Managing Markets for Toxic Assets

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Abstract

We present a model in which banks trade toxic assets to raise funds for investment. The toxic assets generate an adverse selection problem and, as a consequence, the interbank asset market provides insufficient liquidity to finance investment. While the best investments are fully funded, socially efficient projects with modest payoffs are not. Investment is inefficiently low because acquiring funding requires banks to sell high-quality assets for less than their "fair" value. We then consider whether equity injections and asset purchases can improve market outcomes. Equity injections do not improve liquidity and may be counterproductive as a policy for increasing investment. By allowing banks to fund investments without having to sell high-quality assets, equity injections reduce the number of high-quality assets traded and further contaminate the interbank market. Paradoxically, if equity injections are directed to firms with the greatest liquidity needs, the contamination effect causes investment to fall. In contrast, asset purchase programs, like the Public-Private Investment Program, often have favorable impacts on liquidity, investment and welfare.

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1 Introduction

How can government agencies restore liquidity in financial markets when these markets stop functioning? In 2008, the U.S. Treasury Department and the Federal Reserve were confronted with exactly this question. Interbank loan markets were frozen because banks had too many "toxic assets" in their portfolios. According to market observers, these toxic assets could not be easily valued and were therefore illiquid. Banks with many such assets could not sell them easily to raise funds necessary for lending. The Treasury and the Fed responded with a combination of policy actions including asset purchases, loan guarantees, and equity injections.

This paper uses a simple model to analyze the effects of policies in markets for toxic assets. We model the market for toxic assets with an adverse selection problem. Adverse selection seems like a natural modeling assumption for two reasons. First, many market observers emphasize the problem caused by having assets that buyers could not accurately value. Second, it has been well understood since Akerlof [1979] that adverse selection can cause significant market failures, and in extreme cases can cause markets to shut down completely.

In our model, banks possess both liquid and illiquid (toxic) assets. Banks trade the illiquid assets to finance investments. In the absence of government interventions, the interbank market for toxic assets provides insufficient liquidity for banks to efficiently finance investment projects. Investment is inefficiently low because acquiring adequate funding requires banks to sell high quality assets at prices below their "fair" market value. This illiquidity is a direct consequence of the adverse selection problem in the secondary market for toxic assets.

We then consider whether government policies can increase liquidity in the secondary market and improve the allocation of investment. We focus our attention on equity injections and purchases of toxic assets. We find that equity injections do not increase liquidity in markets for toxic assets and, in some cases, may be counterproductive as a policy for increasing investment. In contrast, outright purchases of assets at above-market prices always increases liquidity in the secondary market and, under certain conditions, may be more effective at increasing investment.

The reason equity injections often do not work as well as asset purchases is that new equity capital allows financial institutions to fund investments directly without having to sell as many high-quality assets at unfavorable terms. While this is unambiguously good for a single financial institution at a given price, equity injections have negative feedback effects in equilibrium. Because financial institutions can now withhold high quality assets and instead rely on the new equity to fund investments, there are fewer good assets traded. As a result, equity injections have a *contamination effect* which causes the average quality (and thus the price) of toxic assets in the secondary market to drop. The reduction in price further reduces the incentive to trade, making toxic assets even less liquid. Paradoxically, if equity injections are directed to banks with the greatest needs for liquidity, the contamination effect is so severe that investment falls in equilibrium.

Unlike equity injections, which have a detrimental impact on the market for toxic assets, asset purchase programs, like the Public-Private Investment Program, often have favorable impacts on secondary markets, investment and welfare. Intuitively, while asset purchase plans can be designed to transfer the same amount of money to distressed financial institutions, the banks have to sell assets to get the transfers. Asset purchases thus encourage trading in the secondary market, leading to greater liquidity and improved efficiency.

In our model, a key requirement of a successful asset purchase policy is that the government purchase assets at above-market prices. If the government buys assets at fair-market prices, the policy will have no effect. Because the government needs to purchase assets at above-market prices, successful asset purchases are costly in terms of the Federal Budget. The differential budgetary impact of equity injections and asset purchases is an important part of the analysis. All of our comparative results are obtained under the restriction that both policies have the same budgetary impact.

Modeling toxic assets with adverse selection obviously rules out other potentially important roles for such securities. For instance, in our model, the banks do not care directly about insolvency nor do they care about whether their balance sheet is sufficiently transparent for potential creditors. Other market frictions (bank runs, liquidity constraints, etc.) surely played important roles in shaping the crisis. Because available evidence does not point uniquely to adverse selection as the source of financial market failure, our results should be viewed somewhat narrowly. Our paper presents one possible channel through which loan market failure may have arisen. Our analysis is not meant to preclude other mechanisms.

2 Background

The Emergency Economic Stabilization Act of 2008 was signed on October 3, 2008 by President George W. Bush. The most prominent component of this legislation was the *Troubled Asset Relief Program* better known by its acronym TARP. At the time of the bill's passage, the conventional understanding was that the TARP would be used to purchase distressed assets with the hope that such purchases would restore trading in interbank markets that were essentially frozen. Congress authorized the Treasury to purchase up to $700 billion of troubled assets.

The preamble of the bill laid out the intentions of the legislation as follows: "(t)he Secretary is authorized to establish the Troubled Asset Relief Program to purchase ... troubled assets from any financial institution, on such terms and conditions as are determined by the Secretary." The term "troubled assets" is defined by the law to mean "residential or commercial mortgages and any securities, obligations, or other instruments that are based on or related to such mortgages ... the purchase of which the Secretary determines promotes financial market stability".[1]

The purpose of the TARP was to restore liquidity to asset markets that had essentially ceased to function. In a formal press release, Secretary Henry Paulson described the problem in the asset markets, and his proposed solution as follows:

> When the financial system works as it should, money and capital flow to and from households and businesses to pay for home loans, school loans and investments that create jobs. As illiquid mortgage assets block the system, the clogging of our financial markets has the potential to have significant effects on our financial system and our economy. [...] The federal government must implement a program to remove these illiquid assets that are weighing down our financial institutions and threatening our economy. This troubled asset relief program must be properly designed and sufficiently large to have maximum impact, while including features that protect the taxpayer to the maximum extent possible.[2]

Paulson's description of the problem is exactly what we want to capture in our model.

The original design of the TARP was not greeted with unanimous support from academic economists many of whom argued that capital injections would work better to stabilize conditions in the markets. (See Stiglitz [2008], Krugman [2009], Diamond *et al.* [2008] and Kashyap and Stein [2008].)[3] In addition, two other issues ultimately led to a substantial re-design of

[1]The Emergency Economic Stabilization Act is Division A of Public Law 110–343. The terms of the TARP are included in Title I of this Act.

[2]Statement by Secretary Henry Paulson on Comprehensive Approach to Market Developments, hp-1149, September 19, 2008.

[3]Some economists viewed the original TARP favorably. Quoted by *The Wall Street Journal*, Anna Schwartz said that the problem was "exotic securities that the market does not know how to value. They're toxic because

the TARP. First, it became clear that calculating the "correct" price of troubled assets would be difficult. Unlike purchasing illiquid assets, injecting equity would be easier. Second, direct capital injections were tried with some apparent success in the U.K. As a result, the original TARP plan was abandoned in favor of equity injections.

In early 2009 a second plan to purchase toxic assets was suggested. Like the original TARP, the Public-Private Investment Program or P-PIP aimed to purchase toxic assets from banks. The price for the assets would be determined by private auctions. The purchase would be made with a combination of funds from the TARP, the Federal Reserve and from private investors. It is worth noting that, according to the original description of the P-PIP, banks would determine for themselves which assets they would put up for auction.

3 Model

We consider a static model of trade in illiquid assets. The main actors in the model are financial intermediaries that fund investment opportunities by either drawing on cash reserves or by selling assets. There is a continuum of such intermediaries. We refer to the financial intermediaries simply as "banks." The cash reserves could be either actual cash or could be other liquid assets like Treasury bills. The illiquid assets are the source of the adverse selection problem. We refer to these toxic assets as mortgage backed securities (MBS) though in theory they could be any asset that buyers find difficult to value. We refer to the secondary market as the "interbank market."

Each bank has a portfolio of mortgage backed securities (measure one) and a measure m of liquid assets. The MBS differ in their default probabilities and the current holders of the securities have some experience with how well the underlying assets are doing. This is a feature of actual toxic assets, many of which are simply pools of mortgages that periodically pay-off as the borrowers make mortgage payments. A bank that possesses a given MBS can observe the rate at which the mortgages pay-off and infer whether the mortgages are good or whether they are likely to underperform. Banks also have different investment opportunities or liquidity needs. Banks in need of liquidity face a trade-off between selling assets at a discount or foregoing profitable investment opportunities. Banks are risk-neutral.

you cannot sell them, you don't know what they're worth, your balance sheet is not credible and the whole market freezes up." In her view, purchasing such assets would be a "step in the right direction." Also quoted in *The Wall Street Journal*, Martin Feldstein said "You can't solve the overall economic problem if you can't get the financial institutions lending again. And you can't do that as long as they don't know what the value [of their troubled assets] is, particularly the residential mortgage-backed securities and derivatives based on them." See Beim [2008] for additional discussion.

To keep matters simple, we assume that a given MBS either pays back fully or defaults. Solvent mortgages repay R dollars. If a mortgage defaults, the owner gets zero. The distribution of default probabilities is common knowledge. Some assets are perfectly solvent; some have moderate default risk and some are sure to default. Let δ be the default rate on an individual mortgage asset. We assume that default rates are distributed according to the distribution function $G(\delta)$ with density $g(\delta)$. This distribution characterizes the portfolio of every bank. At the beginning of the period, banks know the default rate of each asset they own (they know which ones are underperforming and which ones are good).

Each bank receives a different idiosyncratic investment opportunity. The maximum scale of the investment opportunity is i. The investment projects are perfectly divisible so a bank can choose to undertake a fraction of its investment opportunity if it wants. While i is the same for all investments, each investment opportunity has a different expected return z. The idiosyncratic return z is distributed across banks according to the distribution function $F(z)$ with density $f(z)$ and mean μ_z. To make the solution non-trivial, we assume that there are at least some projects for which $z > 1$, that is, we assume that $1 - F(1) > 0$.

In addition to the banks, we also include a second set of agents that trade securities. This second group does not fund any investment but instead simply supplies a set of illiquid assets inelastically. We refer to these agents as traders and we assume there is a measure T of such agents each with one toxic asset. The traders can be interpreted as bankrupt firms that are forced to liquidate irrespective of the price. Let the average default rate of the traders be δ^T which may be greater or less than the default rate of the bank assets. The traders primarily play a technical role in the model. Their presence disciplines the market outcomes to an extent and also allows us to calibrate the equilibrium default rate if need be.

Each bank seeks to maximize expected profit. Banks with good projects (i.e., banks with high z) will attempt to assemble liquidity to invest. We assume that $m < i$ and that banks cannot share projects with each other. As a result, the idiosyncratic investment opportunities create a motive for trade in the secondary market for Mortgage Backed Securities to fund investment projects or satisfy liquidity needs. We should emphasize at the outset that we explicitly rule out other sources of financing such as issuing new equity or debt and instead confine our attention to the liquidity role of toxic assets. If the market price of the MBS is low, then few investment projects are funded. Our focus is on the equilibrium behavior of this secondary market and government policies that influence the equilibrium.

3.1 Equilibrium without Government Policy

We begin by analyzing the equilibrium without government intervention. We discuss the effects of government interventions in the next section.

Optimization Problem. The bank's decision problem is to decide which mortgage backed securities to sell and whether to invest in its project. Let $q \geq 0$ be purchases of assets on the secondary market. Let x be investment in a bank's project. Investment can be at most i (the scale of the project) and cannot be negative. Investment is further constrained by the amount of liquidity that the bank has. A bank's liquidity includes its initial holding of liquid assets m plus proceeds from net sales of MBS.

Let p be the price for MBS in the secondary market. When a bank sells an asset, it knows the asset's default rate. When they buy however, banks do not know the default rates of the assets they purchase. Let Δ be the average default probability of the MBS traded. The average default rate Δ is endogenously determined by the equilibrium distribution of asset sales across banks and traders.

To state the bank's optimization problem, notice that if it is optimal for a bank to sell assets of type δ, then it must be optimal to sell all assets with $\delta' > \delta$. As a result, we confine our attention to the choice of an optimal cutoff $\hat{\delta}$. Taking z and Δ as given, a typical bank chooses a cutoff default rate $\hat{\delta}$, investment level x and asset purchases q to maximize

$$R \int_0^{\hat{\delta}} (1-\delta)\, g(\delta)\, d\delta + R(1-\Delta)\, q + zx + \left(m + p\left[1 - G(\hat{\delta})\right] - x - pq\right) \tag{1}$$

subject to

$$x + pq \leq m + p \int_{\hat{\delta}}^{1} g(\delta)\, d\delta \tag{2}$$

$$0 \leq x \leq i \text{ and } 0 \leq q. \tag{3}$$

The first term in the objective (1) reflects the expected payoff of the MBS retained by the bank. These assets are the ones with the lowest default rates. The second term reflects the expected payoff from MBS purchased by the bank. By definition, assets sold on the secondary market have an average default rate Δ thus each MBS purchased provides the bank with an expected payoff of $R(1-\Delta)$. The third term is the payoff from funding the bank's idiosyncratic investment opportunity. The last term is simply the bank's remaining cash: its initial liquidity m, plus the proceeds from sales of MBS, less the money spent funding investment, less purchases of MBS. Denote the solution to the bank's optimization problem by the functions $q(z)$, $x(z)$ and $\hat{\delta}(z)$.

Market Clearing. The model requires the following market clearing conditions in equilibrium. First, the supply of toxic assets sold must be equal to the demand for purchases of toxic assets. Each bank with investment opportunity z supplies all assets with default rates greater than $\hat{\delta}(z)$. Asset demands are not differentiated by default rates since the purchasers cannot distinguish relatively risky assets from relatively safe assets. In addition to the banks assets, the traders also supply T assets inelastically. Thus, integrating over all banks, we can write this market clearing condition as

$$T + \int \left[\int_{\hat{\delta}(z)}^{1} g(\delta)\, d\delta \right] f(z)\, dz = \int q(z)\, f(z)\, dz. \tag{4}$$

Second, aggregate investment can be no greater than total available liquidity,

$$\int x(z)\, f(z)\, dz \leq m. \tag{5}$$

Equilibrium. In equilibrium banks behave optimally and markets clear. We confine our attention to models in which there is sufficient liquidity to achieve the socially optimal allocation of investment. The only thing preventing the optimal level of investment is the adverse selection problem in the market for toxic assets.

Since the social opportunity cost of funding any given project is 1, the socially optimal investment allocation requires that all projects with $z > 1$ are funded. Denote the socially optimal investment level as $I^* = i[1 - F(1)]$. The following assumption ensures that there is sufficient liquidity in aggregate to achieve the social optimum and satisfy the liquidity needs of the traders but that any one bank is liquidity constrained.

Assumption 1. *We assume that $i > m > I^* + RT$.*

Assumption 1, together with optimal behavior on the part of the banks, has several implications. First, banks never over-invest in equilibrium. That is, no bank invests if $z < 1$. Were a bank to undertake a low return project, it could increase its payoff simply by refraining from investing. Second, no bank ever simultaneously sells assets with $\delta < \Delta$ and purchases assets on the secondary market. If a bank were to do so, it could increase its payoff simply by keeping the relatively safe assets and purchasing fewer assets on the secondary market. Finally, if the price of toxic assets p were less than the diversified value of the assets in the secondary market $R(1-\Delta)$, then all banks with $z \leq 1$ would strictly prefer to trade their cash for toxic assets. For the market to clear, banks with $z > 1$ would have to hold more money than required for their investments (since $m > I^*$). This would not happen in equilibrium however. Banks with $z > 1$ could still invest i while selling fewer high-quality assets. Similarly, if the price of toxic assets were to exceed the diversified value, then no bank would willingly purchase these securities. Thus, if some assets are purchased in equilibrium, $p = R(1-\Delta)$. We summarize these observations in the following Lemma. All proofs are in the appendix.

Lemma 1 *The following conditions hold in any equilibrium with trade:*

1. *No socially inefficient project is undertaken, i.e.,* $x(z) = 0$ *whenever* $z < 1$.
2. *Banks never buy assets while selling assets with a below average default rate, i.e.,* $q(z) > 0 \Rightarrow \hat{\delta}(z) = \Delta$ *and* $\hat{\delta}(z) < \Delta \Rightarrow q(z) = 0$.
3. *The price of mortgage backed securities is actuarially fair, i.e.,* $p = R(1-\Delta)$.

With Lemma 1, we can treat $p = R(1-\Delta)$. Thus, Δ is sufficient to describe both the price of the mortgage backed securities and the return on these investments. We can now solve the banks optimization problem with conventional concave programming techniques. The following proposition describes the optimal policy for an individual bank taking the endogenous default rate Δ, and its investment opportunity z as given.

Proposition 1 *Taking* Δ *as given, the optimal policy, is described by a cutoff rule* $\hat{\delta}(z)$ *in* $[0, \Delta]$, *a purchase function* $q(z)$, *and an investment function* $x(z) \in [0, i]$.

1. *For* $z < 1$, $\hat{\delta}(z) = \Delta$, $q(z) \in [0, m/(R(1-\Delta))]$ *and* $x(z) = 0$;
2. *For* $1 \leq z \leq \bar{z}$, $\hat{\delta}(z) = 1 - z(1-\Delta)$, $q(z) = 0$ *and* $x(z) = m + R(1-\Delta)(1 - G(\hat{\delta}(z)))$;
3. *For* $\bar{z} \leq z$, $\hat{\delta}(z) = \bar{\delta}$, $q(z) = 0$ *and* $x(z) = \min\{i, m + R(1-\Delta)\}$.

where $\bar{\delta} = G^{-1}(1 - \min\{\frac{i-m}{R(1-\Delta)}, 1\})$ *and* $\bar{z} = \frac{1-\bar{\delta}(\Delta)}{1-\Delta}$, $\bar{z} \in \left[1, \frac{1}{1-\Delta}\right]$. *Only banks with* $z < 1$ *purchase assets on the secondary market. They are indifferent to the level of purchases.*

Proposition 1 says that banks fall into three categories. First there are banks without profitable investment projects (i.e., $z < 1$). These banks do not need to raise additional liquidity. Instead, they simply sell all of their toxic assets with $\delta \geq \hat{\delta}(z) = \Delta$ on the secondary market. The price they receive for these securities more than compensates them for the sale. These banks are willing to purchase assets sold on the secondary market at the actuarially fair price $p = R[1-\Delta]$.

Second, there are banks with profitable investment projects (i.e., $z > 1$) but not so profitable that they are willing to part with the best assets. For these banks, there is a critical default rate $\hat{\delta}(z) = 1 - z(1 - \Delta)$ at which the return on holding the marginal asset is equal to the marginal benefit of increasing investment. This cutoff depends negatively on the quality of the investment opportunity. Since $z > 1$, the critical default rate is lower than the average default rate so banks in this group sell above average quality assets to finance their investments. As z increases, banks part with higher-quality assets to increase investment.

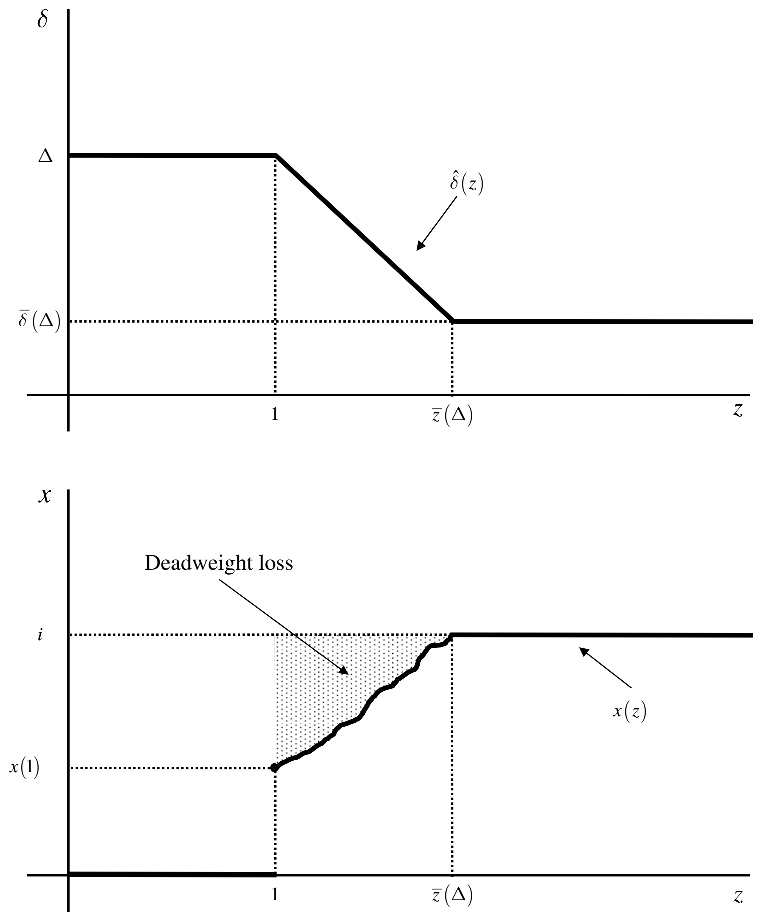

Figure 1: Cutoff Rules and Investment Decisions

Last, there are banks with projects so profitable (or liquidity needs so great) that they invest as much as they can. The critical z at which banks fully fund is $\bar{z}$. If the secondary market value of the toxic assets is sufficiently high, banks can fund their investments without selling all of their securities. This occurs if $R(1 - \Delta) \geq i - m$. If the value of the toxic assets is less than $i - m$, then these banks sell all of their securities and fund as much as possible. Define $\hat{\Delta} = 1 - \left(\frac{i-m}{R}\right)$. For any $\Delta > \hat{\Delta}$, banks with $z > \bar{z}$ will be constrained (that is, they will have $\hat{\delta}(z) = 0$)The cutoff $\hat{\Delta}$ is important for characterizing several of the results below. We refer to equilibria with $\Delta < \hat{\Delta}$ as *interior* equilibria since banks with $z > \bar{z}$ have $\hat{\delta}(z) = \bar{\delta} > 0$ and thus can fully fund their investments. We refer to equilibria with $\Delta > \hat{\Delta}$ as *constrained* equilibria since banks with $z > \bar{z}$ have $\hat{\delta}(z) = 0$ and thus cannot fund their projects fully.

We depict the cutoffs and investment policies as functions of z for a given Δ in Figure 1. The figure is drawn for a default rate for which $\Delta < \hat{\Delta}$ so it is possible for banks to fully fund their investments. For low z, banks simply sell their underperforming assets and do not invest. When $z = 1$, banks begin to undertake their investment opportunities. Investment is discontinuous at $z = 1$. Specifically, investment jumps from zero to $x(1) = m + R(1-\Delta)(1 - G(\Delta))$ since banks are willing to exchange the full value of their liquidity to finance investment at this point. As z increases, the cutoff $\hat{\delta}(z)$ falls and investment rises until $x(z) = i$.

Given a cutoff function $\hat{\delta}(z)$, we define the following objects. Let

$$A \equiv \int_0^\infty \left[\int_{\hat{\delta}(z)}^1 \delta g(\delta)\, d\delta\right] f(z)\, dz + T\delta^T \tag{6}$$

be the total number of assets sold on the secondary market that default and let

$$B \equiv \int_0^\infty \left[\int_{\hat{\delta}(z)}^1 g(\delta)\, d\delta\right] f(z)\, dz + T \tag{7}$$

be total sales of such securities. Recall that the traders supply T assets inelastically with an average default rate of δ^T. We can now define an equilibrium for our model.

Definition 1 *A competitive equilibrium without government policy consists of an aggregate default rate Δ and functions $q(z)$, $x(z)$, and $\hat{\delta}(z)$ such that:*

1. *Taking Δ as given and $p = R(1-\Delta)$, the functions $q(z)$, $x(z)$, and $\hat{\delta}(z)$ maximize (1) subject to the constraints (2) and (3).*
2. *The functions $q(z)$, $x(z)$, and $\hat{\delta}(z)$ satisfy (4) and (5).*
3. *The policy functions $q(z)$, $x(z)$, and $\hat{\delta}(z)$ imply Δ; that is, $\Delta = \frac{A}{B}$ where A is given by (6) and B is given by (7).*

To prove that an equilibrium exists, we now define a fixed-point mapping in the default rate Δ. Let the mapping be $\Gamma : [0,1] \to [0,1]$. Given $\Delta \in [0,1]$, we have the optimal policies $\hat{\delta}(z)$ given in Proposition 1. Let $A(\Delta)$ and $B(\Delta)$ be given by equations (6) and (7). Notice that we have written A and B as functions of the default rate Δ since the optimal cutoff function $\hat{\delta}(z)$ depends on Δ. We then set

$$\Gamma(\Delta) = \frac{A(\Delta)}{B(\Delta)}. \tag{8}$$

Since $A(\Delta) \leq B(\Delta)$, $\Gamma(\Delta) \leq 1$. Figure 2 depicts the mapping $\Gamma(\Delta)$.

A fixed point Δ^* of Γ is an equilibrium default rate. Since the distributions F and G have density functions, Γ is continuous on $[0,1]$ provided that $T > 0$. If there is a positive measure of traders in the market (i.e., if $T > 0$) then Γ is continuous on the closed interval $[0,1]$. However, if $T = 0$, the mapping in (8) is not well defined at $\Delta = 1$. As $\Delta \to 1$, fewer and fewer banks sell good assets and fund projects. One can show that if $T = 0$, then $\lim_{\Delta \to 1} \Gamma(\Delta) = 1$. If $\Delta = 1$, then the only assets "traded" on the secondary market would be those with $\delta = 1$ and banks would be indifferent to such trades. To accommodate this limiting case, we assume that $\Gamma(1) = 1$ when $T = 0$. This trivially guarantees that a shutdown equilibrium always exists when there are no traders in the market. The mapping in Figure 2 is drawn under the assumption that $T = 0$ so $\Delta = 1$ is an equilibrium.

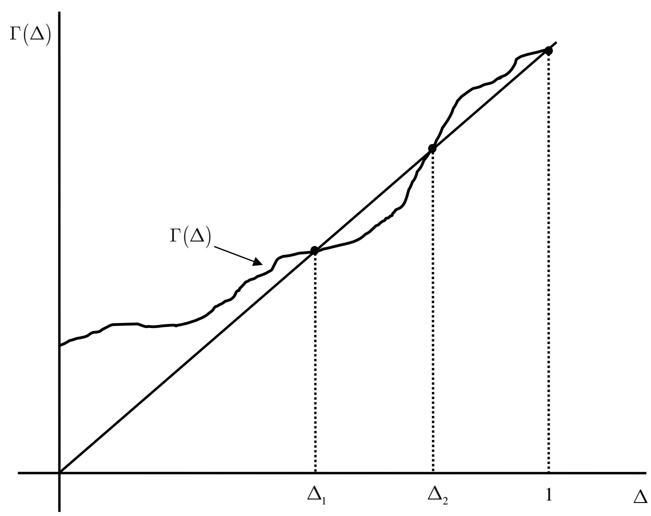

Figure 2: The fixed point Mapping $\Gamma(\Delta)$

If there are multiple equilibria, they are pareto-ranked. All banks are better off with lower Δ. Banks sell toxic assets for which $\delta \geq \hat{\delta}(z)$ and the payoff from these sales is decreasing in Δ. Banks that purchase MBS are indifferent to the value of the equilibrium Δ since the price they pay is actuarially fair. That is, $p = R(1-\Delta)$. Thus all banks must be strictly better off with lower Δ.

The following proposition summarizes the existence of equilibria in the model.

Proposition 2 *Given any distributions F and G with densities f and g,*

1. *there exists at least one equilibrium Δ^*;*
2. *if there are multiple equilibria, lower Δ^* Pareto-dominate higher Δ^*;*
3. *if $T = 0$ then $\lim_{\Delta \to 1} \Gamma(\Delta) = 1$ and $\Delta = 1$ is an equilibrium.*

While it is straight-forward to prove existence of the equilibrium, it is not guaranteed that there is a unique equilibrium. Multiple equilibria may arise because of strategic complementarity in selling good assets. Strategic complementarity is reflected in whether Γ is increasing or decreasing. If a bank anticipates that the average quality of assets on the secondary market is high, the bank is more likely to sell high-quality securities. If banks sell high-quality securities, the average quality on the secondary market will be high. This positive feedback may cause Γ to slope up and may be strong enough to create multiple equilibria. (Figure 4 depicts a numerical example with multiple equilibria.)

Consider an interior equilibrium in which banks with $z > \bar{z}$ can fully fund their projects (i.e., for which $\Delta < \hat{\Delta}$). Differentiating $\Gamma(\Delta)$ with respect to the default rate Δ gives

$$\Gamma_\Delta = \frac{1}{B}\left[\int_1^{\bar{z}} \left[\Delta - \hat{\delta}(z)\right] g(\hat{\delta}(z)) z dF(z) - \frac{i-m}{R(1-\Delta)}(\bar{z}-1)\left[1 - F(\bar{z}(\Delta))\right]\right] \tag{9}$$

The expression for Γ_Δ has natural meaning in terms of the cutoffs $\hat{\delta}(z)$ shown in the top panel in Figure 1. When Δ rises, banks with inefficient projects sell fewer assets but the marginal assets they withhold have the average default rate Δ (i.e., $\hat{\delta}(z) = \Delta$ for $z < 1$) and so there behavior does not influence the implied default rate. Banks with intermediate z's (between 1 and $\bar{z}$) also respond by withholding marginal assets. These assets are higher quality than the average asset sold and thus their behavior causes the implied default rate to rise. The banks with the best projects (i.e., banks with $z > \bar{z}$) still find it in their interest to fully fund the project. As a result, they willingly sell more good assets (assets with the lowest default rates). This effect works to reduce the implied default rate on the secondary market. Whether Δ rises or not depends on the balance of the high z's who sell more and the intermediate z's who sell less. These two effects correspond to the two terms in (9) above.

In a constrained equilibrium ($\Delta > \hat{\Delta}$), banks with $z > \bar{z}$ cannot fully fund their projects and instead sell all of their liquid assets and fund as much as possible. In this case, the increase in Δ does not make them sell more (since they have no more to sell). The derivative

Γ_Δ is therefore missing the negative second term in (9) and thus in a constrained equilibrium, $\Gamma_\Delta > 0$ and increases in Δ unambiguously make the pool worse.

The derivative Γ_Δ evaluated at an equilibrium Δ^* governs the stability of the equilibrium. We refer to equilibria in which $\Gamma_\Delta < 1$ as *stable equilibria*. We refer to equilibria in which $\Gamma_\Delta \geq 1$ as *unstable equilibria*. For there to be multiple equilibria, at least one equilibrium must be unstable. One way to eliminate multiple equilibria is to bound the maximum amount of investors on the margin and limit the minimum number of investors in the market. The following assumption provides conditions under which multiple equilibria cannot occur.

Assumption 2. *The density g satisfies $g(\delta) < \bar{g} = \frac{T}{\mu_z}$.*

The following proposition shows that Assumption 2 is sufficient to guarantee a unique equilibrium.

Proposition 3 *If Assumption 2 holds then the equilibrium is unique and stable.*

Intuitively, the reason multiple equilibria arise is due to the possibility that there may be many banks with intermediate projects with default rates near the cutoff $\hat{\delta}$. For these banks, when Δ increases, they reduce sales of good assets. If there are many such good assets at the margin, the implied Δ may rise considerably (as seen in the first term in (9)). Assumption 2 places a limit on the number of assets that there could be at the margin.

Comparative Statics. Before turning to the equilibrium with government interventions, we consider some comparative statics. We focus on the equilibrium changes in the default rate Δ in a neighborhood of a stable equilibrium as various underlying parameters change. We consider the effect of changes in internal funds m, the scale of investments i, and changes in the distributions F and G. Later, we consider the effect of policy changes on the equilibrium level of investment and welfare.

Consider first how an increase in internal funds m influences the equilibrium. With more internal funds, banks don't need to sell as many illiquid assets. As a result, fewer high-quality assets are sold. This causes the quality to deteriorate and leads to an increase in the equilibrium default rate Δ.

When i increases, banks need more resources to fully fund any given project. Banks with profitable investment opportunities that wish to fully fund their projects must now sell more high-quality assets on the secondary market. (Banks with profitable investment projects that only partially fund their projects are not affected by an increase in the scale of the project.) As a result, the average quality in the pool increases and the equilibrium default rate falls.

When R increases, the value of MBS increases and thus banks get more money when they sell assets. As a result, banks do not need to sell as many high-quality assets. With fewer high-quality assets on the market, the average quality in the secondary market falls and the equilibrium default rate increases. We summarize this discussion in the following proposition:

Proposition 4 *Let the distributions F and G be given and let Δ^* be a stable equilibrium (i.e., $\Gamma_\Delta(\Delta^*) < 1$). Then, in the neighborhood of Δ^*, the equilibrium default rate is increasing when there is*

1. *an increase in internal funds m,*
2. *a decrease in the scale of investment projects i, or*
3. *an increase in the mortgage rate R.*

We also consider the effects of changes in the distribution of investment projects F and the distribution of default rates G. We focus on distributions that are ranked according to first-order stochastic dominance (Rothschild and Stiglitz, 1970).[4] Suppose F' first-order stochastically dominates F. More banks have profitable investment opportunities under F' compared to F and thus supply more high quality assets in the secondary market. As a result, the equilibrium default rate must be lower under F' than under F.

Suppose G' first-order stochastically dominates G. In this case, there are relatively more mortgages with high default probabilities. As a result, there are more low-quality assets in the secondary market and the equilibrium default rate is higher under G' than under G. We summarize this discussion in the following proposition:

Proposition 5 *Let Δ^* be a stable equilibrium given F and G and suppose F' first-order stochastically dominates F and G' first-order stochastically dominates G. Then, in the neighborhood of Δ^*,*

1. *the equilibrium default rate is lower under F' than under F and,*
2. *the equilibrium default rate is higher under G' than under G.*

Proposition 5 shows that the market for toxic assets becomes less liquid when there is an exogenous reduction in the profitability of investment projects across banks. Since banks

[4] Recall that F' first-order stochastically dominates F iff $F'(x) \leq F(x)\ \forall x$. A simple example of first-order stochastic dominance would be a rightward shift of the distribution.

have less reason to sell high quality assets, the average quality of assets falls hence the market becomes less liquid. This is an additional mechanism for generating financial crises in markets for toxic assets.

Aggregate Investment and Welfare. We close this section with a brief discussion of aggregate investment and social welfare. Given the default rate, aggregate investment is simply the sum of investment across banks, that is,

$$I(\Delta) = \int_0^\infty x(z) f(z)\, dz. \tag{10}$$

Social welfare is directly related to aggregate investment. Since the social opportunity cost of any investment x is $1 \times x$, aggregate welfare is simply

$$W(\Delta) = \int_0^\infty (z-1)\, x(z) f(z)\, dz \tag{11}$$

Maximizing welfare requires setting $x(z) = i$ whenever $z > 1$. The allocation of investment is inefficient because banks with intermediate values of z do not fully fund their projects even though it is socially optimal to do so. Specifically, banks with z between 1 and $\bar{z}$ choose $x(z) < i$ whenever $\Delta > 0$. Thus, $I(\Delta) < I^*$ and $W(\Delta) < W^*$. The shaded area in Figure 1 is the deadweight loss in equilibrium.

4 Policy

We now consider the effects of policies in our model. We focus on two types of policies: equity injections and asset purchases. These were two of the main types of policies actually considered during the financial crisis of 2007-2009. We will model equity injections as an increase in m while (as we show below) asset purchases will be treated as a decrease in Δ.

Presumably both types of policies will be costly for the government. Asset purchases will be costly if the government pays more for assets than their "fundamental value." Equity injections will be costly if the government pays too much for the value of the equity. When we analyze policy below, we begin by examining the limiting case in which the equity injections are pure transfers from the government.

We imagine that the government has two possible aims. First, the government might be concerned with efficiency in the allocation of investment. Second, the government might be concerned with the functioning of the interbank market for mortgage backed assets.

4.1 Equity Injections

We model equity injections as an increase in m. This could be due to a forced equity injection or simply due to a direct transfer of funds from a government agency.[5]

Equity injections have two separate effects on allocations. First, they have a direct effect. By providing banks with additional funds, there is more investment for any fixed Δ. This direct effect increases both aggregate investment and welfare. Second, equity injections may have an indirect effect which arises because of changes in the equilibrium default rate Δ. Unlike the direct effect, this indirect effect reduces investment and welfare. As we saw in Proposition 4, increases in m cause Δ to rise because banks rely less on the proceeds of their asset sales. This further contaminates the secondary market and reduces the exchange of toxic assets.

To see these effects, write aggregate investment I as an implicit function of internal funds m.

$$I\left(m, \Delta\left(m\right)\right) = \int_1^{\infty} x\left(z, \Delta\left(m\right), m\right) f\left(z\right) dz. \tag{12}$$

We now consider an incremental increase in internal funds $dm > 0$. We analyze interior equilibria ($\Delta < \hat{\Delta}$) and constrained equilibria ($\Delta > \hat{\Delta}$) separately.

In the first case ($\Delta < \hat{\Delta}$), differentiating (12) with respect to m gives,

$$\frac{\partial I\left(m, \Delta\left(m\right)\right)}{\partial m} = \left[F\left(\bar{z}\right) - F\left(1\right)\right] + I_{\Delta}\Delta_m \tag{13}$$

where $I_{\Delta} \equiv \partial I\left(m, \Delta\right) / \partial \Delta$ and $\Delta_m \equiv \partial \Delta / \partial m$.

Taking Δ as given, banks with low z ($z < 1$) do not change their behavior at all in response to the increase in m. They do not need the extra funds so the new liquidity remains idle and there is no effect on the interbank market or investment arising from these banks. Banks with intermediate projects ($1 \leq z \leq \bar{z}$) do not change their asset sales ($\hat{\delta}\left(z\right)$ is constant) but instead use the additional funds to expand investment. For these banks, each dollar of additional equity results directly in an extra dollar of investment. The increase in investment is directly proportional to the number of projects in this range, $F\left(\bar{z}\right) - F\left(1\right)$. This effect is captured by the first term in (13).

Banks with $z > \bar{z}$, make no change in investment but instead change their asset sales. Prior to the equity injection, they were selling high-quality assets sufficient to fund their investments.

[5] It is likely that the actual TARP had elements of both a government transfer and an equity injection at a fair market price. See Veronesi and Zingales [2009].

After the equity injection, these banks still finance their projects but now sell fewer high quality assets. This second effect is captured by the term $I_\Delta \Delta_m$. By Proposition 4, this effect must be negative. We refer to this side-effect of equity injections as the *contamination effect*. The contamination effect has a detrimental impact on the interbank market whenever $\Delta < \hat{\Delta}$ (i.e., whenever $\bar{\delta} > 0$) and implies that the increase in investment is less than the direct effect $F(\bar{z}) - F(1)$.

Figure 3 shows the effects of an equity injection on the credit markets (the top panel) and on investment (the bottom panel). The upward shift in the investment curve (from $I(m_1, \Delta)$ to $I(m_2, \Delta)$) in the lower panel is the direct effect of having more funds. The movement along the $I(m_2, \Delta)$ curve from Δ_1 to Δ_2 is the contamination effect.

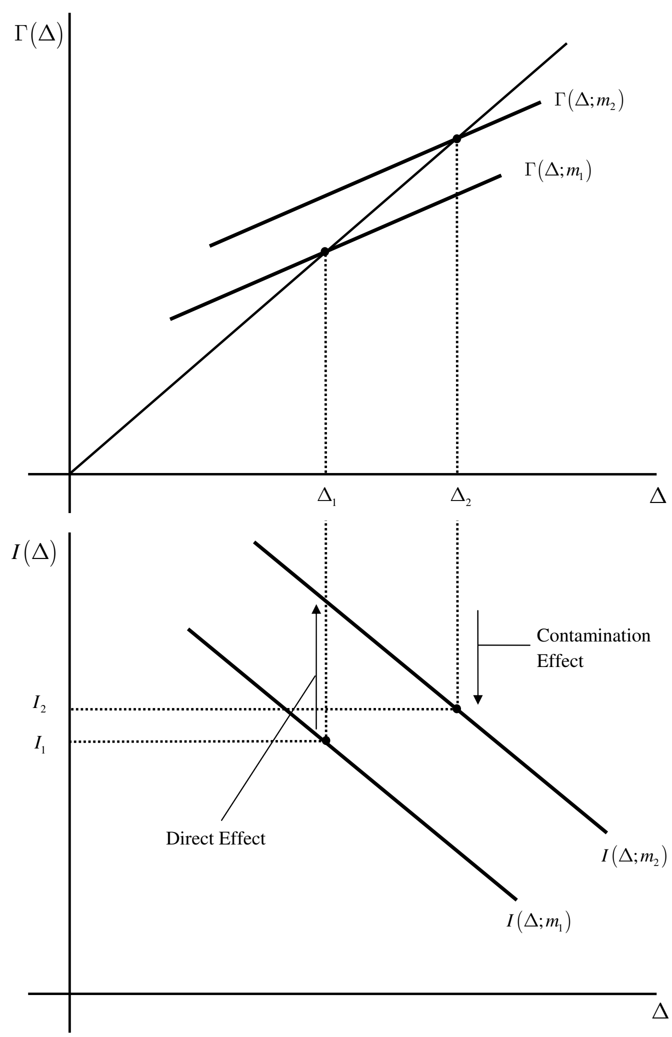

Figure 3: Equity Injections and The Contamination Effect

In the constrained case ($\Delta > \hat{\Delta}$), differentiating (12) with respect to m gives,

$$\frac{\partial I\left(m, \Delta\left(m\right)\right)}{\partial m} = \left[1 - F\left(1\right)\right] \tag{14}$$

As before, given Δ, banks with low z's do not change their behavior. Unlike the first case however, all banks with projects $z \geq 1$ use the additional funds to expand investment dollar-for-dollar. Since all banks with $z \geq 1$ are liquidity constrained, the increase in investment is proportional to $1 - F\left(1\right)$. Unlike the case with $\Delta < \hat{\Delta}$, there is no contamination effect. The contamination effect arises when banks with $z > \bar{z}$ continue to finance their projects but sell fewer good assets. If $\Delta > \hat{\Delta}$ however, the banks with $z > \bar{z}$ are also constrained (they are already selling all of their assets). These banks continue to sell all of their assets after the equity injection. Thus, there is no contamination effect and the equity injection neither improves nor hinders the interbank market. The following proposition summarizes the discussion above.

Proposition 6 *Let Δ^* be a stable equilibrium (i.e., $\Gamma_\Delta(\Delta^*) < 1$) and consider an increase in internal funds $dm > 0$. In the neighborhood of Δ^*, the new equilibrium always features (weakly) greater toxicity in the secondary market (the new equilibrium always has a weakly higher default rate Δ). Also,*

1. *if the initial $\Delta^* > \hat{\Delta}$, the change in investment is $[1 - F(1)]dm$ and there is no improvement in the interbank market for toxic assets (the new equilibrium is $\Delta = \Delta^*$).*
2. *if $\Delta^* < \hat{\Delta}$ then the change in investment is less than $\left[F\left(\bar{z}\left(\Delta^*\right)\right) - F\left(1\right)\right]$ and the interbank market for toxic assets becomes less liquid (the new equilibrium is $\Delta > \Delta^*$).*

The equity injection considered above gave an equal amount of new cash assets to all banks regardless of their liquidity needs. One might think that an equity injection targeted to banks with the best projects would be more effective than a policy which increases equity indiscriminately. In fact, an equity injection targeted toward the banks with the best projects is worse than an indiscriminate equity injection to all banks. Equation (13) shows that investment only rises due to the increased investment of the banks with intermediate projects (those with $1 < z < \bar{z}$). If the new equity were to go only to banks with $z > \bar{z}$, only the second term (the contamination effect) in (13) would be present and investment, welfare and liquidity would all decrease.

4.2 Asset Purchases

The original design of the TARP called for the government to purchase toxic assets and take them off of the banks' balance sheets. The idea was that such an asset purchase would restore liquidity to the inter-bank market provided that it was (according to Secretary Paulson) "properly designed and sufficiently large." In this section we consider the effects of an asset purchase policy in our model.

Before turning to the formal analysis we should point out that there is some reason to anticipate that an asset purchase program might work well in the adverse selection environment. Like the equity injection, an asset purchase program would transfer funds to the banks to the extent that the government overpaid for the assets. Unlike an equity injection however, banks have to sell assets to get the transfer. This increased incentive to trade might well be expected to have beneficial effects. The implicit transfer from an asset purchase program would automatically be concentrated on banks that sold the most assets – presumably the banks with the greatest liquidity needs.

Suppose the government takes actions which lower Δ by an amount $d\Delta < 0$. How costly is this policy? There are two ways to look at this. One way to think about such a policy is that the government provides a subsidy for sales of toxic assets (a negative "Tobin Tax").[6] The cost of such a policy would be the magnitude of the subsidy times the number of assets sold in equilibrium. A second way of thinking about such a policy is that the government purchases assets in sufficient amounts to increase the price. The costs to the government are the same in both cases though the initial out-of-pocket costs to the government are greater in the second case since the government needs to purchase effectively all of the assets sold on the market. We proceed with the second interpretation since it more accurately describes the policy interventions actually considered during the financial crisis.

If the asset purchase program succeeds, there will be a difference between the actual quality of the loan pool and the price of securities. In particular, $p > R(1 - \Gamma(\Delta))$. This was exactly the concern voiced by many academic economists when the TARP and the P-PIP were unveiled. In our model, for the asset purchase program to work, the price *must* exceed the fair market value of the securities. Because the price will typically be higher than what would be justified by the underlying value of the securities, no private banks will purchase assets – only the government will be on the demand side of the market.

[6] Policies that subsidize trade are often beneficial in adverse selection environments. See House and Zhang [2010] for examples of how hiring taxes and subsidies can be used in labor markets with adverse selection.

The fact that $p > R(1 - \Gamma(\Delta))$ presents a slight technical problem because we cannot use Δ to describe both the average quality of the loan pool and the price in the secondary market.[7] We proceed by using Δ to describe the effective price to sellers. That is, if a bank sells an asset in the secondary market, it receives a price equal to $R(1 - \Delta)$. We make this choice so we can continue to use the solutions in Proposition 1. The actual quality on the secondary market will be given by $\Gamma(\Delta) \neq \Delta$. We should expect that $\Gamma(\Delta) > \Delta$ since the government will want to increase in price of securities in the secondary market.

While we could analyze the program by starting with the size of the purchase, we instead analyze the program as if the government were choosing Δ directly. That is, we act as though the government chooses the price on the secondary market and then derive the amount of purchases that would imply that price.

Investment and Welfare. The effects of an asset purchase can be seen by differentiating (10) and (11) with respect to Δ in a neighborhood of an equilibrium $\Delta^* = \Gamma(\Delta^*)$. The effects naturally depend on whether the market is constrained or not. If $\Delta < \hat{\Delta}$ so that the banks with the highest z's can afford to fully fund their investment projects, an incremental change in Δ has the following effects on investment and welfare:

$$I_\Delta = -\int_1^{\bar{z}} R\left[(1 - G(\hat{\delta}(z))) + (1 - \Delta) g(\hat{\delta}(z)) z\right] f(z)\, dz, \tag{15}$$

and

$$W_\Delta = -\int_1^{\bar{z}} (z - 1) R\left[(1 - G(\hat{\delta}(z))) + (1 - \Delta) g(\hat{\delta}(z)) z\right] f(z)\, dz. \tag{16}$$

The change in Δ influences only the banks with intermediate investment projects (the other banks are infra-marginal). For these banks, the change in Δ has two separate impacts on investment. First, as Δ increases, the value of their existing asset sales falls. This is captured by the first terms in the integrals in the equations above. Second, as Δ increases, these banks substitute away from selling high-quality MBS. There are $g(\hat{\delta}(z))$ such securities at the margin. Thus, in addition to the declining value of their sales, banks also choose to sell fewer assets. Both effects work to reduce investment and welfare as Δ increases.

If $\Delta > \hat{\Delta}$ the effects on investment and welfare are given by

$$I_\Delta = -R\left[\int_1^{\bar{z}} \left[(1 - G(\hat{\delta}(z))) + (1 - \Delta) g(\hat{\delta}(z)) z\right] f(z)\, dz + \int_{\bar{z}}^{\infty} f(z)\, dz\right] \tag{17}$$

[7] Among other things it requires that we modify our definition of an equilibrium. The necessary modifications are natural. We omit this modified definition to save space.

and

$$W_{\Delta} = -R\left[\int_{1}^{\bar{z}} (z-1)\left[(1-G(\hat{\delta}(z))) + (1-\Delta)g(\hat{\delta}(z))z\right]f(z)dz + \int_{\bar{z}}^{\infty} (z-1)\, f(z)\, dz\right] \quad (18)$$

As before, the change in Δ influences investment and welfare by reducing the value of sales and the number of sales for the intermediate banks. Unlike the previous case however, the change in Δ now affects the infra-marginal banks as well (banks with $z > \bar{z}$). While these banks do not change the number of assets being sold (they are infra-marginal), an increase in Δ does reduce the value of their sales. As before, these effects work to unambiguously reduce investment and welfare.

Typically, asset purchases would be expected to increase the market price of MBS and reduce Δ. If the asset purchase program succeeded in increasing the price, investment and welfare would both increase according to either (15) and (16) if $\Delta < \hat{\Delta}$ or (17) and (18) if $\Delta > \hat{\Delta}$.

Proposition 7 *A decrease in Δ unambiguously increases welfare and investment.*

The Cost of Asset Purchases. If the government purchases all toxic assets sold on the secondary market, its outlays are $RB\,(1-\Delta)$. The true value of the purchase is $RB\,(1-\Gamma\,(\Delta))$ so the net cost of the policy is $RB\,(\Gamma\,(\Delta)-\Delta)$.

At an equilibrium ($\Gamma\,(\Delta) = \Delta$) the net impact on the government budget is zero. However, if the subsidy causes $\Gamma\,(\Delta)$ to deviate from Δ (which must occur if $\Gamma_{\Delta} \neq 1$) then the costs are positive or negative depending on whether $\Gamma_{\Delta} < 1$ (as in a stable equilibrium) or $\Gamma_{\Delta} > 1$ (as in an unstable equilibrium). For our analysis, we assume the equilibrium is stable.[8]

The budget cost of causing a change in Δ (by an amount $d\Delta$) is $RB\,[\Gamma_{\Delta}-1]\,d\Delta$. The term in square brackets is negative provided that the equilibrium is stable. Presumably, the government will seek to reduce Δ, thus, $d\Delta < 0$ and the overall cost of the asset purchase program is positive. The stronger the positive feedback effects are (as $\Gamma_{\Delta} \to 1$), the smaller budget impact of the asset purchase is.

[8] If the equilibrium is unstable (i.e., if $\Gamma_{\Delta} > 1$) then the government can effectively raise revenue by purchasing toxic assets. In this case, asset purchases have such a strong positive feedback effect that the implied default rate falls by more than the reduction in Δ caused by the asset purchase.

4.3 Comparison

We now compare the effects of the two policies. We will initially make the limiting assumption that the equity injection is a pure transfer of funds to the financial sector. This is absolutely not an accurate description of the actual cost of the TARP program but it is useful as a benchmark case. Furthermore, we assume that the equity injection entails giving money to the traders as well. Thus, for the banks internal funds to rise by dm requires a total injection of $dm\,(1+T)$ dollars (this is tantamount to assuming that the government cannot distinguish between the traders and the optimizing banks).

The cost of the equity injection is $dm\,(1+T)$ and it affects aggregate investment according to equation (13) per dollar injected. We compare this with an asset purchase which has the same cumulative budget cost as the equity injection. The implied change in Δ is

$$dm\,(1+T)\left[RB\left[\Gamma_{\Delta}\left(\Delta\right)-1\right]\right]^{-1}=d\Delta \tag{19}$$

and the change in investment (I_{Δ}) is given by (15) if $\Delta<\hat{\Delta}$ or (17) if $\Delta>\hat{\Delta}$. The government will choose $d\Delta<0$. Since $\Gamma_{\Delta}<1$ at a stable equilibrium, the cost is positive. We consider the two cases $\Delta<\hat{\Delta}$ and $\Delta>\hat{\Delta}$ in turn.

Interior Equilibrium ($\Delta<\hat{\Delta}$). We compare the incremental change in investment from an equity injection to the incremental change in investment from an asset purchase. Since $\Delta<\hat{\Delta}$ we use (13) and (19) to write this comparison as

$$\left[F\left(\bar{z}\right)-F\left(1\right)\right]+I_{\Delta}\Delta_{m}\gtreqless I_{\Delta}\frac{1+T}{RB\left[\Gamma_{\Delta}-1\right]}.$$

If the left hand side is greater than the right hand side, equity injections are more stimulative. If the right hand side is greater, asset purchases are more stimulative. Using (15) to substitute for I_{Δ} we can rewrite this expression as

$$\begin{aligned}F\left(\bar{z}\right)-F\left(1\right)\quad\gtreqless\quad&\frac{\int_{1}^{\bar{z}}(1-G(\hat{\delta}(z)))f\left(z\right)dz}{B\left[1-\Gamma_{\Delta}\right]}\left(1+T\right)\\&+\frac{\int_{1}^{\bar{z}}\left(1-\Delta\right)g(\hat{\delta}(z))zf\left(z\right)dz}{B\left[1-\Gamma_{\Delta}\right]}\left(1+T\right)\\&+\Delta_{m}R\int_{1}^{\bar{z}}\left[(1-G(\hat{\delta}(z)))+\left(1-\Delta\right)g(\hat{\delta}(z))z\right]f\left(z\right)dz\end{aligned} \tag{20}$$

Each term on the right hand side is positive. The first term is the effective transfer to the marginal banks ($1 \leq z \leq \bar{z}$) assuming that there is no change in behavior. The second term reflects increases in asset sales by the marginal banks. As Δ falls, the marginal banks sell more toxic assets and use the proceeds to increase investment. The last term is the (negative) contamination effect.

Using the definition of B (total sales of all toxic assets, equation (7)) we can rewrite the first term in (20) as

$$[F(\bar{z}) - F(1)] \times \frac{\int_1^{\bar{z}} (1 - G(\hat{\delta}(z))) \frac{f(z)}{[F(\bar{z}) - F(1)]} dz}{\left(\frac{1}{1+T}\right) \left[\int_0^{\infty} (1 - G(\hat{\delta}(z))) f(z)\, dz\right] + \left(1 - \frac{1}{1+T}\right)} \times \left(\frac{1}{1 - \Gamma_\Delta}\right)$$

This gives us a natural set of conditions under which an asset purchase will be preferable to an equity injection. The last term in this expression is greater than 1 provided that there is positive feedback in equilibrium (i.e., provided that $\Gamma_\Delta \geq 0$). The middle term is the ratio of the average number of sales for marginal banks to the average number of sales for banks overall (including the traders). If the average sales of marginal banks is at least as great as the average number of sales of a typical bank then the term multiplying $[F(\bar{z}) - F(1)]$ is greater than 1. In this case, the first term on the right side of (20) exceeds the term on the left. Since the other two terms are positive, an asset purchase will increase investment by more than an equally costly equity injection.

Constrained Equilibrium ($\Delta > \hat{\Delta}$). We again compare the (local) change in investment from the two policies. Since $\Delta > \hat{\Delta}$ we use (14) and (19) to write the comparison as

$$\begin{aligned}[1 - F(1)] \quad \gtreqless \quad & [1 - F(1)] \times \left(\frac{1}{1 - \Gamma_\Delta}\right) \frac{\int_1^{\infty} (1 - G(\hat{\delta}(z))) \frac{f(z)}{[1 - F(1)]} dz}{\left(\frac{1}{1+T}\right) \int_0^{\infty} (1 - G(\hat{\delta}(z))) f(z)\, dz + \frac{T}{1+T}} \\ & + \left(\frac{1}{1 - \Gamma_\Delta}\right) \frac{\int_1^{\bar{z}} (1 - \Delta)\, g(\hat{\delta}(z)) z f(z)\, dz}{\left(\frac{1}{1+T}\right) \int_0^{\infty} (1 - G(\hat{\delta}(z))) f(z)\, dz + \frac{T}{1+T}}\end{aligned} \tag{21}$$

where we have used the fact that $G(\hat{\delta}(z)) = 1$ for $z > \bar{z}$ in a constrained equilibrium.

Again, each term on the right is positive. The first term is the effective transfer to the marginal banks assuming that there is no change in behavior and the second term reflects increases in sales by the marginal banks. In a constrained equilibrium however, all banks with $z > 1$ are marginal. Note there is no change in Δ arising from the equity injection (i.e., there is no contamination effect) because we are in a constrained equilibrium. The ratio in the first term is again the ratio of the average number of sales for marginal banks to the average number of sales for banks overall.

Since $\Delta > \hat{\Delta}$ it must be the case that $\Gamma_\Delta \geq 0$. Moreover, if T were zero (no traders) then the term multiplying $[1 - F(1)]$ would always be greater than 1 since all banks with $z \geq 1$ sell more assets than banks with $z < 1$. In this case, the first term on the right side of (21) exceeds the term on the left. Since the other term is positive, an asset purchase again increases investment by more than an equity injection.

Proposition 8 *An asset purchase increases welfare and investment more than an equity injection whenever the following conditions hold:*

1. *There is positive feedback in equilibrium ($\Gamma_\Delta \geq 0$).*
2. *The average sales of the banks with investment $0 < x < i$ is at least as great as the average sales of all banks.*
3. *The cost of equity is at least $1 - \Gamma_\Delta$ per dollar.*

Notice that in the constrained equilibrium, conditions *1* and *2* are guaranteed automatically. Thus, asset purchases perform better when the intrabank market is highly contaminated ($\Delta > \hat{\Delta}$). Also, the conditions in the proposition are sufficient; they are not necessary. In the numerical example below, we demonstrate that even if the cost of equity is substantially less than $1 - \Gamma_\Delta$, asset purchases may still be preferable as a policy option.

A NUMERICAL EXAMPLE. We present a numerical example for illustration. Set $m = .5$, $R = i = 1$ and $T = 0$. We assume that both G and F are truncated normal distributions. Specifically, projects (z) are normally distributed with mean 0.9 and standard deviation 0.5. The density is truncated below $z = 0$ and above $z = 2$. Default rates (δ) are normally distributed with mean 0.1 and standard deviation 0.25. The density is truncated below $\delta = 0$ and above $\delta = 1$.

Figure 4 plots the fixed point mapping $\Gamma(\Delta)$, aggregate investment $I(\Delta)$ and welfare $W(\Delta)$ implied by these parameters. There are three equilibria $\Delta^* \in \{0.4046, 0.7699, 1.000\}$, each depicted by a vertical line. Naturally, since $T = 0$, the shut-down equilibrium ($\Delta = 1$) is one of the equilibria.

We now consider the differential effects of an equity injection and several asset purchase policies in our numerical example. The magnitude of the equity injection is (arbitrarily) chosen to be equal to ten percent of the banks internal funds (so $dm = 0.05$). The asset purchases

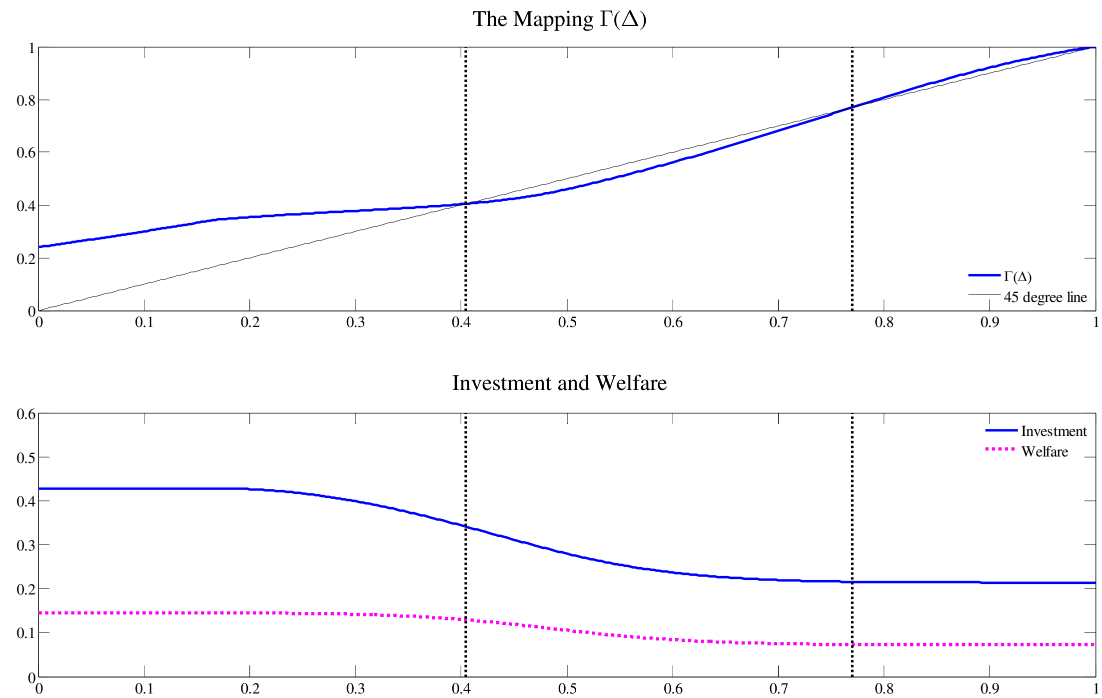

Figure 4: Numerical Example

are less than the equity injections. Specifically, we consider asset purchases of 0.005, 0.010, 0.025 and 0.050. We focus on the stable equilibrium $\Delta = 0.4046$. This is the only stable equilibrium other than the shut-down equilibrium. At this equilibrium, the feedback effect, given by the slope of the fixed point mapping, is $\Gamma_\Delta = 0.3528$.

The results of each policy are presented in Table 1. Asset purchases outperform the equity injection in all cases. Even when the budget cost of the asset purchases is one tenth of the cost of the equity injections, the asset purchase increases investment and welfare by more than the equity injection.

	Default Rate (Δ)	Investment (I)	Welfare (W)	Cost (dm)
Initial Equilibrium	0.4046	0.3411	0.1291	0.000
Equity Injection	0.4166	0.3508	0.1310	0.050
Asset Purchases	0.3797	0.3568	0.1335	0.005
	0.3547	0.3716	0.1370	0.010
	0.2798	0.4070	0.1428	0.025
	0.1551	0.4281	0.1443	0.050

Table 1: Numerical Examples

5 Discussion and Related Literature

Has the Government Purchased Toxic Assets? While asset purchases were initially suggested as a possible course for addressing the crisis, political and institutional concerns forced the government to change course. While the Treasury and the Federal Reserve have engaged in a variety of dramatic market interventions since the crisis began, with only a few exceptions, they have made very limited purchases of toxic assets.

In terms of sheer magnitude, the Federal Reserve's Mortgage Backed Securities Purchase Program dwarfs most of the other credit market interventions. Beginning in January 2009, the Federal Reserve has made weekly purchases of MBS's. The last purchases under this program were made by March 31, 2010. The Federal Reserve now holds roughly $1.25 trillion in agency mortgage backed securities. Unlike non-agency MBS's, agency MBS's are guaranteed as to interest and principal by one of the Government Sponsored Enterprises (GSE's, i.e., Fannie Mae, Freddy Mac or Ginnie Mae).[9] While this asset purchase program is quite large, because these securities are guaranteed against default, these assets would not seem to be "toxic" in the way Secretary Paulson described.

In addition to these purchases, the Federal Government has purchased roughly $100 billion dollars in mortgage backed securities from Bear Stearns and AIG. Acting through legally separate entities (the Maiden Lane LLC's), the New York Federal Reserve has effectively arranged for the purchase of three sets of non-agency mortgage backed securities. These purchases occurred on March and November 2008.

Finally, the P-PIP was intended to pick up where the TARP left off by purchasing "legacy assets" (i.e., toxic assets) of financial institutions. The P-PIP is authorized to purchase up to $40 billion of non-agency mortgage backed securities. The qualifier "non-agency" means that the securities are not guaranteed by any of the GSE's. As of March 31, 2010 however, the P-PIP has purchased only $10.5 billion dollars of residential and commercial mortgage backed securities.[10]

The Cost of Equity Injections. In the model, we assumed that in the worst case, equity injections cost the government the full amount of the injection. It is quite possible however,

[9] While the Fed is free to lend to "any individual, partnership, or corporation" provided that the loans are "indorsed or otherwise secured to the satisfaction of the Federal Reserve bank" under section 13(3) of the Federal Reserve Act, section 14(b) restricts the Fed to purchase only assets "guaranteed as to principal and interest by [an] agency of the United States."

[10] Most of these purchases are residential mortgage backed securities (RMBS). Of the $10.5 billion, roughly $8.8 billion are RMBS. See U.S. Department of the Treasury, 2010, a and b.

that the cost of equity injections is much smaller and may even be zero. If an equity injection can be achieved with zero budget cost, a simple way of ensuring optimal investment would simply be to transfer i to the banks.[11]

The actual equity injections undertaken by the TARP funds were initially seen as moderately costly. A 2009 CBO report estimated that the average subsidy associated with the equity purchases through the Capital Purchase Program was roughly 18 percent (see Congressional Budget Office 2009.a and 2009.b). Other equity purchases by the Treasury were estimated to be somewhat more costly. Additional equity purchases (outside the CPP) of AIG and Citigroup were estimated to be at prices that exceeded the fair value of the equity by 53 percent and 26 percent respectively. By 2010, the CBO estimated that the actual ex post budget impact of the equity injections was substantially lower. These costs are comparable to the cost of equity used in our numerical example earlier.

Related Literature. Our paper adds both to the literature on adverse selection in financial markets and to the so-called financial accelerator literature. Both literatures are very large so we cannot provide an adequate summary here.[12] Instead we briefly mention four recent papers that make contributions closely related to those in our paper.

There are several papers which emphasize the amplifying effects of adverse selection in macroeconomic contexts. Eisfeldt [2004] considers an optimal consumption allocation model in which financial assets are illiquid due to adverse selection. When productivity rises, agents want to invest more and thus sell more high-quality assets. This causes quality in the loan market to improve and amplifies the effects of the productivity shock. Because of the complexity of her framework, many of her most striking results are obtained numerically. Kurlat [2009] also emphasizes the amplifying role of adverse selection. In his model, investors sell old projects to finance new ones. Adverse selection in existing projects thus impedes investment. Like Eisfeldt's model, productivity innovations are amplified by the adverse selection problem. Kurlat also considers the possibility that assets give signals of their type. Learning slows down during downturns which make the adverse selection distortions worse.

Philippon and Skreta [2010] focus on optimal policy in markets where the value of collateral is private information and thus, adverse selection in the loan market cause investment to

[11]Such a strategy would maximize the contamination effect. The only reason a bank would sell a toxic asset would be to get rid of assets with below average quality. As a result, in equilibrium, the secondary market for toxic securities would shut down completely.

[12]For the literature on adverse selection in financial markets, Stiglitz and Weiss [1981], De Meza and Webb [1987] and Mankiw [1986] (and obviously Akerlof, 1970). For the literature on the financial accelerator, see Bernanke and Gertler [1989], Bernanke Gertler and Gilchrist [1999] and the references therein.

be inefficiently low. Building on the Myers and Majluf [1984] framework, they consider a mechanism design setting in which banks finance investments by issuing claims on future payoffs including the future value of their toxic assets. Thus, unlike our setting in which banks sell their assets to raise funds directly, Philippon and Skreta consider a model with risky loans when collateral is toxic. They analyze policies designed to make all banks invest. Their main focus is on the stigma attached to a bank that chooses to participate in a government program. If markets observe the decision to participate, market participants conclude that the financial institution is a bad risk. This negative stigma effect gives banks an incentive to refuse the government program to signal that they are a high type.

Tirole [2010] also uses mechanism design to analyze market failure due to adverse selection. In his model the optimal mechanism is designed to remove the worst assets from the market and leave only the best ones. Like our model, the government must overpay for toxic assets.

Finally, in ongoing work, Camargo and Lester [2010] focus on how quickly markets for toxic assets clear. Like the previous papers (and our paper), they appeal to adverse selection as the source of the market friction. Unlike the other papers, Camargo and Lester use a search-theoretic framework to study the rate at which trades occur in markets for toxic assets. Patient buyers wait for the most desperate sellers to accept low offers and clear out of the market. If the adverse selection friction is strong enough, it may take many periods for the market to clear as patient buyers strategically wait for the impatient buyers to exit.

6 Conclusion

When the value of mortgage backed securities dropped sharply in 2007, banks and financial intermediaries faced a pronounced funding shortage. Many banks with MBS could not find willing buyers at fair prices and as a result could not raise the funds needed to finance investments. We have presented a model of an interbank market for toxic assets. In our model, the funding shortage arises from an adverse selection problem. Assets traded on the market are of the lowest quality which creates a severe liquidity and funding crisis and reduces investment and welfare.

We use the model to analyze two types of government interventions: equity injections and asset purchases. While equity injections typically increase investment, they also reduce liquidity in the interbank market. Because banks have greater access to internal funds, they sell fewer high-quality assets. By reducing the number of high-quality assets traded, equity injections further contaminate the interbank market, reducing prices and liquidity. In an extreme case, if equity injections are directed to banks with the greatest liquidity needs, this contamination effect causes investment and welfare to fall.

In contrast, by increasing the price of toxic assets, asset purchase programs increase liquidity in the interbank market and also increase investment. By bidding up the price of toxic assets, asset purchase plans effectively transfer more funds to the banks that need them the most. At the same time, to get the transfer, banks have to continue to sell high quality assets.

The conclusions of our analysis depend directly on adverse selection as the source of the financial friction. Obviously, there were many other factors operating in 2008 so our analysis cannot be viewed as a complete evaluation of the policies considered by the Treasury and the Federal Reserve. While it is not comprehensive, our analysis has identified an important channel that should be part of the evaluation of policies designed to manage markets for toxic assets.

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Appendix: Proofs of the Propositions

Lemma 1 *The following conditions hold in any equilibrium with trade:*

1. *No socially inefficient project is undertaken, i.e.,* $x = 0$ *whenever* $z < 1$.
2. *Banks never simultaneously purchase assets and sell assets with a below average default rate, i.e.,* $q > 0$ *iff* $\hat{\delta} = \Delta$.
3. *The price of mortgage backed securities is actuarially fair, i.e.,* $p = R(1 - \Delta)$.

Proof: Given Δ, if it is optimal for a bank to sell assets of type δ, then it must be optimal to sell all assets with $\delta' > \delta$. As a result, each bank chooses a cutoff depreciation rate $\hat{\delta}$, and investment level x and asset purchases q to solve the following maximization:

$$\max_{\hat{\delta},x,q}\{R\int_0^{\hat{\delta}}(1-\delta)g(\delta)d\delta + R(1-\Delta)q + zx + (m + p\int_{\hat{\delta}}^1 g(\delta)d\delta - x - pq)\} \tag{22}$$

subject to

$$x + pq \leq m + p\int_{\hat{\delta}}^1 g(\delta)d\delta$$
$$0 \leq x \leq i \text{ and } 0 \leq q.$$

Then the Lagrangian of maximization problem of a bank with z is

$$\begin{aligned} L_z(\hat{\delta}, x, q) &= R\int_0^{\hat{\delta}}(1-\delta)g(\delta)d\delta + R(1-\Delta)q + zx + (m + p\int_{\hat{\delta}}^1 g(\delta)d\delta - x - pq) \\ &\quad + \lambda(m + p\int_{\hat{\delta}}^1 g(\delta)d\delta - x - pq) + \mu(i - x) + \theta x + \eta q, \end{aligned}$$

$\hat{\delta}$:

$$\begin{aligned} R(1-\hat{\delta})g(\hat{\delta}) - pg(\hat{\delta})(1+\lambda) &= 0 \\ R(1-\hat{\delta}) &= p(1+\lambda), \end{aligned} \tag{23}$$

x :

$$\begin{aligned} z - 1 - \lambda - \mu + \theta &= 0 \\ z - 1 &= \lambda + \mu - \theta, \end{aligned} \tag{24}$$

q :

$$\begin{aligned} R(1-\Delta) - p - p\lambda + \eta &= 0 \\ R(1-\Delta) &= p(1+\lambda) - \eta. \end{aligned} \tag{25}$$

1) Suppose that some bank is investing, $x > 0$. In this case, $\theta = 0$ and the equation (24) becomes $z - 1 = \lambda + \mu \geq 0$. Thus, no project with $z < 1$ is ever activated.

2) Assume that a bank simultaneously has $q > 0$ and sells assets for which $\delta < \Delta$. This cannot be optimal since the bank could simply buy ε fewer assets on the market and sell ε fewer good assets. Since $1 - \delta > 1 - \Delta$, the bank is strictly better off.

3) Suppose that some bank purchases MBS. In this case, $\eta = 0$ for this bank, by (25), $R(1-\Delta) = p(1+\lambda)$ which implies that $p \leq R(1-\Delta)$. Since these are aggregate variables, if any bank purchases MBS, it must be the case that $p \leq R(1-\Delta)$. Put differently, if $p > R(1-\Delta)$ then no bank is purchasing MBS.

Now we show that $p = R(1-\Delta)$. Suppose not $p < R(1-\Delta)$. In this case $\lambda > 0$ for every firm. Since $\lambda(m + p\int_{\hat{\delta}}^1 g(\delta)d\delta - x - pq) = 0$, for every firm, $m + p\int_{\hat{\delta}}^1 g(\delta)d\delta = x + pq$. Summing over all of the firms,

$$\int mf(z)dz + \int\left[p\int_{\hat{\delta}(z)}^1 g(\delta)d\delta\right]f(z)dz = \int x(z)f(z)dz + \int pq(z)f(z)dz,$$

then use the asset market clearing,

$$m + p\int \left[\int_{\hat{\delta}(z)}^{1} g(\delta)d\delta\right] f(z)dz = \int x(z)f(z)dz + pT + p\int \left[\int_{\hat{\delta}(z)}^{1} g(\delta)d\delta\right] f(z)dz$$

$$m = \int x(z)f(z)dz + pT.$$

Since no project with $z < 1$ is activated,

$$\int x(z)f(z)dz \leq I^*.$$

By $p < R(1-\Delta) \leq R$

$$m \leq \int x(z)f(z)dz + RT \leq I^* + RT,$$

but $I^* + RT < m$ by assumption. This is a contradiction so $p = R(1-\Delta)$ in equilibrium. ■

Proposition 1 *Taking Δ as given, the optimal policy, is described by a cutoff rule $\hat{\delta}(z,\Delta) \in [0,\Delta]$ and an investment function $i(z,\Delta) \in [0,i]$.*

1. *For $z < 1$, $\hat{\delta}(z,\Delta) = \Delta$ and $x(z,\Delta) = 0$;*
2. *$\hat{\delta}(1,\Delta) = \Delta$ and $x(1,\Delta) \in [0, m + R(1-\Delta)(1-G(\Delta))]$;*
3. *For $1 < z \leq \bar{z}(\Delta)$, $\hat{\delta}(z,\Delta) = 1 - z(1-\Delta)$ and $x(z,\Delta) = m + R(1-\Delta)(1-G(\hat{\delta}(z,\Delta)))$;*
4. *For $\bar{z}(\Delta) < z$, $\hat{\delta}(z,\Delta) = \bar{\delta}(\Delta)$ and $x(z,\Delta) = \min\{i, m + R(1-\Delta)\}$.*

where $\bar{\delta}(\Delta) = G^{-1}(1-\min\{\frac{i-m}{R(1-\Delta)}, 1\})$ and $\bar{z}(\Delta) = \frac{1-\bar{\delta}(\Delta)}{1-\Delta}$, $\bar{z}(\Delta) \in [1, \frac{1}{1-\Delta}]$. Only banks with $z < 1$ purchase assets on the secondary market. They are indifferent to the level of purchases.

Proof: **Case I:** $\lambda = \mu = 0$. Then by (23) we have

$$\hat{\delta} = \Delta$$

and by (24)

$$\begin{aligned} z - 1 &= -\theta \leq 0 \\ z &\leq 1. \end{aligned}$$

Hence, whenever $z \leq 1$, we have $\hat{\delta}(z,\Delta) = \Delta$. In addition to that if $z < 1$ then $\theta > 0$, which means $0 = x(z,\Delta)$. If $z = 1$, then $\theta = 0$,[1] which implies that the investment level, $x(1,\Delta)$, can be any number between 0 and $m + R(1-\Delta)(1-G(\Delta))$. Hence we prove the first and the second part of the proposition.

Case II: $\lambda = 0$ but $\mu > 0$. By (23) and $\mu(i-x) = 0$, we $\hat{\delta}(z,\Delta) = \Delta$ and $x(z,\Delta) = i$. Hence $\theta = 0$, which yields $z \geq 1$ by (24). Given Δ, if there exists a bank that can afford to invest i, i.e., $m + p\int_{\Delta}^{1} g(\delta)d\delta \geq i$, each bank with $z \geq 1$ can afford to invest i by selling only bad assets. Hence, none would need to sell good assets. In other words, no bank ever sells a mortgage with $\delta < \Delta$. Since Δ is the average default rate, if there are mortgages with $\delta > \Delta$ there must be some mortgages with $\delta < \Delta$ unless $\Delta = 1$. If $\Delta = 1$, then $p = 0$ and $m \geq i$, which is a contradiction.

Case III: $\lambda > 0$ but $\mu = 0$. By equation (23) and the equilibrium condition $p = R(1-\Delta)$

$$1 - \hat{\delta} = (1-\Delta)(1+\lambda).$$

Assume $\Delta < 1$. Since $\lambda(1-\Delta) > 0$, we have $\Delta > \hat{\delta}$. By Lemma 1, $q = 0$, hence

$$i \geq m + p\int_{\hat{\delta}}^{1} g(\delta)d\delta = x(z,\Delta) > 0. \tag{26}$$

[1] Note that μ and θ cannot be non-zero at the same time. Otherwise, $x = i - x = 0$. Hence $\lambda = 0$ and $z = 1$ if and only if either $\mu = 0$ or $\theta = 0$, but not both.

Since $x(z,\Delta)\theta = 0$, we have $\theta = 0$, so $z - 1 = \lambda$ by (24). Hence $z > 1$. Then by substituting λ, we have

$$\begin{aligned} 1 - \hat{\delta} &= (1-\Delta)(1+z-1) \\ \hat{\delta}(z,\Delta) &= 1 - z(1-\Delta). \end{aligned}$$

The investment level is

$$\begin{aligned} x(z,\Delta) &= m + p\int_{\hat{\delta}}^{1} g(\delta)d\delta \\ &= m + R(1-\Delta)\int_{\hat{\delta}}^{1} g(\delta)d\delta \\ &= m + R(1-\Delta)(1 - G(\hat{\delta}(z,\Delta))). \end{aligned}$$

The inequality in (26) implies that

$$\begin{aligned} i &\geq m + R(1-\Delta)(1 - G(\hat{\delta}(z,\Delta))) \\ \frac{i-m}{R(1-\Delta)} &\geq 1 - G(\hat{\delta}(z,\Delta)). \end{aligned}$$

If $1 > \frac{i-m}{R(1-\Delta)}$, then

$$\begin{aligned} \frac{i-m}{R(1-\Delta)} &\geq 1 - G(\hat{\delta}(z,\Delta)) \\ \hat{\delta}(z,\Delta) &\geq G^{-1}(1 - \frac{i-m}{R(1-\Delta)}). \end{aligned}$$

If $1 \leq \frac{i-m}{R(1-\Delta)}$, then

$$\begin{aligned} 1 &\geq 1 - G(\hat{\delta}(z,\Delta)) \\ \hat{\delta}(z,\Delta) &\geq 0. \end{aligned}$$

Hence $\hat{\delta}(z,\Delta)$ is bounded below by

$$\bar{\delta}(\Delta) := G^{-1}(1 - \min\{\frac{i-m}{R(1-\Delta)}, 1\})$$

then we have

$$\begin{aligned} \hat{\delta}(z,\Delta) &\geq \bar{\delta}(\Delta) \\ 1 - z(1-\Delta) &\geq \bar{\delta}(\Delta) \\ \frac{1-\bar{\delta}(\Delta)}{1-\Delta} &\geq z. \end{aligned}$$

Define $\bar{z}(\Delta) = \frac{1-\bar{\delta}(\Delta)}{1-\Delta}$. Therefore, for all $1 < z < \bar{z}(\Delta)$, $\hat{\delta}(z,\Delta) = 1 - z(1-\Delta)$ and $x(z,\Delta) = m + R(1-\Delta)(1 - G(\hat{\delta}(z,\Delta)))$..If $\bar{\delta}(\Delta) = 0$ then $\bar{z}(\Delta) = \frac{1}{1-\Delta}$. For all $z \geq \frac{1}{1-\Delta}$, $\hat{\delta}(z,\Delta) = 0$ and $x(z,\Delta) = m + R(1-\Delta)$.

Case IV: $\lambda > 0$ and $\mu > 0$. By Lemma 1 and $\mu(i - x) = 0$,

$$m + p\int_{\hat{\delta}}^{1} g(\delta)d\delta = i$$

To hold this equality, we need

$$\begin{aligned} m + R(1-\Delta)\underbrace{\int_{0}^{1} g(\delta)d\delta}_{=1} &\geq i \\ 1 &\geq \frac{i-m}{R(1-\Delta)}. \end{aligned}$$

We can solve for $\bar{\delta}$:

$$m + R(1-\Delta)\int_{\bar{\delta}}^{1} g(\delta)d\delta = i$$

or

$$1 - G(\bar{\delta}) = \frac{i-m}{R(1-\Delta)} < 1$$

$$\bar{\delta}(\Delta) = G^{-1}(1 - \frac{i-m}{R(1-\Delta)}) > 0.$$

This means $\theta = 0$, so we get $z - 1 = \lambda + \mu$ by (24). By (23), we have

$$1 - \bar{\delta} = (1-\Delta)(1+\lambda).$$

Then

$$\begin{aligned} z - 1 &= \lambda + \mu \\ z - 1 &= \frac{1-\bar{\delta}}{1-\Delta} - 1 + \mu \\ z &= \frac{1-\bar{\delta}}{1-\Delta} + \mu \\ z &= \bar{z}(\Delta) + \mu. \end{aligned}$$

Hence, $z > \bar{z}(\Delta)$, $\hat{\delta}(z,\Delta) = \bar{\delta}(\Delta) > 0$ and $x(z,\Delta) = i$. ∎

Proposition 2 *Given any distributions $F$and G with densities f and g,*

1. *there exists at least one equilibrium Δ^*;*
2. *if there are multiple equilibria, lower values of Δ^* Pareto-dominate higher values of Δ^*;*
3. *if $T = 0$ then $\lim_{\Delta\to 1}\Gamma(\Delta) = 1$ and $\Delta = 1$ is an equilibrium.*

Proof: There are two cases to consider: $T > 0$ and $T = 0$. The proof for $T > 0$ is straight-forward so we begin with this case. If $T > 0$ then

$$\Gamma(\Delta) = \frac{\int_0^\infty (\int_{\hat{\delta}(z,\Delta)}^1 \delta g(\delta)d\delta) f(z)dz + T\delta_T}{\int_0^\infty (\int_{\hat{\delta}(z,\Delta)}^1 g(\delta)d\delta) f(z)dz + T}$$

is well-defined and continuous for all $\Delta \in [0,1]$. Moreover, $\Gamma(\Delta) \in [0,1]$ for all $\Delta \in [0,1]$. Thus, by Brouwer's Fixed Point Theorem, there exists at least one fixed point.

If $T = 0$, the mapping $\Gamma(\Delta)$ is well-defined and continuous for all $\Delta \in [0,1)$ but is not well-defined for $\Delta = 1$. To accommodate this, we show that $\lim_{\Delta\to 1}\Gamma(\Delta) = 1$. We then assign $\Gamma(1) = 1$ for this case. To show this, note first that $\lim_{\Delta\to 1}\hat{\delta}(z,\Delta) = 1$ for all z. This is trivial for $z \le 1$ since $\hat{\delta}(z,\Delta) = \Delta$ in this range. For any $z > 1$, and any $1 > \varepsilon > 0$, take Δ_z such that $1 > \Delta_z > \max\{1 - \frac{\varepsilon}{z}, \hat{\Delta}\}$ where $\hat{\Delta}$ is defined in the text. Since $\Delta_z > \hat{\Delta}$, $\bar{\delta}(\Delta_z) = 0$ and $\bar{z}(\Delta_z) = \frac{1}{1-\Delta_z}$. Thus, $z < \bar{z}(\Delta_z)$ for any such Δ_z (this follows since $1 - \frac{1}{z} < 1 - \frac{\varepsilon}{z}$). Since $1 < z < \bar{z}(\Delta_z)$ we have $\hat{\delta}(z,\Delta_z) = 1 - z(1-\Delta_z)$. Moreover, $1 - \frac{\varepsilon}{z} < \Delta_z \implies z(1-\Delta_z) < \varepsilon \implies 1 - \hat{\delta}(z,\Delta_z) < \varepsilon$, which is the desired result. We now show that $\lim_{\Delta\to 1}\Gamma(\Delta) = 1$. The mapping is,

$$\Gamma(\Delta) = \frac{A(\Delta)}{B(\Delta)} = \frac{\int_0^\infty (\int_{\hat{\delta}(z,\Delta)}^1 \delta g(\delta)d\delta) f(z)dz}{\int_0^\infty (\int_{\hat{\delta}(z,\Delta)}^1 g(\delta)d\delta) f(z)dz}.$$

Note that as $\Delta \to 1$, $A(\Delta) \to 0$ and $B(\Delta) \to 0$. Using L'Hopital's rule,

$$\lim_{\Delta\to 1}\Gamma(\Delta) = \lim_{\Delta\to 1}\frac{A_\Delta(\Delta)}{B_\Delta(\Delta)} = \lim_{\Delta\to 1}\frac{\int_0^\infty \hat{\delta}(z,\Delta)g(\hat{\delta}(z,\Delta))\frac{\partial\hat{\delta}(z,\Delta)}{\partial\Delta}f(z)dz}{\int_0^\infty g(\hat{\delta}(z,\Delta))\frac{\partial\hat{\delta}(z,\Delta)}{\partial\Delta}f(z)dz}$$

$$\lim_{\Delta\to 1}\Gamma(\Delta) = \lim_{\Delta\to 1}\frac{\Delta\int_0^1 g(\Delta)f(z)dz + \int_1^{\bar{z}(\Delta)}(1-z(1-\Delta))g(1-z(1-\Delta))zf(z)dz}{\int_0^1 g(\Delta)f(z)dz + \int_1^{\bar{z}(\Delta)} g(1-z(1-\Delta))zf(z)dz} = 1.$$

We now define the function $\Gamma(1) = 1$. ∎

Proposition 3 *If Assumption 2 holds then the equilibrium is unique and stable.*

Proof: Average default rate implied by Δ is

$$\Gamma(\Delta) = \frac{A}{B} = \frac{\int_0^\infty (\int_{\hat{\delta}(z,\Delta)}^1 \delta g(\delta) d\delta) f(z) dz + T\delta_T}{\int_0^\infty (\int_{\hat{\delta}(z,\Delta)}^1 g(\delta) d\delta) f(z) dz + T}.$$

Differentiating w.r.t. Δ gives

$$\begin{aligned}\Gamma_\Delta(\Delta) &= \frac{-1}{B}\int_0^\infty \hat{\delta}(z,\Delta) g(\hat{\delta}(z,\Delta)) \frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta} f(z) dz + \frac{A}{B^2}\int_0^\infty g(\hat{\delta}(z,\Delta)) \frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta} f(z) dz \\ &= \frac{-1}{B}\int_0^\infty \hat{\delta}(z,\Delta) g(\hat{\delta}(z,\Delta)) \frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta} f(z) dz + \frac{1}{B}\Gamma(\Delta)\int_0^\infty g(\hat{\delta}(z,\Delta)) \frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta} f(z) dz\end{aligned}$$

$$\Gamma_\Delta(\Delta) = \frac{1}{B}\int_0^\infty \left[\Gamma(\Delta) - \hat{\delta}(z,\Delta)\right] g(\hat{\delta}(z,\Delta)) \frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta} f(z) dz.$$

Consider $\frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta}$. For $z \leq 1$ this derivative is 1. For $1 \leq z \leq \bar{z}(\Delta)$ the derivative is z. For $\bar{z}(\Delta) \leq z$ the derivative is $\frac{\partial \bar{\delta}(\Delta)}{\partial \Delta}$. Rewriting the derivative we have

$$\begin{aligned}\Gamma_\Delta(\Delta) &= \frac{1}{B}\left\{\int_0^1 [\Gamma(\Delta) - \Delta] g(\Delta) f(z) dz + \int_1^{\bar{z}(\Delta)} [\Gamma(\Delta) - \Delta + (z-1)(1-\Delta)] g(\hat{\delta}(z,\Delta)) z f(z) dz \right.\\ &\left. + \int_{\bar{z}(\Delta)}^\infty \left[\Gamma(\Delta) - \bar{\delta}(\Delta)\right] g(\bar{\delta}(\Delta)) \frac{\partial \bar{\delta}(\Delta)}{\partial \Delta} f(z) dz \right\}.\end{aligned}$$

Since we evaluate it at a fixed point $\Gamma(\Delta) = \Delta$, we have

$$\Gamma_\Delta(\Delta) = \frac{1}{B}\left\{\int_1^{\bar{z}(\Delta)} (z-1)(1-\Delta) g(\hat{\delta}(z,\Delta)) z f(z) dz + \int_{\bar{z}(\Delta)}^\infty \left[\Delta - \bar{\delta}(\Delta)\right] g(\bar{\delta}(\Delta)) \frac{\partial \bar{\delta}(\Delta)}{\partial \Delta} f(z) dz \right\}.$$

Recall that $\bar{\delta}(\Delta)$ is implicitly defined by the condition $1 - G(\bar{\delta}(\Delta)) = \min\left\{\frac{i-m}{R(1-\Delta)}, 1\right\}$. If $\frac{i-m}{R(1-\Delta)} \leq 1$, there is a point at which firms could finance everything if they sold enough securities. If $\frac{i-m}{R(1-\Delta)} > 1$, the price of the assets $R(1-\Delta)$ is so low that they would not be able to finance their project even if they sold everything. In the first case, differentiating w.r.t. Δ gives

$$-g(\bar{\delta}(\Delta)) \frac{\partial \bar{\delta}(\Delta)}{\partial \Delta} = R\frac{i-m}{[R(1-\Delta)]^2} = \frac{1 - G(\bar{\delta}(\Delta))}{1-\Delta}$$

which gives

$$\frac{\partial \hat{\delta}(z,\Delta)}{\partial \Delta} = \frac{\partial \bar{\delta}(\Delta)}{\partial \Delta} = \frac{-1}{g(\bar{\delta}(\Delta))}\frac{1 - G(\bar{\delta}(\Delta))}{1-\Delta} < 0$$

and, when $\bar{\delta} = 0$,

$$\frac{\partial \bar{\delta}(\Delta)}{\partial \Delta} = 0.$$

Thus, $\frac{i-m}{R(1-\Delta)} \leq 1$

$$\Gamma_\Delta(\Delta) = \frac{1}{B}\left\{\int_1^{\bar{z}(\Delta)} (z-1)(1-\Delta) g(\hat{\delta}(z,\Delta)) z f(z) dz - \frac{\left[\Delta - \bar{\delta}(\Delta)\right]}{1-\Delta}\left[1 - G(\bar{\delta}(\Delta))\right]\left[1 - F(\bar{z}(\Delta))\right]\right\}$$

using $\bar{z}(\Delta) - 1 = \frac{\Delta - \bar{\delta}(\Delta)}{1-\Delta}$ we have,

$$\Gamma_\Delta(\Delta) = \frac{1}{B}\left\{\int_1^{\bar{z}(\Delta)} (z-1)(1-\Delta) g(\hat{\delta}(z,\Delta)) z f(z) dz - (\bar{z}(\Delta) - 1)\left[1 - G(\bar{\delta}(\Delta))\right]\left[1 - F(\bar{z}(\Delta))\right]\right\}$$

while if $\frac{i-m}{R(1-\Delta)} > 1$

$$\Gamma_\Delta(\Delta) = \frac{1}{B}\left\{\int_1^{\bar{z}(\Delta)} (z-1)(1-\Delta)g(\hat{\delta}(z,\Delta))zf(z)dz\right\}.$$

Since the last term when $\frac{i-m}{R(1-\Delta)} \leq 1$ is negative we have

$$\Gamma_\Delta(\Delta) < \frac{1}{B}\left\{\int_1^{\bar{z}(\Delta)} (z-1)(1-\Delta)g(\hat{\delta}(z,\Delta))zf(z)dz\right\}.$$

Since $B > T$, then

$$\Gamma_\Delta(\Delta) < \frac{1}{T}\left\{\int_1^{\bar{z}(\Delta)} (z-1)(1-\Delta)g(\hat{\delta}(z,\Delta))zf(z)dz\right\}.$$

Since z is always less than $\bar{z}(\Delta)$, we can replace them by $\bar{z}(\Delta)$

$$\begin{aligned}
\Gamma_\Delta(\Delta) &< \frac{1}{T}\left\{\int_1^{\bar{z}(\Delta)} \frac{\Delta-\bar{\delta}(\Delta)}{1-\Delta}(1-\Delta)g(\hat{\delta}(z,\Delta))zf(z)dz\right\} = \frac{\Delta-\bar{\delta}(\Delta)}{T}\left\{\int_1^{\bar{z}(\Delta)} g(\hat{\delta}(z,\Delta))zf(z)dz\right\} \\
&< \frac{1}{T}\left\{\int_1^{\bar{z}(\Delta)} g(\hat{\delta}(z,\Delta))zf(z)dz\right\} < \frac{1}{T}\left\{\int_0^{\infty} \bar{g}zf(z)dz\right\} < \frac{\bar{g}}{T}\mu_z.
\end{aligned}$$

For the equilibrium to be unique, we want $\Gamma_\Delta(\Delta)$ to be less than 1. Provided that g and T satisfy

$$\bar{g} < \frac{T}{\mu_z}$$

we have $\Gamma_\Delta(\Delta) < 1$, hence the equilibrium is unique. ■

Proposition 4 *Let the distributions F and G be given and let Δ^* be a stable equilibrium (i.e., $\Gamma_\Delta(\Delta^*) < 1$). Then, in the neighborhood of Δ^*, the equilibrium default rate is increasing when there is*

1. *an increase in internal funds m,*
2. *a decrease in the scale of investment projects i, or*
3. *an increase in the mortgage rate R.*

Proof: 1. When m increases, $\bar{\delta}(\Delta)$ increases (weakly, see above). As a result, $\bar{z}(\Delta)$ decreases (weakly). As a result, $\hat{\delta}(z,\Delta)$ weakly increases. Differentiating gives

$$\begin{aligned}
\frac{d\Gamma(\Delta)}{d\hat{\delta}(z,\Delta)} &= \frac{-\hat{\delta}(z,\Delta)g(\hat{\delta}(z,\Delta))}{\int_0^\infty(\int_{\hat{\delta}(z,\Delta)}^1 g(\delta)d\delta)f(z)dz} + \frac{\int_0^\infty(\int_{\hat{\delta}(z,\Delta)}^1 \delta g(\delta)d\delta)f(z)dz}{[\int_0^\infty(\int_{\hat{\delta}(z,\Delta)}^1 g(\delta)d\delta)f(z)dz]^2}g(\hat{\delta}(z,\Delta)) \\
&= \frac{\Gamma(\Delta)-\hat{\delta}(z,\Delta)}{\int_0^\infty(\int_{\hat{\delta}(z,\Delta)}^1 g(\delta)d\delta)f(z)dz}g(\hat{\delta}(z,\Delta)).
\end{aligned}$$

This is positive at a fixed point. (i.e., where $\Gamma(\Delta) = \Delta$). Hence the fixed point, Δ^*, increases (weakly). (requires a stable equilibrium).

2. When i decreases, $\bar{\delta}(\Delta)$ increases (weakly, see above). This increases both $\Gamma(\Delta)$ and Δ^*. ■

Proposition 5 *Let Δ^* be a stable equilibrium given F and G and suppose F' first-order stochastically dominates F and G' first-order stochastically dominates G. Then, in the neighborhood of Δ^*,*

1. *the equilibrium default rate is lower under F' than under F and*
2. *the equilibrium default rate is higher under G' than under G.*

Proof: 1. When F shifts to the right, then Γ decreases. Hence Δ^* decreases. 2. When G shifts to the right, then for every fixed $\hat{\delta}$, there are more bad assets on the market. In addition, $\bar{\delta}(\Delta)$ increases (weakly). Both effects make the market worse off and $\Gamma(\Delta)$ and Δ^* rise. ■

Proposition 6 *Let Δ^* be a stable equilibrium (i.e., $\Gamma_\Delta(\Delta^*) < 1$) and consider an increase in internal funds $dm > 0$. In the neighborhood of Δ^*, the new equilibrium always features (weakly) greater toxicity in the secondary market (the new equilibrium always has a weakly higher default rate Δ). Also,*

1. *if the initial $\Delta^* > \hat{\Delta}$, the change in investment is $[1 - F(1)]dm$ and there is no improvement in the interbank market for toxic assets (the new equilibrium is $\Delta = \Delta^*$).*
2. *if $\Delta^* < \hat{\Delta}$ then the change in investment is less than $[F(\bar{z}(\Delta^*)) - F(1)]$ and the interbank market for toxic assets becomes less liquid (the new equilibrium is $\Delta > \Delta^*$).*

Proof: We first write aggregate investment Ias an implicit function of internal funds m,

$$\begin{aligned} I(m,\Delta(m)) &= \int_0^\infty x(z,\Delta(m),m)f(z)dz \\ &= \int_0^1 0f(z)dz + \int_1^{\bar{z}(\Delta(m),m)} x(z,\Delta(m),m)f(z)dz + \int_{\bar{z}(\Delta(m),m)}^\infty \min\{i, m+R(1-\Delta)\}f(z)dz \\ &= \int_1^{\bar{z}(\Delta(m),m)} x(z,\Delta(m),m)f(z)dz + \int_{\bar{z}(\Delta(m),m)}^\infty \min\{i, m+R(1-\Delta)\}f(z)dz. \end{aligned}$$

If $i \leq m + R(1-\Delta)$ then

$$I(m,\Delta(m)) = \int_1^{\bar{z}(\Delta(m),m)} x(z,\Delta(m),m)f(z)dz + \int_{\bar{z}(\Delta(m),m)}^\infty if(z)dz.$$

We take the derivative of I with respect to m,

$$\begin{aligned} \frac{\partial I}{\partial m} &= \int_1^{\bar{z}(\Delta(m),m)} [1 + \frac{\partial x(z,\Delta(m),m)}{\partial \Delta}\frac{\partial \Delta}{\partial m}]f(z)dz \\ &\quad + x(\bar{z},\Delta(m),m)f(\bar{z})[\frac{\partial \bar{z}}{\partial m} + \frac{\partial \bar{z}}{\partial \Delta}\frac{\partial \Delta}{\partial m}] \\ &\quad - if(\bar{z})[\frac{\partial \bar{z}}{\partial m} + \frac{\partial \bar{z}}{\partial \Delta}\frac{\partial \Delta}{\partial m}]. \end{aligned}$$

Since $i = x(\bar{z},\Delta(m),m)$ we can cancel the last two terms to get:

$$\begin{aligned} \frac{\partial I}{\partial m} &= \int_1^{\bar{z}(\Delta(m),m)} [1 + \frac{\partial x(z,\Delta(m),m)}{\partial \Delta}\frac{\partial \Delta}{\partial m}]f(z)dz \\ &= \int_1^{\bar{z}(\Delta(m),m)} f(z)dz + \frac{\partial \Delta}{\partial m}\int_1^{\bar{z}(\Delta(m),m)} \frac{\partial x(z,\Delta(m),m)}{\partial \Delta}f(z)dz. \end{aligned}$$

Since $\frac{\partial x(z,\Delta(m),m)}{\partial \Delta}$ is zero when $z < 1$ and $z > \bar{z}$, we have

$$\frac{\partial I(m,\Delta(m))}{\partial m} = [F(\bar{z}) - F(1)] + I_\Delta \Delta_m \tag{27}$$

where $I_\Delta \equiv \partial I(m,\Delta)/\partial \Delta$ and $\Delta_m \equiv \partial \Delta/\partial m$.

If $i > m + R(1-\Delta)$ then

$$I(m,\Delta(m)) = \int_1^{\bar{z}(\Delta(m),m)} x(z,\Delta(m),m)f(z)dz + \int_{\bar{z}(\Delta(m),m)}^\infty (m + R(1-\Delta))f(z)dz$$

Then we have,

$$\begin{aligned} \frac{\partial I}{\partial m} &= \int_1^{\bar{z}(\Delta(m),m)} [1 + \frac{\partial x(z,\Delta(m),m)}{\partial \Delta}\frac{\partial \Delta}{\partial m}]f(z)dz \\ &\quad + x(\bar{z},\Delta(m),m)f(\bar{z})[\frac{\partial \bar{z}}{\partial m} + \frac{\partial \bar{z}}{\partial \Delta}\frac{\partial \Delta}{\partial m}] \\ &\quad + \int_{\bar{z}(\Delta(m),m)}^\infty (1 - R\frac{\partial \Delta}{\partial m})f(z)dz \\ &\quad - (m + R(1-\Delta))f(\bar{z})[\frac{\partial \bar{z}}{\partial m} + \frac{\partial \bar{z}}{\partial \Delta}\frac{\partial \Delta}{\partial m}] \end{aligned}$$

Since $m + R(1-\Delta) = x(\bar{z}(\Delta(m), m), \Delta(m), m)$ we can cancel the second and fourth terms to get:

$$\begin{aligned}\frac{\partial I}{\partial m} &= \int_1^{\bar{z}(\Delta(m),m)} [1 + \frac{\partial x(z, \Delta(m), m)}{\partial \Delta} \frac{\partial \Delta}{\partial m}] f(z) dz \\ &= \int_1^{\infty} f(z) dz + \frac{\partial \Delta}{\partial m} \int_1^{\bar{z}(\Delta(m),m)} (\frac{\partial x(z, \Delta(m), m)}{\partial \Delta} - R) f(z) dz\end{aligned}$$

Since $\frac{\partial x(z,\Delta(m),m)}{\partial \Delta}$ is zero when $z < 1$ and $z > \bar{z}$, we have

$$\frac{\partial I(m, \Delta(m))}{\partial m} = [1 - F(1)] + (I_\Delta - R)\Delta_m \tag{28}$$

Note, for $1 \leq z \leq \bar{z}$, $\frac{\partial x(z,\Delta(m),m)}{\partial \Delta} = -R[1 - G(\hat{\delta}(z, \Delta)) + (1-\Delta) g(\hat{\delta}(z, \Delta)) z]$, which is always negative. Hence I_Δ is always negative. We now need Δ_m. Use,

$$\Delta(m) = \Gamma(\Delta(m), m)$$

Differentiating gives

$$\Delta_m = \Gamma_\Delta \Delta_m + \Gamma_m$$

or

$$\Delta_m = \frac{\Gamma_m}{1 - \Gamma_\Delta}$$

Now, we investigate Γ_m.

$$\Gamma(\Delta(m), m) = \frac{A}{B} = \frac{\int_0^\infty (\int_{\hat{\delta}(z,\Delta(m),m)}^1 \delta g(\delta) d\delta) f(z) dz}{\int_0^\infty (\int_{\hat{\delta}(z,\Delta(m),m)}^1 g(\delta) d\delta) f(z) dz}$$

$$\begin{aligned}\Gamma_m(\Delta(m), m) &= \frac{-\int_0^\infty \hat{\delta}(z, \Delta(m), m) g(\hat{\delta}(z, \Delta(m), m)) \frac{\partial \hat{\delta}(z,\Delta(m),m)}{\partial m} f(z)}{B} \\ &\quad + \frac{A}{B^2} \int_0^\infty g(\hat{\delta}(z, \Delta(m), m)) \frac{\partial \hat{\delta}(z, \Delta(m), m)}{\partial m} f(z) \\ &= \frac{1}{B} \int_0^\infty [\Gamma(\Delta) - \hat{\delta}(z, \Delta(m), m)] g(\hat{\delta}(z, \Delta(m), m)) \frac{\partial \hat{\delta}(z, \Delta(m), m)}{\partial m} f(z)\end{aligned}$$

To get $\frac{\partial \hat{\delta}(z,\Delta(m),m)}{\partial m}$, recall $\hat{\delta}$

$$\hat{\delta}(z, \Delta(m), m) = \begin{cases} \Delta & \text{for } z \leq 1 \\ \Delta - (z-1)(1-\Delta) & \text{for } 1 \leq z \leq \bar{z}(\Delta) \\ \bar{\delta}(\Delta, m) & \text{for } \bar{z}(\Delta) \leq z \end{cases}$$

where $1 - G(\bar{\delta}(\Delta, m)) = \min\{\frac{i-m}{R(1-\Delta)}, 1\}$. The derivative of $\hat{\delta}$ with respect to m is

$$\frac{\partial \hat{\delta}(z, \Delta(m), m)}{\partial m} = \begin{cases} 0 & \text{for } z \leq 1 \\ 0 & \text{for } 1 \leq z \leq \bar{z}(\Delta) \\ \frac{\partial \bar{\delta}(\Delta(m),m)}{\partial m} = \begin{cases} \frac{1}{R(1-\Delta)} \frac{1}{g(\bar{\delta}(\Delta,m))} & \text{if } \frac{i-m}{R(1-\Delta)} < 1 \\ 0 & \text{if } \frac{i-m}{R(1-\Delta)} \geq 1 \end{cases} & \text{for } \bar{z}(\Delta) \leq z \end{cases}$$

Now we plug $\frac{\partial \hat{\delta}(z,\Delta(m),m)}{\partial m}$ into Γ_m :

$$\begin{aligned}\Gamma_m(\Delta(m), m) &= \frac{1}{B} \times [\\ &\quad \int_0^1 [0] g(\hat{\delta}(z, \Delta(m), m)) [0] f(z) dz + \\ &\quad \int_1^{\bar{z}(\Delta(m),m)} (z-1)(1-\Delta) g(\hat{\delta}(z, \Delta(m), m)) [0] f(z) dz \\ &\quad \int_{\bar{z}(\Delta(m),m)}^\infty [\Delta - \bar{\delta}(\Delta(m), m)] g(\bar{\delta}(\Delta(m), m)) \frac{\partial \bar{\delta}(\Delta(m), m)}{\partial m} f(z) dz]\end{aligned}$$

Hence,

$$\Gamma_m(\Delta(m),m) = \frac{1}{B}\int_{\bar{z}(\Delta(m),m)}^{\infty}[\Delta - \bar{\delta}(\Delta(m),m)]g(\bar{\delta}(\Delta(m),m))\frac{\partial\bar{\delta}(\Delta(m),m)}{\partial m}f(z)dz$$

If $i \leq m + R(1-\Delta)$ then,

$$\Gamma_m(\Delta(m),m) = \frac{1}{B}[\Delta - \bar{\delta}(\Delta(m),m)]\frac{[1-F(\bar{z}(\Delta(m),m))]}{R(1-\Delta)} > 0 \tag{29}$$

If $i > m + R(1-\Delta)$ then, the derivative is zero, $\Gamma_m = 0$.

If the initial equilibrium default rate $\Delta_1 > \hat{\Delta} = 1 - \frac{i-m}{R}$ ($i > m + R(1-\Delta_1)$) then $\Gamma_m = 0$, so $\Delta_m = 0$. In other words, there is no improvement in the interbank market for toxic assets (the new equilibrium is $\Delta = \Delta_1$). By (28) we have $I_m = 1 - F(1)$. In this case, all banks with good projects ($1 \leq z$) do not change their asset sales but instead use the additional funds to expand investment. For these firms, each dollar of additional equity results directly in an extra dollar of investment. The increase in investment is directly proportional to the number of projects in this range, $1 - F(1)$.

If the initial default rate $\Delta_1 < \hat{\Delta}$ then $\Gamma_m > 0$, so $\Delta_m > 0$. Since he second term in (27) is negative $I_\Delta \Delta_m < 0$ (by Proposition 4), the change in investment is less than $F(\bar{z}(\Delta_1)) - F(1)$. In this case, banks with $z > \bar{z}$, make no change in investment directly but instead change their asset sales. Prior to the equity injection, they were reluctantly selling high quality assets in sufficient amounts to fund their investment projects. After the equity injection, these banks still finance their projects but now sell fewer high quality assets. These banks are already selling a number of good assets to ensure that they have sufficient capital to accommodate their investment projects. After the equity injection, these banks do not need to rely on the secondary markets as much and as a result, they sell fewer good assets. Therefore, the interbank market for toxic assets becomes even less liquid (the new equilibrium is $\Delta > \Delta_1$. ■

Proposition 7 *A decrease in Δ unambiguously increases welfare and investment.*

Proof: To understand I_Δ and W_Δ, we consider two cases: $\Delta < \hat{\Delta}$ ("interior") or $\Delta > \hat{\Delta}$ ("corner").

Interior solution: In an interior solution ($\Delta < \hat{\Delta}$),

$$I(\Delta) = \int_1^{\bar{z}(\Delta)} x(z;\Delta)f(z)dz + \int_{\bar{z}(\Delta)}^{\infty} if(z)dz$$

$$W(\Delta) = \int_1^{\bar{z}(\Delta)} (z-1)x(z;\Delta)f(z)dz + \int_{\bar{z}(\Delta)}^{\infty} (z-1)if(z)dz$$

Note that $\frac{\partial x(z;\Delta)}{\partial \Delta}$ is equal to $-R[(1-G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]$ for $1 \leq z \leq \bar{z}(\Delta)$, otherwise it is zero. Differentiating both I and W w.r.t. Δ gives:

$$\begin{aligned} I_\Delta(\Delta) &= \int_1^{\bar{z}(\Delta)} \frac{\partial x(z;\Delta)}{\partial \Delta} f(z)dz \\ &= -\int_1^{\bar{z}(\Delta)} R[(1-G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz, \end{aligned} \tag{30}$$

$$\begin{aligned} W_\Delta(\Delta) &= \int_1^{\bar{z}(\Delta)} (z-1)\frac{\partial x(z;\Delta)}{\partial \Delta} f(z)dz \\ &= \int_1^{\bar{z}(\Delta)} (z-1)R[(1-G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz. \end{aligned} \tag{31}$$

Corner solution: Alternatively, if this is a corner solution ($\Delta > \hat{\Delta}$) then

$$I(\Delta) = \int_1^{\bar{z}(\Delta)} x(z;\Delta)f(z)dz + \int_{\bar{z}(\Delta)}^{\infty} [m + R(1-\Delta)]f(z)dz$$

$$W(\Delta) = \int_1^{\bar{z}(\Delta)} (z-1)x(z;\Delta)f(z)dz + \int_{\bar{z}(\Delta)}^{\infty} (z-1)[m + R(1-\Delta)]f(z)dz$$

Differentiating w.r.t. Δ gives

$$\begin{aligned} I_\Delta(\Delta) &= \int_1^{\bar z(\Delta)} \frac{\partial x(z;\Delta)}{\partial \Delta} f(z)dz - R\int_{\bar z(\Delta)}^{\infty} f(z)dz \\ &= -R\{\int_1^{\bar z(\Delta)} [(1-G(\hat\delta(z,\Delta))) + (1-\Delta)g(\hat\delta(z,\Delta))z]f(z)dz + [1-F(\bar z(\Delta))]\}, \end{aligned} \tag{32}$$

$$\begin{aligned} W_\Delta(\Delta) &= \int_1^{\bar z(\Delta)} (z-1)\frac{\partial x(z;\Delta)}{\partial \Delta} f(z)dz - R\int_{\bar z(\Delta)}^{\infty} (z-1)f(z)dz \\ &= -R[\int_1^{\bar z(\Delta)} (z-1)[(1-G(\hat\delta(z,\Delta))) + (1-\Delta)g(\hat\delta(z,\Delta))z]f(z)dz + \int_{\bar z(\Delta)}^{\infty} (z-1)f(z)dz]. \end{aligned} \tag{33}$$

By (30), (32), (31) and (32), welfare and investment increase as a result of a decrease in Δ. ■

Proposition 8 *An asset purchase increases welfare and investment more than an equity injection whenever the following conditions hold:*

1. *There is positive feedback in equilibrium ($\Gamma_\Delta \geq 0$).*
2. *The average sales of the banks with investment $0 < x < i$ is at least as great as the average sales of all banks.*
3. *The cost of equity is at least $1 - \Gamma_\Delta$ per dollar.*

Proof: Suppose the government takes actions which lower Δ by an amount $d\Delta < 0$. There are tow possibilities: (1) the government subsidizes the sale through a direct payment, or (2) the government purchases assets to increase the price. The costs are the same in both cases. If the government purchases everything then its outlays are

$$\begin{aligned} &R(1-\Delta)\left(\int_0^\infty \left[\int_{\hat\delta(z,\Delta)}^1 g(\delta)d\delta\right] f(z)dz + T\right) \\ &\textit{or} \\ &R(1-\Delta)B, \end{aligned}$$

while the value of the purchase is the repayment R times the number of firms that do not default, that is,

$$R\left(\int_0^\infty \left[\int_{\hat\delta(z,\Delta)}^1 (1-\delta)g(\delta)d\delta\right] f(z)dz + (1-\delta^T)T\right).$$

The net budget impact is (outlays less value)

$$\begin{aligned} D &= R(1-\Delta)B - R\left(\int_0^\infty \left[\int_{\hat\delta(z,\Delta)}^1 (1-\delta)g(\delta)d\delta\right] f(z)dz + T(1-\delta^T)\right) \\ &= R(1-\Delta)B - R\left(B - \int_0^\infty \left[\int_{\hat\delta(z,\Delta)}^1 \delta g(\delta)d\delta\right] f(z)dz - T\delta^T\right) \\ &= RB\left((1-\Delta) - 1 + \frac{\int_0^\infty [\int_{\hat\delta(z,\Delta)}^1 \delta g(\delta)d\delta] f(z)dz + T\delta^T}{\int_0^\infty [\int_{\hat\delta(z,\Delta)}^1 g(\delta)d\delta] f(z)dz + T}\right) \\ &= RB\left((1-\Delta) - 1 + \Gamma(\Delta)\right) \end{aligned}$$

$$D = RB[\Gamma(\Delta) - \Delta].$$

Consider the (local) budget cost of causing a deviation in Δ (by an amount $d\Delta$) in the neighborhood of an equilibrium $\Gamma(\Delta) = \Delta$. This is

$$dD = \left(\frac{R\partial B}{\partial \Delta}[\Gamma(\Delta) - \Delta] + RB[\Gamma_\Delta(\Delta) - 1]\right) d\Delta$$

$$dD = RB[\Gamma_\Delta(\Delta) - 1]d\Delta.$$

First consider an interior solution ($\Delta < \hat{\Delta}$). An equity injection of dm results in changes in investment via

$$\frac{\partial I(m, \Delta(m))}{\partial m} = [F(\bar{z}) - F(1)] + \frac{\partial I(m)}{\partial \Delta}\frac{\partial \Delta}{\partial m}$$

Since the government cannot distinguish between traders and banks, for the banks internal funds to go up by dm requires a total injection of $dm(1+T)$. To make the total costs equal in both cases, we take

$$(1+T)dm = RB[\Gamma_\Delta(\Delta) - 1]d\Delta$$

	Money Injections	Asset Purchase
Increase in I	$[F(\bar{z}) - F(1) + I_\Delta \Delta_m]dm$	$I_\Delta d\Delta = I_\Delta \frac{1+T}{RB[\Gamma_\Delta - 1]}dm$

Hence we need to compare

$$F(\bar{z}) - F(1) + I_\Delta \Delta_m \gtreqless I_\Delta \frac{1+T}{RB[\Gamma_\Delta - 1]}$$

$$\begin{aligned}
F(\bar{z}) - F(1) &\gtreqless I_\Delta \left[\frac{1+T}{RB[\Gamma_\Delta - 1]} - \Delta_m\right] \\
&\gtreqless I_\Delta \left[\frac{1+T}{RB[\Gamma_\Delta - 1]} - \frac{1}{B}\frac{[\Delta - \bar{\delta}][1 - F(\bar{z})]}{(1-\Gamma_\Delta)R(1-\Delta)}\right] \\
&\gtreqless I_\Delta \frac{1}{RB[\Gamma_\Delta - 1]}\left[1 + T + (\frac{\Delta - \bar{\delta}}{1-\Delta})[1 - F(\bar{z})]\right].
\end{aligned}$$

Since $\bar{z}(\Delta) = 1 + \frac{\Delta - \bar{\delta}(\Delta)}{1-\Delta}$, we have

$$F(\bar{z}) - F(1) \gtreqless I_\Delta \frac{1}{RB[\Gamma_\Delta - 1]}\left[1 + T + (\bar{z} - 1)[1 - F(\bar{z})]\right].$$

After plugging I_Δ (from (30)), the right-hand side becomes

$$\frac{\int_1^{\bar{z}}[(1 - G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz}{B[1-\Gamma_\Delta]}\left[1 + T + (\bar{z} - 1)[1 - F(\bar{z})]\right]$$

$$\begin{aligned}
= \quad & \frac{\int_1^{\bar{z}(\Delta)}(1 - G(\hat{\delta}(z,\Delta)))f(z)dz}{B[1-\Gamma_\Delta]}(1+T) \\
&+ \frac{\int_1^{\bar{z}(\Delta)}(1-\Delta)g(\hat{\delta}(z,\Delta))zf(z)dz}{B[1-\Gamma_\Delta]}(1+T) \\
&+ \frac{\int_1^{\bar{z}(\Delta)}[(1 - G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz}{B[1-\Gamma_\Delta]}(\bar{z} - 1)(1 - F(\bar{z})).
\end{aligned}$$

Since the second and third terms are positive, if one can show the first term is greater than $F(\bar{z}) - F(1)$, then assets purchases increase welfare and investment more than equity injections. The first term can be written as follows:

$$\begin{aligned}
\frac{(1+T)\int_1^{\bar{z}(\Delta)}(1 - G(\hat{\delta}(z,\Delta)))f(z)dz}{[1-\Gamma_\Delta]B} &= \frac{[F(\bar{z}) - F(1)]\int_1^{\bar{z}(\Delta)}[1 - G(\hat{\delta}(z))]\frac{f(z)}{F(\bar{z})-F(1)}dz}{\int_0^\infty[1 - G(\hat{\delta}(z))]f(z)dz + T}\frac{1+T}{1-\Gamma_\Delta} \\
&= [F(\bar{z}) - F(1)]\frac{\int_1^{\bar{z}(\Delta)}[1 - G(\hat{\delta}(z))]\frac{f(z)}{F(\bar{z})-F(1)}dz}{\frac{1}{1+T}\int_0^\infty[1 - G(\hat{\delta}(z))]f(z)dz + \frac{T}{1+T}}\frac{1}{1-\Gamma_\Delta} \\
&= [F(\bar{z}) - F(1)]\frac{\int_1^{\bar{z}(\Delta)}[1 - G(\hat{\delta}(z))]\frac{f(z)}{F(\bar{z})-F(1)}dz}{\frac{1}{1+T}\int_0^\infty[1 - G(\hat{\delta}(z))]f(z)dz + (1 - \frac{1}{1+T})}\frac{1}{1-\Gamma_\Delta}.
\end{aligned}$$

Given $1 > \Gamma_\Delta > 0$, if the average number of sales by intermediate z's ($\int_1^{\bar{z}(\Delta)}[1 - G(\hat{\delta}(z))]\frac{f(z)}{F(\bar{z})-F(1)}dz$) is at least as great as the average number of sales for a typical bank in the market overall then we have

$$\frac{(1+T)\int_1^{\bar{z}(\Delta)}(1-G(\hat{\delta}(z,\Delta)))f(z)dz}{[1-\Gamma_\Delta]B} > F(\bar{z}) - F(1).$$

In other words, asset purchases dominate.

Now consider a corner solution ($\Delta > \hat{\Delta}$). An equity injection of dm results in changes in investment given by

$$\frac{\partial I(m,\Delta(m))}{\partial m} = 1 - F(1)$$

(No contamination effect) As before, we make the total costs equal in both cases:

$$(1+T)dm = RB[\Gamma_\Delta(\Delta) - 1]d\Delta$$

	Money Injections	Asset Purchase
Increase in I	$[1-F(1)]dm$	$I_\Delta d\Delta = I_\Delta \frac{1+T}{RB[\Gamma_\Delta - 1]}dm$

Hence we need to compare

$$1 - F(1) \gtreqless I_\Delta \frac{1+T}{RB[\Gamma_\Delta - 1]}$$

After plugging I_Δ (from (32)), the right-hand side becomes

$$\frac{\int_1^{\bar{z}}[(1-G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz + \int_{\bar{z}}^{\infty} f(z)dz}{B[1-\Gamma_\Delta]}[1+T]$$

$$\frac{\{\int_1^{\bar{z}}[(1-G(\hat{\delta}(z,\Delta))) + (1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz + [1-F(\bar{z}(\Delta))]\}[1+T]}{B[1-\Gamma_\Delta]}$$

Since $\hat{\delta}(z,\Delta) = 0$ for $\bar{z} < z$, we have

$$\frac{[1+T]\int_1^{\infty}(1-G(\hat{\delta}(z,\Delta)))f(z)dz}{B[1-\Gamma_\Delta]} + \frac{[1+T]\int_1^{\bar{z}}[(1-\Delta)g(\hat{\delta}(z,\Delta))z]f(z)dz}{B[1-\Gamma_\Delta]}$$

$$(\frac{1+T}{1-\Gamma_\Delta})\frac{[1-F(1)]\int_1^{\infty}(1-G(\hat{\delta}(z,\Delta)))\frac{f(z)}{1-F(1)}dz}{\int_0^{\infty}(1-G(\hat{\delta}(z,\Delta)))f(z)dz + T} + (\frac{1+T}{1-\Gamma_\Delta})\frac{\int_1^{\bar{z}}(1-\Delta)g(\hat{\delta}(z))zf(z)dz}{\int_0^{\infty}(1-G(\hat{\delta}(z)))f(z)dz + T}$$

As above, since the second term is positive, if one can show the first term is greater than $1-F(1)$, then assets purchases increase welfare and investment more than equity injections. The first term can be written as follows:

$$[1-F(1)]\frac{\int_1^{\infty}[1-G(\hat{\delta}(z))]\frac{f(z)}{1-F(1)}dz}{\frac{1}{1+T}\int_0^{\infty}[1-G(\hat{\delta}(z))]f(z)dz + \frac{T}{1+T}}\frac{1}{1-\Gamma_\Delta}$$

$$[1-F(1)]\frac{\int_1^{\infty}[1-G(\hat{\delta}(z))]\frac{f(z)}{1-F(1)}dz}{\frac{1}{1+T}\int_0^{\infty}[1-G(\hat{\delta}(z))]f(z)dz + (1-\frac{1}{1+T})}\frac{1}{1-\Gamma_\Delta}$$

Given $1 > \Gamma_\Delta > 0$, if the average number of sales for banks with$z$'s above 1 ($\int_1^{\infty}[1-G(\hat{\delta}(z))]\frac{f(z)}{1-F(1))}dz$) is at least as great as the average number of sales for a typical bank in the market overall then asset purchases dominate. ■