Forecast variance in simultaneous equation models: analytic and Monte Carlo methods

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FORECAST VARIANCE IN SIMULTANEOUS EQUATION MODELS: ANALYTIC AND MONTE CARLO METHODS

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ABSTRACT

Five alternative techniques have been applied to measure the degree of uncertainty associated with the forecasts produced by a macro-model of the French economy, the Mini-DMS developed at INSEE. They are bootstrap, analytic simulation on coefficients, Monte Carlo on coefficients, parametric stochastic simulation and re-estimation, a residual-based procedure. Due to the complexity and the size of the model (nonlinear and with more than 200 equations), several associated technical problems had to be solved. The remarkable convergence of results which has been obtained for all the main endogenous variables suggests that forecast confidence intervals are likely to be quite reliable for this model.

1. INTRODUCTION

It is often said that everyone can produce a forecast, but only the statistician can associate a corresponding confidence interval.
In the case of forecasts obtained by simulating a macroeconomic model, it is the responsibility of the economist who builds the model to produce forecasts that are as accurate as possible, while the econometrician should, among other things, estimate the associated degree of uncertainty: e.g., standard errors, confidence intervals, or confidence regions of forecasts.
Moving from the ideal framework of what *should be done* to what *is actually done* in daily practice, the econometrician is soon faced with a major problem:

Even if the model is small and linear, the exact sampling distributions of the estimators and of forecasts are so complicated that even the most patient researcher would probably never tackle analytically the case of a structural model with more than two simultaneous equations.

He will therefore slowly accept the idea that some approximation must be introduced. And this will not be particularly troublesome to him since, in fact, he is measuring a degree of uncertainty, and a *minor* uncertainty in measuring the uncertainty is surely unimportant, *provided it is really minor.*
Once he has abandoned the idea of deriving the exact sampling distribution of forecasts, and accepted the need to introduce some approximation, he is immediately faced with a long series of alternative choices, often embarrassing.

1) A first possibility is to replace the sampling distribution with a normal approximation; the approximation will be asymptotically exact under general and widely accepted conditions and is therefore expected to provide accurate results if the sample is *sufficiently large.*
2) An alternative possibility is to resort to sampling experiments (Monte Carlo).

In the first case, the econometrician can choose between two alternatives proposed by the econometric literature.

1.(1) Full analytical methods: they were originally designed for linear systems (e.g., Goldberger et al., 1961) but even in case of models containing nonlinearities, these methods can be applied to solve a large part of the problem (Calzolari, 1981, 1987).

1.(2) Mixed methods, partially analytical and partially based on numerical simulation procedures (analytic simulation): conceptually equivalent to the full analytical methods, they allow for a considerable reduction of computational complexity and are suitable for application even to medium-large size models (Bianchi and Calzolari, 1980).

In the case where Monte Carlo methods are adopted, a choice must first be made between the following two alternatives.

2.(1) Pure simulation techniques, which require a series of replicated solutions of the model after appropriate pseudo-random shocks have been applied to all the variables and parameters of the model that are affected by uncertainty (e.g., Haitovsky and Wallace, 1972, Cooper and Fischer, 1974, or Fair, 1980).

2.(2) Simulation and re-estimation techniques, which require a series of replicated solutions of the model, each of them followed by re-estimation of the model's parameters, followed in turn by another simulation with the new parameters. This leads unavoidably to a further consideration.

2.(2.1) Should simulation be parametric? In other words, should the econometrician first assume that the model involves some kind of multinormal error process and then operate consequently (see e.g., Schink, 1971)?

2.(2.2) Alternatively, should he avoid any assumption on the underlying error process and simply re-sample in each replication only from the available set of estimation residuals with some nonparametric technique like the bootstrap (Efron, 1979)?

The list of possible alternatives is not yet finished, since all macroeconomic models used in practice for forecasting purposes have a dynamic structure,

2.(2.1.1) and 2.(2.2.1) should simulation over the sample period (either parametric or nonparametric) be static;

2.(2.1.2) and 2.(2.2.2) or should simulation over the sample period be dynamic (and re-estimation performed accordingly)? In other words, should the lagged endogenous variables be kept fixed in all Monte Carlo replications at their historical values both in the simulation and in the estimation phase (like exogenous variables), or should we solve the model dynamically and set lagged endogenous variables at their computed values also in the estimation phase?

There is no doubt that any method obtained by any combination of alternatives would have some merits and appear more appealing in some respects than the others, and less appealing from other points of view. What is sure is that if the econometrician, after considerable trouble, decides to adopt one of the possible methods and applies it to the model he is analyzing, he will then remain in doubt as to whether the application of some other method might lead to completely different results (and not simply to *minor* differences), as has been often evidenced in comparative studies (Bianchi and Calzolari, 1982, 1983; Freedman and Peters, 1984; and Peters and Freedman, 1986).

This argument leads us to the idea that underlies this paper. In the case at hand we are dealing with a model (Mini-DMS, see Fouquet et al., 1978, or Brillet, 1981) that is used in practice for the purpose of forecasting the macroeconomic variables of the French economy. Mini-DMS is, like all models of this kind, so complex that no analytical investigation of the sampling distribution is practically feasible, so we intend to apply all the methods deriving from all the possible combinations listed above.

If the different methods will evidence dramatic differences of results, it will be clear that nothing definitive can be said about the degree of uncertainty involved in this model's forecasts. If, on the contrary, all the methods evidence only small differences, very few doubts will remain that those results are in fact quite reliable measurements of the degree of forecasts uncertainty.

To anticipate the end of the story, we can say that a remarkable coincidence of results has been obtained from all the methods applied to the Mini-DMS macro-model of the French economy.

2. NOTATIONS FOR THE CASE OF STATIC MODELS

We first deal with the static model or, if the model is dynamic, with the case of one-period ex post forecasts, where lagged endogenous variables are treated as predetermined exactly like the exogenous variables. We can therefore represent the simultaneous equation model as

$$f(y_t,x_t,a) = u_t \qquad t = 1,2,...,T \tag{1}$$

where f is the $m \times 1$ vector of structural form functions, assumed to be continuously differentiable with respect to all variables and parameters, y_t is the $m \times 1$ vector of endogenous variables at time t, x_t is the vector of exogenous variables at time t, and a is the $s \times 1$ vector of all the unknown structural coefficients in the model. The $m \times 1$ vector of random error terms at time t, $u_t = (u_{1t},u_{2t},...,u_{mt})'$, is assumed to be independently and identically distributed with zero means and finite covariance matrix Σ, which is completely unknown, apart from being symmetric and positive definite.

If the model is linear both in the variables and in the coefficients, its structural form can be represented as

$$Ay_t + Bx_t = u_t \tag{2}$$

where A is the $m \times m$ matrix of structural coefficients of endogenous variables, and B is the matrix of coefficients of exogenous variables. A and B are sparse matrices, if the model's equations are overidentified (as usually happens in practice), whose unknown terms are the elements of the vector a, in the notation (1).

It is usually assumed that a simultaneous equation system like (1) uniquely defines the values of the elements of y_t once values for the coefficients, the predetermined variables, and the disturbance terms are given (at least in some range); in the case of the linear model (2), this is equivalent to assuming nonsingularity of the matrix A. This means that the structural form equations (1) implicitly define a system of reduced form equations

$$y_t = y(x_t,a,u_t) \tag{3}$$

where the vector of functional operators y is generally unknown in the case of nonlinear models.

If the model is linear, equation (3) becomes simply

$$y_t = \Pi x_t + v_t; \qquad \Pi = -A^{-1}B; \qquad v_t = A^{-1}u_t. \tag{4}$$

3. EX POST FORECASTS AND FORECAST ERRORS

Let $\hat{a}$ be the estimate of the vector a, obtained by applying to the data for y_t and x_t in the sample period $t = 1,2,...,T$ some suitable estimation method, and let us use the model to produce a forecast in a forecast period h not belonging to $1,2,...,T$. We shall deal only with ex post forecasts, that is, assuming exact knowledge of all the predetermined variables in the forecast period x_h. Under the assumption of serial independence, the disturbances at time h, u_h, are independent of the disturbance terms in the sample period and, therefore, u_h and $\hat{a}$ are two independent random variables.

The usual forecast supplied by the model is obtained by inserting, in the structural form equations (1), the estimated vector $\hat{a}$ and the values of the predetermined variables x_h (supposed exact, for the purposes of this paper, as already observed), dropping the disturbance term, and solving the resulting system

(5) $$f(\hat{y}_h, x_h, \hat{a}) = 0$$

by means of some numerical method. In terms of the (unknown) reduced form, this means that the vector of forecasts at time h can be represented as

(6) $$\hat{y}_h = y(x_h, \hat{a}, 0).$$

The vector of the *true* values of endogenous variables in the forecast period can be represented, using the unknown reduced form, as

(7) $$y_h = y(x_h, a, u_h).$$

The vector of forecast errors is the difference between $\hat{y}_h$ and y_h. It is now convenient to introduce an auxiliary vector, $\bar{y}_h$, defined as the vector of forecasts that would be produced by the model if the structural coefficients were known with certainty; in other words $\bar{y}_h$ is the solution of the model *free of errors* at time h .

(8) $$f(\bar{y}_h, x_h, a) = 0$$

that is, in terms of reduced form,

(9) $$\bar{y}_h = y(x_h, a, 0).$$

Returning to the vector of forecast errors, we now have

(10) $$\hat{y}_h - y_h = [\hat{y}_h - \bar{y}_h] + [\bar{y}_h - y_h] = [y(x_h, \hat{a}, 0) - y(x_h, a, 0)] + [y(x_h, a, 0) - y(x_h, a, u_h)].$$

In the case of the linear model, equation (10) assumes the well-known form

(11) $$\hat{y}_h - y_h = [\hat{\Pi} x_h - \Pi x_h] + [\Pi x_h - (\Pi x_h + v_h)] = [\hat{\Pi} - \Pi] x_h - A^{-1} u_h.$$

In both cases, the vector of forecast errors is the sum of two random vectors: the first is a function of several variables, among which only the vector of estimated coefficients, $\hat{a}$, is random (since x_h is assumed to be known); the second is also a function of several variables, among which only the vector of structural disturbances, u_h, is random. Since, by assumption, $\hat{a}$ and u_h are independent, so also are the two components of the vector of forecast errors.

Therefore the two components can be separately analyzed and, in particular, an estimate of the variances of the forecast errors can be obtained by summing the estimated variances of the two components.

What we have stated above is not exactly true if lagged endogenous variables are present among the predetermined variables; in this case the two terms of the sum are both functions of the (random) lagged endogenous variables. The above considerations, however, still hold *conditional* on a given value of the lagged endogenous variables (the historical value, in case of one-period forecasts).

4. THE EFFECT OF THE RANDOM ERROR TERMS

To analyze the component $(\bar{y}_h - y_h)$, which is a function of the random structural disturbances, stochastic simulation (with several variants) is usually proposed as the basic computational method. By means of replicated solutions of the model, each time introducing a vector of pseudo-random disturbances in place of u_h, it is possible to compute approximate values of the conditional means and variances of the elements of $(\bar{y}_h - y_h)$. Of course, if the model is linear, the mean of this component is zero and the covariance matrix of its elements is

(12) $$A^{-1}\Sigma A'^{-1} \quad \textit{and} \quad \hat{A}^{-1}\hat{\Sigma}\hat{A}'^{-1} \quad \textit{is its estimate,}$$

where $\hat{\Sigma}$ is in all cases computed from the structural form residuals, usually as

(13) $$\hat{\Sigma} = T^{-1}\sum_{t=1}^{T} \hat{u}_t \hat{u}_t'$$

but it might also take into account the problem of degrees of freedom, analogously to the linear regression model (e.g., Klein, 1969).

4.1. Parametric stochastic simulation

It is the stochastic simulation procedure that is most widely used in the literature (e.g., Corker et al., 1983; Fisher and Salmon, 1986; and Hall, 1985). The procedure is as follows.

1) A vector of pseudo-random numbers $\tilde{u}_h$, with multivariate normal distribution, zero mean and the available covariance matrix $\hat{\Sigma}$ is generated. The method of Nagar (1969) can be applied if $\hat{\Sigma}$ is positive definite; if $\hat{\Sigma}$ is not of full rank, the method of McCarthy (1972,a) can be used.
2) The vectors $\tilde{u}_h$ are inserted into the model, where the structural coefficients are maintained fixed at their originally estimated values, and the model is solved in the forecast period, h, obtaining for the endogenous variables the vector $\tilde{y}_h$.

Stages 1 and 2 are repeated and sample variances of the elements of $\tilde{y}_h$ are computed.

If finite moments of the first two orders exist, a very large number of replications would lead, in practice, to the exact values of means and variances, if the parameters of the model (the vector a and the covariance matrix of the structural disturbances) were known with certainty. As, however, we assume only estimates of these parameters, stochastic simulation will lead to an estimate of the means and variances of the elements of $(\bar{y}_h - y_h)$.

4.2. Residual-based procedure

A residual-based procedure has recently been proposed by Brown and Mariano (1984) for the one-period forecasts in static models. The procedure utilizes complete enumeration of the residuals over the sample period. It consists of replicating the solution of the model in the forecast period, h, exactly T times, using the T vectors of estimated residuals $\hat{u}_1, \hat{u}_2, ..., \hat{u}_T$, and then computing sample variances of the elements of the T vectors of solutions. The computational steps are quite similar to those of the parametric method described above. In the first step, however, we use one vector of estimated residuals rather than using a generator of pseudo-random numbers; moreover, steps 1 and 2 are repeated *exactly* T times (rather than an arbitrary number of times), using each time a different vector of residuals.

In a way, the residual-based procedure could be considered a particular case of bootstrap. Application of the same procedure in the case of linear models gives the same results as computing the reduced form variances through the calculation of the mean squared errors of a static simulation over the sample period, which, in its turn, is well known to produce the same results as equation (12).

4.3. Local linearization of the model

Equation (12) can be used also for nonlinear models, provided that a local linearization has been first performed in the neighborhood of the solution point in the forecast period.

In the linear model, A^{-1} is the Jacobian matrix of first derivatives of the elements of y_h with respect to the elements of u_h. Analogously, in the nonlinear model the linearization procedure first requires the computation of $\partial y_h / \partial u_h'$; then the covariance matrix of $(\bar{y}_h - y_h)$ follows as $(\partial y_h / \partial u_h')\Sigma(\partial y_h' / \partial u_h)$.

Of course linearization involves approximation. If exact results (or better approximations) are required, linearization and parametric stochastic simulation can be used together to form a Monte Carlo method with control variates (see Calzolari and Sterbenz, 1986).

5. THE EFFECT OF ERRORS IN ESTIMATED COEFFICIENTS

5.1. Analytical method and analytic simulation

The full analytical method, or its equivalent numerical version called analytic simulation (Bianchi and Calzolari, 1980), relies on the property, well-known in large sample theory, that asymptotic normality of sample statistics can be maintained through transformations, even nonlinear, provided they are continuously differentiable. If we assume that, as T increases, asymptotically

$$\sqrt{T}(\hat{a} - a) \sim N(0, \Psi) \tag{14}$$

(and $\hat{\Psi}$ is a consistent estimate of Ψ) then, asymptotically,

$$\sqrt{T}(\hat{y}_h - \bar{y}_h) = \sqrt{T}[y(x_h, \hat{a}, 0) - y(x_h, a, 0)] \sim N(0, J_h \Psi J_h') \tag{15}$$

(δ-method, see Rao, 1973, p.388) where the Jacobian matrix J_h has dimensions $m \times s$ and contains the first order derivatives of the elements of y with respect to the elements of a, computed at the point $(x_h, a, 0)$.

It is important to note that equation (14) has been proven in the literature for a variety of estimation methods under quite general conditions, provided that the model is static (if nonlinear), or linear (if dynamic). The most general case of a model that is at the same time nonlinear and dynamic has not yet been fully investigated in the literature. In such a case, assumption (14) is still widely accepted, but mainly on the basis of heuristic considerations (e.g., Gallant, 1977, pp.73-74; or Hatanaka, 1978, fn.8); simulation experiments are also encouraging (e.g., Prucha, 1984).

If the computation of (15) is performed at the point $(x_h, \hat{a}, 0)$ and $\hat{\Psi}$ is used instead of the unknown Ψ , then $\hat{J}_h \hat{\Psi} \hat{J}_h'$ is a consistent estimator of $J_h \Psi J_h'$; the division by the sample period length, T, leads to the result we are looking for, the estimate of the covariance matrix of a multinormal distribution that approximates the small sample distribution of the random vector $(\hat{y}_h - \bar{y}_h)$.

Continuity and differentiability of the elements of the (unknown) vector of reduced form functional operators y is ensured by the implicit function theorem, which also provides the means for a full analytical computation of the first order derivatives

$$\partial y / \partial a' = - (\partial f / \partial y')^{-1} (\partial f / \partial a') \tag{16}$$

since the structural form operators vector f is known; of course, some numerical solution method must be first used to get the deterministic solution of the model at time h.

For medium or large scale models, it can be simpler to perform the computation of the above derivatives with finite differences (analytic simulation).

If the model is linear, recalling equation (4) and making use of the formula proposed in Nissen (1968), the above method can be made more explicit as

(17) $$\sqrt{T}(\hat{y}_h - \bar{y}_h) = \sqrt{T}(\hat{\Pi} - \Pi)x_h = \sqrt{T}\,vec[I_m(\hat{\Pi} - \Pi)x_h] = \sqrt{T}(x_h' \otimes I_m)vec(\hat{\Pi} - \Pi)$$

where I_m is the $m \times m$ unit matrix.

Equation (17) represents a linear combination of the elements of $(\hat{\Pi} - \Pi)$ with fixed coefficients (the values of the predetermined variables, which are supposed known), so that the asymptotic covariance matrix of $\sqrt{T}(\hat{y}_h - \bar{y}_h)$ can be computed with no difficulty as soon as the asymptotic covariance matrix of $\sqrt{T}\,vec(\hat{\Pi} - \Pi)$ has been computed;, this can be done with the method proposed by Goldberger et al. (1961) (see also Schmidt, 1974, for the extension to dynamic simulation forecasts). A method for obtaining a simple explicit representation of this component of the forecast error as a function of the structural coefficients errors is given in Calzolari (1981, 1987).

Some first results of the application of the analytic simulation technique to the Mini-DMS model have been given in a recent paper by the authors (1984).

5.2. Monte Carlo on structural coefficients

This method, described in Fair (1980), can be summarized as follows. Let $\hat{\Psi}/T$ be the available estimate of the covariance matrix of the structural coefficients $\hat{a}$.

1) A vector $\tilde{a}$ of pseudo random numbers, with mean equal to the available vector $\hat{a}$, and covariance matrix equal to the available $\hat{\Psi}/T$, is generated.
2) The pseudo-random coefficients vector $\tilde{a}$ replaces the original estimates $\hat{a}$ and the model is solved deterministically in the forecast period h, obtaining the vector of pseudo-forecasts $\tilde{y}_h$.

The process is repeated from step 1 to 2 and the desired results follow from the computation of the sample variances of the elements of all the $\tilde{y}_h$ computed in the various replications.

A difficulty may arise in the generation of the pseudo-random vectors $\tilde{a}$. The usual generation methods are, in fact, based on Choleski's triangularization of the matrix $\hat{\Psi}/T$ and the most widely used triangularization algorithms require such a matrix to be positive definite (see Cooper and Fischer, 1974, or Nagar, 1969, for example). Unfortunately, this is not always the case. For example when, in a large scale model, the length of the time series does not allow the application of system estimation methods, the matrix $\hat{\Psi}/T$ must be built block by block (see, for example, Brundy and Jorgenson, 1971, p.215, for instrumental variables LIVE estimation) and it is not necessarily of full rank. This problem clearly does not arise if only the diagonal blocks of the $\hat{\Psi}/T$ matrix are taken into account, as in the work of Cooper and Fischer (1974), Haitovsky and Wallace (1972), and Fair (1980). We also shall adopt this strategy in the experiments based on this Monte Carlo method: rather than performing the triangular decomposition of the singular covariance matrix, we shall take into account only the diagonal blocks of the matrix $\hat{\Psi}/T$.

From experiments performed on other models (Bianchi and Calzolari, 1982), this method seems to be more sensitive to outliers than parametric stochastic simulation and re-estimation or bootstrap simulation and re-estimation. However, the kind of *instability* in the convergence of this Monte Carlo process, often encountered with other models, has not caused problems in this case.

5.3. Parametric stochastic simulation and re-estimation

This method can be summarized as follows (see Schink, 1971 for more details). Let $\hat{\Sigma}$ be the available estimate of the covariance matrix of the structural disturbances.

1) T vectors of pseudo-random numbers, $\tilde{u}_t$, $t = 1,2,...,T$ (each of which having multinormal distribution, zero means, and covariance matrix equal to the available $\hat{\Sigma}$), are generated. As for section 4.1, the method by Nagar (1969) can be applied if $\hat{\Sigma}$ is positive definite; if $\hat{\Sigma}$ is not of full rank, the method by McCarthy (1972,a) can be used.
2) The vectors $\tilde{u}_t$ are inserted into the model, where the structural coefficients are maintained fixed at their originally estimated values, and the model is solved over all the sample period, obtaining for the endogenous variables the vectors $\tilde{y}_t$, $t = 1,2,...,T$.
3) The vectors $\tilde{y}_t$ are treated as a new set of observations of the endogenous variables and are used to re-estimate the model, thus obtaining a new vector, $\tilde{a}$, of pseudo-estimated coefficients.
4) The coefficients $\tilde{a}$ are inserted into the model to produce, with deterministic solution, a vector of pseudo-forecasts at time h, $\tilde{y}_h$.

The process is repeated from step 1 to 4 and the desired results follow from the computation of the sample variances of the elements of all the $\tilde{y}_h$ computed in the various replications.

Some complications arise from the treatment of lagged endogenous variables in the simulation phase 1 (in other words simulation can be *static* or *dynamic*) and in the re-estimation phase 3 (they can be maintained *static*, i.e. fixed at their historical value, or their computed value can be chosen). This problem is discussed in Schink (1971, pp.101-108). In the experiments here performed two different combinations have been adopted: the *static-static* and the *dynamic-dynamic*; cross combinations have not been applied, first of all because they are conceptually not very appealing, and secondly because Schink himself (1971) commented on their bad performances in practical applications.

This method is reported in the literature as being frequently used to derive small sample distributions of estimators for simultaneous equation systems, when analytical methods are not available. The main theoretical limitation is in the possible nonexistence of finite moments in the small sample distribution of the structural form or reduced form coefficients (these last directly related to forecasts); this topic is discussed on theoretical grounds in McCarthy (1972,b) and Mariano (1982).

As pointed out in McCarthy (1972,b, p.761), ". . . *it should be noted that the non-existence of moments has some implications for those engaged in Monte Carlo studies. Outliers can be expected. Computation of mean squared forecast errors and the mean* *squared errors of the restricted reduced form coefficient estimates will not converge as the number of Monte Carlo runs increases. These computations really will not yield meaningful information. Throwing out the outliers in making these calculations is also of questionable value. What is accomplished by throwing them out? . . .".*

Bianchi and Calzolari (1982, 1983) investigate this problem for some macroeconomic models used in practice for forecasting purposes. They showed that the problem of outliers, and the consequent non-convergence of the procedure as the number of replications increases, is not just *theoretically possible*, but may be encountered in practical applications on real world models, even if more rarely than with Monte Carlo on coefficients. It has not been encountered, however, in the experiments on Mini-DMS model.

5.4. Bootstrap simulation and re-estimation

This method, described in Efron (1979), has been recently used by Freedman and Peters (1984) and Peters and Freedman (1986) for the purposes we are interested in.

Like stochastic simulation, bootstrap is a procedure for estimating standard errors by resampling the data with a Monte Carlo approach. The process is exactly the same as the one described in the previous subsection; the only difference is that in the first step, rather than sampling from some assumed parametric distribution (i.e., from a multinormal with zero means and covariance matrix equal to the available $\hat{\Sigma}$), the T vectors $\tilde{u}_t$ are the results of T draws, made at random with replacement, from the T vectors $\hat{u}_t$ $(t = 1,2,...,T)$ of the structural estimation residuals over the sample period. Steps 2, 3, and 4, and the way in which the sample variances of $\tilde{y}_h$ are computed, are exactly the same as described in the previous case.

Using this procedure, the only distributional assumption concerning the disturbances in the system is that the disturbances should be independent and identically distributed over time. Of course, in order to preserve the stochastic relationships of the estimated equations, the pattern of the disturbances across equations does not change in the experiment.

As for the parametric method, simulation over the sample period can be either static or dynamic, as already described in section 5.3.

As usual in these sorts of methods, bootstrap and re-estimation is expected to be less efficient, but more robust, than the method based on parametric stochastic simulation. It is certainly less efficient if the process that generates the random error terms has been correctly specified (multivariate normal) because in such a case the parametric simulation makes use of more *a priori* information. It is expected to be more robust because the estimated residuals are suitable for the simulation phase even if the distribution underlying the error terms is not the assumed one. This problem is investigated in detail in Brown and Mariano (1984), as far as the first moment of the distribution of forecasts is concerned; extension of their result to second order moments seems feasible.

6. A BRIEF NOTE ON MINI-DMS MODEL FOR THE FRENCH ECONOMY

The Mini-DMS model (Brillet, 1981) constitutes a smaller version of the Dynamic Multi Sectorial model of the French economy (Fouquet et al., 1978) built in 1974-1976 at INSEE (National Institute for Statistics and Economic Studies) to be used as a medium term forecasting tool, in particular for national planning studies conducted through the Commissariat General au Plan (General Planning Agency). Largely reduced in size (the present version contains 220 equations, 65 of which are behavioral, as compared to more than 2400 for the larger version), Mini-DMS nevertheless preserves the same economic structure as well as most of the theoretical mechanism of the original model.

In its present state, the Mini-DMS model can be considered as being half way between a forecasting tool and a model for economic policy decisions. Its acceptable forecasting qualities, as well as its rather detailed set of decisional variables, can lead to its use for simple enough macro-economic studies, and for carrying out mathematical economic experiments.

Estimates of the structural coefficients of the model, on the sample period 1962-1980, were obtained by means of a straightforward extension of Brundy and Jorgenson's (1971) instrumental variables method (limited information) to the case of nonlinear models. The method was applied iteratively, till convergence was reached, so that the final estimates of coefficients are not affected by the choice of the values of the initial coefficients values. In each iteration, the instrumental variables are computed as the deterministic solution values of the system (which is the simplest choice, although not the *best* in the class of nonlinear estimators as is explained in Amemiya, 1983). Since the number of stochastic equations in the model was considerably larger than the sample period length, the estimate of the covariance matrix of the disturbance process was singular, and the standard system estimation methods could not be applied.

To represent the equations of the model, a Fortran code suitable for Gauss-Seidel solution method and for instrumental variables estimation has been used.

7. NOTATION FOR THE CASE OF DYNAMIC MODELS

Let us now represent the structural econometric model, linear or nonlinear in the variables as well as in the coefficients, explicitly distinguishing between exogenous and lagged endogenous variables

$$f(y_t, y_{t-1}, x_t, a) = u_t \qquad t = 1,2,...,T. \tag{18}$$

We proceed analogously to sections 2 and 3 assuming that a simultaneous equation system like (18) implicitly defines a single inverse relationship (reduced form) for relevant values of the coefficients, the predetermined variables, and any values of the disturbance terms:

(19) $$y_t = g(y_{t-1}, x_t, a, u_t).$$

Let the model be used to forecast at times 1, 2, ..., h, not belonging to the sample estimation period (the time subscripts related to the forecast periods should not be confused with the time subscripts related to the sample period). Given the values of the endogenous variables at time 0, y_0, and the values of the exogenous variables in the forecast periods, $x_1, x_2, ..., x_h$, then the values of the endogenous variables in the forecast periods can be obtained recursively as:

(20)
$$\begin{cases} y_1 = g(y_0, x_1, a, u_1) \\ y_2 = g(y_1, x_2, a, u_2) = g[g(y_0, x_1, a, u_1), x_2, a, u_2] = g^{(2)}(y_0, x_1, x_2, a, u_1, u_2) \\ \vdots \\ y_h = g(y_{h-1}, x_h, a, u_h) = g[g(\ldots), x_h, a, u_h] = g^{(h)}(y_0, x_1, \ldots, x_h, a, u_1, \ldots, u_h). \end{cases}$$

Equations (20) are used by model builders to produce multiperiod forecasts with dynamic simulation. The model builder first introduces values for y_0 and $x_1, x_2, ..., x_h$, assumed exact, sets the random error terms $u_1, u_2, ..., u_h$ to their expected value (zero), introduces the vector of estimated coefficients $\hat{a}$, and solves the dynamic system (18) at time 1, 2, ..., h. Using the recursive reduced form notation (20), forecasts are obtained as

(21)
$$\begin{cases} \hat{y}_1 = g(y_0, x_1, \hat{a}, 0) \\ \hat{y}_2 = g(\hat{y}_1, x_2, \hat{a}, 0) = g[g(y_0, x_1, \hat{a}, 0), x_2, \hat{a}, 0] = g^{(2)}(y_0, x_1, x_2, \hat{a}, 0, 0) \\ \vdots \\ \hat{y}_h = g(\hat{y}_{h-1}, x_h, \hat{a}, 0) = g[g(\ldots), x_h, \hat{a}, 0] = g^{(h)}(y_0, x_1, \ldots, x_h, \hat{a}, 0, \ldots, 0). \end{cases}$$

Analogously to section 3, the vector of forecast errors at the end of the dynamic simulation period is

(22)
$$\begin{aligned} \hat{y}_h - y_h &= g^{(h)}(y_0, x_1, \ldots, x_h, \hat{a}, 0, \ldots, 0) - g^{(h)}(y_0, x_1, \ldots, x_h, a, u_1, \ldots, u_h) \\ &= [g^{(h)}(y_0, x_1, \ldots, x_h, \hat{a}, 0, \ldots, 0) - g^{(h)}(y_0, x_1, \ldots, x_h, a, 0, \ldots, 0)] \\ &\quad + [g^{(h)}(y_0, x_1, \ldots, x_h, a, 0, \ldots, 0) - g^{(h)}(y_0, x_1, \ldots, x_h, a, u_1, \ldots, u_h)]. \end{aligned}$$

Having assumed exact knowledge of all the predetermined variables involved in our forecast (the starting endogenous variables y_0 and all the exogenous), the two components of the forecast error vector are independent, since the former depends on $\hat{a}$ (estimated on the sample period), while the latter depends on $u_1, ..., u_h$. We can, therefore, calculate the variances or the covariance matrices of the two components separately, and sum them to get the final results.

The second component, which is due to the random error terms $u_1, ..., u_h$, can be computed as in the static case by means of replicated parametric stochastic simulations on the forecast period, or by the mixed use of local linearization and parametric stochastic simulation (Monte Carlo with control variates). Extending the residual-based procedure seems plausible, but has not been performed in this paper.

Also for the first component the same methods applied for the static case are simply a straightforward extension to dynamic simulation.

8. RESULTS ON MINI-DMS MODEL FOR THE FRENCH ECONOMY

As already mentioned, the model has been estimated by means of instrumental variables (limited information) on the sample period 1962-1980. Static forecasts are related to the first year outside the sample estimation period, while dynamic simulation forecasts are related to the last of a 5-year simulation (1981-1985).

The tables present standard errors of forecasts (in percentage form) for some of the main endogenous variables of the model.

Q1	= added value in industrial sector
Q2	= added value in the other sectors
Q	= total added value
UT	= degree of capacity utilization
SALT	= wage rate
PU1	= price index of industrial products
PU2	= price index of other products
PDRE	= unemployment
PC	= consumption price index
TPRO1	= profit rate in industrial sector
TPRO2	= profit rate in other sectors
CFG	= government balance
CFX	= trade balance
PIBZ	= gross national product
X1	= exports of industrial products
C	= consumption
I	= investment

M = imports
N = employment

Each column of percentage standard errors has been computed by a different method. The first six columns in each table are related to the standard errors of the component of the forecast errors due to errors in estimated structural coefficients. The computation methods are indicated as follows (in parentheses some notes on the computation time of each experiment are displayed; CPU time is related to an IBM/3083).

Anal = Analytical method, or analytic simulation (2 minutes)
MCc = Monte Carlo on coefficients (1000 replications, 1% discarded; 10 minutes)
Psss = Parametric stochastic simulation and re-estimation in *static-static* version (500 replications, 5% discarded; 200 minutes)
Pssd = Parametric stochastic simulation and re-estimation in *dynamic-dynamic* version (500 replications, 6% discarded; 200 minutes)
Boots = Bootstrap and re-estimation (*static-static*, 500 replications, 8% discarded; 190 minutes)
Bootd = Bootstrap and re-estimation (*dynamic-dynamic*, 500 replications, 9% discarded; 190 minutes).

The last columns in each table are related to the standard errors of the component of the forecast errors due to the structural random error terms. The computation methods are indicated as follows.

Llin = Local linearization of the model in the neighborhood of the solution point in the forecast period (2 minutes)
Pss = Parametric stochastic simulation (static in one-period forecast, dynamic in multiperiod forecast, 1000 replications, 1% discarded; 10 minutes)
Rb = Residual-based procedure (static simulation only; 2 minutes).

A remarkably large number of replications had to be discarded in the four methods that require solutions of the model over the whole sample period. The fault is not of the methods themselves, but of the behavior of the model in the first years of the sample period, where convergence of the Gauss-Seidel solution algorithm is harder to achieve.

Table 1

One-period forecast at 1981. Static simulation. Summary table of differences among methods. Percentage standard errors computed by alternative methods.

	From coefficients						From error terms		
	Anal	*Psss*	*Pssd*	*Boots*	*Bootd*	*MCc*	*Llin*	*Pss*	*Rb*
Q1	.956	1.03	1.15	1.01	1.06	1.13	1.44	1.58	1.85
Q2	.414	.443	.479	.490	.506	.468	.555	.597	.658
Q	.530	.569	.635	.587	.616	.623	.767	.845	.965
UT	.799	.862	1.07	.952	.994	.872	1.52	1.56	1.93
SALT	1.77	2.02	1.99	2.11	1.88	2.00	2.86	3.06	4.33
PU1	1.40	1.56	1.64	1.63	1.53	1.85	3.64	4.14	8.29
PU2	1.20	1.30	1.36	1.39	1.33	1.56	2.15	2.34	3.22
PDRE	2.22	2.43	2.45	2.29	2.40	2.42	3.97	3.92	4.29
PC	1.38	1.52	1.53	1.53	1.42	1.63	3.00	3.23	5.00
TPRO1	15.9	17.0	16.9	17.9	20.8	16.2	36.2	39.6	96.7
TPRO2	9.25	8.94	9.24	9.90	8.79	7.98	21.1	22.9	55.0
CFG	15.3	17.1	17.5	16.6	16.2	16.5	30.2	30.7	40.3
CFX	22.1	26.6	28.1	29.5	25.2	24.9	53.4	55.1	51.7
PIBZ	.454	.491	.551	.505	.542	.552	.618	.664	.748
X1	1.60	1.69	2.17	1.84	2.06	1.81	2.57	2.61	3.56
C	.429	.530	.573	.464	.543	.575	.864	.947	1.27
I	2.53	2.86	2.95	4.07	3.14	2.61	3.96	4.34	6.12
M	1.55	1.75	2.26	1.85	1.95	1.48	3.18	3.23	3.23
N	.303	.330	.356	.336	.349	.343	.619	.631	.694

For some variables like TPRO1, TPRO2, CFG, and CFX it would be more appropriate to give standard errors of forecasts in absolute values, rather than in percentage form, given the nature of the variables themselves. They are nevertheless given in percentage form for homogeneity with the other variables.

With very few exceptions we have results that are sufficiently similar for each variable changing the algorithm, and this makes us sufficiently confident of the reliability of our results.

Let us first consider the one-period forecast errors. Concerning the error on coefficients, we can first observe that the variability among methods seems quite low: the relative difference between the lowest and highest values of the statistic for one specific variable stays generally under a 20% limit, much smaller than for the cases considered in Freedman and Peters (1984), or in Bianchi and Calzolari (1982, 1983). This would lead to the belief that the best method for producing these results, from a practical point of view, should in fact be the cheapest in terms of computations (analytic simulation or Monte Carlo on coefficients); of course this conclusion is valid only for this particular model.

Table 2

Forecast period 1985. Dynamic simulation from 1981. Summary table of differences among methods. Percentage standard errors computed by alternative methods.

	From coefficients						From error terms	
	Anal	*Psss*	*Pssd*	*Boots*	*Bootd*	*MCc*	*Llin*	*Pss*
Q1	2.81	3.02	3.28	3.40	3.36	3.42	2.49	2.69
Q2	1.18	1.35	1.28	1.61	1.51	1.37	1.10	1.21
Q	1.42	1.53	1.62	1.89	1.86	1.74	1.37	1.52
UT	1.34	1.46	1.48	2.15	1.38	1.55	1.57	1.58
SALT	8.05	8.46	9.64	8.38	7.75	8.70	5.38	5.71
PU1	5.04	5.28	5.75	5.79	4.98	6.42	4.91	5.97
PU2	6.51	6.77	7.00	6.71	5.86	8.10	4.15	4.53
PDRE	6.25	7.01	6.80	6.68	7.68	7.01	6.35	6.52
PC	6.27	6.61	6.55	6.42	5.55	7.49	4.45	4.77
TPRO1	86.8	97.0	87.9	92.6	94.7	89.6	80.1	90.5
TPRO2	12.6	11.7	12.5	16.2	10.1	11.6	15.7	17.2
CFG	43.3	45.0	56.6	51.3	46.5	54.0	55.8	57.0
CFX	34.3	35.5	39.1	63.2	38.6	40.8	38.5	39.8
PIBZ	1.19	1.26	1.38	1.68	1.66	1.50	1.22	1.32
X1	5.21	5.52	6.13	5.92	5.58	6.58	3.77	3.95
C	2.48	2.54	2.76	2.65	2.71	2.35	2.29	2.48
I	8.85	8.75	9.68	15.5	9.46	9.06	7.34	8.40
M	3.13	3.31	3.92	4.72	4.02	3.35	3.58	3.39
N	1.17	1.24	1.31	1.48	1.51	1.41	1.22	1.30

From an economic point of view, the conclusions concerning the validity of the model can then be drawn from any of the methods: we could say that the model seems to be acceptably precise both on quantities and prices, the percentage error on global activity (PIBZ) being much lower than its individual elements, exports (X1), household consumption (C), productive investment (I), and imports (M), and also lower than the weighted average of the error on products Q1 (industrial), Q2 (non-industrial), and the non-market one.

Now if we look into the relative values of the statistics, some constant features stand out:

- Analytic simulation usually gives the lowest value (which could mean that it is somewhat downward biased). This may be due to the linearization feature introduced in the process, and in particular to the subsequent elimination of the bias.
- Parametric stochastic simulation-static (Psss) never gives the highest value; we can see in fact that it is usually dominated by the dynamic version of the same statistic; this could be expected, as the large number of positive autocorrelations in the model gives a larger variance in the dynamic case to the set of ex post simulated variables, and thus to the estimated coefficients and the forecasted values.
- Static bootstrap statistics (Boots) are generally higher than equivalent parametric stochastic simulation ones, but the same observation does not hold if we consider the dynamic case; no clear explanation for these phenomena can be found.
- As to Monte Carlo on coefficients (MCc), let us say only that its statistics are quite comparable with the others, except perhaps for prices, where they appear somewhat higher.

The interpretation of the uncertainty due to the direct influence of the error terms is much simpler: the first two methods give comparable statistics, quite higher than the statistics related to coefficients errors discussed above; they are even more than double sometimes, especially when prices are involved. The most peculiar feature is the much greater degree of uncertainty (frequently at least 50% higher) that is measured using the residual-based procedure (Rb). This might be explained at least partially by an error on the homoskedasticity of residuals hypothesis.

Let us now consider the dynamic forecast errors. Looking first at the errors on coefficients, we can make the same general observation as before: the statistics do not change much with the method used, although the gap is somewhat widened (most frequently to 30%); this, however, is not so large as to make a big difference in the eyes of a decision maker.

Considering the hierarchy among methods:

- The first method still gives the lowest statistics.
- In the parametric stochastic simulation case, the order is less clear than before: we can imagine that the dynamic method estimates should be better adapted to dynamic forecasts, for instance by giving more importance to dynamic estimated equations. It would be interesting to see if this observation holds with longer forecast periods.
- The fact that the bootstrap static method (Boots) gives generally higher values appears now clearer than before (it comes first 8 times out of 19); in the case of firms' investment (I) it is more than 50% larger than the dynamic version of bootstrap, while the other methods show variations lower than 10%.
- The results of the Monte Carlo on coefficients method (MCc) appear to be somewhat higher than average.

From an economic point of view, the results still do not look too bad, especially if we take into account the fact that the decision maker considers more frequently the rates of growth of variables than their actual values, and that we can expect the errors shown to be strongly and positively autocorrelated.

As to the uncertainty coming from the error terms, we can see that it appears lower on the whole than the one coming from coefficients. As observed in Bianchi, et al. (1984), the identity of the error on coefficients over time, coupled with the strong positive autocorrelation of some relations, introduces an autocorrelation on the errors due to coefficients that is of course absent here. And as before, the first statistic gives lower values than the other.

9. CONCLUDING REMARKS

Two conclusions may be drawn from the experiments discussed in this paper. Both of them hold for the model here considered, and not necessarily in general.

1) The question we asked initially was how confident the decision maker can be in the forecasting properties of this model, and how much reliance he can place on the measurement of the degree of uncertainty associated with the forecasts. The differences observed in the statistics are small enough for this model to suggest that, in changing the method, the decision maker should not alter significantly his judgment.
2) The numerical values of the standard errors may appear somewhat surprising. They tend to be larger than expected for forecasts one-period ahead. For example, for the first variable, Q1, which is the added value in the industrial sector, considering the two components we have a total standard error approximately equal to 2% of the predicted value, which implies a confidence interval of forecasts approximately equal to ±4%, which is rather large. The forecaster is probably able to predict with a smaller confidence interval, but this is usually because forecasts, in practice, are not based only on the model, but make use of additional a priori information not included in the model. This reasonably explains why forecasts are usually better than they might be expected from the rough analysis of the model's forecasting properties.

   It seems to be the opposite for the results of dynamic simulation forecasts 5 years ahead. Here the standard errors probably seem to be too small for such a long forecasting horizon. But this is again a consequence of the assumptions we make; we are in fact considering forecasts conditional on the exact knowledge of all the predetermined variables, and predetermined variables in forecasting activity are one of the main unknowns of the problem. What we are obtaining should, therefore, be considered similar to lower bounds for the confidence intervals of our forecasts.

APPENDIX 1 - Static notation

(A1.1)
$$\left[\begin{array}{l} C_t = a_1 + a_2P_t + a_3P_{t-1} + a_4(W1 + W2)_t + u_{1t} \\ I_t = a_5 + a_6P_t + a_7P_{t-1} + a_8K_{t-1} + u_{2t} \\ W1_t = a_9 + a_{10}(Y + T - W2)_t + \\ \qquad + a_{11}(Y + T - W2)_{t-1} + a_{12}t + u_{3t} \\ Y_t = C_t + I_t + G_t - T_t \\ P_t = Y_t - W1_t - W2_t \\ K_t = K_{t-1} + I_t \end{array}\right.$$

Klein - I model
m = 6 equations
n = 8 predetermined variables

(A1.2)
$$Ay_t + Bx_t = u_t \qquad t = 1, 2, ..., T$$

(A1.3)
$$A = \begin{bmatrix} 1 & 0 & -a_4 & 0 & -a_2 & 0 \\ 0 & 1 & 0 & 0 & -a_6 & 0 \\ 0 & 0 & 1 & -a_{10} & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$
dimensions m × m

(A1.4)
$$B = \begin{bmatrix} -a_1 & -a_3 & -a_4 & 0 & 0 & 0 & 0 & 0 \\ -a_5 & -a_7 & 0 & -a_8 & 0 & 0 & 0 & 0 \\ -a_9 & -a_{11} & a_{10} & 0 & -a_{10} & -a_{11} & -a_{12} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
dimensions m × n

(A1.5)
$$\Pi = -A^{-1}B$$

(A1.6)
$$y_h = \Pi x_h + A^{-1}u_h.$$

(A1.7)
$$\bar{y}_h = \Pi x_h$$

(A1.8)
$$\hat{y}_h = \hat{\Pi} x_h.$$

(A1.9)
$$\hat{y}_h - y_h = (\hat{y}_h - \bar{y}_h) + (\bar{y}_h - y_h) = (\hat{\Pi} - \Pi)x_h - A^{-1}u_h.$$

(A1.10)
$$V(\bar{y}_h - y_h) = A^{-1}\Sigma A'^{-1}$$

(A1.11) $$\hat{y}_h - \bar{y}_h = (\hat{\Pi} - \Pi)x_h = I_m(\hat{\Pi} - \Pi)x_h = vec\,[I_m(\hat{\Pi} - \Pi)x_h] = (x_h' \otimes I_m)[vec\,\hat{\Pi} - vec\,\Pi]$$

(A1.12) $$\gamma = \begin{bmatrix} vec\,A \\ vec\,B \end{bmatrix}$$

(A1.13) $$\sqrt{T}\,[\hat{\gamma} - \gamma] \xrightarrow[T \to \infty]{} N(0, \Phi)$$

The matrix Φ has dimensions $(mm + mn) \times (mm + mn)$

(A1.14) $$J = \frac{\partial(vec\,\Pi)}{\partial\gamma'} = \frac{\partial(vec\,\Pi)}{\partial[(vec\,A)' ; (vec\,B)']} = -[\Pi' ; I_n] \otimes A^{-1}$$

(A1.15) $$\sqrt{T}\,(vec\,\hat{\Pi} - vec\,\Pi) \xrightarrow[T \to \infty]{} N(0, \Omega)$$

(A1.16) $$\Omega = J\,\Phi\,J' = \left\{[\Pi'; I_n] \otimes A^{-1}\right\} \Phi \left\{[\Pi'; I_n]' \otimes A'^{-1}\right\}$$ *dimensions* $mn \times mn$

(A1.17) $$(x_h' \otimes I_m)\,(\hat{\Omega}/T)\,(x_h \otimes I_m)$$ *Goldberger, Nagar, and Odeh* (1961)

Alternative method (Calzolari, 1981). Let $a = (a_1, a_2, ...a_s)'$ be the vector $(s \times 1)$ of the unknown structural coefficients to be estimated (and let $\hat{a}$ be its estimate); let be $C = [A\ ;\ B]$, matrix of dimensions $[m \times (m + n)]$, and let be $C_0, C_1, C_2, ..., C_s$, each of dimensions $[m \times (m + n)]$, matrices such that

(A1.18) $$C = C_0 + C_1 a_1 + C_2 a_2 + ... + C_s a_s$$

(A1.19) $$e.g. \quad C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(A1.20) $$\left[\begin{array}{l} vec\,C = vec\,C_0 + (vec\,C_1)a_1 + ... + (vec\,C_s)a_s \\ vec\,\hat{C} = vec\,C_0 + (vec\,C_1)\hat{a}_1 + ... + (vec\,C_s)\hat{a}_s \end{array}\right.$$

(A1.21) $$\left[\begin{array}{l} vec\,C = vec\,C_0 + [vec\,C_1; vec\,C_2; ....; vec\,C_s]a \\ vec\,\hat{C} = vec\,C_0 + [vec\,C_1; vec\,C_2; ....; vec\,C_s]\hat{a} \end{array}\right.$$

(A1.22) $$\underset{T \to \infty}{plim}\,\hat{\Pi} = \Pi$$

(A1.23) $$\underset{T \to \infty}{plim}\,\hat{y}_h = \bar{y}_h$$

(A1.24) $$\hat{y}_h - \bar{y}_h = (\hat{\Pi} - \Pi)x_h = -\left[\hat{A}^{-1}\hat{B}x_h - A^{-1}Bx_h\right] =$$

(subtract and add, inside brackets, $A^{-1}\hat{B}x_h$)

$$= -\left[-A^{-1}\hat{B}x_h + A^{-1}\hat{B}x_h + \hat{A}^{-1}\hat{B}x_h - A^{-1}Bx_h\right] = -\left[A^{-1}\hat{A}(-\hat{A}^{-1}\hat{B})x_h + A^{-1}\hat{B}x_h - A^{-1}A(-\hat{A}^{-1}\hat{B})x_h - A^{-1}Bx_h\right]$$

$$= -\left[A^{-1}\hat{A}\hat{\Pi}x_h + A^{-1}\hat{B}x_h - A^{-1}A\hat{\Pi}x_h - A^{-1}Bx_h\right] = -\left\{A^{-1}[\hat{A}; \hat{B}]\begin{bmatrix}\hat{\Pi}x_h \\ x_h\end{bmatrix} - A^{-1}[A; B]\begin{bmatrix}\hat{\Pi}x_h \\ x_h\end{bmatrix}\right\} = -A^{-1}(\hat{C} - C)\begin{bmatrix}\hat{\Pi}x_h \\ x_h\end{bmatrix}$$

$$= -A^{-1}(\hat{C} - C)\begin{bmatrix}\hat{\Pi} \\ I_n\end{bmatrix}x_h = -\left\{[x'_h(\hat{\Pi}'; I_n)] \otimes A^{-1}\right\}(vec\,\hat{C} - vec\,C) = -\left\{[x'_h(\hat{\Pi}'; I_n)] \otimes A^{-1}\right\}[vec\,C_1; vec\,C_2; ....; vec\,C_s]\,(\hat{a} - a)$$

$$= -\left\{A^{-1}C_1\begin{bmatrix}\hat{\Pi} \\ I_n\end{bmatrix}x_h ; A^{-1}C_2\begin{bmatrix}\hat{\Pi} \\ I_n\end{bmatrix}x_h ; ... ; A^{-1}C_s\begin{bmatrix}\hat{\Pi} \\ I_n\end{bmatrix}x_h\right\}(\hat{a} - a) = -A^{-1}\left\{C_1\begin{bmatrix}\hat{y}_h \\ x_h\end{bmatrix}; C_2\begin{bmatrix}\hat{y}_h \\ x_h\end{bmatrix}; ... ; C_s\begin{bmatrix}\hat{y}_h \\ x_h\end{bmatrix}\right\}(\hat{a} - a) = -A^{-1}\hat{F}_h(\hat{a} - a)$$

having defined

(A1.25) $$\hat{F}_h = \left\{C_1\begin{bmatrix}\hat{y}_h \\ x_h\end{bmatrix}; C_2\begin{bmatrix}\hat{y}_h \\ x_h\end{bmatrix}; ... ; C_s\begin{bmatrix}\hat{y}_h \\ x_h\end{bmatrix}\right\}$$

(A1.26) $$\underset{T \to \infty}{plim}\ \hat{F}_h = \bar{F}_h = \left\{ C_1 \begin{bmatrix} \bar{y}_h \\ x_h \end{bmatrix} ;\ C_2 \begin{bmatrix} \bar{y}_h \\ x_h \end{bmatrix} ;\ \dots ;\ C_s \begin{bmatrix} \bar{y}_h \\ x_h \end{bmatrix} \right\}$$

(A1.27) $$\sqrt{T}(\hat{y}_h - \bar{y}_h) = \left[-A^{-1}\hat{F}_h \right] \sqrt{T}(\hat{a} - a)$$

(A1.28) $$\sqrt{T}(\hat{y}_h - \bar{y}_h) \xrightarrow[T \to \infty]{} N(0, A^{-1}\bar{F}_h \Psi \bar{F}_h' A'^{-1})$$

(A1.29) $$\hat{F}_h = - \begin{bmatrix} 1 & \hat{P}_h & P_{h-1} & \hat{W}1_h + W2_h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \hat{P}_h & P_{h-1} & K_{h-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (\hat{Y} + T - W2)_h & (Y + T - W2)_{h-1} & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(A1.30) $$\hat{A}^{-1}\hat{F}_h(\hat{\Psi}/T)\hat{F}_h'\hat{A}'^{-1}$$

(A1.31) $$C_k = \frac{\partial C}{\partial a_k} = \frac{\partial [A\ ;\ B]}{\partial a_k}$$

(A1.32) $$A\bar{y}_h + Bx_h = [A\ ;\ B] \begin{bmatrix} \bar{y}_h \\ x_h \end{bmatrix} = C \begin{bmatrix} \bar{y}_h \\ x_h \end{bmatrix}$$

(A1.33) $$\bar{F}_h = \frac{\partial [A\bar{y}_h + Bx_h]}{\partial a'}$$

(A1.34) $$A\bar{y}_h + Bx_h \equiv 0$$

(A1.35) $$\frac{\partial (A\bar{y}_h + Bx_h)}{\partial a'} + \frac{\partial (A\bar{y}_h + Bx_h)}{\partial \bar{y}_h'} \cdot \frac{\partial \bar{y}_h}{\partial a'} = 0$$

(A1.36) $$\bar{F}_h + A\frac{\partial \bar{y}_h}{\partial a'} = 0$$

(A1.37) $$-A^{-1}\bar{F}_h = \frac{\partial \bar{y}_h}{\partial a'}$$

APPENDIX 2 - Dynamic notation

(A2.1) $$Ay_t + Bx_t + Dy_{t-1} = u_t \qquad t = 1,2,\dots,T$$

(A2.2) $$\Pi_1 = -A^{-1}B$$

(A2.3) $$\Pi_0 = -A^{-1}D$$

(A2.4) $$\Pi = [\Pi_1\ ;\ \Pi_0] = -A^{-1}[B\ ;\ D]$$

(A2.5) $$y_t = \Pi_1 x_t + \Pi_0 y_{t-1} + A^{-1}u_t = \Pi_1 x_t + \Pi_0 \Pi_1 x_{t-1} + \Pi_0^2 y_{t-2} + A^{-1}u_t + \Pi_0 A^{-1}u_{t-1} = \sum_{r=0}^{q-1} \Pi_0^r \Pi_1 x_{t-r} + \Pi_0^q y_{t-q} + \sum_{r=0}^{q-1} \Pi_0^r A^{-1} u_{t-r}$$

Forecasting periods: 1, 2, ..., h (no relationship with the sample estimation period)

(A2.6) $$\begin{cases} \hat{y}_1 = \hat{\Pi}_1 x_1 + \hat{\Pi}_0 y_0 \\ \hat{y}_2 = \hat{\Pi}_1 x_2 + \hat{\Pi}_0 \hat{y}_1 \\ \vdots \\ \hat{y}_h = \hat{\Pi}_1 x_h + \hat{\Pi}_0 \hat{y}_{h-1} \end{cases}$$

(A2.7) $$\begin{cases} \bar{y}_1 = \Pi_1 x_1 + \Pi_0 y_0 \\ \bar{y}_2 = \Pi_1 x_2 + \Pi_0 \bar{y}_1 \\ \vdots \\ \bar{y}_h = \Pi_1 x_h + \Pi_0 \bar{y}_{h-1} \end{cases}$$

(A2.8) $$\bar{y}_h = \sum_{i=0}^{h-1} \Pi_0^i \Pi_1 x_{h-i} + \Pi_0^h y_0$$

(A2.9) $$\hat{y}_h = \sum_{i=0}^{h-1} \hat{\Pi}_0^i \hat{\Pi}_1 x_{h-i} + \hat{\Pi}_0^h y_0$$

(A2.10) $$y_h = \sum_{i=0}^{h-1} \Pi_0^i \Pi_1 x_{h-i} + \Pi_0^h y_0 + \sum_{i=0}^{h-1} \Pi_0^i A^{-1} u_{h-i}$$

(A2.11) $$\hat{y}_h - y_h = (\hat{y}_h - \bar{y}_h) + (\bar{y}_h - y_h) = \left[\sum_{i=0}^{h-1} \left(\hat{\Pi}_0^i \hat{\Pi}_1 - \Pi_0^i \Pi_1 \right) x_{h-i} + \left(\hat{\Pi}_0^h - \Pi_0^h \right) y_0 \right] - \left[\sum_{i=0}^{h-1} \Pi_0^i A^{-1} u_{h-i} \right]$$

(A2.12) $$V(\bar{y}_h - y_h) = \sum_{i=0}^{h-1} \left[\Pi_0^i A^{-1} \Sigma A'^{-1} \Pi_0'^{i} \right]$$

(A2.13) $$w_h = \begin{bmatrix} y_0 \\ x_1 \\ \vdots \\ x_h \end{bmatrix}$$

(A2.14) $$\hat{H}_h = \left[\hat{\Pi}_0^h ; \ \hat{\Pi}_0^{h-1} \hat{\Pi}_1 ; \ \hat{\Pi}_0^{h-2} \hat{\Pi}_1 ; \ \dots ; \ \hat{\Pi}_0 \hat{\Pi}_1 ; \ \hat{\Pi}_1 \right]$$ *dimensions* $m \times (m + hn)$

(A2.15) $$\hat{y}_h = \hat{H}_h w_h$$

(A2.16) $$\bar{y}_h = H_h w_h.$$

(A2.17) $$\hat{y}_h - \bar{y}_h = (\hat{H}_h - H_h) w_h = vec \left[I_m (\hat{H}_h - H_h) w_h \right] = (w_h' \otimes I_m) \left[vec\, \hat{H}_h - vec\, H_h \right]$$

(A2.18) $$\hat{\bar{M}}_h = \left[0_{mm,mn} ; \ \sum_{i=0}^{h-1} \left((\Pi_0^{h-1-i})' \otimes \hat{\Pi}_0^i \right) \right]$$

(A2.19) $$\hat{\bar{N}}_k = \left[I_n \otimes \Pi_0^k ; \ \sum_{j=0}^{k-1} \left((\Pi_0^{k-1-j} \hat{\Pi}_1)' \otimes \hat{\Pi}_0^j \right) \right]$$

(A2.20) $$N_0 = \left[I_{mn} ; \ 0_{mn,mm} \right]$$

(A2.21) $$\hat{\bar{Q}}_h = \begin{bmatrix} \hat{\bar{M}}_h \\ \hat{\bar{N}}_{h-1} \\ \hat{\bar{N}}_{h-2} \\ \vdots \\ N_0 \end{bmatrix}$$

(A2.22) $$\sqrt{T}\,(vec\, \hat{H}_h - vec\, H_h) = \hat{\bar{Q}}_h \sqrt{T}\, vec\,(\hat{\Pi} - \Pi) \quad where \quad \sqrt{T}\, vec\,(\hat{\Pi} - \Pi) \xrightarrow[T \to \infty]{} N(0, \Omega)$$ *Schmidt* (1973, 1974)

(A2.23) $$\sqrt{T}\,(\hat{y}_h - \bar{y}_h) \xrightarrow[T \to \infty]{} N\left\{ 0, \left[(w_h' \otimes I_m) Q_h \Omega Q_h' (w_h \otimes I_m) \right] \right\}$$

Alternative method (Calzolari, 1987).

(A2.24) $$C = [A ; B ; D] = C_0 + C_1 a_1 + C_2 a_2 + \dots + C_s a_s$$

(A2.25) $$\hat{F}_{h-i} = \left\{ C_1 \begin{bmatrix} \hat{y}_{h-i} \\ x_{h-i} \\ \hat{y}_{h-i-1} \end{bmatrix} ; \ C_2 \begin{bmatrix} \hat{y}_{h-i} \\ x_{h-i} \\ \hat{y}_{h-i-1} \end{bmatrix} ; \ \dots ; \ C_s \begin{bmatrix} \hat{y}_{h-i} \\ x_{h-i} \\ \hat{y}_{h-i-1} \end{bmatrix} \right\}$$

(A2.26) $$\sqrt{T} [\hat{\Pi} - \Pi] \begin{bmatrix} x_{h-i} \\ \hat{y}_{h-i-1} \end{bmatrix} = -A^{-1} \hat{F}_{h-i} \sqrt{T} (\hat{a} - a)$$

(A2.27)
$$\sqrt{T}(\hat{y}_h - \bar{y}_h) = \sqrt{T}\left[\sum_{i=0}^{h-1}\hat{\Pi}_0^i\hat{\Pi}_1 x_{h-i} + \hat{\Pi}_0^h y_0 - \bar{y}_h\right] = \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i\hat{\Pi}_1 x_{h-i} + \sqrt{T}\left[\hat{\Pi}_0^h y_0 - \bar{y}_h\right]$$

$$= \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i\hat{\Pi}_1 x_{h-i} + \sqrt{T}\left[(\hat{\Pi}_0\bar{y}_{h-1} - \bar{y}_h) + (\hat{\Pi}_0^2\bar{y}_{h-2} - \hat{\Pi}_0\bar{y}_{h-1})\right.$$

$$\left. + (\hat{\Pi}_0^3\bar{y}_{h-3} - \hat{\Pi}_0^2\bar{y}_{h-2}) + \ldots + (\hat{\Pi}_0^{h-1}\bar{y}_1 - \hat{\Pi}_0^{h-2}\bar{y}_2) + (\hat{\Pi}_0^h y_0 - \hat{\Pi}_0^{h-1}\bar{y}_1)\right]$$

$$= \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i\hat{\Pi}_1 x_{h-i} + \sqrt{T}\sum_{i=0}^{h-1}\left[\hat{\Pi}_0^{i+1}\bar{y}_{h-i-1} - \hat{\Pi}_0^i\bar{y}_{h-i}\right] = \sqrt{T}\sum_{i=0}^{h-1}\left[\hat{\Pi}_0^i\hat{\Pi}_1 x_{h-i} + \hat{\Pi}_0^{i+1}\bar{y}_{h-i-1} - \hat{\Pi}_0^i\bar{y}_{h-i}\right]$$

$$= \sqrt{T}\sum_{i=0}^{h-1}\left\{\hat{\Pi}_0^i[\hat{\Pi}_1; \hat{\Pi}_0]\begin{bmatrix}x_{h-i}\\ \bar{y}_{h-i-1}\end{bmatrix} - \hat{\Pi}_0^i\bar{y}_{h-i}\right\} = \sqrt{T}\sum_{i=0}^{h-1}\left\{\hat{\Pi}_0^i\hat{\Pi}\begin{bmatrix}x_{h-i}\\ \bar{y}_{h-i-1}\end{bmatrix} - \hat{\Pi}_0^i\bar{y}_{h-i}\right\}$$

(A2.28)
$$\sqrt{T}(\hat{y}_h - \bar{y}_h) = \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i\hat{\Pi}\begin{bmatrix}x_{h-i}\\ \bar{y}_{h-i-1}\end{bmatrix} - \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i\Pi\begin{bmatrix}x_{h-i}\\ \bar{y}_{h-i-1}\end{bmatrix} = \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i(\hat{\Pi} - \Pi)\begin{bmatrix}x_{h-i}\\ \bar{y}_{h-i-1}\end{bmatrix}$$

$$= \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i(\hat{\Pi} - \Pi)\begin{bmatrix}x_{h-i}\\ \hat{y}_{h-i-1}\end{bmatrix} - \sqrt{T}\sum_{i=0}^{h-1}\hat{\Pi}_0^i(\hat{\Pi} - \Pi)\begin{bmatrix}0\\ \hat{y}_{h-i-1} - \bar{y}_{h-i-1}\end{bmatrix}$$

(A2.29)
$$\left[\sum_{i=0}^{h-1}\hat{\Pi}_0^i A^{-1}\hat{F}_{h-i}\right]\sqrt{T}(\hat{a} - a)$$

(A2.30)
$$\left[\sum_{i=0}^{h-1}\Pi_0^i A^{-1}\bar{F}_{h-i}\right]\Psi\left[\sum_{i=0}^{h-1}\Pi_0^i A^{-1}\bar{F}_{h-i}\right]'$$

Table 3
Calzolari (1979)

Klein–Goldberger (revised) model. Bias of deterministic simulation at 1965 (all values are multiplied by 1000)

	Observed value	One-step simulation at 1965 bias	Dynamic simulation from 1960 bias	Dynamic simulation from 1957 bias
Cd	59600	−6.393 (0.567)	130.5 (8.66)	601.5 (37.6)
Ct	308100	−6.919 (0.614)	225.3 (17.1)	1207.0 (74.8)
R	20600	−1.292 (0.115)	35.04 (2.47)	173.4 (10.7)
H	7650	−0.1783 (0.016)	7.612 (0.482)	34.70 (2.02)
Im	32700	−1.370 (0.373)	−34.84 (5.19)	128.6 (16.1)
X	539200	−15.02 (1.33)	482.9 (29.5)	2099.0 (124.0)
h	103800	−3.441 (0.305)	111.7 (6.42)	426.3 (24.7)
W	315100	−7.453 (0.659)	257.5 (15.8)	1129.0 (66.2)
w	5492	−0.0788 (0.007)	2.461 (0.372)	23.61 (1.64)
r	4640	0.0	0.0	0.0
I	50000	0.0	20.57 (1.48)	104.2 (5.97)
D	55700	0.0	0.0	0.0
Rs	4380	0.0	0.0	0.0
Pc	51500	145.7 (5.34)	2301.0 (110.0)	7689.0 (462.0)
Nw	68000	−3.614 (0.320)	108.6 (7.02)	494.5 (29.4)
Y	374200	−27.63 (2.45)	604.7 (40.9)	2857.0 (178.0)
p	1228	−0.0524 (0.022)	−2.149 (0.223)	3.020 (0.572)
Sc	12900	154.5 (5.51)	2045.0 (95.8)	6626.0 (399.0)
π	95300	113.6 (4.26)	1870.0 (91.2)	6402.0 (386.0)
πr	33600	−28.37 (1.05)	−282.3 (13.2)	−794.4 (47.3)

Table 4
Brillet, Calzolari, and Panattoni (1986)

One-period forecast at 1981. Static simulation

Percentage deviations from deterministic forecasts

	Determin. Forecast	*Pss*	*Rb*	*Rba*	*Mode*
Q1	258326.0	0.100	0.286	0.114	0.056
QO2	536543.0	0.064	0.073	0.059	-0.008
UT	.8080540	0.087	0.273	0.101	0.043
SALT	66.72890	0.061	-0.580	-0.063	0.106
PU1	2.514670	-0.464	-1.520	-0.713	0.011
PU2	3.079440	-0.054	-0.665	-0.131	-0.009
ROM	2296130	0.047	-0.705	-0.075	0.014
CM	1981490.	0.039	-0.580	-0.086	-0.028
C1	216234.0	0.309	0.573	0.345	0.142
C2	494296.0	0.100	0.187	0.112	0.007
N1	4581.480	0.057	0.104	0.065	0.024
N2	13025.80	0.035	-0.090	0.034	-0.040
PDRE	1636.860	-0.158	-0.155	-0.173	0.247
OEFM	58.04020	-1.110	4.140	-2.004	12.603
W1	1.900640	0.036	-0.635	-0.089	0.074
W2	1.534870	0.073	-0.567	-0.050	0.118
PVA1	2.468430	0.054	-0.631	-0.052	-0.032
PVA2	2.839010	0.032	-0.784	-0.034	-0.077
PC	2.758950	-0.180	-0.938	-0.301	-0.075
TEP	.1125830	0.490	-0.647	0.281	0.331
IL2M	59275.80	0.039	-0.513	0.038	0.061
BF1	60955.80	-1.196	-11.177	-5.562	0.003
BF2	95328.70	2.853	17.230	8.570	1.124
AUT1	99158.80	1.093	5.278	3.888	0.223
AUT2	220742.0	-2.460	-15.167	-6.697	-0.849
TPRO1	.0298494	3.763	18.860	9.928	0.977
TPRO2	.0487430	-2.163	-11.816	-5.714	-1.178
DF1	336288.0	0.113	0.361	0.134	0.079
DF2	576187.0	0.057	0.032	0.048	-0.020
PIB	3148560.	-0.079	-0.941	-0.224	-0.049
CFG	-83026.60	-1.391	-4.776	-2.018	-0.757
CFM	142544.0	0.725	-1.923	0.108	0.665
CFX	-34661.10	3.629	8.762	2.062	0.665
PIBZ	1132290.	0.062	0.117	0.060	-0.001
PPIB	2.780710	-0.154	-1.087	-0.303	-0.047
CFF	62106.10	-2.343	8.615	3.599	-1.179
I1	35797.10	0.510	2.159	1.091	0.411
I2	83001.00	-0.512	-2.539	-1.369	-0.284
M1	180087.0	0.079	-0.021	0.026	0.011
M2	73167.80	0.109	0.126	0.110	0.690
X1	199824.0	-0.047	0.522	0.060	0.019
C	710529.0	0.164	0.305	0.183	0.048
I	118798.0	-0.204	-1.333	-0.627	-0.075
M	253255.0	0.088	0.021	0.034	0.034
N	17607.30	0.041	-0.039	0.042	-0.023

Pss = Percentage difference between the deterministic solution forecast and the estimate of the conditional mean, computed with parametric stochastic simulation (40000 couples of replications with antithetic variates)

Rb = Percentage difference between the deterministic solution forecast and the estimate of the conditional mean, computed with the residual-based procedure.

Rba = Percentage difference between the deterministic solution forecast and the estimate of the conditional mean, computed with the antithetic residual-based procedure.

Mode = Percentage difference between the deterministic solution forecast and the estimate of the most likely joint value.

Table 5
Bianchi, Calzolari, and Corsi (1980)

Variab. Name	Computed Value	Stochastic Simulation Number of Replications 50	500	5000	50000	Analytic Simulation
Cd	55.33	2.78	2.48	2.44	2.42	2.42
X	530.1	9.05	8.44	8.52	8.54	8.53
W	310.8	5.24	4.77	4.73	4.77	4.78
Pc	41.97	6.44	6.21	6.16	6.11	6.11
p	1.225	.040	.035	.036	.036	.036

Table 6
Bianchi and Calzolari (1980)

KLEIN-GOLDBERGER REVISED MODEL. ONE-PERIOD FORECASTS FOR 1965

	(1) Observ.	(2) Comput.	(3) Forecast Error	(4) I compon. Variance	(5) II compon. Variance (Lineariz.)	(6) II compon. Variance (Stoc. Sim.)	(7) Approx. Comput. $E(\hat{Y}_t - Y_t \mid X_t, \hat{A})$	(8) Std. Dev. of (7)	(9) Asymp. St. Er. of Forecast Error
C_d	59.6	55.3	−4.27	1.24	5.88	5.86	−.006	.005	2.67
C_n	308.1	303.4	−4.73	2.17	13.0	12.9	.002	.008	3.89
R	20.6	22.4	1.84	.302	1.42	1.42	−.004	.003	1.31
H	7.65	7.55	−0.10	.0095	.0627	.0628	.00009	.0006	.269
I_m	32.7	30.4	2.33	.489	1.23	1.23	.0003	.002	1.31
X	539.2	530.1	−9.15	12.4	72.7	72.8	−.007	.02	9.22
h	101.8	101.6	−2.18	.688	5.46	5.46	−.01	.005	2.48
W	315.1	310.8	−4.34	3.85	22.9	22.8	.002	.01	5.17
w	5.492	5.518	.0256	.00048	.00613	.00612	.0001	.0002	.081
r	4.64	4.89	.255	.0198	.127	.126	−.0004	.0008	.383
I	50.0	46.7	3.27	1.13	5.61	5.63	.007	.005	2.60
D	55.7	59.1	3.39	.489	1.57	1.56	.001	.003	1.43
r_s	4.38	4.63	.253	.0237	.119	.119	.00007	.0008	.378
P_c	51.5	42.0	9.53	12.4	37.4	37.4	.14	.01	7.06
N_w	68.0	67.9	−.103	.501	1.96	1.95	−.005	.003	1.57
Y	374.2	369.9	−4.29	6.74	37.0	37.0	.01	.01	6.61
p	1.228	1.225	−.003	.00047	.00129	.00128	.0001	.00008	.042
S_c	12.9	1.9	11.0	12.6	36.0	36.0	.15	.01	6.97
H	95.3	86.2	9.10	10.0	33.8	33.9	.11	.01	6.62
H_r	33.6	33.8	.165	.322	1.10	1.10	−.03	.002	1.19

Table 7
Calzolari and Sterbenz (1986)

KLEIN-GOLDBERGER REVISED MODEL (SEE KLEIN (1969))
REDUCED FORM VARIANCES AT 1965; 100 MONTE CARLO REPLICATIONS

	Static simulation: Det. sol. y (1)	Red. form variance $\widehat{Var}(y)$ (2)	Lin. red. form var. (3)	Std. dev. of $\widehat{Var}(y)$ MC with CV (4)	Std. dev. of $\widehat{Var}(y)$ MC (5)	Dynamic simulation from 1962: Det. sol. y (1)	Red. form variance $\widehat{Var}(y)$ (2)	Lin. red. form var. (3)	Std. dev. of $\widehat{Var}(y)$ MC with CV (4)	Std. dev. of $\widehat{Var}(y)$ MC (5)
Cd	55.326	5.8656	5.8755	.78424e−2	.57639	50.535	14.157	14.145	.75652e−1	1.3748
Cn	303.37	12.949	12.962	.14919e−1	1.6229	292.29	78.436	78.245	.34568	10.259
R	22.439	1.4211	1.4211	.76771e−3	.21917	21.074	3.0744	3.0715	.91812e−2	.34493
H	7.5463	.62654e−1	.62657e−1	.30255e−4	.10618e−1	7.1196	.15510	.15467	.35269e−3	.26052e−1
Im	30.372	1.2284	1.2334	.21174e−2	.20584	30.557	6.3078	6.3772	.80744e−1	.98031
X	530.05	72.677	72.714	.84397e−1	9.2919	507.62	276.91	276.40	.89720	33.596
100h	101.62	5.4648	5.4621	.63251e−2	.74862	101.27	7.2397	7.2116	.55033e−1	1.0058
W	310.76	22.845	22.869	.19377e−1	2.7963	298.29	87.848	87.775	.25755	11.850
w	5.5176	.61212e−2	.61254e−2	.49043e−5	.94735e−3	5.5235	.65039e−1	.65240e−1	.24899e−3	.62783e−2
r	4.8949	.12662	.12662	.0	.15088e−1	5.2895	.30117	.30117	.0	.32614e−1
I	46.725	5.6141	5.6141	.0	.95519	42.606	13.686	13.694	.97518e−2	2.0129
D	59.086	1.5656	1.5656	.0	.18740	59.086	1.5656	1.5656	.0	.19909
rs	4.6329	.11942	.11942	.0	.15510e−1	4.6329	.11942	.11942	0	.14223e−1
Pc	41.971	37.401	37.374	.38070	3.9640	38.210	195.66	184.72	7.0648	27.930
Nw	67.897	1.9618	1.9638	.31957e−2	.26099	67.728	6.7180	6.6893	.37458e−1	.78873
Y	369.91	36.883	36.965	.10420	4.8109	353.75	181.44	180.29	1.3935	22.517
p	1.2251	.12727e−2	.12851e−2	.53471e−5	.16181e−3	1.2701	.11487e−1	.11622e−1	.14917e−3	.11884e−2
Sc	1.8752	36.056	36.010	.42848	4.0374	1.2160	170.92	161.65	5.7300	22.348
H	86.199	33.705	33.785	.32431	4.0198	81.863	141.70	134.70	4.2800	18.542
Hr	33.765	1.1038	1.1014	.13621e−1	.13339	31.660	5.6125	5.3850	.13523	.63715

Table 8
Bianchi and Calzolari (1982)

Table 13.7 Klein-I model. FIML estimates 1921–1941. One-period forecasts at 1948

	(1)	(2)	(3)	380 repl. (4.1)	390 repl. (4.2)	500 repl. (4.3)	1000 repl. (4.4)	(5)
C	76.5	1.4	1.3	17	9.2	8.2	6.1	2.3
I	6.27	1.0	1.0	12	6.8	6.1	4.5	1.9
W_1	58.2	1.4	1.3	15	5.8	5.2	3.9	2.2
Y	91.0	2.3	2.2	27	16	14	11	4.2
P	29.1	1.2	1.2	16	10	9.2	6.8	2.2
K	204	1.0	1.0	12	6.8	6.1	4.5	1.9

Table 13.8 Klein-I model. LIVE estimates 1921–1939. One-period forecasts at 1940

	(1)	(2)	(3)	310 repl. (4.1)	320 repl. (4.2)	440 repl. (4.3)	1000 repl. (4.4)	(5)
C	65.6	1.0	1.1	1.3	2.0	2.7	2.3	2.0
I	2.97	0.67	0.64	0.98	1.4	2.6	2.4	1.5
W_1	45.9	0.80	0.84	0.95	1.3	3.9	1.7	1.8
Y	74.3	1.6	1.6	2.2	3.3	5.2	4.6	3.4
P	20.4	1.1	1.0	1.5	2.2	3.5	3.0	1.9
K	204	0.67	0.64	0.98	1.4	2.6	2.4	1.5

Table 13.9 Klein-Goldberger revised model 2SLS-4PC estimates. One-period forecasts at 1965

	(1)	(2)	10 repl. (3.1)	50 repl. (3.2)	100 repl. (3.3)	40 repl. (4.1)	200 repl. (4.2)	1000 repl. (4.3)	(5)
C_d	55.3	1.1	0.72	6.4	4.6	1.7	1.4	16.0	2.4
C_n	303.0	1.5	1.5	6.1	4.5	1.4	2.2	15.0	3.6
R	22.4	0.55	0.54	1.5	1.2	0.38	0.58	2.9	1.2
H	7.55	0.10	0.06	0.59	0.42	0.08	0.15	1.4	0.25
I_m	30.4	0.70	0.73	15.0	25.0	0.77	6.5	87.0	1.1
X	530.0	3.5	3.4	54.0	39.0	3.6	31.0	136.0	8.5
h 100	102.0	0.83	1.4	29.0	21.0	1.1	4.3	50.0	2.3
W	311.0	2.0	1.9	25.0	18.0	1.9	4.5	67.0	4.8
w	5.52	0.02	0.02	0.82	0.59	0.03	0.21	1.1	0.08
r	4.89	0.14	0.17	0.14	0.13	0.13	0.15	0.14	0.36
I	46.7	1.1	1.2	1.0	1.0	1.1	1.1	1.1	2.4
D	59.1	0.70	0.85	0.84	0.72	0.89	0.72	0.70	1.3
r_s	4.63	0.15	0.19	0.14	0.14	0.18	0.15	0.15	0.35
P_c	42.0	3.5	5.3	14.0	11.0	5.3	6.1	76.0	6.1
N_w	67.9	0.21	1.0	36.0	26.0	1.3	6.5	51.0	1.4
Y	369	2.6	1.9	35.0	25.0	2.6	7.1	65.0	6.1
p	1.22	0.022	0.035	6.2	4.4	0.043	0.52	5.0	0.036
S_c	1.90	3.6	5.3	18.0	14.0	5.7	7.5	75.0	6.0
H	86.2	3.2	4.8	9.7	7.6	4.4	4.0	75.0	5.8
H_r	33.8	0.52	0.83	6.4	4.7	0.95	1.9	1.8	1.1

Table 9
Freedman and Peters (1984)

Table 4. Bootstrap Forecast Experiment for Equation (18) in Straight Forecasting Mode. Estimation Is by 3SLS. There Are 400 Replications

	(1) Sample Mean Actuals Y^*_{i995}	(2) Sample Mean Forecasts $\hat{Y}^*_{i995}$	(3) Standard Deviation $Y^*_{i995} - \hat{Y}^*_{i995}$	(4) The RMS G-N-O Standard Error
S_K	.0207	.00995	.0647	.0417
S_L	.301	.301	.0370	.0257
S_E	.0117	.0179	.0344	.0275
$\log(p_K/p_M)$	−2.38	−2.56	1.65	1.08
$\log(p_L/p_M)$	.500	.492	.456	.314
$\log(p_E/p_M)$	−2.71	−2.74	1.02	.678

Table 10
Peters and Freedman (1986)

Table 1. Bootstrap forecast experiment for equation (9). Estimation is by one-step *gls*. There are 100 bootstrap replications.

Region	(1) Sample Mean Actuals $q^*_{r,1995}$	(2) Sample Mean Forecasts $\hat{q}^*_{r,1995}$	(3) Standard Deviation $q^*_{r,1995} - \hat{q}^*_{r,1995}$	(4) Delta SE	(5) RMS Delta SE	(6) RMS Bootstrap SE
1	.40	.41	.12	.063	.053	.094
2	.12	.14	.14	.050	.055	.11
3	.31	.32	.12	.045	.055	.090
4	.77	.78	.11	.068	.055	.093
5	.44	.44	.11	.043	.053	.079
6	.96	.97	.14	.073	.059	.10
7	.69	.69	.12	.065	.054	.095
8	.94	.94	.11	.084	.058	.091
9	.55	.56	.13	.098	.057	.097
10	.63	.63	.11	.061	.051	.087

Table 11
Calzolari and Corsi (1977)

Year	IAB	PVAI
1961	212.0	.0168
1962	190.4	.0173
1963	170.4	.0200
1964	162.9	.0223
1965	172.9	.0202
1966	178.5	.0195
1967	165.2	.0204
1968	166.2	.0202
1969	156.9	.0189
1970	150.6	.0203
1971	145.5	.0249
1972	134.8	.0253
1973	130.8	.0253
1974	129.9	.0359
1975	120.4	.0444
1976	119.0	.0486

REDUCED FORM STANDARD DEVIATIONS OVER TIME.
1000 REPLICATIONS

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