Quantifying Flexibility Real Options Calculus

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Lecture Notes

Abstract

We expose a real options theory as a tool for quantifying the value of the operating flexibility of real assets. Additionally, we have pointed out that this theory is an appropriated methodology for determining optimal operating policies, and provide an example of successful application of our approach to power industries, specifically to valuate the power plant of electricity. In particular by increasing the volatility of prices will eventually lead to higher assets values.

I. Introduction

As is well known not a long time has passed from the date when Black and Scholes had published their breaking paper [1]. But their model has had a significant impact on the developments of derivatives market. After that much interest in option pricing has been generated from the development of new options markets. The rapid development of theory and consequently its diverse application to the option pricing problem occurred after that. Since throughout the paper we will deal with the concept of call options, let us first say something fundamental about the term “Call Options”. We will need throughout the whole paper to use this term in many different modes. Let us assume that there are two parties. These parties need to sign an agreement that is a financial contract between them. So, we have the buyer and the seller that have in common an option. It is an option to buy shares of stock at a specified time in the future. Often it is simple called an “option”. The buyer has the right but not the obligation to buy a given quantity of stock at a given price on or before a given date from the seller of the option of a particular commodity or financial instrument. The seller or also called as “writer” is obligated to sell the commodity or financial instrument should the buyer so decide. The buyer pays the fee, named also “premium” for this right. There are markets in call options on stocks, commodities, currencies, stock indexes, futures and interest rates. Specific options are priced differently but their common features can be restricted to the following definition. The given quantity is fixed and is usually either 100 units or 1,000 units. The given price is known as the exercise price or strike price (K). The given date is known as the expiry date (T). Often the stock underlying the option is referred to as the underlying with the price $S(t)$. The “option” price (value) we denote as $C(t)$.

Exchange traded stock options are listed with three, six and nine months of life and various strike prices. According their features of the end transactions we will have the following different possibilities.
In-the-money is an option whose strike price is below the current stock price.
At-the-money is an option whose strike price is close to the current stock price.
Out-of-the-money an option whose strike price is above the current stock price.
Most exchanges continually list options and there are all three types of options for each expiry cycle.
American option is that defined earlier, can be exercised on or before maturity.
European option can be only exercised on the maturity date.
Usually their prices only slightly differ.
Intrinsic value is the difference between the current stock price and the strike price.
Time value is the difference between the option price and the current stock price and is the money the investor has at risk if the stock price stays constant.

Trees and Black-Scholes Approach

Let us discuss now the so-called tree approach to option pricing and its connection to continuous Black-Scholes one. Later on we assume that the interest rates $r \neq 0$.
Due to asymmetry and no-arbitrage one can see that

$$\max(S-K,0) \le C(t) \le S(t)$$

that is presented at the graph. 1

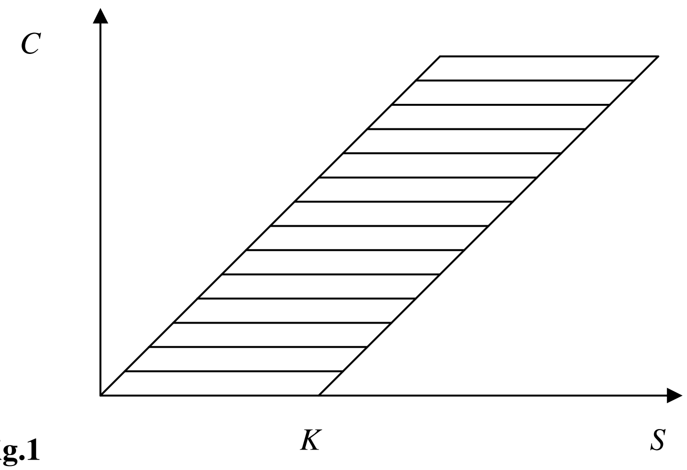
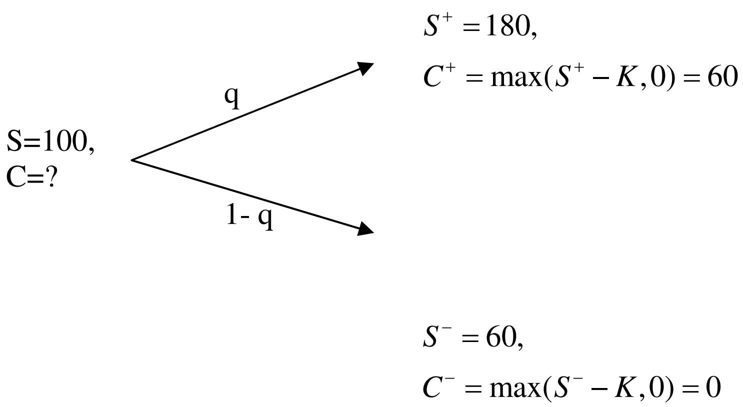


Fig.1

The parameters necessary to calculate the option price *C* are *K*, the strike price; *S(0), r*
S(T) = S $\pm$ *dS,* (spot price) with no probability needed for jumping up or down.

Black-Scholes pricing formula through Cox, Ross & Rubinstein tree.

For better explaining this approach let us consider a simple example of the binomial tree for stock dynamics with $r = 0.08$, $K = \$120$. So, the diagram for obtaining the value of called option C is



Next we can build a portfolio: Long N shares at \$100 (current price). Borrow \$$B$ amount of money at r = 0.25. The net out-of-pocket cost is $NS - B$. We consider this portfolio as a replica of the option C:

$$C \cong (NS - B)$$

This portfolio gives the same return as the call option at the cycle end. So now we have the tree

$$\begin{tikzcd}[column sep=large] & NS^{+} - (1+r)B \cong C^{+} \\ C \cong NS - B \arrow[ur] \arrow[dr] & \\ & NS^{-} - (1+r)B \cong C^{-} \end{tikzcd}$$

As can be easily calculated the solutions at the end of the cycle are

$$N = \frac{C^{+} - C^{-}}{S^{+} - S^{-}} = \frac{60 - 0}{160 - 60} = 0.5 = \Delta$$
$$B = \frac{S^{-}C^{+} - S^{+}C^{-}}{S^{+} - S^{-}} . \frac{1}{1+r} = \frac{NS^{-} - C^{-}}{1+r} = \frac{0,5.60 - 0}{1.25} = 24$$

The replica of the call on the cycle is long N = 0.5 shares at S = \$100 and borrowing B = \$24 at r = 0.25, then we have obtained

$$C = NS - B = 50 - 24 = 26 \equiv \frac{pC^{+} + (1-p)C^{-}}{1+r} = \frac{\langle C_f \rangle}{1+r}$$

with

$$p = \frac{(1+r)S - S^{-}}{S^{+} - S^{-}} = 0.54$$

Being a risk-neutral probability (due to no-arbitrage). Denoting $S^{+} = uS, \quad S^{-} = dS$

Then the following relations $R^{+} = u-1, R^{-} = d-1$ are the returns, while the risk-neutral evaluation is

$$<R>= pR^{+} + (1-p)R^{-} = r$$

That is also written as a following relation

$$p = \frac{(1+r)-d}{u-d}$$

All are pretending to be in the risk-neutral world and the risk is irrelevant.

Multiple periods

If this is the case, let us subdivide the time to expiry $\tau = T - t$ into n equal subintervals, $h = \tau / n$ and then the expected terminal option value is

$$C = \hat{E}\left\{\max(S-K,0)\right\} =$$

$$= \frac{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^{j}(1-p)^{n-j} \max(u^{j}d^{n-j}S-K,0)}{(1+r)^{n}}$$

The factor $[n!/ j!(n-j)!]p^{j}(1-p)^{n-j}$ is the binomial probability that the stock will take j upward jumps in n steps, each with (risk-neutral) probability p. The second factor $\max(u^{j}d^{n-j}S-K,0)$ gives the call option value at expiry conditional on the stock following j ups and n-j downs. Let C be in-the-money option for m ups then

$$u^{m}d^{n-m}S > K$$

and

$$C = S\Phi[m,n,p'] - \frac{K}{(1+r)^{n}}\Phi[m,n,p] \qquad (2)$$

with

$$\Phi[m,n,p] \equiv \sum_{j=m}^{n} (n!/ j!(n-j)!) p^{j}(1-p)^{n-j}$$

the binomial *DF* (probability at least m ups out of n steps) and

$$p' = \left(\frac{u}{1+r}\right)p$$

For $h = \tau / n \to 0$ we have $\Phi[\bullet] \to N[\bullet]$. Let us denote as usual [3]

$$u = \exp(\sigma \sqrt{h}), \quad d = 1/u, \quad p = \frac{1}{2}\{1 + (\mu / \sigma)\sqrt{h}\}, \quad \mu = \ln r - \sigma^2 / 2$$

After some evaluation we finally come to the B-S formula

$$C = SN(x) - Ke^{-r\tau} N(x - \sigma\sqrt{\tau}) \equiv S\Delta - B$$

With the following definition of

$$x = \frac{\ln(S / K(1+r)^{-\tau}) + \sigma^2 \tau / 2}{\sigma\sqrt{\tau}}$$

The symbol $\equiv$ that appears here is due to the dynamical continuous rehedging. As is known *t*he B-S equation without dividends reads

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - \frac{\partial C}{\partial \tau} - rC = 0 \qquad (1)$$

with the terminal condition

$$C(S, \tau = 0, K) = \max(S - K, 0)$$

and the boundary conditions

$$C(0,\tau,K) = 0, \quad C(S,\tau,K) \to S, \quad \text{as } S \to \infty$$

Now let us obtain some conclusion from all these approaches. If a constant continuous compound dividend yield is present ($S \to Se^{-D\tau}$) then we have

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - \frac{\partial C}{\partial \tau} - rC = 0$$

And consequently

$$C(S,\tau,r,\sigma,K,D) = Se^{-D\tau}N(x) - Ke^{-r\tau}N(x - \sigma\sqrt{\tau})$$
$$x = \{\ln(S/K) + (r - D + \sigma^2/2)\tau\} / \sigma\sqrt{\tau}$$

Multi-stock description

Consider a certain derivative security that depends on n state variables and time, t. We make the assumption that there a total of at least n+1 traded securities (including the one under consideration) whose prices depend on some or all of the n state variables. In practice this is not unduly restrictive. The traded securities may be options with different strike prices and exercise dates, forward contracts, futures contracts, bonds, stocks, and so on. We assume that no dividends or other income is paid by n+1 traded securities.

1. The short selling of securities with full use of proceeds is permitted.
2. There are no transactions costs and taxes
3. All securities are perfectly divisible
4. There are no riskless arbitrage opportunities
5. Security trading is continuous

The n state variables are assumed to follow continuous-time Ito diffusion processes. We denote the ith state variable by $\theta_i (1 \le i \le n)$ and suppose that

$$d\theta_i = m_i \theta_i dt + s_i \theta_i dz_i \tag{2}$$

Where dz is a Wiener process and the parameters m and s are the expected growth rate in θ_i. The m and s can be functions of any of the n state variables and time. This is not restrictive. A non dividend paying security by reinvesting the dividends in the security. Other notation used as follows

ρ_{ik}: Correlation between dz_i and $dz_k (1 \le i, k \le n)$

f_j: Price of the j-th traded security $(1 \le i, k \le n)$

r: Instantaneous (i.e. very short term) risk –free rate

One of the f_j is the price of the security under consideration. The short-term risk-free rate, r, may be one of the n state variables.

Since the n+1 traded securities are all dependent on the θ_i it follows from Ito´s lemma in Appendix 12A that the f_j follow diffusion processes:

$$df_j = \mu_j f_j dt + \sum \sigma_{ij} f_j dz_i \tag{3}$$

Where

$$\mu_j f_j = \frac{\partial f_j}{\partial t} + \sum_i \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \tag{4}$$

$$\sigma_{ij} f_j = \frac{\partial f_j}{\partial \theta_i} s_i \theta_i \tag{5}$$

In these equations μ_j is the instantaneous mean rate of return provided by f_j and σ_{ij} is the component of the instantaneous standard deviation of the rate of return provided by f_j, which may be attributed to the θ_i

Since there are n+1 traded securities and n Wiener processes in Equation (3), it is possible to form an instantaneously riskless portfolio, P, using the securities. Define k_j as the amount of the j th security in the portfolio, so that

$$\Pi = \sum_j k_j f_j \tag{6}$$

The k_j must be chosen so that the stochastic components of the returns from the securities are eliminated. From Eq. (3) this means that

$$\sum_j k_j \sigma_{ij} f_j = 0 \tag{7}$$

For $1 \le i \le n$. The return from the portfolio is the given by

$$d\Pi = \sum_j k_j \mu_j f_j dt$$

The cost of setting up the portfolio is $\sum_j k_j f_j$. If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

$$\sum_j k_j \mu_j f_j = r \sum_j k_j f_j \tag{8}$$

Or

$$\sum_j k_j f_j (\mu_j - r) = 0 \tag{9}$$

Eq.(7) and (9) can be regarded as n+1 linear equations in the k_j´s . The k_j´s are not all zero. From a well known theorem in linear algebra, th homogeneous equations (7) and (9) can be consistent only if

$$f_j(\mu_j - r) = \sum_i \lambda_i \sigma_{ij} f_j$$

(10)
Or

$$\mu_j - r = \sum_i \lambda_i \sigma_{ij} \tag{11}$$

For some $\lambda_i (1 \le i \le n)$, which are dependent only on the state variables and time. This proves the result in Equation (12.13?).
Substituting from equations (4) and (5) into equation (10) , we obtain

$$\frac{\partial f_j}{\partial t} + \sum_i \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} - r f_j = \sum \lambda_i \frac{\partial f_j}{\partial \theta_i} s_i \theta_i$$

Which reduces to

$$\frac{\partial f_j}{\partial t} + \sum_i \theta_i \frac{\partial f_j}{\partial \theta_i}(m_i - \lambda_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} = r f_j$$

(12)

Dropping the subscripts to f, we deduce that any security whose price , f, is contingent on the state variables $\theta_i (1 \le i \le n)$ and time ,t, satisfies the second order differential equation

$$\frac{\partial f}{\partial t}+\sum_i \theta_i \frac{\partial f}{\partial \theta_i}(m_i-\lambda_i s_i)+\frac{1}{2}\sum_{i,k}\rho_{ik}s_i s_k \theta_i \theta_k \frac{\partial^2 f}{\partial \theta_i \partial \theta_k}=rf$$
(13)

Application

Two underlying traded assets V and S.

When we have this case, the equation (13) assumes the form

$$\frac{1}{2}\sigma_v^2 V^2 F_{vv}+\rho_{vs}\sigma_v\sigma_s VSF_{vs}+\frac{1}{2}\sigma_s^2 S^2 F_{ss}+(1-D_v)VF_v+$$
$$+(r-D_s)SF_s-F_\tau=rF$$

This equation has been used by Magrabe [4] to evaluate an option to exchange *S* by *V* and Myers & Majd [5] for option to abandon with *V* being a value of the project and *S* its uncertain salvage value.

Option to exchange *S* by *V*.

In this case *F(V,S)* is a homogeneous of degree 1 function, i.e.

$$F(cV,cS)=cF(V,S)$$

and

$$F(V,S,\tau)=SF(V/S,1,\tau)\rightarrow f\equiv F/S,\, X\equiv V/S,\, K=1$$
$$r=D_s,\quad s^2=\sigma_v^2+\sigma_s^2-2\rho_{vs}\sigma_v\sigma_s$$

Therefore the equation for F now reads

$$\frac{1}{2}S^2X^2F_{xx}+(D_s-D_v)XF_x-D_sF-F_\tau=0$$

whereby

$$C(X,\tau,s,D_s,D_v)=Xe^{-D_v\tau}N(d)-1\cdot e^{-D_s\tau}N(d-s\sqrt{\tau})$$

and finally we have obtained that

$$F(V,S,\tau) = Ve^{-D_v\tau}N(d) - Se^{-D_s\tau}N(d - s\sqrt{\tau})$$
$$d = \{\ln(V/S) + (D_s - D_v + s^2/2)\tau\}/s\sqrt{\tau}$$

Here $Se^{-D_s\tau}$ is a future price for the uncertain variable with yield D_s.
One of possible application of the model is considering firm's operations as a series of European options to exchange the uncertain variable production cost (*S*) for the uncertain revenue (*V*).

Real Capital Investment Opportunities as Collections of Options on Real Assets (real options)

For this approach let us assume we now have the conventional NPV (net present value) technique, i.e. the risk-less world:
For one period we will have :

$$NPV = \frac{C_1}{1+r} - I \qquad \text{with}$$

C_1 being a cash flow at the end of the period and I an investment at the beginning.
For T periods we can calculate :

$$NPV = \sum_{i=1}^{T} \frac{C_i}{(1+r)^i} - \sum_{i=0}^{T} \frac{O_i}{(1+r)^i}$$

where

C_i is a cash inflow at the end of period i

O_i is an analogous cash outflow

In case that we are in front of a **Risk-adverse world**, the standard procedure gives

$$NPV = \sum_{i=1}^{T} \frac{\hat{C}_i}{(1+r)^i} - I \equiv \sum_{i=1}^{T} \frac{E\{C_i\}}{(1+k)^i} - I$$

$k = r + p$

$p = E\{r_i\} - r = \lambda \operatorname{cov}(r_i, r_m)$ is a discount risk premium

$\lambda = \dfrac{E\{r_m\} - r}{\operatorname{var}(r_m)}$ is the market price of risk

r_m is the market risk

The basic inadequacy of this approach and other discount cash flow (DCF) approaches to capital budgeting is that they ignore management's flexibility to adapt and revise later decisions, viz. to review its implicit operating strategy. The traditional NPV approach makes implicit assumptions concerning an *expected scenario* of cash flows and presumes management's commitment to a definite *operating strategy.* In doing this, an expected pattern of cash flows over a specified project life is discounted at a risk-adjusted rate to arrive at the project's NPV (what is reflected in the above formulae). This rate is usually derived from the prices of a twin traded financial security. Only projects with positive NPV are to be accepted.

In the real world of uncertainty and competition the realization of cash flows would differ from what management originally expected. As new information arrives and uncertainty about future cash flows is gradually resolved, management may find that existing (or created) flexibility to depart from the original project design allows it to revise the initial operating strategy. For instance, management may be able to abandon, defer, expand, contract, or some other way, alter a project at various stages of its life. This flexibility introduces specific elements similar to those of financial options, in particular asymmetric distribution. Then the true expected value, or *expanded expected NPV* incorporates managerial operating flexibility and strategic adaptability. It exceeds the static or passive expected NPV by an option premium reflecting that flexibility.

Quantifying Flexibility. Real Option Calculus.

Ability to create a risk-less replicating portfolio – if the underlying asset is traded – or to obtain a "certainty-equivalent" expected growth rate by subtracting an appropriate risk premium ($\lambda\sigma$), allows one more convenient valuation in a "risk-neutral world", where

$$F = e^{-r\tau} \hat{E}(F_T)$$

the risk-neutral expectation of a future option payoff (at maturity *T*), F_T, can be discounted at the rate, *r,* i.e. (European calls)

where

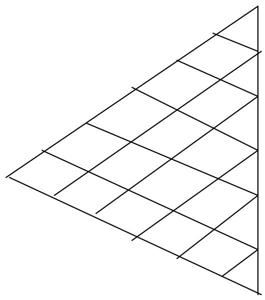
$$F_T = \max(S_T - E, 0)$$

and

$$\frac{dS}{S} = \alpha dt + \sigma dW$$

for S traded $\alpha = r - \delta$, and δ is the dividend yield.

The triangle



$$S_T = \sum_{i=1}^{N} \Delta S_i$$

generates a sample of trajectories with the error ~ $s/\sqrt{n}$. The major drawback – it is limited to European - type options with no early exercise or intermediate decisions.

Finite-Difference Methods

For this method to apply we will use the following Kolmogorov – BS equation

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + (r-\delta)SF_S + F_t - rF = 0$$

where σ, r are constant. And the boundary conditons

$F(S,T) = \max(E - S, 0)$ (terminal for put)
$F(0,T) = E$ (lower boundary)
$F(S,T)/S \to 0$ (upper boundary)

Let us consider K-BS equation and derive a finite-diff. approximation. Let *F(s,t) = F(ih,jq)* then we have

$$F_S \simeq (F_{i+1,j} - F_{i-1,j})/2h + O(h^2)$$
$$F_{SS} \simeq (F_{i+1,j} - 2F_{i,j} + F_{i-1,j})/h^2 + O(h^2)$$
$$F_t \simeq (F_{i,j+1} - F_{i,j})/q + O(q)$$

$$c_i^+ F_{i+1,j} + c_i^0 F_{i,j} + c_i^- F_{i-1,j} = F_{i,j+1}$$

and the ***implicit scheme*** gives

$$c_i^{\pm} = -\frac{1}{2}(\sigma^2 i \pm r)iq\,, \quad c_i^0 = 1 + (\sigma^2 i + r)iq$$

$$\mathbb{C}\,\mathbb{F}_j = \mathbb{F}_{j+1} \quad \Rightarrow \quad \mathbb{F}_j = \mathbb{C}^{-1}\mathbb{F}_{j+1}$$

This is a system of (3xN) linear equations such that and so on all down till $j = 0$.

The explicit scheme.

Further, when we use the F_S and F_{SS} representations let us make the substitution $j \to j+1$. After some calculations we have got the next equation

$$F_{i,j} = (1+rq)^{-1}\left(p_i^+ F_{i+1,j+1} + p_i^0 F_{i,j+1} + p_i^- F_{i-1,j+1}\right) \qquad (*)$$

and the coefficients are determined by the next set of relations

$$\begin{aligned} p_i^{\pm} &= \frac{1}{2}\left(\sigma^2 i \pm r\right) iq = -c_i^{\pm} \\ p_i^0 &= 1 - \sigma^2 i^2 q = 1 - (p_i^+ + p_i^-) \\ i &= (0,\ldots,M), \qquad j = (0,\ldots,N) \end{aligned}$$

for each time step backward (lattice approach). The coefficients $p_i^{\pm}$ and p_i^0 are the risk-neutral probabilities that the state variable, S, being in state i at time j will jump up (to state i+1), jump down (to state i-1) or stay in the same state (i) by the next period (time j+1) respectively and all of them should be non-negative or otherwise an instability may arise.
So the explicit scheme gives the equation that says that "***the current option price is obtained from the <u>expected</u> one period future option values (using the probabilities in a trinomial tree), discounted back at the risk-less rate in a risk-neutral world***"

Some simplifications for obtaining solutions

1. the first simplification we can do is a log-transformation of S, viz. $X = lnS$ that leads to the following equation

$$\frac{1}{2}\sigma^2 F_{XX} + (r - \frac{1}{2}\sigma^2)F_X + F_t - rF = 0 \qquad (**)$$

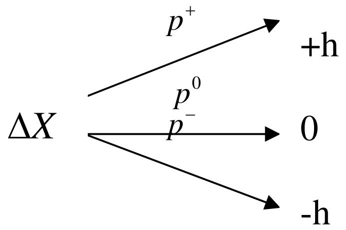
with

$$p^{\pm} = \tfrac{1}{2}\left[(\sigma/h)^2 \pm (r-\sigma^2/2)/h\right]q \quad \geq 0$$
$$p^0 = 1-(\sigma/h)^2 q \quad \geq 0$$

So the p's variables are independent of the state and can be chosen always to be non-negative:

$$q \leq (h/\sigma)^2, \qquad h \leq \sigma^2 \left|r-\sigma^2/2\right|^{-1}$$

and X follows a trinomial jump process



and

$$E(\Delta X) = p^+h + p^-(-h) = (r-\sigma^2/2)q$$
$$Var(\Delta X) = E(\Delta X^2) - [E(\Delta X)]^2 = p^+h^2 + p^-(-h)^2 -$$
$$(r-\sigma^2/2)^2 q^2 = \sigma^2 q - (r-\sigma^2/2)^2 q^2$$

Unfortunately we have obtained Var(DX) < $q\sigma^2$ for the variance of the continuous process.

2. The second simplification consists in the following scheme. Removing the term rF by $f(X,\tau) = e^{r\tau}F(S,\tau)$ and using the implicit diff. scheme (at *j*) we come to (*) with (**).

3. The third one, is done by playing with the probabilities, considering the finite-difference equation as a phenomenological one, then we have

$$p^{\pm} = \tfrac{1}{2}\left[(\sigma/h)^2 \pm (r-\sigma^2/2)/h\right]q + \tfrac{1}{2}\left[(r-\sigma^2/2)q/h\right]^2$$
$$p^0 = 1-(\sigma/h)^2 q - \left[(r-\sigma^2/2)q/h\right]^2$$

and

$$E(\Delta X) = (r - \sigma^2 / 2)q \quad \text{and} \quad Var(\Delta X) = q\sigma^2$$

4. The next simplification will be done by a more general transformation

$$X = \sigma^{-1}\left\{\ln S - \left(r - \sigma^2/2\right)\tau\right\}, \quad f(X,t) = e^{r\tau}F(S,t)$$

(log., detrended and normalized transformation) gives

$$\frac{1}{2} f_{XX} + f_t = 0 \qquad \text{(heat eq.)}$$

or

$$f_{i,j} = (1+q)^{-1}\left(p^+ f_{i+1,j+1} + p^0 f_{i,j+1} + p^- f_{i-1,j+1}\right)$$

with

$$p^{\pm} = q/2h^2, \quad p^0 = 1 - q/h^2 \quad \to \ q/h^2 \le 1$$

now

$$E(\Delta X) = (p^+ - p^-)h = 0, \quad Var(\Delta X) = (p^+ + p^-)h^2 = q$$

therefore in the case $h^2 = q$ we have

$$p^{\pm} = \frac{1}{2}, \quad p^0 = 0$$

the binomial tree results (a particular case of the binomial tree).

Special binomial approach

At this stage we can establish the goal: to design method applicable both to the valuation of complex financial options and to the valuing of capital budgeting projects with multiple real options.

In our case the underlying asset, V the gross present value of the expected cash flows from immediately undertaking the real project (rather than S).

Assume

$$\frac{dV}{V} = \alpha dt + \sigma dW$$

With the following parametric definition

α the instantaneous expected return on the project,
β the instantaneous standard deviation, $q=\Delta t$ then
$X=lnV$ follows an arithmetic Brownian motion.
If we analyze the case under risk neutrality, for which $\alpha = r$, we obtain As it can be easily seen, the increments, ΔX, are independent, identically and normally distributed with

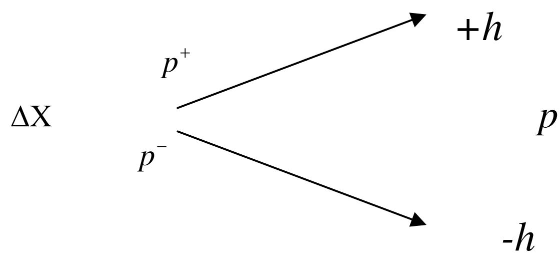
$$\Delta X = \ln\left(V_{t+\Delta t}/V_t\right) = (r-\sigma^2/2)\Delta t + \sigma\Delta W$$

$$E(\Delta X) = (r-\sigma^2/2)\Delta t \quad \text{and} \quad Var(\Delta X) = \sigma^2\Delta t$$

Denote $q=\sigma^2\Delta\tau$, then

$$E(\Delta X) = \mu q \quad \text{and} \quad Var(\Delta X) = q, \qquad \mu = \frac{r}{\sigma} - \frac{1}{2}$$

Consider the discrete process with the same mean and
Var and $q=\sigma^2\tau/N$ the binomial tree is



$p^+ \equiv p$

Then $2ph-h=\mu q, \;\rightarrow\; p=(1+\mu q/H)/2$or

$$E(\Delta X) = ph + (1-p)(-h) = 2ph - h$$

$$Var(\Delta X) = ph^2 + (1-p)h^2 - (2ph-h)^2 = h^2 - (2ph-h)^2$$

$$h^2 - (\mu q)^2 = q, \quad \rightarrow h = \sqrt{q + (\mu q)^2} \geq mq$$

or

$$\left|\mu q / h\right| \leq 1, \quad \rightarrow \quad 0 \leq p \leq 1$$

and there are not external constraints for stability.

Implementation

For this aim we will follow the next four main steps:

1. The standard parameters are specified affecting option values

$$V, r, \sigma^2, T, \delta$$

(Any "dividend yield"), set of costs outlays I,s and the number of subintervals N
The cash flows, *CF*, and their timing (if discrete) and the type, timing and other characteristics of the embedded real options are specified as well.

2. Calculations of the algorithm parameters;

time step $q = \sigma^{\mathbf{2}} T / N$,
drift $\mu = (r - \delta)/ \sigma^2 - 1/2$

value step $h = \sqrt{q + (\mu q)^2}$

probability $p = \frac{1}{2}(1 + \mu q / h)$

3. Determination of terminal values (at $j = N$) for each state i,:

$X(i) = X_0 + ih$ and $F(i) \equiv R(i)$ then

$V(i) = e^{ih}$ and $R(i) = \max\left(V(i), 0\right)$

4. Backward iterative process: for each step j ($j = N,\ldots,1$) and every second state i,

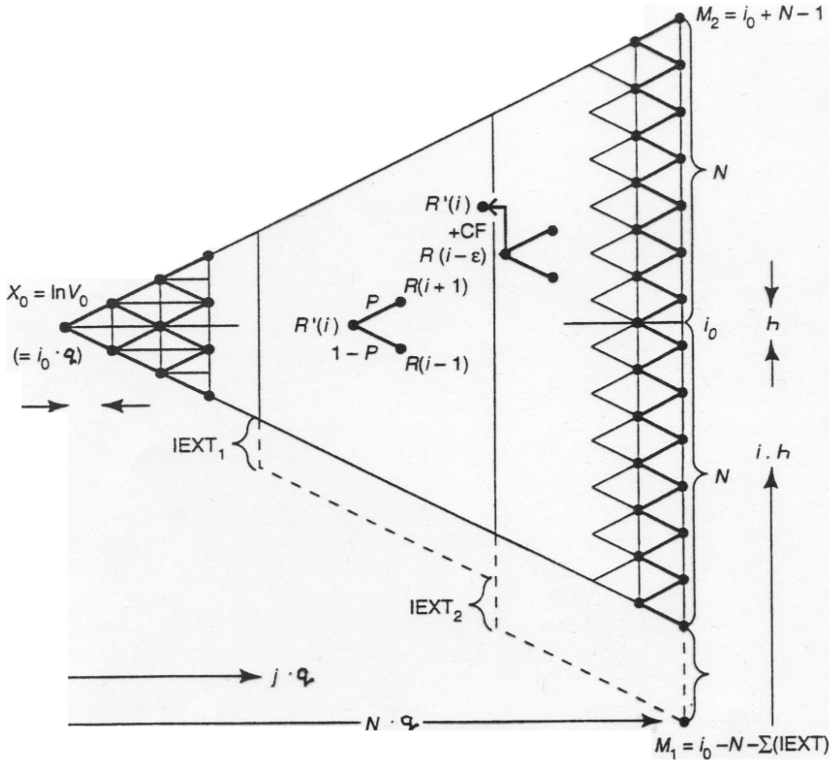
$$R'(i) = e^{-rq/\sigma^2}\left[pR(i+1) + (1-p)R(i-1)\right]$$

calculate opportunity values using information from step (j +1) as

• Adjustments for cash flows (dividends):
At each cash inflow (ex-dividend) time, determine downward extension of triangular path and shift (ε) for each state i: $R'(i) = R(i - \varepsilon) + CF$. At each cash outflow (exercise) time:

$$R'(i) = R(i) - I$$

• Adjustments for multiple real options:
The project can be outlined as shown in the picture:



Three investment outlays I_1 , I_2 , and I_3 during the building stage, then possible cash inflows. The following Options are available : 1) to wait up to T_1 years; 2) to abandon early by forgoing a preplanned outlay I_2 , 3) to contract the scale of operations by *c%* thereby saving part I'_3 of a planned outlay I_3 , 4) to expand production be *e%* , making an extra outlay I_4 , 5) to switch the project from current to its best future alternative use or to abandon for its salvage value (S).

Switch use (abandon for salvage S): $R' = \max(R, S)$

expand by *e%* by investing additional : $R' = R + \max(eV - I_4, 0)$

contract by *c%,* saving : $R' = R + \max(I'_3 - cV, 0)$

abandon by defaulting on : $R' = \max(R - I_2, 0)$

defer until next period: $R' = \max(e^{-r\tau} E(R_{j+1}), R_j)$

- Adjustments for exogenous competitive arrivals (jumps) must be made at appropriate times.

Applications. Power Plant

At this part we will follow the work [10].Valuing a power plant using real option theory has two main purposes in competitive markets:

1. Accurately determine its value.
2. To facilitate the use of risk management tools developed for financial markets in order to hedge both asset value and earnings. (For instance, a power plant can be hedged using forward electricity contracts.

Ignoring non-fuel costs, the net profit per hour for a power plant is

$$NP = q\left(P^E - HP^F\right)$$

where the involved variables are

q is dispatch (output) level *(MW)*

P^E is electricity spot price *($/MWH)*

P^F is the input fuel spot price *($/MMBtu)*

H is the plant heat rate *(MMBtu fuel per MWh electricity)*

The quantity $(P^E - HP^E)$ is the <u>*spark spread*</u> (sp-sp) and

- If *sp-sp* > 0 then q must be maximal
- If *sp-sp* < 0 then shut down

i.e. the instantaneous plant pay-off per unit capacity is

$$\max(P^E - HP^F, 0) = -HP^F + \max(P^E, HP^F)$$

an option to exchange one asset HP^F for another P^E, or a call option on the asset P^E with exercise price HP^F.

There are three types of generating plants (units):

1. **base-load** (low input costs) "in-the –money" option with low price enough to work
2. **mid-load**
3. **peakers** (high input costs) "out-of-the-money" option with high price needed to work

This is a good-but-not-enough approach (linear option to exchange) for it may misprice the plant value and mislead on the optimal operating policy. The following important characteristics (restrictions) are not involved:

1. Minimum on (up) and off (down) times.
2. Minimum ramp (start-up) time (*e.g. heating the boiler*)
3. Minimum generation level
4. Response rate constraints (*time required to effect a discrete change in the dispatch level)*
5. Non-constant heat rate (*heat rate H varies with the generation level)*
6. Variable start-up cost (*cost to start-up depends on the time spent off-line)*

Stochastic dynamic programming is a tool to solve the problem, viz. to calculate plant values and optimal operating policies.

Two tasks

- Developing a lattice for the underlying stochastic variables
- Backward dynamic programming to compute the value & the optimal operating policies.

For a method, see in Hull & White 1993 for "path dependent" option evaluation. Also this approach is good for energy pipelines and storage facilities.

Model: <u>evaluation</u> of thermal power units over a short-term horizon (a week).

Time spacing (decision making interval) is 1 hour: $0,1,2,\ldots,T$

Table 14

Parameter	description	units
t_{on}	minimum up time	hours
t_{off}	minimum down time	hours
t_{cold}	additional time over t_{off} with variable cost	hours
t_{ramp}	time required to bring a unit on line	hours
q_{min}	minimum dispatch level	MW
q_{max}	maximum dispatch level	MW
$H(q)$	heat rate	MMBtus/ MWh

Operating state constraints

s is the operating state of the plant consisting of plant condition and its duration

N is the total number of the plant states: $1 \leq s \leq N$ with

$$N \geq t_{off} + t_{cold} + t_{ramp} + t_{on}$$

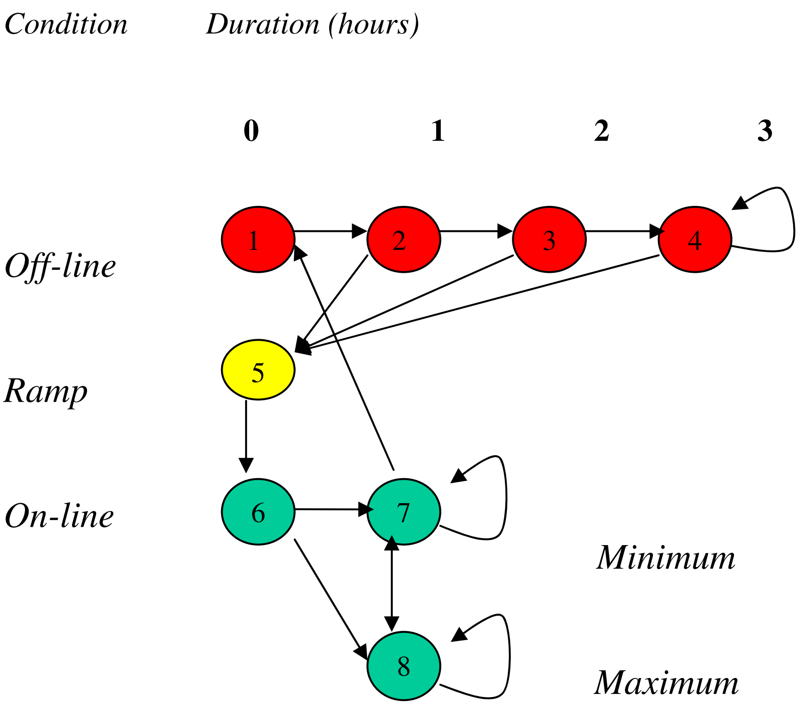
Table 15

Plant condition	States
Off-line	$1 \leq s \leq t_{off} + t_{cold}$
Ramp (unable to sell power)	$t_{off} + t_{cold} \leq s \leq t_{off} + t_{cold} + t_{ramp}$
On-line	$t_{off} + t_{cold} + t_{ramp} \leq s \leq t_{off} + t_{cold} + t_{ramp} + t_{on}$

State transition diagram

Example:

$$t_{on} = 2, \quad t_{off} = 2, \quad t_{ramp} = 1, \quad t_{cold} = 2$$



State 1: a plant has just gone off-line

State 2: a plant has been off-line for an hour
State 3: a plant remains off-line 2 hours
State 4: off-line for 3 or more hours
State 5: start up
State 6: on-line at the minimum dispatch level
State 7: on-line for one or more hours (2 hours t_{on}) at the minimal dispatch level
State 8: on-line for one or more hours at the maximum dispatch level

Possible transitions $s \to s'$ shown at the diagram mean that $s' \in A(s)$.

Price Processes

Let us assume the prices are discrete: P^E_{jt} and P^F_{jt} with index *j* specifying the point in the price space. Assume also that the spot price of electricity follows a mean-reverting geometric Brownian motion process:

$$d \ln P^E = \left(p_E - a_E \ln P^E \right) dt + \sigma_E dW_E$$

where p_E is a drift parameter, a_E is the mean reversion rate, σ_E is the volatility and dW_E is the Wiener generator. In the Hull book it's shown that a trinomial tree may be used to represent this process. This tree is determined by the price space and the transition probabilities we have discussed. The drift term is assumed time-dependent to calibrate to an observed forward price curve. In the simplest model the fuel prices are put constant, however this restriction can be removed by considering two-factor model. Some more simplification of the model is to assume the plant heat rate to be constant and playing with only one stochastic process, the spark spread directly.

Costs and Revenues

Each operating state has an associated cost or revenue.
We assume the following form

$$f_{jt}(q,s) = \begin{cases} -K_{fix} & 1 \le s \le t_{off} + t_{cold} + t_{ramp} \\ q\left(P^E_{jt} - H(q)P^F_{jt}\right) - K_{fix} & otherwise \end{cases}$$

Here K_{fix} is the fixed cost in all states. Transition costs (from state *s* to state *s*`) may be accounted for.

Dispatch and response rate constraints

$$\begin{aligned} &q = 0 && if\ s \in \left(1, \ldots, t_{off} + t_{cold} + t_{startup}\right) \\ &q_{\min} \le q \le q_{\max} && otherwise \end{aligned}$$

This is denoted as $q \in B(s)$. These constraints impose no restriction on how fast a plant can change its dispatch level: if it is on-line, it can be dispatched at any output level. The third dimension to the state descriptor (in addition to plant condition and duration) is two discrete levels of the plant dispatch, *min* and *max.* There is a restriction for that transition, say one hour.

Solution method

Here we face the optimization problem that may be formulated in a set of time periods. An optimal policy with n periods remaining may be determined by selecting the policy that maximizes the sum of net revenue in period n plus the expected net revenue in the subsequent $n-1$ remaining periods. The optimal policy for this problem is to solve

$$F_{jt}(s) = \max_{q \in B(s)} f_{jt}(q,s) + \sum_{j'} p_{jt}^{j'} \left\{ \max_{s' \in A(s)} \left[F_{j',t+1}(s') - g_{jt}(s,s') \right] \right\}$$

Here $F_{jt}(s)$ denotes the value of the power plant over the period t to T conditional on being in energy Price State j at time t and operating state s ; $p_{jt}^{j'}$ represents the probability of moving from price state j at time t to price state j' at time $t+1$. This equation states that the value of the plant over the remaining periods (from time t to T) is the sum of the net revenue in period t and the expected value of the power plant from time $t+1$ to T which is conditional on the plant operating state at time $t+1$. We select the operating state s' that results in the maximum plant value (net the state transition cost), conditional that it is feasible transition from state *s*.

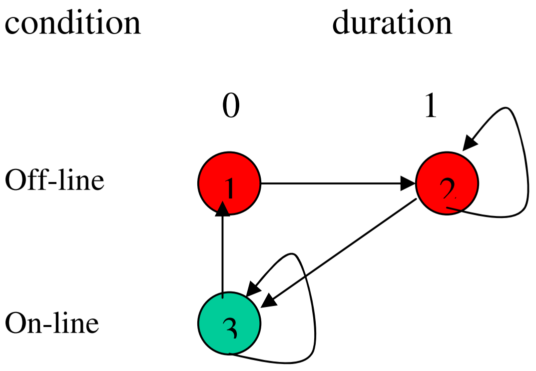
This maximization determines the optimal operating state transition policy for the plant. The plant value at time *0,* $F_{0,0}(s)$ is obtained by solving the equation recursively, backward from time T for all possible Price States j and operating States s , to time *0* which has only a single known price state. In addition to plant value, a key result of the solution is the optimal operating policy that consists of the optimal plant output in each on-line state as a function if price state and time, and the optimal state transition strategy as a function of the current operating state, price state, and time. The optimal operating policy should be used by the plant operators to maximize the plant value. Usually the optimal state transition strategy may be expressed in terms of a set of exercise boundaries. For example, if the current state is on-line, the optimal transition in the next period will be to remain on-line for all values of the spark spread greater than a certain critical value and go off-line for all spark spreads that are less. In general those boundaries vary through time.

Simple example:

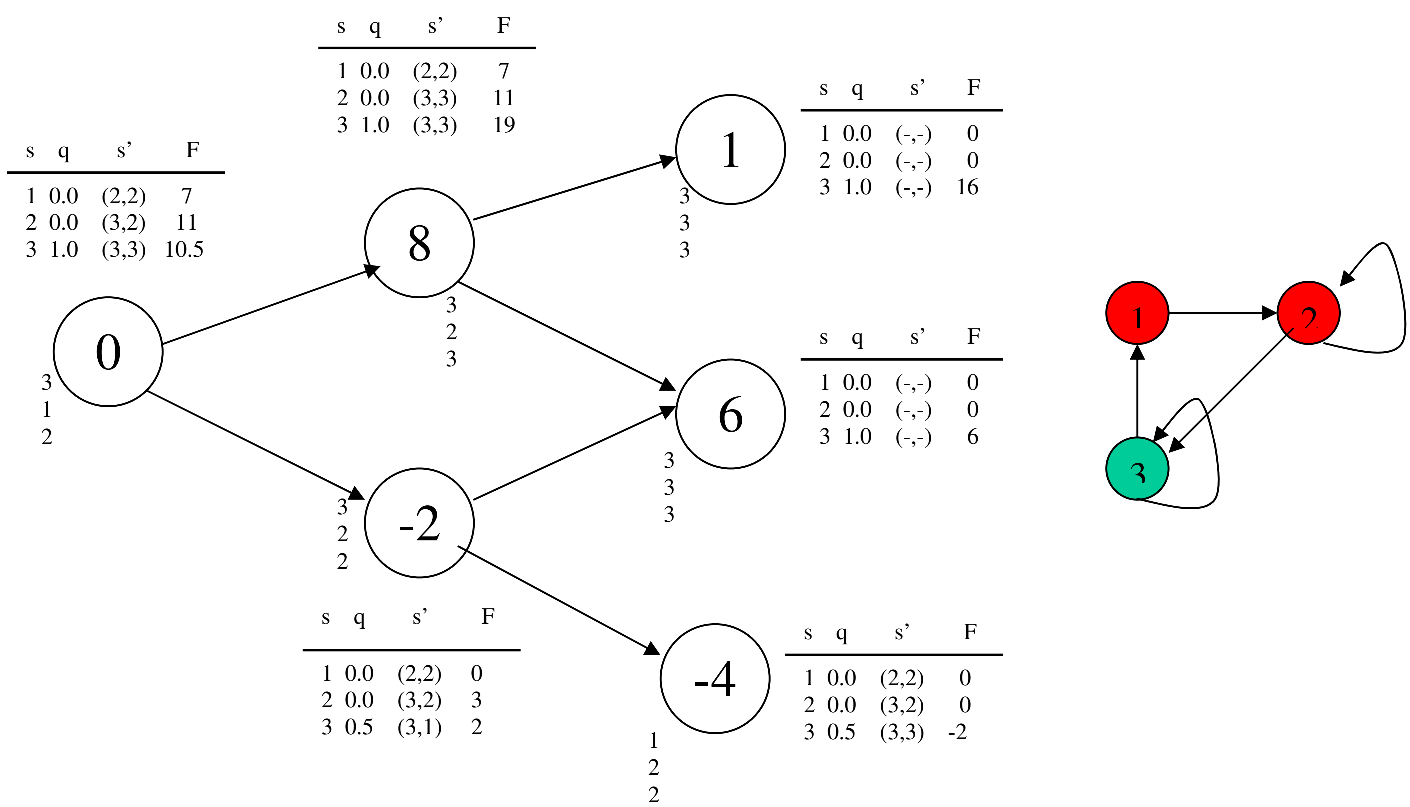
$$t_{on} = 2, \quad t_{off} = 1, \quad t_{cold} = t_{startup} = 0$$

H = *const.*, K_{fix} , q_{min} = 0.5 *MW,* q_{max}= 1. See the diagram (???????)

Feasible operating state transitions



Plant value and optimal operating policy: NV + Option



s	q	s'	F
1	0.0	(2,2)	7
2	0.0	(3,2)	11
3	1.0	(3,3)	10.5

s	q	s'	F
1	0.0	(2,2)	7
2	0.0	(3,3)	11
3	1.0	(3,3)	19

s	q	s'	F
1	0.0	(-,-)	0
2	0.0	(-,-)	0
3	1.0	(-,-)	16

s	q	s'	F
1	0.0	(-,-)	0
2	0.0	(-,-)	0
3	1.0	(-,-)	6

s	q	s'	F
1	0.0	(2,2)	0
2	0.0	(3,2)	3
3	0.5	(3,1)	2

s	q	s'	F
1	0.0	(2,2)	0
2	0.0	(3,2)	0
3	0.5	(3,3)	-2

State transition decisions should take into account not just immediate net revenue but also the opportunity cost in terms of future decision-making flexibility; the simple exchange option approach does not consider this. This phenomenon explains why electricity prices have gone to zero or even negative for short time periods in some markets.

Conclusions

We saw how real options theory may be applied to value power generation assets. In particular, the model we develop is capable of handling constraints related to minimum on-and off-times, ramp times, minimum dispatch levels and response rates. The optimal operating policy also may be very much affected.

Real options theory supplies a methodology for quantifying the value of the operating flexibility of real assets and for determining optimal operating policies.

It is possible to improve greatly the effectivity of operating options and to reveal "hidden" asset value. Understanding the sources of asset value and its sensitivity to fuel and electricity prices is also critical for companies seeking to determine a suitable hedging policy through either forward sales or other derivatives contracts. Effective applications of real options theory demands that managers become familiar with its underlying assumptions in order to understand its strengths and weaknesses as well. The pay-off for companies is the ability to effectively leverage a company's assets to achieve an optimal trade-off between risk and payoff.

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