Bubbles and crashes in finance: A phase transition from random to deterministic behaviour in prices.

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Abstract

We develop a rational expectations model of financial bubbles and study ways in which a generic risk-return interplay is incorporated into prices. We retain the interpretation of the leading Johansen-Ledoit-Sornette model, namely, that the price must rise prior to a crash in order to compensate a representative investor for the level of risk. This is accompanied, in our stochastic model, by an illusion of certainty as described by a decreasing volatility function. As the volatility function goes to zero, crashes can be seen to represent a phase transition from stochastic to deterministic behaviour in prices.

Keywords: financial crashes, super-exponential growth, illusion of certainty, housing-bubble.

1 Introduction

Rational expectations models were introduced with the work of Blanchard and Watson to account for the possibility that prices may deviate from fundamental levels [1]. We take as our main starting point the somewhat controversial subject of log-periodic precursors to financial crashes [2]-[11], with a fundamental aim of our approach being relatively easy

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calibration of our model to empirical data. Additional background on log-periodicity and complex exponents can be found in [12]. A first-order approach in [3] and subsequent extensions in [13] state that prior to a crash the price must exhibit a super-exponential growth in order to compensate a representative investor for the level of risk. However, this approach concentrates solely on the drift function and ignores the underlying volatility fluctuations which typically dominate financial time series [14]. Similar in spirit to [3], we derive a second-order condition which incorporates volatility fluctuations and enables us to combine insights from a rational expectations model with a stochastic model [15]-[16].

Our model gives two important characterisations of bubbles in economics. Firstly, a rapid super-exponential growth in prices. Secondly, an illusion of certainty as described by a decreasing volatility function prior to the crash. As the volatility function goes to zero bubbles and crashes can be seen to represent a phase transition from stochastic to purely deterministic behaviour in prices. This clarifies the oft cited link in the literature between phase transitions in critical phenomena and financial crashes. Further, this recreates the phenomenology of the Sornette-Johansen paradigm: namely that prices resemble a deterministic function prior to a crash. We explore a number of different applications of our model and the potential relevance to recent events is striking.

The layout of this paper is as follows. In Section 2 we introduce the basic model and derive the crash-size distribution, the post-crash dynamics, simple estimates of fundamental-value and speculative-bubble components. Section 3 describes an empirical application to the UK housing bubble of the early to late 2000s [17]. Section 4 is a brief conclusion.

2 The model

In this section we give an alternative formulation of the model solution in [3]. This leads naturally to a stochastic generalisation of the original model, which is then solved in full to give empirical predictions for the distribution of crash-sizes, post-crash dynamics, fundamental values and the level of over-pricing.

We offer an alternate derivation of the basic model in [3] as follows. Let $P(t)$ denote the price of an asset at time t. Our starting point is the equation

$$P(t) = P_1(t)(1-\kappa)^{j(t)}, \tag{1}$$

where $P_1(t)$ satisfies

$$dP_1(t) = \mu(t)P_1(t)dt + \sigma(t)P_1(t)dW_t, \tag{2}$$

where W_t is a Wiener process and $j(t)$ is a jump process satisfying

$$j(t) = \begin{cases} 0 & \text{before the crash} \\ 1 & \text{after the crash.} \end{cases} \tag{3}$$

When a crash occurs $\kappa\%$ is automatically wiped off the value of the asset. Prior to a crash $P(t) = P_1(t)$ and $X_t = \log(P(t))$ satisfies

$$dX_t = \tilde{\mu}(t)dt + \sigma(t)dW_t + \ln[(1-\kappa)]dj(t), \tag{4}$$

where $\tilde{\mu} = \mu(t) - \sigma^2(t)/2$. If a crash has not occurred by time t, we have that

$$E[j(t+\delta) - j(t)] = h(t)dt + o(dt), \tag{5}$$
$$\text{Var}[j(t+\delta) - j(t)] = h(t)dt + o(dt), \tag{6}$$

where $h(t)$ is the hazard rate. We compare (4) with the prototypical Black-Scholes model for a stock price:

$$dX_t = \mu dt + \sigma dW_t, \tag{7}$$

and use (7) as our model for "fundamental" or purely stochastic behaviour in prices.

The first-order condition see e.g. [1], [3], suggests that $\tilde{\mu}(t)$ in (4) grows in order to compensate a representative investor for the risk associated with a crash. The instantaneous drift associated with (4) is

$$\tilde{\mu}(t) + (\ln(1-\kappa))h(t). \tag{8}$$

For (7) the instantaneous drift is μ. Setting (8) equal to μ, it follows that in order for bubbles and non-bubbles to co-exist

$$\tilde{\mu}(t) = \mu - (\ln(1-\kappa))h(t). \tag{9}$$

If we ignore volatility fluctuations by setting $\sigma(t) = \sigma$, then our pre-crash model for an asset price becomes

$$dX_t = (\mu - \ln(1-\kappa)h(t))dt + \sigma dW_t. \tag{10}$$

However, this is actually a rather poor empirical model [18], failing to adequately account for the volatility fluctuations in (4). Under a Markowitz interpretation, means represent returns and variances/standard deviations represent risk. Suppose that in (4) $\sigma(t)$ adapts in an analogous way to $\mu(t)$ so as to compensate a representative investor for bearing additional levels of risk. The instantaneous variance associated with (4) is

$$\sigma^2(t) + (\ln(1-\kappa))^2 h(t). \tag{11}$$

For (7) the instantaneous variance is σ^2. Setting (11) equal to σ^2, the second-order condition for co-existence of bubbles and non-bubbles becomes

$$\sigma^2(t) = \sigma^2 - (\ln(1-\kappa))^2 h(t). \tag{12}$$

(12) illustrates an *illusion of certainty* – a decrease in the volatility function – which arises as part of a bubble process. Intuitively, in order for a bubble to occur not only must returns increase but the volatility must also decrease. If this does not happen (7) with an instantaneous variance of σ^2 would represent a more attractive and less risky investment than a market described by (10) and bubbles could not occur. We use (7) as a model of a 'fundamental' or purely stochastic regime, as in Black-Scholes theory. From (12), our model for prices under a bubble regime becomes

$$dX_t = [\mu - \ln(1-\kappa)h(t)]dt + \sqrt{\sigma^2 - (\ln(1-\kappa))^2 h(t)}dW_t. \tag{13}$$

The simplest $h(t)$ considered in [3] is

$$h(t) = B(t_c - t)^{-\alpha}, \tag{14}$$

where it is assumed that $\alpha \in (0,1)$ and t_c is a critical time when the hazard function becomes singular, by analogy with phase transitions in statistical mechanical systems [19]. Here, we choose on purely statistical grounds

$$h(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta + t^\beta}, \tag{15}$$

which is the form corresponding to a log-logistic distribution and is intended to capture the essence of the previous approach as the hazard rate has both a relatively simple form and, for $\beta > 1$, has a non-trivial mode at $t = \alpha(\beta-1)^{\frac{1}{\beta}}$, with modal point $(\beta-1)^{1-\frac{1}{\beta}}/\alpha$. For these reasons, the log-logistic distribution is commonly used in statistics [20]. The log-logistic distribution has probability density

$$f(x) = \frac{\beta\alpha^{\beta}x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^2}, \tag{16}$$

on the positive half-line. The cumulative distribution function is

$$F(x) = 1 - \frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}} \tag{17}$$

The model (13) with $h(t)$ given by (15) has the solution

$$X_t = X_0 + \mu t + \upsilon \ln\left(1 + \frac{t^{\beta}}{\alpha^{\beta}}\right) + \int_0^t \sqrt{\sigma^2 - \upsilon^2 \frac{\beta t^{\beta-1}}{\alpha^{\beta}+t^{\beta}}} dW_u. \tag{18}$$

where $\upsilon = -\ln(1-\kappa)$ with $\upsilon > 0$. From (18) the conditional densities can be written as

$$X_t|X_s \sim N(\mu_{t|s}, \sigma^2_{t|s}), \tag{19}$$

where

$$\mu_{t|s} = X_s + \mu(t-s) + \upsilon \ln\left(\frac{\alpha^{\beta}+t^{\beta}}{\alpha^{\beta}+s^{\beta}}\right), \tag{20}$$

$$\sigma^2_{t|s} = \sigma^2(t-s) - \upsilon^2 \ln\left(\frac{\alpha^{\beta}+t^{\beta}}{\alpha^{\beta}+s^{\beta}}\right). \tag{21}$$

Under the fundamental equation (7) these expressions are simply $\mu_{t|s} = X_s + \mu(t-s)$ and $\sigma^2_{t|s} = \sigma^2(t-s)$. Thus, we see that under the bubble model the incremental distributions demonstrate a richer behaviour over time.

The fundamental or purely stochastic non-bubble model (7) corresponds to the case that $\kappa = 0$, or equivalently that $\upsilon = 0$. We can test for bubbles by testing the null hypothesis $\upsilon = 0$ (no bubble) against the alternative hypothesis $\upsilon > 0$ (bubble). This can be simply done using a (one-sided) t-test since maximum likelihood estimates, and estimated standard errors, can be easily calculated numerically from (19). A range of further implications of our bubble model can be derived as we describe below.

Crash-size distribution. Suppose that prices are observed up to and including time t and that a crash has not occurred by time t. The crash-size distribution resists an analytical description but a Monte Carlo algorithm to simulate the crash-size C is straightforward and reads as follows:

1. Generate u from $U \sim$ Log-logistic(α, β) with the constraint $u \geq t$.

2. $C \sim \kappa e^{Z}$,

where

$$Z \sim N\left(X_t + \mu u + v \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right), \sigma^2 u - v^2 \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right)\right) \tag{22}$$

We note that simulating u from the log-logistic distribution is straight-forward and from (17) possible via inversion using

$$F^{-1}(x) = \alpha\left(\frac{x}{1-x}\right)^{\frac{1}{\beta}} \quad \text{or} \quad F^{-1}(x) = \left(\frac{\alpha^\beta + t^\beta}{1-x} - \alpha^\beta\right)^{\frac{1}{\beta}} \quad \text{with constraint } u \geq t.$$

Post-crash increase in volatility. Before a crash equation (18) applies. After a crash, the price reverts to the fundamental price dynamics (7). Suppose the crash occurs at time C. At $t = C$ we have that

$$\mathrm{Var}(X_{t+h}|X_t) = \sigma^2 h, \tag{23}$$

but for $t < C$

$$\mathrm{Var}(X_{t+h}|X_t) = \sigma^2 h - v^2 \ln\left(\frac{\alpha^\beta + (t+h)^\beta}{\alpha^\beta + (t)^\beta}\right) \tag{24}$$

Thus, our model predicts an increase in volatility following a crash given by

$$\kappa^2 \ln\left(\frac{\alpha^\beta + (t+h)^\beta}{\alpha^\beta + (t)^\beta}\right). \tag{25}$$

Fundamental values. The above model suggests a simple approach to estimate fundamental value. Under the fundamental dynamics (7)

$$P_F(t) := E(P(t)) = P(0)e^{\mu t}, \tag{26}$$

and we use (26) to estimate fundamental value in our empirical application in Section 7. This approach recreates the widespread phenomenology of approximate exponential growth in economic time series (see e.g. Chapter 7 in [21]).

*Estimated bubble component.*Define

$$H(t) = \int_0^t h(u)du. \tag{27}$$

Under the fundamental model $E(P(t))$ is given by (26). Under the bubble model, since

$X_t = \log(P_t)$ satisfies

$$X_t \sim \mathrm{N}\left(X_0 + \tilde{\mu}t + vH(t), \sigma^2 t - v^2 H(t)\right), \tag{28}$$

it follows that

$$P_B(t) := E(P(t)) = P(0)e^{\mu t + \left(v - \frac{v^2}{2}\right)H(t)}. \tag{29}$$

This motivates the following estimate for the proportion of observed prices which can be attributed to a speculative bubble:

$$1 - \frac{1}{T}\int_0^T \frac{P_F(t)}{P_B(t)}\,dt = 1 - \frac{1}{T}\int_0^T \left(1 + \frac{t^\beta}{\alpha^\beta}\right)^{-(v - v^2/2)} dt. \tag{30}$$

3 Empirical application

As an empirical application we consider the UK housing bubble from 2002-2007 by modelling a monthly time series of average UK house prices. The null hypothesis of no bubble is a test of the hypothesis $v = 0$. This can be tested using a one-sided t-test – dividing the estimate $\hat{v}$ by its estimated standard error and comparing to a normal distribution. For this data set we obtain a t-statistic of 3.66 and a p value of 0.0001 to give strong evidence of a bubble in this data.

From our fit of the bubble model (18) we use $P_F(t) = P(0)e^{\mu t}$ in (26) as a simple estimate of fundamental value. A plot of UK house prices together with estimated fundamental values and associated 95% confidence intervals is shown in Figure 1. Prices appear to be well in excess of fundamental values, with prices lying above the upper confidence limits of the estimated fundamental values throughout the sample (Figure 1 to the left of the vertical line). We then estimate fundamental value for the years 2008-2009 using data from 2002-2007 only and compare with the actual historically observed prices. That is, we use our model to provide estimates of fundamental value *out of sample*. The results are shown to the right of the vertical line in Figure 1 and show prices reverting towards fundamental values – moving inside the confidence intervals constructed for fundamental value. From the second half of 2008 observed prices are statistically indistinguishable from estimated fundamental value. Finally, the estimated speculative bubble component is 0.202 suggesting that the bubble accounts for around 20% of the observed prices. This compares reasonably with similar estimates of 12-25% in [22] and 28-53% in [23].

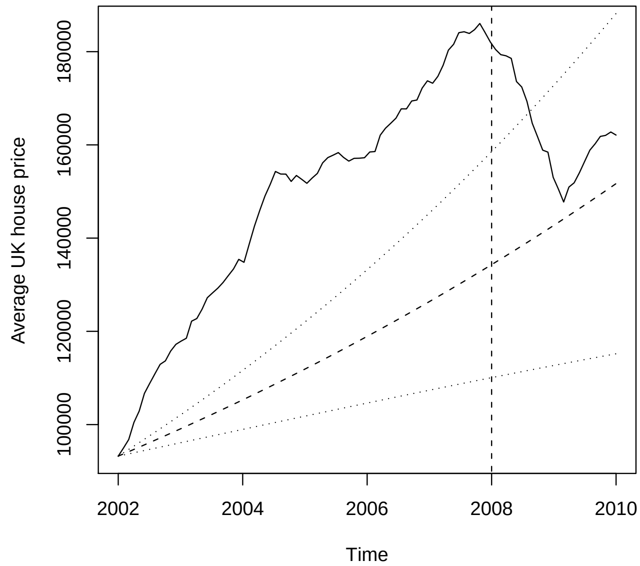

Figure 1: Plot of average UK house-prices and estimated fundamental value (dashed line) and associated 95% confidence intervals (dots). Estimation takes place over the period 2002-2007 (to the left of the dashed vertical line). Out-of-sample estimates of fundamental value are then compared to historically observed prices (to the right of the dashed vertical line).

4 Conclusions

This paper has provided a stochastic version of the model in [3]. Crash precursors are a super-exponential growth accompanied by an "illusion of certainty", characterised by a decrease in the volatility function prior to the crash. A range of potential applications to economics were discussed including statistical tests for bubbles, crash-size distributions, predictions of a post-crash increase in volatility and simple estimates of fundamental-value and speculative-bubble components. As a brief empirical application we consider the UK housing bubble in the early to mid 2000s. Prices appear to be in excess of fundamental levels, with the speculative bubble component accounting for around 20% of observed prices. In addition, prices are seen to revert towards estimated fundamental values out of sample over the period 2008-2009.

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