodes-Kropf (2002) address the possibility of signaling in auctions, but their work is concerned with bidders signaling their private information to other bidders so as to affect future strategic interactions. Our setting more closely resembles that of Lamping (2007) in that the signaling behavior is motivated by the structure of the auction game itself: bidders are interested in signaling their private information to the seller in order to influence the choice of winner.

Finally, suppose the seller lacks commitment power but observes each bidder's risk of default at the outset. Competition among bidders will drive offers up to the point at which all the rent is extracted from the bidder with the second-lowest risk of default. Since the seller cannot commit to rejecting offers past this point, the lowest risk bidder obtains the contract and enjoys sizable information rents. Thus, when the seller is informed, his value for a commitment device is positive since commitment power would enable him to make a (credible) take-it-or-leave-it offer and extract all the rent.

Moreover, when the seller lacks commitment power, he prefers to be uninformed. That is, the seller's value for the information is actually negative. By observing each bidder's default risk at the outset, the seller introduces a bias in favor of low risk bidders. The lowest risk bidder capitalizes on this bias by bidding less aggressively than he otherwise would.[3]

The remainder of the paper is organized as follows. Section 2 introduces the model. In Section 3, we assume the seller can commit to any mechanism and establish that the seller can capture the entire surplus whether or not he obtains information about each bidder's default risk prior to allocating the contract. In Section 4, we assume the seller lacks commitment power and determine the extent to which gathering information affects the share of the surplus captured by the seller. Conclusions are offered in Section 5.

[3]Milgrom and Weber (1982b) and Cantillon (forthcoming) also examine the negative impact of asymmetries on price competition. Milgrom and Weber consider the case in which one bidder is better informed than the other about the value of the object, while Cantillon addresses the more general case in which bidder types are drawn from different distributions.

2 The Model

A seller is to auction off a contract to one of n risk-neutral bidders ($n \geq 2$). In order to simplify the analysis and focus on the role of default risk, we assume every bidder values the contract at $v \in \mathbb{R}$, where v is common knowledge.

Bidder i's private information is his type, $\theta_i \in \left[\underline{\theta}, \overline{\theta}\right] \subset [0, 1]$, which represents his exogenously determined probability of delivering on his bid offer given that he is awarded the contract. Bidder i's default risk can then be represented as $1 - \theta_i$. We assume the θ_i's are independently and identically distributed according to a commonly known cumulative distribution function (cdf) $F(\theta)$ with $F\left(\underline{\theta}\right) = 0$ and $F\left(\overline{\theta}\right) = 1$.

Assumption 1 *The distribution $F(\theta)$ is continuous over $\left[\underline{\theta}, \overline{\theta}\right]$ and has positive density f.*

To participate in the auction, each bidder submits a bid independently and simultaneously in addition to posting a letter of credit for the amount of the predetermined default penalty. Let $b_i \in \mathbb{R}$ denote the bid submitted by bidder i and $c \in \mathbb{R}$ denote the default penalty.

If bidder i does not win the contract, he ends up with his reservation utility of zero. However, if he is awarded the contract and delivers on his bid offer, his utility is $v - b_i$. If, instead, he defaults on his offer, he loses the contract and pays the default penalty, c. Therefore, bidder i's expected utility from contracting with the seller is

$$\theta_i \left(v - b_i\right) - \left(1 - \theta_i\right) c.$$

The seller's reservation utility is $v_0 \in \mathbb{R}$. If the seller contracts with bidder i and bidder i delivers on his bid offer, the seller forgoes v_0 but gains b_i. If, instead, bidder i defaults, the seller retains v_0 and collects c. Hence, the seller's expected utility from contracting with bidder i is

$$\theta_i b_i + (1 - \theta_i)(v_0 + c).$$

The following assumption is imposed so as not to rule out the possibility of a mutually beneficial trade:

Assumption 2 (participation condition) *v is strictly greater than v_0.*

3 Rent Extraction under Commitment

Although v is common across bidders, the surplus is greatest when the seller contracts with the bidder whose risk of default is lowest. However, if the seller is uninformed as to the risk associated with each bidder, he may not be able to select the bidder with the lowest risk. For instance, suppose the seller were to simply announce a price of v. If default were costless, every bidder would be willing to purchase the contract at that price. Since the seller forfeits v if the bidder defaults, the seller would prefer to contract with the bidder who is least likely to do so, but without additional information, the seller would be unable to identify who that bidder is.

We are interested in which mechanisms resolve this problem and maximize the surplus appropriated by the seller. In this section, we assume the seller can commit to any mechanism. Under *commitment*, the seller can credibly announce any allocation rule. That is, the seller is committed to allocating the contract in accordance with the announced rule even if he prefers a different selection ex post.

We pay special attention to two tools the seller has at his disposal: imposing a default penalty and gathering information about each bidder's risk level. The seller can impose a default penalty by requiring bidders to either post a letter of credit or put up collateral. For example, PDVSA required that each bidder present a \$250,000 letter of credit along with his bid; if the winning bidder failed to sign the contract by the predetermined date, PDVSA could cash his letter of credit (Bids, 1992). The seller can also gather information about each bidder's risk level ex ante and use the information to exclude high risk bidders. eBay facilitates this practice by making each user's profile publicly available (eBay, 2005b); in practice, sellers of high value items often elect to cancel bids from bidders whose rating or experience falls below a certain threshold (eBay, 2005c).

In the following section, we examine how the seller can use this information to appropriate a greater share of the surplus while in Section 3.2, we investigate how much the seller should be willing to pay for this information.

3.1 The Full Information Case

Suppose the seller is able to costlessly observe the vector of types prior to allocating the contract. With the information in hand, the seller can extract the entire surplus by approaching the bidder with the lowest risk of default and making him a take-it-or-leave-it offer: the contract at price v with no default penalty other than the reappropriation of the contract.

If the bidder with the lowest risk has type θ, then the seller's expected utility is $\theta v + (1 - \theta)v_0$. The seller has no incentive to exclude high risk bidders since $\underline{\theta} v + (1 - \underline{\theta})v_0$ is at least as high as the seller's reservation utility, v_0.

Moreover, a risk-neutral seller gains nothing by imposing a default penalty of $c > 0$. If the seller did impose such a penalty, a bidder with type θ would refuse any offer in which the price exceeded $v - \left(\frac{1-\theta}{\theta}\right) c$. That is, the bidder would adjust for the expected penalty. Suppose the seller were to make the following offer to the bidder with the lowest risk of default: the contract at a price of $v - \left(\frac{1-\theta}{\theta}\right) c$ with a penalty of c and the reappropriation of the contract if default occurs. Under this scheme, the seller's expected utility would still be $\theta v + (1-\theta)v_0$.

3.2 The Incomplete Information Case

We have established that observing the types allows the seller to appropriate the entire surplus. Thus, if it is costless to obtain this information, it is clearly in the seller's interest to do so. But what if there were a cost associated with gathering information about each bidder's risk of default? How much should the seller be willing to pay for the information?

Suppose the seller does not observe the vector of types and assume he allocates the contract via a standard first-price sealed-bid auction with a reserve price of b_* and a default penalty of c. Let $P_i(b)$ represent bidder i's probability of winning given that he bids b. Since the seller awards the contract to the bidder offering the highest bid, $P_i(b)$ is simply the probability that b is the highest bid submitted if $b > b_*$ and zero otherwise. It follows that if bidder i has type θ and bids b, his expected utility is

$$U_i(b, \theta) \equiv \left[\theta(v - b) - (1-\theta)c\right] P_i(b).$$

Let $\beta_i : \mathbb{R} \times \left[\underline{\theta}, \overline{\theta}\right] \rightarrow \mathbb{R}_+$ represent bidder i's equilibrium bidding strategy (possibly a mixed strategy), where $\beta_i\left(b, \theta\right)$ is the density function that determines bidder i's probability of bidding b in equilibrium when his type is θ. Bidder i's problem is to select β_i such that for any type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and any bid b in the support of $\beta_i(\cdot, \theta)$, the following two conditions are satisfied:

$$U_i(b, \theta) \geq U_i(x, \theta) \quad \forall x \in \mathbb{R}$$

and

$$U_i(b, \theta) \geq 0.$$

We proceed by reformulating the bidder's problem so as to directly apply the standard independent private values results. Let

$$\tilde{U}_i(b, \theta) \equiv \left[v - \left(\frac{1-\theta}{\theta}\right) c - b\right] P_i(b).$$

Since θ is positive, $\tilde{U}_i(b, \theta)$ is a monotonic transformation of $U_i(b, \theta)$. Therefore, bidder i's problem is to select β_i such that for any type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and any bid b in the support of $\beta_i(\cdot, \theta)$, the following two conditions are satisfied:

$$\tilde{U}_i(b, \theta) \geq \tilde{U}_i(x, \theta) \quad \forall x \in \mathbb{R}$$

and

$$\tilde{U}_i(b, \theta) \geq 0.$$

Note that by interpreting $v - \left(\frac{1-\theta}{\theta}\right) c$ as the bidder's type, we can map this formulation into the standard independent private values framework and invoke Maskin and Riley (1986) and Riley and Samuelson (1981) to obtain the following result:

Proposition 1

(i) Suppose $c \geq 0$ *and let*

$$\underline{t} \equiv \inf \left\{ \theta \in \left[\underline{\theta}, \overline{\theta}\right] : v - \left(\frac{1-\theta}{\theta}\right) c > b_* \right\}.$$

There exists a unique equilibrium in which any bidder with type $\theta \in [\underline{t}, \overline{\theta}]$ *bids according to*

$$b(\theta) = v - \left(\frac{1-\theta}{\theta}\right) c - \left(\frac{\int_{\underline{t}}^{\theta} \frac{1}{x^2} F^{n-1}(x) dx}{F^{n-1}(\theta)}\right) c$$

and any bidder with type $\theta \in [\underline{\theta}, \underline{t})$ *bids* $b(\theta) < b_*$.[4]

(ii) Suppose $c \leq 0$ *and let*

$$\overline{t} \equiv \sup \left\{ \theta \in \left[\underline{\theta}, \overline{\theta}\right] : v - \left(\frac{1-\theta}{\theta}\right) c > b_* \right\}.$$

There exists a unique equilibrium in which any bidder with type $\theta \in [\underline{\theta}, \overline{t}]$ *bids according to*

$$b(\theta) = v - \left(\frac{1-\theta}{\theta}\right) c + \left(\frac{\int_{\theta}^{\overline{t}} \frac{1}{x^2} F^{n-1}(x) dx}{F^{n-1}(\theta)}\right) c$$

and any bidder with type $\theta \in (\overline{t}, \overline{\theta}]$ *bids* $b(\theta) < b_*$.

The first two terms of the bidding functions represent the bidder's expected value for the contract while the third term represents the degree to which the bidder shades his bid. The default penalty c enters into the shading term because c is the source of heterogeneity in the valuations: the greater the magnitude of c, the greater the range of valuations, and the greater the margin by which each bidder can shade his bid.

[4] Despite the multiplicity of bids satisfying the condition $b(\theta) < b_*$, we assert that the equilibrium is unique in the sense that these bids can be classified as "non-participating."

If c is positive, the bidding function is increasing in θ, and therefore, the winner is the bidder with the lowest risk of default. This is in contrast to the results in Zheng (2001) and Hansen and Lott (1991), where the winner is the bidder with the highest default risk. The difference arises from the structure of penalty payments. In Zheng and Hansen and Lott, penalty payments vary across bidders; the bidder facing the smallest penalty payment can afford to bid more aggressively but also finds defaulting more attractive. In our model, the penalty payment is constant across bidders, but the expected penalty payment varies: bidders with a low risk of default are less likely to incur the penalty payment, and therefore, their expected penalty payment is smaller. As a result, low risk bidders can afford to bid more aggressively. Since default risk is exogenous in our model, these higher bids do not raise the probability of defaulting, and the seller obtains the desired combination of high bids and low risk.

Despite this, the bidder does enjoy some information rents as represented by the shading term. Since the seller is not able to appropriate the entire surplus, there is some value in gathering information about each bidder's default risk in advance. However, as the penalty payment approaches zero, the information rents also approach zero. Therefore, if $\underline{t} = \underline{\theta}$, then as c approaches zero, the outcome of this auction approaches the outcome in the full information case.

Corollary 1 *If the seller has commitment power, the value of information about each bidder's default risk is negligible.*

Note that there is a discontinuity at $c = 0$. If the seller sets the penalty payment to zero, then every bidder bids v, but since the seller remains uninformed, he cannot distinguish between high risk and low risk bidders. Therefore, c must be positive in order to generate heterogeneity in valuations such that low risk bidders have a higher value for the contract than high risk bidders do. However, if c is too large, competition among bidders becomes less intense, and the seller's share of the surplus is reduced. This may explain why the letter of credit required by PDVSA was for such a small amount in relation to the value of the operations contracts.

As long as the information is free, the seller is better off (though marginally so) observing the vector of types and extracting the entire surplus. However, if the cost associated with acquiring the information exceeds the cost of administering the auction, then the seller is better off not observing the types and holding the auction.

4 Rent Extraction under No Commitment

The analysis in Section 3 assumed the seller had commitment power. In the full information case, the seller committed to walking away if the bidder with the lowest default risk refused the take-it-or-leave-it offer. In the incomplete information case, the seller committed to awarding the contract to the bidder submitting the highest bid even if another bidder offered a more attractive combination of bid and risk.

In this section, we assume the seller lacks commitment power. That is, we assume the seller always selects the offer he likes best even if that offer is not the one specified by the allocation rule. In this context, we ask how much the information is worth to the seller and how the seller can introduce a default penalty in order to generate a more favorable outcome.

4.1 The Game Form

In the absence of commitment, the seller allocates the contract to the bidder whose combination of bid and risk maximizes the seller's expected utility. Consider the following auction game:

1. Each bidder submits a bid independently and simultaneously.

2. The seller contracts with the bidder whose offer maximizes the seller's expected utility provided that the offer is not less than the seller's reservation utility, v_0. That is, bidder i wins the contract if

   $$E(\theta_i|b_i)\,b_i + [1 - E(\theta_i|b_i)]\,(v_0 + c) >$$

   $$E(\theta_j|b_j)\,b_j + [1 - E(\theta_j|b_j)]\,(v_0 + c) \quad \forall j \neq i$$

   and

   $$E(\theta_i|b_i)\,b_i + [1 - E(\theta_i|b_i)]\,(v_0 + c) \geq v_0.$$

   Ties are resolved by a random draw with equal probability.

3. Once the contract is allocated to bidder i, he delivers his bid offer with probability θ_i and defaults with probability $1 - \theta_i$. The seller's payoff is b_i if the bid is delivered and $v_0 + c$ otherwise. Similarly, bidder i's payoff is $v - b_i$ if he delivers his bid and $-c$ if he defaults. All other bidders get zero. The auction game is then over.

We call this auction game a *first-score* auction, where the term "score" refers to the combination of bid and risk. For instance, bidder i's score is given by $E(\theta_i|b_i)\,b_i + [1 - E(\theta_i|b_i)](v_0 + c)$. As indicated in the timeline above, the contract is allocated to the bidder with the highest score provided that that score is at least v_0.

The first-score auction is much like a first-price auction in that the winning bidder pays his bid. However, unlike a first-price auction, the winner is the bidder who offers the most attractive combination of bid and risk. This allocation rule allows the seller to reject an offer made by the highest bidder and allocate the contract to another bidder whose pairing of bid and risk is more attractive than the pairing offered by the highest bidder. Since there is no ex-post regret, the seller does not require commitment power in order to adhere to this rule.

Our approach will be to investigate the perfect Bayesian equilibria of the first-score auction in both the full information case and the incomplete information case and then compare the results.

4.2 The Incomplete Information Case

We begin by examining the case in which the seller does not observe the vector of types. In Section 3.2 we found that the seller could get infinitely close to appropriating the entire surplus by administering a standard first-price auction with a positive but arbitrarily small default penalty. In this section, we ask whether the seller's lack of commitment power causes him to forfeit a share of the surplus. In other words, how much should the seller be willing to pay for a commitment device?

Once again, we let $P_i(b)$ represent bidder i's probability of winning with a bid of b. Although the seller accounts for default risk in selecting a winner, he observes only the bids offered, and therefore, each bidder's probability of winning the auction is a function of his bid alone. In addition, let $\beta_i : \mathbb{R} \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ represent bidder i's equilibrium bidding strategy (possibly a mixed strategy), where $\beta_i(b, \theta)$ is the density function that determines bidder i's probability of bidding b in equilibrium when his type is θ.

Lemma 1 *Suppose there exist $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and some b in the support of $\beta_i(\cdot, \theta)$ such that $P_i(b) > 0$.*

(i) If $c > 0$, then for all $\hat{\theta} > \theta$ and all $\hat{b}$ in the support of $\beta_i(\cdot, \hat{\theta})$, it is the case that $P_i(\hat{b}) > 0$.

(ii) If $c < 0$, then for all $\hat{\theta} < \theta$ and all $\hat{b}$ in the support of $\beta_i(\cdot, \hat{\theta})$, it is the case that $P_i(\hat{b}) > 0$.

Proof: See Appendix. □

If the default penalty is positive, the expected penalty faced by low risk bidders is less than the expected penalty faced by high risk bidders, and any bid which is profitable when the bidder draws type θ must also be profitable when the bidder draws any type greater than θ. Hence, if the bidder finds it profitable to participate in the auction when his type is θ, he will also find it profitable to participate when his type is greater than θ. Lemma 1 indicates that when c is positive, there exists a threshold, $\underline{t}_i$, for each bidder such that if the type drawn exceeds this threshold, the bidder's probability of winning is positive but if the type drawn falls short of this threshold, the bidder's probability of winning is zero.

In contrast, if the default penalty is negative, low risk bidders are less likely to enjoy the *benefit* of defaulting. That is, drawing a lower type raises the expected benefit from defaulting, and as a result, any bid b which is profitable when the bidder draws type θ must also be profitable when the bidder draws any type lower than θ. Lemma 1 indicates that when c is negative, there exists a threshold, $\bar{t}_i$, for each bidder such that if the type drawn falls below this threshold, the bidder's probability of winning is positive but if the type drawn is higher than this threshold, the bidder's probability of winning is zero.

Since the available surplus decreaes with the bidder's risk of default, we are interested in equilibria in which the seller can identify which bidders have the lowest risk of default. Thus, we elect to focus on separating equilibria of the first-score auction. In the following lemma, we show that in any separating equilibrium, if c is positive, bids are strictly increasing in type for $\theta_i > \underline{t}_i$, but if c is negative, bids are strictly decreasing in type for $\theta_i < \bar{t}_i$.

Lemma 2 *Let $\beta_i : \mathbb{R} \times \left[\underline{\theta}, \overline{\theta}\right] \to \mathbb{R}_+$ be a separating equilibrium bidding strategy. Suppose there exist $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and some b in the support of $\beta_i(\cdot, \theta)$ such that $P_i(b) > 0$.*

(i) If $c > 0$, $\hat{\theta} > \theta$, and $\hat{b}$ is in the support of $\beta_i(\cdot, \hat{\theta})$, then $\hat{b} > b$.

(ii) If $c < 0$, $\hat{\theta} < \theta$, and $\hat{b}$ is in the support of $\beta_i(\cdot, \hat{\theta})$, then $\hat{b} > b$.

Proof: See Appendix. □

In a separating equilibrium, the seller can fully extract types from bids. Lemma 2 suggests that if c is positive, then for each bidder a higher bid signals lower risk. Since reducing risk reduces the expected penalty, low risk bidders have a higher expected value for the contract. As a result, bidders can credibly signal their lower risk by offering higher bids. On the other hand, if c is negative, Lemma 2 indicates that a higher bid signals higher risk. Since a negative default penalty functions as a reward for defaulting, high risk bidders enjoy a higher expected reward, which raises their expected value for the contract. Since high risk bidders can afford to bid more than their low risk counterparts, higher bids signal higher risk.

In the following lemma, we establish that the incentive to separate breaks down if the magnitude of c is too high.

Lemma 3 *Let $\beta_i : \mathbb{R} \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ be a separating equilibrium bidding strategy. Suppose there exist $\theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}]$, b in the support of $\beta_i(\cdot, \theta)$, and $\hat{b}$ in the support of $\beta_i(\cdot, \hat{\theta})$ such that $P_i(b) > 0$ and $P_i(\hat{b}) > 0$. Let $\hat{\theta} > \theta$.*

(i) If $c > 0$, then b and $\hat{b}$ satisfy

$$\frac{\hat{\theta}\hat{b} - \theta b}{\hat{\theta} - \theta} > v_0 + c.$$

(ii) If $c < 0$, then b and $\hat{b}$ satisfy

$$\frac{\hat{\theta}\hat{b} - \theta b}{\hat{\theta} - \theta} \leq v_0 + c$$

(with strict inequality if $\theta > 0$).

Proof: See Appendix. □

Lemma 2 stipulates that if c is positive, higher types submit higher bids. However, if c is too large, the seller stands to gain if the bidder defaults. In this case, the seller prefers to contract with a high risk bidder despite the fact that high risk bidders submit lower bids. Incentive compatibility is then violated for low risk bidders since these bidders clearly gain if they are able to both reduce their bids and raise their probability of winning. Therefore, if c is too large, the incentive to separate breaks down, and there is no separating equilibrium.

Separation also fails when c is too small. Lemma 2 indicates that if c is negative, higher types submit lower bids. However, if the magnitude of c is too large, the seller is more concerned with the amount he has to pay in the event of a default than with the bid itself. In this case, the seller prefers to contract with a low risk bidder despite the fact that low risk bidders submit lower bids. As a result, high risk bidders have an incentive to mimic their low risk counterparts, and the incentive to separate breaks down. Hence, if c is too small, there is no separating equilibrium.

We proceed by solving for the symmetric separating equilibria of the first-score auction. The following lemma establishes that equilibrium bidding strategies must be pure.

Lemma 4 *Let $\beta : \mathbb{R} \times \left[\underline{\theta}, \overline{\theta}\right] \to \mathbb{R}_+$ be a symmetric separating equilibrium bidding strategy. Suppose $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and there exists some b in the support of $\beta(\cdot, \theta)$ such that $P(b) > 0$.*

(i) If $c > 0$, then $\beta(\cdot, \theta)$ must be a pure strategy.

(ii) If $c < 0$ and $\theta \neq 0$, then $\beta(\cdot, \theta)$ must be a pure strategy.

Proof: See Appendix. $\square$

Suppose $c > 0$. Lemma 1 indicates that there exists a $\underline{t} \in [\underline{\theta}, \overline{\theta}]$ such that bidders with types greater than $\underline{t}$ are awarded the contract with positive probability and bidders with types less than $\underline{t}$ are never awarded the contract. Lemma 4, in turn, indicates that in any symmetric separating equilibrium, the bidding strategy is pure when the bidder's type exceeds $\underline{t}$. Therefore, the equilibrium bidding strategy of any bidder with type $\theta \in (\underline{t}, \overline{\theta}]$ can be represented by $b : (\underline{t}, \overline{\theta}] \to \mathbb{R}$, where $b(\theta) \equiv \{b : \beta(b, \theta) \neq 0\}$.

Similarly, if $c < 0$, Lemma 1 indicates that there exists a $\overline{t} \in [\underline{\theta}, \overline{\theta}]$ such that bidders with types less than $\overline{t}$ are awarded the contract with positive probability and bidders with types greater than $\overline{t}$ are never awarded the contract. Lemma 4 then indicates that in any symmetric separating equilibrium, the bidding strategy is pure when the bidder's type is less than $\overline{t}$. Therefore, the equilibrium bidding strategy of any bidder with type $\theta \in [\underline{\theta}, \overline{t})$ can be represented by $b : [\underline{\theta}, \overline{t}) \to \mathbb{R}$.

The following two lemmas determine the functional form of b.

Lemma 5 *In any symmetric separating equilibrium, the bidding strategy b is continuous over*

(i) $\left(\underline{t}, \overline{\theta}\right]$ *if* $c > 0$,

(ii) $\left[\underline{\theta}, \overline{t}\right)$ *if* $c < 0$ *and* $\underline{\theta} > 0$, *and*

(iii) $\left(\underline{\theta}, \overline{t}\right)$ *if* $c < 0$ *and* $\underline{\theta} = 0$.

Proof: See Appendix. □

Lemma 6 *In any symmetric separating equilibrium, a bidder with type θ bids*

(i) $b(\theta) = v - \left(\frac{1-\theta}{\theta}\right) c - \frac{c \int_{\underline{t}}^{\theta} \frac{1}{x^2} F^{n-1}(x) dx}{F^{n-1}(\theta)}$ *if* $c > 0$, $\underline{t} > 0$, *and* $\theta \in (\underline{t}, \overline{\theta}]$

(ii) $b(\theta) = v - \left(\frac{1-\theta}{\theta}\right) c + \frac{c \int_{\theta}^{\overline{t}} \frac{1}{x^2} F^{n-1}(x) dx}{F^{n-1}(\theta)}$ *if* $c < 0$ *and* $\theta \in (\underline{\theta}, \overline{t})$.

Proof: See Appendix. □

The bidding function specified by Lemma 6 is identical to the bidding function specified by Proposition 1. By Lemma 3, the contract goes to the bidder with the lowest risk when the penalty is positive and to the bidder with the highest risk when the penalty is negative. Since Lemma 2 stipulates that bids increase in type when the penalty is positive and decrease in type when the penalty is negative, the contract always goes to the bidder submitting the highest bid – which is precisely the allocation rule in a first-price auction.

However, unlike the bidding function in the first-price auction, the bidding function specified by Lemma 6 cannot be sustained for any $c \in \mathbb{R}$. Lemma 3 indicates that the magnitude of c cannot be too large. The following lemma imposes additional restrictions on c and establishes the existence of a symmetric separating equilibrium when those restrictions are met.

Lemma 7 *Suppose the seller sets a reserve score of $tv + (1-t)v_0$, where $t \in [\underline{\theta}, \overline{\theta}]$ and a default penalty of $c = \gamma(v - v_0)$, where $\gamma \in \mathbb{R}$.*

(i) If $\gamma > t$, then there is no symmetric separating equilibrium.

(ii) If $\gamma \in (0, t]$, then there exists a symmetric separating equilibrium in which bidders bid according to

$$\begin{aligned} b(\theta) &= v - \left(\tfrac{1-\theta}{\theta}\right) c - \frac{c\int_t^{\theta} \frac{1}{x^2} F^{n-1}(x)dx}{F^{n-1}(\theta)} && \text{if } \theta \in [t, \overline{\theta}] \\ b(\theta) &< b(t) && \text{otherwise} \end{aligned}$$

and the seller's off-equilibrium-path beliefs are such that $E(\theta_i | b_i) \in (\underline{\theta}, t]$ if $b_i < b(t)$.

(iii) If $\gamma < 0$ and $t = \underline{\theta} > 0$, then there exists a symmetric separating equilibrium in which bidders bid according to

$$\begin{aligned} b(\theta) &= v - \left(\tfrac{1-\theta}{\theta}\right) c && \text{if } \theta = \underline{\theta} \\ b(\theta) &< v - \left(\tfrac{1-\theta}{\theta}\right) c - \left(\tfrac{\theta - \underline{\theta}}{\theta}\right)(v - v_0) && \text{otherwise} \end{aligned}$$

and the seller's off-equilibrium-path beliefs are such that $E(\theta_i | b_i) = \underline{\theta}$ if $b_i < b(\underline{\theta})$.

(iv) If $\gamma < 0$ and either $t = \underline{\theta} = 0$ or $t > \underline{\theta}$, then there is no symmetric separating equilibrium.

Proof: See Appendix. $\square$

Part (i) follows directly from Lemma 3. Parts (iii) and (iv) all but rule out the existence of symmetric separating equilibria when the default penalty is negative. When $c < 0$, high risk bidders can afford to bid more than low risk bidders. Nevertheless, high risk bidders prefer not to outbid their low risk counterparts because doing so would identify them as high risk thereby mitigating the increase in the probability of winning that would normally accompany an increase in bid.

Turning our attention to the equilibrium outlined in part (ii), we find that, as was the case with the first-price auction, the first-score auction delivers the desired combination of high bid and low risk. Moreover, the seller can reduce the bidder's information rents and capture a greater share of the surplus by making the default penalty arbitrarily small. However, when the seller lacks commitment power, he is unable to reject offers that exceed his reservation utility, v_0. That is, v_0 is the only reserve score the seller can sustain.

Proposition 2 *Suppose the seller sets a reserve score of v_0 and a default penalty of $c \in (0, \underline{\theta}(v - v_0)]$. There exists a symmetric separating equilibrium in which bidders bid according to*

$$b(\theta) = v - \left(\frac{1-\theta}{\theta}\right) c - \frac{c \int_{\underline{\theta}}^{\theta} \frac{1}{x^2} F^{n-1}(x) dx}{F^{n-1}(\theta)}$$

if $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and the seller's off-equilibrium-path beliefs are such that $E\left(\theta_i | b_i\right) = \underline{\theta}$ if $b_i < b(\underline{\theta})$.

Proof: See Appendix. □

Proposition 2 indicates that the seller's inability to reject offers that exceed v_0 does not preclude him from capturing the entire surplus; to do so, he need only make the default penalty arbitrarily small. Therefore, there is no cost associated with a lack of commitment power. That is, the value of a commitment device is zero.

Corollary 2 *When the seller does not observe each bidder's default risk in advance, he gains nothing from the ability to commit.*

In the following section, we examine whether this result continues to apply when the seller does observe the bidders' types in advance.

4.3 The Full Information Case

In Section 3.1 we found that if the seller is informed and has commitment power, he can capture the entire surplus by identifying the bidder with the lowest risk of default and making him a take-it-or-leave-it offer: the contract at price v with no default penalty other than the reappropriation of the contract. In this section we ask whether an informed seller without commitment power can also capture the entire surplus.

We begin by examining the first-score auction outlined in Section 4.1 under the assumption that the seller observes the vector of types in advance.[5] Since the seller awards the contract to the bidder whose bid-risk combination is most attractive, each bidder's probability of winning is a function of both his bid and his type. Let $P_i(b, \theta)$ represent bidder i's probability of winning the contract with a bid of b given that his type is θ. It follows that if bidder i has type θ and bids b, his expected utility is

$$U_i(b, \theta) \equiv [\theta(v - b) - (1 - \theta)c]\, P_i(b, \theta).$$

[5] Given that v is common knowledge and the seller observes each bidder's default risk, one might ask what purpose an auction serves in this context. Our immediate purpose is to isolate the effect of observing the types on the strategic interaction among bidders by holding the game form constant and varying the seller's information. However, it should be noted that in reality the seller is often uninformed about v, and this uncertainty may be sufficient reason to hold an auction. Our model casts light on the value (if any) of obtaining information about default risk within an auction framework.

Let $\beta_i : \mathbb{R} \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ represent bidder i's equilibrium bidding strategy (possibly a mixed strategy), where $\beta_i(b, \theta)$ is the density function that determines bidder i's probability of bidding b in equilibrium when his type is θ. Bidder i's problem is to select β_i such that for any type $\theta \in [\underline{\theta}, \overline{\theta}]$ and any bid b in the support of $\beta_i(\cdot, \theta)$, the following two conditions are satisfied:

$$U_i(b, \theta) \geq U_i(x, \theta) \quad \forall x \in \mathbb{R}$$

and

$$U_i(b, \theta) \geq 0.$$

We proceed by reformulating the bidder's problem using bidder i's score as the choice variable. This reformulation permits us to directly apply the standard independent private values results. Let

$$s \equiv \theta b + (1 - \theta)(v_0 + c) \tag{4.1}$$

denote the score offered by a bidder with type θ who bids b. Note that since both the bidder and seller observe θ ex ante, the bidder's choice of bid unambiguously determines his score. Let $Q_i(s)$ represent bidder i's probability of winning with a score of s. Since the seller awards the contract to the bidder with the highest score, $Q_i(s)$ is simply the probability that s is the highest score offered if $s \geq s_*$ and zero otherwise. Finally, let $\sigma_i : \mathbb{R} \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ be an equilibrium score strategy (possibly a mixed strategy), where $\sigma_i(s, \theta)$ is the density function that determines bidder i's probability of offering s in equilibrium when his type is θ.

Using this notation, we can now reformulate the bidder's problem. Bidder i selects σ_i such that for any type $\theta \in [\underline{\theta}, \overline{\theta}]$ and any score s in the support of $\sigma_i(\cdot, \theta)$, the following two conditions are satisfied:

$$\hat{U}_i(s, \theta) \geq \hat{U}_i(x, \theta) \quad \forall x \in \mathbb{R}$$

and

$$\hat{U}_i(s, \theta) \geq 0,$$

where

$$\hat{U}_i(s, \theta) \equiv [\theta v + (1 - \theta) v_0 - s] \, Q_i(s).$$

Note that by interpreting $\theta v + (1 - \theta) v_0$ as the bidder's type, s as the bidder's bid, and s_* as the reserve price, we can map this formulation into the standard independent private values framework. Therefore, we can invoke Maskin and Riley (1986) and Riley and Samuelson (1981) to obtain the following result:

Lemma 8 *Let*

$$\theta_* \equiv \inf \left\{ \theta \in [\underline{\theta}, \overline{\theta}] : \theta v + (1 - \theta) v_0 > s_* \right\}.$$

There exists a unique equilibrium in which any bidder with type $\theta \in [\theta_*, \overline{\theta}]$ offers a score of

$$s(\theta) = \theta v + (1 - \theta) v_0 - (v - v_0) \frac{\int_{\theta_*}^{\theta} F^{n-1}(x) dx}{F^{n-1}(\theta)}$$

and any bidder with type $\theta \in [\underline{\theta}, \theta_*)$ offers a score less than s_*.

Since the seller lacks commitment power, the only reserve score that can be sustained is $s_* = v_0$, and since $\underline{\theta}v + (1-\underline{\theta})v_0 \geq v_0$, no type is excluded. Therefore, the equilibrium score function is given by

$$s(\theta) = \theta v + (1-\theta)v_0 - (v - v_0)\frac{\int_{\underline{\theta}}^{\theta} F^{n-1}(x)dx}{F^{n-1}(\theta)}$$

for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$.

The equilibrium score function indicates that higher types offer higher scores, and therefore, the contract will be awarded to the bidder with the lowest risk of default. However, this does not imply that higher types offer higher bids. Using the fact that $s(\theta) = \theta b(\theta) + (1-\theta)(v_0 + c)$ in equilibrium, we derive the equilibrium bidding function specified in the following proposition.

Proposition 3 *There exists a unique equilibrium in which any bidder with type* $\theta \in [\underline{\theta}, \overline{\theta}]$ *bids according to*

$$b(\theta) = v - \left(\frac{1-\theta}{\theta}\right)c - \left(\frac{v - v_0}{\theta}\right)\frac{\int_{\underline{\theta}}^{\theta} F^{n-1}(x)dx}{F^{n-1}(\theta)}.$$

Once again, the first two terms of the bidding function represent the bidder's expected value for the contract while the third term represents the degree to which the bidder shades his bid. However, in this case, c does not appear in the shading term, and as a result, the seller is unable to reduce bidder rents by manipulating the default penalty.

To gain a better understanding of the shading term, we set aside the first-score auction and consider a Bertrand game in which bidders make competing offers. In the absence of commitment, the seller is unable to reject any offer which exceeds his reservation value but cannot be matched by any other bidder. Hence, the best the seller can do is allocate the contract to the bidder with the lowest risk of default, who, in turn, delivers a score equal to $\theta_2 v + (1-\theta_2)v_0$, where θ_2 is the type corresponding to the bidder with the second-lowest risk of default.

Let θ_1 be the type of the bidder whose risk of default is lowest and let p be the selling price of the contract. By setting the bidder's score, $\theta_1 p + (1-\theta_1)(v_0 + c)$, equal to $\theta_2 v + (1-\theta_2)v_0$, we obtain

$$p = v - \left(\frac{1-\theta_2}{\theta_2}\right) c + \left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) c - \left(\frac{\theta_1 - \theta_2}{\theta_1}\right)(v - v_0). \tag{4.2}$$

The first two terms represent the maximum price the bidder with the second-lowest risk of default is willing to pay for the contract, while the third and fourth terms represent the extent to which the bidder with the lowest risk of default can undercut his closest competitor and still win the auction. The third term is driven by the reduction in the expected penalty payment that results from selecting a bidder who is less likely to default. In contrast, the fourth term is associated with the increase in the seller's expected utility that results from selecting a bidder who is more likely to deliver the selling price.

By rewriting equation (4.2) as

$$p = v - \left(\frac{1-\theta_1}{\theta_1}\right) c - \left(\frac{\theta_1 - \theta_2}{\theta_1}\right)(v - v_0),$$

we see that the third term in equation (4.2) is absorbed by the second term of the bidding function in Proposition 3. Hence, the last term of equation (4.2) corresponds to the shading term in the bidding function, which indicates that a bidder in a first-score auction shades his bid to the extent that the seller prefers him over a more risky competitor because he is more likely to deliver his bid.

The existence of this term is contingent on the seller's knowledge of bidder types. In order for a bidder to capitalize on his low risk status, the seller must be aware that the bidder's risk of default is low. More generally, the seller's knowledge of bidder types generates an asymmetry across bidders which dampens price competition. Manipulating the default penalty no longer stimulates price competition because low risk bidders have no need to signal their types. Reducing the penalty does raise bids but only by enough to counteract the associated reduction in the seller's expected penalty revenue.

Before we sum up our results, note that this Bertrand game delivers the same expected utility to the seller as a first-score auction does. Consider a *second-score* auction, in which the object is allocated to the bidder offering the highest score but that bidder delivers only the second-highest score offer. Clearly, the outcome of the Bertrand game is the same as the outcome of the second-score auction. By invoking the revenue equivalence results of Riley and Samuelson (1981), we establish that the second-score auction delivers the same expected utility to the seller as the first-score auction. Therefore, the seller is indifferent between having the bidders submit competing offers and holding a first-score auction.

Proposition 4 *When the seller lacks commitment power and observes each bidder's default risk in advance, the first-score auction, second-score auction, and Bertrand game deliver the same expected utility to the seller.*

Regardless of which of the three mechanisms the seller implements, his knowledge of the bidders' types precludes him from capturing the entire surplus. This implies that in the absence of commitment, the seller prefers not to gather information about each bidder's default risk prior to allocating the contract. Moreover, if circumstances are such that the seller is informed about bidder types in advance, he benefits from being able to commit to a take-it-or-leave-it offer.

Corollary 3 *If the seller lacks commitment power, the value of information about each bidder's default risk is negative.*

Corollary 4 *When the seller observes each bidder's default risk in advance, the value of a commitment device is positive.*

5 Conclusion

After the close of an auction, the winning bidder may find that he is unable to deliver his bid offer. This paper has addressed the question of how the seller maximizes his share of the surplus in the presence of exogenously determined default risk.

It has been shown that the value of gathering information about default risk is negligible when the seller has commitment power. This follows from the fact that the seller can capture nearly the entire surplus even when he is uninformed about each bidder's risk of default. The introduction of a default penalty causes low risk bidders to have a higher expected value for the contract than do their high risk counterparts, which translates into bidders with lower risk submitting higher bids. Moreover, by making the penalty arbitrarily small, the seller minimizes the spread in valuations, thereby stimulating price competition, and minimizing the rents captured by the winning bidder.

On the other hand, when the seller lacks commitment power, the value of gathering information is negative. When the seller is uninformed about each bidder's risk of default, low risk bidders are compelled to signal their low risk status by submitting higher bids. But when the seller elects to inform himself about each bidder's risk, low risk bidders no longer have the need to signal their status. Moreover, they can capitalize on the fact that the seller is biased in their favor by bidding less aggressively than they otherwise would.

This paper has also examined the value of commitment. When the seller is informed about each bidder's risk, the seller's value for a commitment device is positive. In the absence of commitment, low risk bidders capitalize on the bias in their favor by bidding less aggressively. However, when the seller has commitment power, he can make a take-it-or-leave-it offer to the bidder with the lowest risk of default and extract all the surplus. In contrast, when the seller is uninformed, he is able to capture nearly all the surplus – independent of whether or not he has commitment power – by introducing a small but positive default penalty.

A pair of interesting points should be highlighted before closing. First, the existing literature has indicated that high bids are accompanied by high risk. We have shown that to the extent that default risk is exogenously determined, the seller can reverse this relationship by introducing a default penalty. Though the default penalty is constant across bidders, the expected penalty is higher for bidders with sizable exogenous risk and lower for bidders with low exogenous risk. As has been shown in the literature, a lower penalty leads to more aggressive bidding, but it is now the lower risk bidders who face a lower penalty and deliver higher bids.

Second, the best outcome for the seller occurs when the penalty payment is positive but small. This result is driven by the assumption that the value of the contract is common across bidders and may not hold when this assumption is relaxed. Nevertheless, it suggests a tradeoff between raising the penalty so as to achieve a stronger correlation between high expected value and low risk and lowering the penalty so as not to weaken price competition by widening the existing spread of valuations.

One of the stronger assumptions made is that each bidder's risk of default does not vary with the magnitude of the bid submitted. In the future, it may be useful to relax this assumption and allow the likelihood of default to rise with the bid. One way to do this would be to modify existing models in the literature. In these models default is driven by uncertainty about the value of the contract, where the assumption is that the probability of a high realization of the contract's value is independent of which bidder wins the contract. By allowing the probability to vary across bidders, we can incorporate the notion of exogenous sources of risk into existing frameworks and allow the risk of default to vary with the bid submitted and be determined within the context of the model.

A Appendix: Proofs

Proof of Lemma 1: We first show that the lemma does not apply when θ equals zero and then prove the lemma.

Suppose that in equilibrium, bidder i plays the (possibly mixed) strategy $\beta_i : \mathbb{R} \times \left[\underline{\theta}, \overline{\theta}\right] \rightarrow \mathbb{R}_+$. Suppose further that $\hat{b}$ is in the support of $\beta_i(\cdot, \hat{\theta})$, b is in the support of $\beta_i(\cdot, \theta)$, and $P_i(b) > 0$.

By individual rationality,

$$[\theta(v-b) - (1-\theta)c]P_i(b) \geq 0. \tag{A.1}$$

If $\theta = 0$, inequality (A.1) would reduce to $-cP_i(b) \geq 0$. Since $P_i(b) > 0$, c cannot be positive, and since the lemma only applies to cases in which either $c > 0$ and $\hat{\theta} > \theta$ or $c < 0$ and $\hat{\theta} < \theta$, it cannot be the case that $\theta = 0$.

We will now proceed to prove the lemma. Given inequality (A.1) and the assumption that $P_i(b) > 0$, it must be the case that $\theta(v-b) - (1-\theta)c \geq 0$. Since $\theta \neq 0$, we can divide by θ, multiply by $\hat{\theta}$, and add $[(\hat{\theta} - \theta)/\theta]c$ to both sides to obtain

$$\hat{\theta}(v-b) - (1-\hat{\theta})c \geq \left(\frac{\hat{\theta} - \theta}{\theta}\right)c.$$

If either $c > 0$ and $\hat{\theta} > \theta$ or $c < 0$ and $\hat{\theta} < \theta$, then

$$[\hat{\theta}(v-b) - (1-\hat{\theta})c]P_i(b) > 0.$$

By incentive compatibility,

$$\begin{aligned}
[\hat{\theta}(v-\hat{b}) - (1-\hat{\theta})c]P_i(\hat{b}) &\geq [\hat{\theta}(v-b) - (1-\hat{\theta})c]P_i(b) \\
&> 0.
\end{aligned}$$

Therefore, it must be the case that $P_i(\hat{b}) > 0$. $\square$

Proof of Lemma 2: Following the proof of Lemma 1, Lemma 2 does not apply when θ equals zero. Therefore, we assume that $\theta > 0$ for the remainder of this proof.

Since we are constraining ourselves to separating equilibria, $\theta \neq \hat{\theta}$ implies $b \neq \hat{b}$. Hence, it is sufficient to show that if either $c > 0$ and $\hat{\theta} > \theta$ or $c < 0$ and $\hat{\theta} < \theta$, then $\hat{b} \geq b$.

Incentive compatibility requires that

$$[\theta(v-b) - (1-\theta)c]P_i(b) \geq [\theta(v-\hat{b}) - (1-\theta)c]P_i(\hat{b}) \tag{A.2}$$

and

$$[\hat{\theta}(v-\hat{b}) - (1-\hat{\theta})c]P_i(\hat{b}) \geq [\hat{\theta}(v-b) - (1-\hat{\theta})c]P_i(b). \tag{A.3}$$

If $\hat{\theta} \neq 0$, we can combine inequalities (A.2) and (A.3) to obtain

$$\begin{aligned} \tfrac{c}{\theta}\left[P_i(\hat{b}) - P_i(b)\right] &\geq (v+c)\left[P_i(\hat{b}) - P_i(b)\right] + bP_i(b) - \hat{b}P_i(\hat{b}) \\ &\geq \tfrac{c}{\hat{\theta}}\left[P_i(\hat{b}) - P_i(b)\right], \end{aligned}$$

which implies that $c(\hat{\theta} - \theta)[P_i(\hat{b}) - P_i(b)] \geq 0$. If either $c > 0$ and $\hat{\theta} > \theta$ or $c < 0$ and $\hat{\theta} < \theta$, it must be the case that $P_i(\hat{b}) \geq P_i(b) > 0$. If $\hat{\theta} = 0$ and $c < 0$, inequality (A.3) reduces to

$$-cP_i(\hat{b}) \geq -cP_i(b),$$

and again it must be the case that $P_i(\hat{b}) \geq P_i(b) > 0$.

Given this, inequality (A.2) can be rewritten as

$$\hat{b} - b \geq [\theta(v-b) - (1-\theta)c]\frac{P_i(\hat{b}) - P_i(b)}{\theta P_i(\hat{b})}.$$

The right-hand side of the inequality above is nonnegative since by definition of equilibrium, $[\theta(v-b)-(1-\theta)c]P_i(b) \geq 0$. Therefore, it must be the case that $\hat{b} \geq b$. □

Proof of Lemma 3: Suppose $c > 0$. Since the individual rationality constraint (inequality (A.1)) is violated when $c > 0$, $P_i(b) > 0$, and $\theta = 0$, θ cannot be equal to zero. Since $\hat{\theta} > \theta > 0$, we can follow the proof of Lemma 2 and assert that

$$c\left(\hat{\theta}-\theta\right)\left[P_i(\hat{b})-P_i(b)\right] \geq 0. \tag{A.4}$$

$P_i(b)$ cannot equal $P_i(\hat{b})$ because if $P_i(b) = P_i(\hat{b})$, incentive compatibility (inequalities (A.2) and (A.3)) would require that $b = \hat{b}$, which would violate the assumption that the equilibrium is separating. Therefore, it must be the case that $P_i(\hat{b}) > P_i(b)$, meaning that the score associated with a bid of $\hat{b}$ is greater than the score associated with a bid of b:

$$\hat{\theta}\hat{b} + (1-\hat{\theta})(c+v_0) > \theta b + (1-\theta)(c+v_0).$$

Rearranging terms yields the condition outlined in part (i) of the lemma.

Now suppose $c < 0$. If $\theta = 0$, then inequality (A.1) reduces to $-cP_i(b) \geq -cP_i(\hat{b})$, which implies that $P_i(b) \geq P_i(\hat{b})$. Therefore, the score associated with a bid of b is greater than or equal to the score associated with a bid of $\hat{b}$:

$$\theta b + (1-\theta)(c+v_0) \geq \hat{\theta}\hat{b} + (1-\hat{\theta})(c+v_0).$$

Rearranging terms yields the condition outlined in part (ii).

If $c < 0$ and $\theta \neq 0$, we can invoke inequality (A.4) and assert that $P_i(b) \geq P_i(\hat{b})$. Once again, $P_i(b)$ cannot equal $P_i(\hat{b})$ because inequalities (A.2) and (A.3) would imply that $b = \hat{b}$, which would violate the assumption that equilibrium is separating. Therefore, $P_i(b) > P_i(\hat{b})$ and the score associated with b is strictly greater than the score associated with $\hat{b}$:

$$\theta b + (1-\theta)(c+v_0) > \hat{\theta}\hat{b} + (1-\hat{\theta})(c+v_0).$$

Rearranging terms yields the condition outlined in part (ii) with strict inequality. $\square$

Proof of Lemma 4: Suppose $c > 0$. By Lemma 1, there exists $\underline{t} \in \left[\underline{\theta}, \overline{\theta}\right]$ such that bidders with types in $(\underline{t}, \overline{\theta}]$ win the auction with positive probability, while bidders with types in $[\underline{\theta}, \underline{t})$ win with probability zero. Hence, a bidder whose type is in $(\underline{t}, \overline{\theta}]$ beats any bidder whose type is in $[\underline{\theta}, \underline{t})$.

Suppose $\theta, \hat{\theta} \in (\underline{t}, \overline{\theta}]$ and $\theta > \hat{\theta}$. Let b be in the support of $\beta(\cdot, \theta)$ and $\hat{b}$ be in the support of $\beta(\cdot, \hat{\theta})$. By Lemma 3, b and $\hat{b}$ must satisfy

$$\frac{\theta b - \hat{\theta}\hat{b}}{\theta - \hat{\theta}} > c + v_0,$$

which implies that the score offered by a bidder with type θ is larger than the score offered by a bidder with type $\hat{\theta}$:

$$\theta b + (1-\theta)(c+v_0) > \hat{\theta}\hat{b} + (1-\hat{\theta})(c+v_0).$$

Hence, a bidder with type θ beats a bidder with type $\hat{\theta}$.

By combining these two results, we find that if a bidder's type is $\theta \in (\underline{t}, \overline{\theta}]$, the bidder wins the auction if the type of every other bidder is strictly less than θ and loses the auction if there exists a bidder whose type is strictly greater than θ. Since the distribution of types,

F, has positive density, f, we can say that a bidder with type $\theta \in (\underline{t}, \overline{\theta}]$ wins the auction with probability $F^{n-1}(\theta)$.

Let $\theta \in (\underline{t}, \overline{\theta}]$ and $\beta(\cdot, \theta)$ be a mixed strategy. Suppose the support of $\beta(\cdot, \theta)$ includes not only b but also $b' \neq b$. Since a bidder with type θ should be indifferent among bids in the support of $\beta(\cdot, \theta)$, it must be the case that

$$[\theta(v-b) - (1-\theta)c]F^{n-1}(\theta) = [\theta(v-b') - (1-\theta)c]F^{n-1}(\theta).$$

Since $F^{n-1}(\theta) > 0$ ($P_i(b)$ is strictly positive by assumption), the bidder is not indifferent between b and b' – a contradiction.

An analogous argument can be used to show that $\beta(\cdot, \theta)$ must be a pure strategy when $c < 0$ and $\theta \neq 0$. $\square$

Proof of Lemma 5: Suppose $c > 0$ but $b : (\underline{t}, \overline{\theta}] \to \mathbb{R}$ is not continuous at some $\theta \in (\underline{t}, \overline{\theta}]$. Then, for some $\epsilon > 0$, there is no $\delta > 0$ such that

$$\hat{\theta} \in (\underline{t}, \overline{\theta}] \text{ and } \left|\hat{\theta} - \theta\right| < \delta \Rightarrow \left|b(\hat{\theta}) - b(\theta)\right| < \epsilon.$$

By Lemma 2, b is strictly increasing on $(\underline{t}, \overline{\theta}]$. Therefore, we can restate the discontinuity condition as follows: there exists $\epsilon > 0$ such that either

$$b(\theta) - b(\hat{\theta}) \geq \epsilon, \quad \forall \hat{\theta} \in (\underline{t}, \theta) \tag{A.5}$$

or

$$b(\hat{\theta}) - b(\theta) \geq \epsilon, \quad \forall \hat{\theta} \in \left(\theta, \overline{\theta}\right]. \tag{A.6}$$

Incentive compatibility for a bidder with type θ requires that

$$[\theta(v - b(\theta)) - (1-\theta)c]F^{n-1}(\theta) \geq [\theta(v - b(\hat{\theta})) - (1-\theta)c]F^{n-1}(\hat{\theta})$$

for all $\hat{\theta} \in (\underline{t}, \overline{\theta}]$. If condition (A.5) holds, then

$$\left[\theta(v - b(\hat{\theta})) - (1-\theta)c\right]\left[F^{n-1}(\theta) - F^{n-1}(\hat{\theta})\right] \geq \theta F^{n-1}(\theta)\epsilon \tag{A.7}$$

for all $\hat{\theta} \in (\underline{t}, \theta)$. Since F is continuous on $\left[\underline{\theta}, \overline{\theta}\right]$, F^{n-1} is also continuous on $\left[\underline{\theta}, \overline{\theta}\right]$, which implies that for all $\epsilon' > 0$, there exists $\delta' > 0$ such that

$$\hat{\theta} \in (\underline{t}, \theta) \text{ and } \theta - \hat{\theta} < \delta' \Rightarrow F^{n-1}(\theta) - F^{n-1}(\hat{\theta}) < \epsilon'.$$

That is, $F^{n-1}(\theta) - F^{n-1}(\hat{\theta})$ can be brought arbitrarily close to zero by selecting a $\hat{\theta}$ sufficiently close to θ. Furthermore, since $\theta \in (\hat{\theta}, \overline{\theta}]$ and $\epsilon > 0$ is fixed, $\theta F^{n-1}(\theta)\epsilon$ is both positive and fixed. Therefore, inequality (A.7) is violated for some $\hat{\theta}$ sufficiently close to θ, and condition (A.5) cannot hold.

Now suppose condition (A.6) holds. Incentive compatibility for a bidder with type $\hat{\theta}$ requires that

$$[\hat{\theta}(v - b(\hat{\theta})) - (1-\hat{\theta})c]F^{n-1}(\hat{\theta}) \geq [\hat{\theta}(v - b(\theta)) - (1-\hat{\theta})c]F^{n-1}(\theta)$$

for all $\hat{\theta} \in (\underline{t}, \overline{\theta}]$. By imposing condition (A.6), we obtain

$$\begin{aligned}\left[\hat{\theta}(v - b(\theta)) - (1-\hat{\theta})c\right]\left[F^{n-1}(\hat{\theta}) - F^{n-1}(\theta)\right] &\geq \hat{\theta}F^{n-1}(\hat{\theta})\epsilon \\ &> \theta F^{n-1}(\theta)\epsilon\end{aligned} \tag{A.8}$$

for all $\hat{\theta} \in (\theta, \overline{\theta}]$. As before, $F^{n-1}(\hat{\theta}) - F^{n-1}(\theta)$ can be brought arbitrarily close to zero by selecting a $\hat{\theta}$ sufficiently close to θ, and since $\theta F^{n-1}(\theta)\epsilon$ is both positive and fixed, inequality (A.8) is violated for some $\hat{\theta}$ sufficiently close to θ.

Since neither condition (A.5) nor condition (A.6) holds when $c > 0$, b must be continuous over $(\underline{t}, \overline{\theta}]$. An analogous argument can be used to verify parts (ii) and (iii). $\square$

Proof of Lemma 6: Suppose $c > 0$ and $\underline{t} > 0$. Our approach will be to first establish a boundary condition by showing that

$$\lim_{\theta \to \underline{t}^+} b(\theta) = v + c - \frac{c}{\underline{t}}$$

and then use this condition to derive the bid function.

By Lemma 5, b is continuous, and hence, the limit exists. Suppose $\lim_{\theta \to \underline{t}^+} b(\theta) \neq v + c - c/\underline{t}$. Then there exists an $\epsilon > 0$ such that either

$$\lim_{\theta \to \underline{t}^+} b(\theta) > v + c - \frac{c}{\underline{t}} + \epsilon \tag{A.9}$$

or

$$\lim_{\theta \to \underline{t}^+} b(\theta) < v + c - \frac{c}{\underline{t}} - \epsilon. \tag{A.10}$$

Suppose Condition (A.9) holds and consider a bidder with type $\theta = \underline{t} + \delta$, where δ is small and positive. By Lemma 2, the bidding function b is strictly increasing for all $\theta > \underline{t}$, and hence, the bidder submits a bid $b(\theta) > v + c - c/\underline{t} + \epsilon$. Furthermore, since $\theta > \underline{t}$, the bidder's probability of winning is $F^{n-1}(\theta) > 0$. Thus, the bidder's expected utility is

$$\begin{aligned} [\theta(v - b(\theta)) - (1-\theta)c]\, F^{n-1}(\theta) &< [\theta(-c + c/\underline{t} - \epsilon) - (1-\theta)c]\, F^{n-1}(\theta) \\ &= [(\delta/\underline{t})c - (\underline{t} + \delta)\epsilon]\, F^{n-1}(\underline{t} + \delta). \end{aligned}$$

If δ is sufficiently close to zero, $(\delta/\underline{t})c - (\underline{t} + \delta)\epsilon$ is negative, and individual rationality is violated.

Now suppose Condition (A.10) holds and $\underline{t} > \underline{\theta}$. Consider a bidder with type $\theta = \underline{t} - \delta$, where δ is small and positive. Since $\theta < \underline{t}$, the bidder does not win the auction and earns utility of zero. By Lemma 5, b is continuous. Therefore, there exists a $x > \underline{t}$ such that $b(x) < v + c - c/\underline{t} - \epsilon$. If the bidder deviates to $b(x)$, his expected utility is

$$\begin{aligned}
[\theta(v - b(x)) - (1-\theta)c]\, F^{n-1}(x) &> [\theta(-c + c/\underline{t} + \epsilon) - (1-\theta)c]\, F^{n-1}(x) \\
&= [(\underline{t} - \delta)\epsilon - (\delta/\underline{t})c]\, F^{n-1}(x).
\end{aligned}$$

If δ is sufficiently close to zero, $(\underline{t} - \delta)\epsilon - (\delta/\underline{t})c$ is positive, making the deviation to $b(x)$ profitable.

And finally, suppose Condition (A.10) holds and $\underline{t} = \underline{\theta}$. Consider a bidder with type $\theta = \underline{t}$. Since $\underline{t} = \underline{\theta}$, his probability of winning the auction is zero; therefore, he earns utility of zero. By Lemma 5, b is continuous. Therefore, there exists a $x > \underline{t}$ such that $b(x) < v + c - c/\underline{t} - \epsilon$. If the bidder deviates to $b(x)$, his expected utility is

$$\begin{aligned}
[\theta(v - b(x)) - (1-\theta)c]\, F^{n-1}(x) &> [\theta(-c + c/\underline{t} + \epsilon) - (1-\theta)c]\, F^{n-1}(x) \\
&= (\underline{t}\epsilon)\, F^{n-1}(x) \\
&> 0.
\end{aligned}$$

Therefore, the deviation to $b(x)$ profitable.

We have shown that $\lim_{\theta \to \underline{t}^+} b(\theta) = v + c - c/\underline{t}$. We now proceed with our derivation of the bidding function, $b(\cdot)$.

Consider a bidder with type $\theta \in (\underline{t}, \overline{\theta}]$. If the bidder offers a bid of $b(x)$, where $x \in (\underline{t}, \overline{\theta}]$, then the bidder's expected utility is

$$U(x, \theta) \equiv [\theta(v - b(x)) - (1-\theta)c]\, F^{n-1}(x).$$

In equilibrium, $b(\cdot)$ must satisfy the following conditions for all $\theta \in (\underline{t}, \overline{\theta})$:

$$\text{Global IC}: \quad U(\theta,\theta) \geq U(x,\theta), \quad \forall x \in (\underline{t},\overline{\theta}];$$

and

$$\text{Local IC}: \quad U_x(\theta,\theta) = 0.$$

Since b is continuous, Global IC implies Local IC for all $\theta \in (\underline{t},\overline{\theta})$. Taking the derivative of $U(x,\theta)$ with respect to x, substituting θ for x, and setting the resulting expression equal to zero yields

$$\frac{db(\theta)}{d\theta}F^{n-1}(\theta) + b(\theta)\frac{dF^{n-1}(\theta)}{d\theta} = \left(v + c - \frac{c}{\theta}\right)\frac{dF^{n-1}(\theta)}{d\theta}.$$

Integrating both sides and evaluating the integrals from $\underline{t}$ to θ yields

$$b(\theta)F^{n-1}(\theta) - \lim_{x\to\underline{t}^+} b(x)F^{n-1}(x) =$$

$$\left(v + c - \tfrac{c}{\theta}\right)F^{n-1}(\theta) - \left(v + c - \tfrac{c}{\underline{t}}\right)F^{n-1}(\underline{t}) - c\int_{\underline{t}}^{\theta} \tfrac{1}{x^2}F^{n-1}(x)dx.$$

Using the boundary condition ($\lim_{x\to\underline{t}^+} b(x) = v + c - c/\underline{t}$) and solving for $b(\theta)$ gives the bidding function

$$b(\theta) = v - \left(\frac{1-\theta}{\theta}\right)c - \frac{c\int_{\underline{t}}^{\theta}\frac{1}{x^2}F^{n-1}(x)dx}{F^{n-1}(\theta)}$$

for $\theta \in (\underline{t},\overline{\theta})$. Since $b(\cdot)$ is continuous over $(\underline{t},\overline{\theta}]$, the bidding function specified gives the equilibrium bid for type $\overline{\theta}$ as well.

An analogous argument can be used to show that if $c < 0$, the bidding function is

$$b(\theta) = v - \left(\frac{1-\theta}{\theta}\right)c + \frac{c\int_{\theta}^{\overline{t}}\frac{1}{x^2}F^{n-1}(x)dx}{F^{n-1}(\theta)}$$

for $\theta \in (\underline{\theta}, \bar{t})$. By Lemma 5, if $\underline{\theta} > 0$, $b(\cdot)$ is continuous over $[\underline{\theta}, \bar{t})$, and therefore the bidding function gives the equilibrium bid for a bidder with type $\underline{\theta}$ as well. $\square$

Proof of Lemma 7: We will prove each part in succession.

Proof of (i). Let $\gamma > t$ and suppose there exists a symmetric separating equilibrium. Since $c > 0$, Lemma 1 indicates that there exists a threshold type such that if the type drawn exceeds this threshold, the bidder's probability of winning is positive but if the type drawn is less than this threshold, the bidder's probability of winning is zero. Let $\underline{t} \in \left[\underline{\theta}, \overline{\theta}\right]$ represent this threshold. Lemma 3 requires that

$$\frac{d\left[\theta b(\theta)\right]}{d\theta} > c + v_0 \tag{A.11}$$

for all $\theta \in \left(\underline{t}, \overline{\theta}\right]$. By applying the bidding function outlined in Lemma 6 and making a substitution using $c = \gamma(v - v_0)$, we obtain

$$\begin{aligned}
\lim_{\theta \to \underline{t}^+} \tfrac{d[\theta b(\theta)]}{d\theta} &= \lim_{\theta \to \underline{t}^+} \theta b'(\theta) + b(\theta) \\
&= v - \left(\tfrac{1-\underline{t}}{\underline{t}}\right) c \\
&= c + v_0 + \left(\tfrac{1}{\gamma} - \tfrac{1}{\underline{t}}\right) c.
\end{aligned}$$

If $\underline{t} < \gamma$, then $\lim_{\theta \to \underline{t}^+} \theta b'(\theta) + b(\theta)$ is strictly less than $c + v_0$. Since $\theta b'(\theta) + b(\theta)$ is continuous over $\left(\underline{t}, \overline{\theta}\right]$, there exists an $\epsilon > 0$ such that

$$\left(\underline{t} + \epsilon\right) b'\left(\underline{t} + \epsilon\right) + b\left(\underline{t} + \epsilon\right) < c + v_0,$$

which clearly contradicts inequality (A.11). Therefore, it must be the case that $\underline{t} \geq \gamma$.

Suppose $\underline{t}$ is strictly greater than γ and let $b = c + v_0$. Note that the score associated with b is equal to $c + v_0$ for any belief the seller may hold about b. Since

$$\begin{aligned} c + v_0 &= \gamma v + (1-\gamma)v_0 \\ &> tv + (1-t)v_0, \end{aligned}$$

the score associated with b exceeds the minimum score set by the seller. Consider a bidder with type $\theta \in (\gamma, \underline{t})$. Since the bidder's type is strictly less than the threshold type $\underline{t}$, the bidder is never awarded the contract in equilibrium and earns zero utility. However, if the bidder bids b, his probability of winning is $F^{n-1}(\underline{t})$ since his score exceeds the minimum score but falls short of the scores offered by bidders with types in $\left(\underline{t}, \overline{\theta}\right]$. Therefore, his expected utility is

$$\begin{aligned} \left[\theta(v-b) - (1-\theta)c\right] F^{n-1}(\underline{t}) &= (\theta-\gamma)(v-v_0)F^{n-1}(\underline{t}) \\ &> 0. \end{aligned}$$

Since b delivers positive utility, it is a profitable deviation for the bidder. Hence, it cannot be the case that $\underline{t} > \gamma$.

Finally, suppose $\underline{t} = \gamma$. Since $\underline{t} = \gamma$ and $c = \gamma(v - v_0)$,

$$\begin{aligned} \lim_{\theta \to \underline{t}^+} b(\theta) &= v - \left(\tfrac{1-\underline{t}}{\underline{t}}\right) c \\ &= c + v_0. \end{aligned}$$

Let $b = c + v_0 - \epsilon$, where $\epsilon > 0$. The score associated with b is given by

$$\begin{aligned} E(\theta|b)b + \left[1 - E(\theta|b)\right](c+v_0) &= E(\theta|b)\left[b - (c+v_0)\right] + (c+v_0) \\ &= E(\theta|b)[-\epsilon] + \gamma v + (1-\gamma)v_0. \end{aligned}$$

For any belief $E(\theta|b) \in \left[\underline{\theta}, \overline{\theta}\right]$, there exists an ϵ such that the score associated with b is greater than the minimum score, $tv + (1-t)v_0$. We fix ϵ accordingly. Now consider a bidder with type $\theta = \underline{t} - \delta$, where $\delta > 0$. Since the bidder's type is strictly less than the threshold type, the bidder is never awarded the contract in equilibrium and earns zero utility. However, if the bidder bids b, his probability of winning is $F^{n-1}(\underline{t})$ since his score exceeds the minimum score but falls short of the scores offered by bidders with types in $\left(\underline{t}, \overline{\theta}\right]$. Therefore, his expected utility is

$$\begin{aligned}
[\theta(v-b)-(1-\theta)c]\, F^{n-1}(\underline{t}) &= [(\underline{t}-\delta)(v-v_0+\epsilon)-c]\, F^{n-1}(\underline{t}) \\
&= [\gamma\epsilon-\delta(v-v_0+\epsilon)]\, F^{n-1}(\underline{t})\,.
\end{aligned}$$

Since ϵ is positive and fixed, we can select δ such that it satisfies $\gamma\epsilon-\delta(v-v_0+\epsilon)>0$. Since b delivers positive expected utility to a bidder with type $\underline{t}-\delta$, it is a profitable deviation for that bidder. Hence, it cannot be the case that $\underline{t}=\gamma$.

Proof of (ii). Our approach will be to derive the equilibrium expected utility for a bidder with type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and then show that there is no profitable deviation available to that bidder.

Consider a bidder with type $\theta = 0$. Since such a bidder defaults with probability one, the bidder's utility is $-c$ if he wins the auction and zero if he does not. Since his reservation utility is zero, individual rationality requires that the bidder never be awarded the contract in equilibrium.

Consider a bidder with type $\theta \in [\underline{\theta}, t)$ and suppose $\theta > 0$. Since the equilibrium is separating, the seller can infer θ from $b(\theta)$, and the bidder's score is given by

$$\begin{aligned} s(\theta) &= \theta b(\theta) + (1-\theta)(c+v_0) \\ &< \theta b(t) + (1-\theta)(c+v_0) \\ &\leq t b(t) + (1-t)(c+v_0) \\ &= tv + (1-t)v_0. \end{aligned}$$

The first inequality follows from the fact that $b(\theta) < b(t)$ if $\theta \in [\underline{\theta}, t)$. The second inequality follows from the fact that $b(t) \geq c + v_0$ if $c \leq t(v - v_0)$. Since $s(\theta)$ is less than the minimum score, the bidder will not be awarded the contract. Hence, any bidder with type $\theta \in [\underline{\theta}, t)$ earns utility zero in equilibrium.

Consider a bidder with type $\theta \in \left[t, \overline{\theta}\right]$. Since $b(\theta)$ is strictly increasing over $\left[t, \overline{\theta}\right]$ and $b(t) \geq c + v_0$, scores are strictly increasing in θ over $\left[t, \overline{\theta}\right]$. Therefore, the bidder's probability of winning the contract is $F^{n-1}(\theta)$, and his expected utility can be written as

$$\left[\theta(v - b(\theta)) - (1-\theta)c\right] F^{n-1}(\theta).$$

We now show that no bidder has an incentive to deviate to another bid on the equilibrium path. Suppose a bidder has type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. If this bidder deviates to a bid $b(x)$, where $x \in [\underline{\theta}, t)$, then his resulting utility is zero, which does not improve upon his equilibrium expected utility. Suppose a bidder has type $\theta \in \left[t, \overline{\theta}\right]$. If he deviates to a bid of $b(x)$, where $x \in \left[t, \overline{\theta}\right]$, then his expected utility is

$$\left[\theta(v - b(x)) - (1-\theta)c\right] F^{n-1}(x).$$

Substituting for $b(x)$ and taking the derivative with respect to x yields

$$c\left(\frac{\theta - x}{x}\right)\frac{dF^{n-1}(x)}{dx},$$

which is positive for $x < \theta$ and negative for $x > \theta$. Therefore, bidding $b(x)$ delivers lower expected utility than bidding $b(\theta)$. Finally, suppose a bidder has type $\theta \in [\underline{\theta}, t)$. If he deviates to a bid of $b(x)$, where $x \in \left[t, \overline{\theta}\right]$, then his expected utility is

$$\begin{aligned} \left[\theta(v - b(x)) - (1-\theta)c\right] F^{n-1}(x) &\leq \left[\theta(v - b(t)) - (1-\theta)c\right] F^{n-1}(x) \\ &= c\left(\tfrac{\theta - t}{t}\right) F^{n-1}(x) \\ &< 0, \end{aligned}$$

where the first inequality follows from the fact that $b(\cdot)$ is increasing over $\left[t, \overline{\theta}\right]$. Since the expected utility associated with bidding $b(x)$ is less than zero, $b(x)$ is not a profitable deviation.

We now show that no bidder has an incentive to deviate to a bid off the equilibrium path. Consider the deviating bid $b > b(\overline{\theta})$. A bidder with type $\theta = 0$ has no incentive to deviate to b since his utility is $-c$ if he wins the auction and zero otherwise. Suppose the bidder has type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, where $\theta > 0$. If he does not win the auction, his utility is zero, and he gains nothing. If he does win the auction, his expected utility is $\theta(v-b)-(1-\theta)c$, which is strictly less than his expected utility from bidding $b(\overline{\theta})$. Since $b(\overline{\theta})$ is not a profitable deviation, b is not a profitable deviation either.

Now consider the deviating bid $b < b(t)$. If $E\left(\theta_i | b_i\right) \in (\underline{\theta}, t]$ when $b_i = b$, then the score associated with b is

$$\begin{aligned} E(\theta|b)b + (1 - E(\theta|b))(c + v_0) &< E(\theta|b)b(t) + (1 - E(\theta|b))(c + v_0) \\ &\leq tb(t) + (1-t)(c+v_0) \\ &= tv + (1-t)v_0. \end{aligned}$$

Since the score associated with b is less than the minimum score, the contract is never awarded to a bidder offering a bid of b. Therefore, b cannot be a profitable deviation.

Proof of (iii). Our approach will be to derive the equilibrium expected utility for a bidder with type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and then show that there is no profitable deviation available to that bidder.

If the bidder draws type $\underline{\theta}$, he wins the contract with probability one but earns zero utility in expectation. If the bidder draws type $\theta > \underline{\theta}$, his equilibrium score is

$$\begin{aligned} s(\theta) &= \theta b(\theta) + (1-\theta)(c+v_0) \\ &< \theta \left[v - \left(\tfrac{1-\theta}{\theta}\right) c - \left(\tfrac{\theta - \underline{\theta}}{\theta}\right)(v - v_0)\right] + (1-\theta)(c+v_0) \\ &= \underline{\theta} v + (1-\underline{\theta}) v_0. \end{aligned}$$

Since $s(\theta)$ is less than the minimum score, the bidder does not win the contract and therefore, earns utility zero.

We now show that no bidder has an incentive to deviate to another bid on the equilibrium path. Suppose a bidder has type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. If this bidder deviates to a bid $b(x)$, where $x > \underline{\theta}$, he earns utility zero, which does not improve upon his equilibrium expected utility. Now suppose a bidder has type $\theta > \underline{\theta}$. If the bidder deviates to $b\left(\underline{\theta}\right)$, he wins the auction with probability one and earns expected utility of

$$\begin{aligned} \theta(v - b(\underline{\theta})) - (1-\theta)c &= \left(\tfrac{\theta - \underline{\theta}}{\underline{\theta}}\right) c \\ &< 0. \end{aligned}$$

Since deviating to $b(\underline{\theta})$ delivers negative utility in expectation, it is not a profitable deviation.

Finally, we show that no bidder has an incentive to deviate to a bid off the equilibrium path. Consider a bidder with type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and a deviating bid $b > b(\underline{\theta})$. If the bidder does not win the auction, his utility is zero, and he gains nothing. If the bidder does win the auction, his expected utility is

$$\begin{aligned}\theta(v-b)-(1-\theta)c &< \theta(v-b(\underline{\theta}))-(1-\theta)c \\ &\leq 0.\end{aligned}$$

Since $b > b(\underline{\theta})$ does not deliver positive expected utility, it is not a profitable deviation.

Now consider the deviating bid $b < b(\underline{\theta})$. Since the seller's off-equilibrium-path beliefs are such that $E(\theta_i|b_i) = \underline{\theta}$ when $b_i < b(\underline{\theta})$, the score associated with b is

$$\begin{aligned}\underline{\theta}b+(1-\underline{\theta})(c+v_0) &< \underline{\theta}b+(1-\underline{\theta})(c+v_0) \\ &= \underline{\theta}v+(1-\underline{\theta})v_0.\end{aligned}$$

Since bidding b delivers a score that falls short of the minimum score, the bidder is not awarded the contract and earns utility zero. Therefore, $b < b(\underline{\theta})$ is not a profitable deviation.

Proof of (iv). Since $\gamma < 0$, c is negative. By Lemma 1, there exists a threshold type such that if the type drawn exceeds this threshold, the bidder's probability of winning is zero but if the type drawn is less than this threshold, the bidder's probability of winning is positive.

Suppose $t > \underline{\theta}$. Consider a bidder with type $\underline{\theta}$, where $\underline{\theta} > 0$. If the bidder's probability of winning is positive in equilibrium, then individual rationality requires that his equilibrium bid satisfy $b\left(\underline{\theta}\right) \leq v - \left(\frac{1-\underline{\theta}}{\underline{\theta}}\right)c$. Therefore, the bidder's equilibrium score is

$$\begin{aligned}s(\underline{\theta}) &= \underline{\theta}b\left(\underline{\theta}\right)+\left(1-\underline{\theta}\right)\left(c+v_0\right) \\ &\leq \underline{\theta}v+\left(1-\underline{\theta}\right)v_0 \\ &< tv+(1-t)v_0.\end{aligned}$$

Since the bidder's score is strictly less than the minimum score, there is no symmetric separating equilibrium in this case. Now consider a bidder with type $\underline{\theta} = 0$. The bidder's score is $c + v_0$ independent of the bid he submits. Since this score is less than the minimum score, there is no symmetric separating equilibrium in this case either. Therefore, there is no symmetric separating equilibrium when $t > \underline{\theta}$.

Suppose $t = \underline{\theta} = 0$. In this case, the minimum score is v_0. Once again, consider a bidder with type $\underline{\theta} = 0$. Since the bidder's score is $c + v_0 < v_0$, there is no symmetric separating equilibrium when $t = \underline{\theta} = 0$. $\square$

Proof of Proposition 2: Our approach will be to derive the equilibrium expected utility for a bidder with type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and then show that there is no profitable deviation available to that bidder.

Suppose $\underline{\theta} > 0$ and consider a bidder with type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. Since $b(\theta)$ is strictly increasing over $\left[\underline{\theta}, \overline{\theta}\right]$ and $b(\underline{\theta}) \geq c + v_0$, scores are strictly increasing in θ over $\left[\underline{\theta}, \overline{\theta}\right]$. Therefore, the bidder's probability of winning the contract is $F^{n-1}(\theta)$, and his expected utility can be written as

$$[\theta(v - b(\theta)) - (1 - \theta)c]\, F^{n-1}(\theta).$$

We proceed by showing that no bidder has an incentive to deviate to another bid on the equilibrium path. Suppose a bidder has type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. If this bidder deviates to a bid $b(x)$, where $x \in \left[\underline{\theta}, \overline{\theta}\right]$, then his expected utility is

$$[\theta(v - b(x)) - (1 - \theta)c]\, F^{n-1}(x).$$

Substituting for $b(x)$ and taking the derivative with respect to x yields

$$c\left(\frac{\theta - x}{x}\right)\frac{dF^{n-1}(x)}{dx},$$

which is positive for $x < \theta$ and negative for $x > \theta$. Therefore, bidding $b(x)$ delivers lower expected utility than bidding $b(\theta)$.

We now show that no bidder has an incentive to deviate to a bid off the equilibrium path. Suppose a bidder with type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ deviates to a bid $b > b(\overline{\theta})$. If he does not win the auction, his utility is zero, and he gains nothing. If he does win the auction, his expected utility is $\theta(v-b)-(1-\theta)c$, which is strictly less than his expected utility from bidding $b(\overline{\theta})$. Since $b(\overline{\theta})$ is not a profitable deviation, b is not a profitable deviation either. Now suppose the bidder deviates to a bid $b < b(\underline{\theta})$. Since $E\left(\theta_i|b_i\right) = \underline{\theta}$ when $b_i = b$, the score associated with b is

$$\begin{aligned} E(\theta|b)b + (1 - E(\theta|b))(c+v_0) &< E(\theta|b)b(\underline{\theta}) + (1 - E(\theta|b))(c+v_0) \\ &\leq \underline{\theta}b(\underline{\theta}) + (1-\underline{\theta})(c+v_0) \\ &= \underline{\theta}v + (1-\underline{\theta})v_0. \end{aligned}$$

Since the score associated with b is less than the lowest score offered in equilibrium, the contract is never awarded to a bidder offering a bid of b. Therefore, b cannot be a profitable deviation.

References

[1] Bids Listed for PDVSA's Field Reactivation Program (1992), *Oil & Gas Journal*, 90(11), 36.

[2] Branco, F. (1997), The Design of Multidimensional Auctions, *RAND Journal of Economics*, 28(1), 63-81.

[3] Cantillon, E. (forthcoming), The Effect of Bidders' Asymmetries on Expected Revenue in Auctions, *Games and Economic Behavior*.

[4] Chalot, J. P. (1996, October 31), Personal Interview, Petróleos de Venezuela, S.A.

[5] Che, Y. -K. (1993), Design Competition through Multidimensional Auctions, *RAND Journal of Economics*, 24(4), 668-680.

[6] Congressional Budget Office (1997, September), Impending Defaults by Winning Bidders in the FCC's C Block Auction: Issues and Options, CBO Memorandum.

[7] eBay (viewed 2005a, April 6), Bids over $15,000, http://pages.ebay.com/help/buy/bid-largebids.html.

[8] eBay (viewed 2005b, April 2), Evaluating a Member's Reputation, http://pages.ebay.com/help/feedback/evaluating-feedback.html.

[9] eBay (viewed 2005c, April 2), Item number: 5568932078, http://cgi.ebay.com/ws/eBayISAPI.dll?ViewItem&item=5568932078&
rd=1&sspagename=STRK

[10] Goeree, J. (2003), Bidding for the Future: Signaling in Auctions with an Aftermarket, *Journal of Economic Theory*, 108(2), 345-364.

[11] Hansen, R. G. and J. R. Lott, Jr. (1991), The Winner's Curse and Public Information in Common Value Auctions: Comment, *American Economic Review*, 81(1), 347-361.

[12] Harstad, R. M. and M. H. Rothkopf (1995), Withdrawable Bids as Winner's Curse Insurance, *Operations Research*, 43(6), 983-994.

[13] Katzman, B. and M. Rhodes-Kropf (2002), The Consequences of Information Revealed in Auctions, mimeo, Columbia University.

[14] Lamping, J. (2007), Ignorance Is Bliss: Matching in Auctions with an Uninformed Seller, mimeo, University of Colorado at Boulder.

[15] Maskin, E. and J. Riley (1986), Existence and Uniqueness of Equilibrium in Sealed High Bid Auctions, Discussion Paper 407, University of California, Los Angeles.

[16] Milgrom, P. and R. Weber (1982), The Value of Information in a Sealed Bid Auction, *Journal of Mathematical Economics*, 10(1), 105-114.

[17] Rezende, L. (2006). Biased Procurement Auctions, mimeo, University of Illinois.

[18] Riley, J. and W. Samuelson (1981), Optimal Auctions, *American Economic Review*, 71(3), 381-392.

[19] Spulber, D. F. (1990), Auctions and Contract Enforcement, *Journal of Law, Economics, and Organization*, 6(2), 325-344.

[20] Waehrer, K. (1995), A Model of Auction Contracts with Liquidated Damages, *Journal of Economic Theory*, 67(2), 531-555.

[21] Zheng, C. Z. (2000), Optimal Auction in a Multidimensional World, Discussion Paper 1282, Northwestern University, Center for Mathematical Studies in Economics and Management Science.

[22] Zheng, C. Z. (2001), High Bids and Broke Winners, *Journal of Economic Theory*, 100(1), 129-171.