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Abstract

This paper identifies the necessary and sufficient condition for the transitivity of the majority rule when individuals are never indifferent between two distinct alternatives (linear domain). By introducing the concept of the relevant population for each set of alternatives, a unique condition can be stated, and substitute the variety of conditions that already exist in the literature.

Keywords: majority rule, transitivity

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1 Introduction

The (simple) majority rule does not consistently yield transitive social preferences (Condorcet paradox). This fact led economic theory to work in identifying the set of cases in which (simple) majority rule works fine. Many important results have been stated so far. Apart from the popular sufficiency conditions, such as single-peaked (Black, 1948) and value-restricted preferences (Sen, 1966), sets of necessary conditions for the well-behaving of the (simple) majority rule have been offered (Inada, 1969 and Sen, 1969).

This paper identifies a unique condition, that is both necessary and sufficient for the transitivity of the (simple) majority rule in the linear domain (all preference profiles of linear orders) in an economically meaningful manner.

2 Analysis

Suppose that there exists a set of individuals $N = \{1, 2, ..., n\}$, $\#N \geq 3$ and odd. Each $i \in N$ has complete, transitive and strict (linear) preferences on a finite set of alternatives X, $\#X \geq 3$, which can be represented by the linear order P_i on X. That is, the only real restriction that we impose on individual preferences is the fact that no indifferences between any two alternatives are allowed. Other than that, any linear order on X is permitted.

Assume that the Social Welfare Function (SWF)[1] of (simple) majority is applied to aggregate social preferences.

Define the SWF of (simple) majority as follows:

$$xP_my \text{ if and only if } \#\{i \in N|xP_iy\} > \#\{i \in N|yP_ix\}$$

We define as θ_X the preference profile of the society N on a set of alternatives X and $z^t_{\theta_X}$ the number of individuals that have the same $t-type$ preferences on X (the preferences of these individuals are represented by the same linear order).

Given the environment that is described above, define the following.

Definition 1 *Individuals i and j are mutually exclusive in X if and only if for every two distinct x and y from X either xP_iy and yP_jx or yP_ix and xP_jy.*

Definition 2 *Given a set of alternatives X, a relevant population $\tilde{N}_X$ is a subset of N that a) does not include any mutually exclusive individuals in X and b) $N \backslash \tilde{N}_X$ is either the union of $\frac{\#N\backslash\tilde{N}_X}{2}$ disjoint pairs of mutually exclusive individuals or empty.*

Since $N\backslash\tilde{N}_X$ contains individuals that, in pairs, cancel out each other votes, it easily follows that if $\#\{i \in N|xP_iy\} > \#\{i \in N|yP_ix\}$ then $\#\{i \in \tilde{N}_X|xP_iy\} > \#\{i \in \tilde{N}_X|yP_ix\}$ and vice versa for any $\tilde{N}_X$. This allows us to re-phrase the definition of the (simple) majority rule in the following way:

$$xP_my \text{ if and only if } \#\{i \in \tilde{N}_X|xP_iy\} > \#\{i \in \tilde{N}_X|yP_ix\} \text{ for any } \tilde{N}_X$$

Definition 3 *The relevant preference profile of X, $\tilde{\theta}_X$, is the preference profile of any relevant population $\tilde{N}_X$ (it is trivial to see that the preference profiles of all relevant populations are equal).*

Definition 4 *A preference profile θ_X is balanced if and only if each $x \in X$ is the best or the worst choice for less than half of the population.*

We can now state the theorem.

Theorem 1 *The (simple) majority rule yields transitive social preferences in the linear domain if and only if for every triplet $A = \{x, y, z\} \subseteq X$, the relevant preference profile of A is not balanced.*

Proof. If for every triple $A = \{x, y, z\} \subseteq X$, $\tilde{\theta}_A$ is not balanced we have that for any $\tilde{N}_A$, $\#\tilde{N}_A > 0$ (since $\#N$ is odd we must have non-empty relevant populations) and $x \in A$ is the best or the worst choice for more than half of any $\tilde{N}_A$. Following the definition of a relevant population we understand that the types of preferences in $\tilde{\theta}_A$ is 3 (at most). If they are 1 or 2, (simple) majority yields transitive preferences (the Condorcet paradox requires at

[1] Arrow, 1951

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least three types of preferences). If $\tilde{\theta}_A$ contains 3 types of preferences, and $x \in A$ is the best choice for more than half of any $\tilde{N}_A$ we have that $xP_{sm}y, xP_{sm}z$ (which is enough to guarantee transitivity in the present environment). Obviously, if $x \in A$ is the worst choice for more than half of any $\tilde{N}_A$, some $y \in A$ will be the best for more than half of any $\tilde{N}_A$ leading, by the above, to transitive social preferences. Now, we have to prove that if (simple) majority yields transitive social preferences, then for every triple $A = \{x, y, z\} \subseteq X$, $\tilde{\theta}_A$ must not be balanced. To do that we shall assume that there exists $A = \{x, y, z\} \subseteq X$, such that $\tilde{\theta}_A$ is balanced and (simple) majority yields transitive social preferences. It must be the case that the types of preferences in $\tilde{\theta}_A$ is exactly 3 (in all other cases there exists $x \in A$ that is best or worst for more than $\frac{\#\tilde{N}_A}{2}$ individuals for any $\tilde{N}_A$). That is, in each level of preferences (best, intermediate, worst) all elements of A appear in exactly one type (we have "Codorcet paradox" style preferences). Since all $z^1_{\theta_A}, z^2_{\theta_A}, z^3_{\theta_A} < \frac{\#N_A}{2}$, for any $\tilde{N}_A$, we get that $z^1_{\theta_A} + z^2_{\theta_A} > z^3_{\theta_A}$, $z^2_{\theta_A} + z^3_{\theta_A} > z^1_{\theta_A}$, $z^1_{\theta_A} + z^3_{\theta_A} > z^2_{\theta_A}$, or in other words, cyclical social preferences and no transitivity. This concludes the proof. ■

3 Concluding remarks

Recent literature on majority decisions has been, mainly, focused in providing characterizations of the majority rule (e.g. Campbell and Kelly, 2000) and in identifying its "good" properties (Dasgupta and Maskin, 1998). The present paper, reviews the issue of transitivity of the majority rule and establishes a unique necessary and sufficient condition. By introducing the concept of the relevant population for every set of alternatives, we can now easily approach the problem, from an economically meaningful way.

The idea for the result is simple, and based on construction or elimination of Condorcet cycles. As far as triplets of alternatives (x, y, z) are concerned, the above condition states that one of the three alternatives in (x, y, z) is either strictly preferred to the other two, or strictly worse than the other two, for a majority of voters, after eliminating any pair of voters that have exactly offsetting preference orderings. It is easy to see that, if the identified condition holds for any triplet (x, y, z), majority voting implies transitivity almost by construction, because there is an alternative that is worst or best in any pairwise comparison with the two other alternatives. Therefore, it must be that pairwise majority voting generates transitive preferences.

For the counterpart, one can show that if there exists such a triplet for which none of the alternatives is ranked first or last by a majority of voters of the relevant population, then this triplet can be used to construct a Condorcet cycle. This argument follows from the observation that after eliminating off-setting (mutually exclusive) voters from considerations, at most three different profiles remain. If there are only one or two profiles, transitivity is guaranteed, so lack of transitivity requires three profiles, and they must form a Condorcet cycle. The result is then immediate.

4 References

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