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# An Unbalanced Multi-Industry Growth Model with Constant Returns: A Turnpike Approach

by

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**Abstract**

Recent industry-based empirical studies among countries demonstrate that individual industry's per capita capital stock and output grow at industry's own steady state growth rate. The industry growth rate is highly correlated to industry's technical progress measured by total factor productivity TFP) of the industry. Let us refer to this phenomenon as "unbalanced growth among industries." Very few research concerned with this phenomenon has been done yet. Some exceptions are Echevarria (1997), Kongsamut, Rebelo and Xie (2001), and Acemoglu and Guerrieri (2008) among others. However their models and analytical methods are different from mine. Applying the theoretical method developed by McKenzie and Scheinkman in turnpike theory, I now construct a multi-sector optimal growth model with a industry specific Hicks-neutral technical progress and show that each sector's per capita capital stock and output grow at the rate of the sector's technical progress.



**JEL Classification:** O14,O21,O24,O41

## 1 Introduction

Since the seminal papers by Romer (1986) and Lucas (1988), economics has witnessed a strong revival of interest in growth theory under the name of "Endogenous growth theory." Neoclassical optimal growth models have been applied as benchmarks and studied intensively since the late 1960. However, these analytical models have a serious drawback: they are based on highly aggregated macro-production functions and cannot explain the important empirical evidence that I discuss in the following section. Recent industry-based empirical studies among countries clearly demonstrate that growth in an individual industry's per capita capital stock and output grow at industry's own growth rate, which is closely related to its technical progress measured by total factor productivity (TFP) of the industry. For example, per capita capital stock and output of the agriculture industry grow at 5% per annum along its own steady-state, whereas they grow at 10% annually in the manufacturing industry, also paralleling the industry's steady state. Let us refer this phenomenon as "unbalanced growth among industries." The attempt to understand this phenomenon has generated a strong theoretical demand for constructing a multi-sector growth model, yet very little progress has been made so far. Some exceptions are Echevarria (1997), Kongsamut, Rebelo and Xie (2001), and Acemoglu and Guerrieri (2008). However their models and analytical methods are different from mine. Setting up an optimal growth model with three sectors: primary, manufacturing and service, Echevarria (1997) has applied a numerical analysis to solve the model. Kongsamut, Rebelo and Xie (2001) has constructed the similar model to the one of Echevarria (1997), while they have investigated the model under a much stronger assumption than her: each sector produces goods with the same technology. On the other hand, Acemoglu and Guerrieri (2008) has studied the model with two intermediate-goods sectors and single final-goods sector. Note that the last two models will share a common character: consumption goods and capital goods are identical. Contrastingly, my model presented here exhibits a sharp contrast with them. Since I assume that each good is produced with a different technology, consumption goods and capital goods are completely different goods. As I will demonstrate later, this feature of the model will make the characteristics of the model far complicated.

The optimal growth model with heterogeneous capital goods has been studied intensively since the early 1970' under the title of turnpike theory by McKenzie (1976, 1982, 1983 and 1986) and Scheinkman (1976). Turnpike theory shows that any optimal path converges asymptotically to the corresponding optimal steady state path without initial stock sensitivity. In other words, the turnpike property implies that the per capita capital stock and output of each industry eventually converge to an industry-specific constant ratio. Therefore, turnpike theory too, cannot explain the empirical phenomenon: unbalanced growth among industries. McKenzie (1998) has articulated this point: "Almost all the attention to asymptotic convergence has been concentrated on convergence to balanced paths, although it is not clear that optimal balanced growth path will exist. This type of path is virtually impossible to believe in, if the model is disaggregated beyond the division into human capital and physical capital, and new goods and new methods of production appear from time to time." An additional point is that the turnpike result established in a reduced form model has not been fully applied to a structural neoclassical optimal growth model. A serious obstacle in applying the results from the reduced form model is that the transforming of a neoclassical optimal growth model into a reduced form model will not yield a strictly concave reduced form utility function, but just a concave one. In this context, McKenzie (1983) has extended the turnpike property to the case in which the reduced form utility function is not strictly concave, that is, there is a flat segment on the surface, which contains an optimal steady state. This flat segment is often referred to as the Neumann-McKenzie facet. Yano (1990) has studied a neoclassical optimal growth model with heterogeneous capital goods in a trade theoretic context. However, in case of the Neumann-Mckenzie facet with a positive dimension, Yano explicitly assumed the "dominant diagonal block condition" concerned with the reduced form utility function ( see Araujo and Scheinkman (1978) and McKenzie (1986)). Thus, he still did not fully exploit the structure of the neoclassical optimal growth model, especially the dynamics of the path on the Neumann-McKenzie facet, to obtain the turnpike property.

By applying the theoretical method developed in turnpike theory, this study seeks to fill the gap between the results derived by the theoretical research explained above and the empirical evidence from recent studies at the industry level among countries. First, I will set up a multi-industry optimal growth model, in which each industry exhibits the Hicks-neutral technical progress with an industry specific rate. This model will be regarded as a multi-industry optimal growth version of the Solow model with the Hicks neutral technical progress. Second, I will rewrite the original model into a per capita efficiency unit model. Third, I will transform the efficiency unit model into a reduced form model, after which the method developed in turnpike theory will be applied. The neighborhood turnpike theorem demonstrated in McKenzie (1983) indicates that any optimal path will be trapped in a neighborhood of the corresponding optimal steady state path when discount factors are sufficiently close to 1, and the neighborhood can be minimized by choosing a discount factor arbitrarily close to 1. I will demonstrate the local stability theorem by applying the logic used by Scheinkman (1976): a stable manifold extends over today's capital stock plane. As we see later, the dynamics of the Neumann-McKenzie facet are important in demonstrating both the theorems. Combining the neighborhood turnpike and the local stability produces the complete turnpike property: any optimal path converges to a corresponding optimal steady state when discount factors are sufficiently close to 1.

For establishing both theorems, we assume generalized capital intensity conditions, which are the generalized versions of those in a two-sector model. The complete turnpike property means that each sector's optimal per capita capital stock and output converge to its own steady state path with the rate of technical progress determined by the industry's TFP.

The paper is organized in the following manner: In Section 2, I will provide a several empirical facts based on the recent database at the industry level among countries. In Section 3, the model and assumptions are presented and show some existence theorem. In Section 4, the Neumann-McKenzie facet is introduced and the Neighborhood Turnpike Theorem is demonstrated. The results obtained in Section 4 will be used repeatedly in the proofs of main theorems. In Section 5, I show the complete turnpike theorem. Some comments are given in Section 6.

## 2 Empirical Facts

The past few years have witnessed many efforts to collect and archive the industry-level database among countries. One such a database is now easily accessed on the Web: *the EU-Klems Growth and Productivity Database*[1], covering 28 countries and 71 industries from 1970 to 2005, that also contains their GDPs and TFPs. Growth

[1] URL http://www.euklems.net

accounting has been used to analyze each country's economic growth. One of the more interesting applications is to analyze the economic growth of the industries. Let us assume that the production function of the i$^{th}$ industry in a country is given as:

$$Y_i(t) = A_i(t)F^i(K_{1i}(t), K_{2i}(t), \cdots, K_{ni}(t), L_i(t)),$$

where $Y_i$ : t$^{th}$ period capital goods output of the i$^{th}$ industry, $K_{ji}$ : i$^{th}$ capital goods used in the j$^{th}$ industry in the t$^{th}$ period, $A_t^i$ : t$^{th}$ period output-augmented technical-progress (the Hicks neutral technical-progress), and $L_i(t)$ : t$^{th}$ period labor input of the i$^{th}$ industry. If $\theta_j$ stands for the factor share of the j$^{th}$ input factor of the i$^{th}$ industry, then we may derive the following relation concerned with the i$^{th}$ industry;

$$\frac{\dot{A_i}}{A_i} = \frac{\dot{Y_i}}{Y_i} - \left(\sum_{j=1}^{n} \theta_{ji}\frac{\dot{K_{ji}}}{K_{ji}} + \theta_{0i}\frac{\dot{L_i}}{L_i}\right).$$

Based on this equation, we are able to calculate TFPs of 20 industries of a country[2]. Figure 1 shows the relationship between the average per capita growth rate of the US GDP and the average TFP growth rate of the US TFP at the industry

[2]The 20 industries of the US are listed as follows:

1:TOTAL INDUSTRIES 2 :AGRICULTURE, HUNTING, FORESTRY AND FISHING,

3:MINING AND QUARRYING, 4:TOTAL MANUFACTURING,

5:FOOD , BEVERAGES AND TOBACCO, 6:TEXTILES, TEXTILE , LEATHER AND FOOTWEAR,

7:WOOD AND OF WOOD AND CORK 8:PULP, PAPER, PAPER , PRINTING AND PUBLISHING,

9:CHEMICAL, RUBBER, PLASTICS AND FUEL 10:OTHER NON-METALLIC MINERALS, level from 1970 to 2005. Figure 2 presents the same data for the Japanese economy. Note that a 45-degree lines is drawn in both the figures. If an industry were on the 45-degree line, it would imply that the industry's per capita GDP would grow at its TFP growth rate. Observing Figures 1 and 2, we may conclude that in both countries most industries cluster around the 45-degrees line, however, some industries lie far-above or below it.

[Figure 1-Figure 2]

We may summarize these facts as follows:

1) Each industry has its own steady state with a industry-specific growth rate.

2) The steady state level and its growth rate are highly related to its own TFP.

These facts cannot be explained by New growth theory, since it is based entirely on the highly aggregated macro production functions. Thus, we must construct an

11:BASIC AND FABRICATED METALS,

12:MACHINERY, NEC 13:ELECTRICAL AND OPTICAL EQUIPMENT,

14:TRANSPORT EQUIPMENT, 15:MANUFACTURING NEC AND RECYCLING,

16:ELECTRICITY, GAS AND WATER SUPPLY,

17:CONSTRUCTION 18:WHOLESALE AND RETAIL TRADE,

19:HOTELS AND RESTAURANTS,

20:TRANSPORT , STORAGE AND COMMUNICATION.

For the Japanese Economy, three more industries are added.

industry based multi-industry growth model. On the other hand, although turnpike theory is based on the multi-industry model, it has a drawback too. The theory means that each industrial sector with different initial stocks will eventually converge to its own optimal steady state with the common balanced growth rate. In other words, each industry's per capita stock will converge to a certain constant ratio. Thus, turnpike theory cannot explain why each industry's per capita stock grows at its own rate determined by the industry's TFP.

OECD (2003) also studied the industry-level productivity growth in detail and reported the following results, which are consistent with our observations discussed above.

- A large contribution to overall productivity growth patterns comes from productivity changes within industries, rather than as a result of significant shifts of employment across industries.
- TFP depends on country-specific and/industry-specific factors.

From the above discussion, it is imperative to set up a multi-industry optimal growth model with technical progress and demonstrate that each sector's per-capita capital and output will grow at the industry-specific growth rate determined by the industry's TFP.

## 3 The Model and Assumption

Our model is a discrete-time and multi-sector version of the standard neoclassical optimal growth model with the Hicks-neutral technical progress and in the following sections, I will use the term "sector" instead of "industry," which is more commonly used in turnpike theory:

$$Max \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^{t} u(C(t))$$

$$subject\ to: \ k_i(0) = \overline{k}_i$$

$$Y_i(t) + (1-\delta_i)K_i(t) - K_i(t+1) = 0 \tag{1}$$

$$C(t) = A_0(t)F^0(K_{10}(t), K_{20}(t), \cdots\cdot, K_{n0}(t), L_0(t)), \tag{2}$$

$$Y_i(t) = A_i(t)F^i(K_{1i}(t), K_{2i}(t), \cdots\cdot, K_{ni}(t), L_i(t)), \tag{3}$$

$$\sum_{i=0}^{n} L_i(t) = L(t), \tag{4}$$

$$\sum_{j=0}^{n} K_{ij}(t) = K_i(t), \tag{5}$$

where$i = 1, 2, ..., n, t = 0, 1, 2, ....$, and the notation is as follows:

| | |
|---|---|
| $r$ | = a subjective rate of discount, r≥g, |
| $C(t) \in R_+$ | = the total goods consumed at t, |
| $Y_i(t) \in R_+$ | = the t$^{th}$ period capital goods output of the i$^{th}$ sector, |
| $K_i(t) \in R_+$ | = the t$^{th}$ period capital stock of the i$^{th}$ sector, |
| $K_i(0) \in R_+$ | = an initial capital stock of the i$^{th}$ sector, |
| $F^j(\cdot) : R_+^{n+1} \longmapsto R_+$ | = a production function of the j$^{th}$ sector, |
| $L_i(t)$ | = the t$^{th}$ period labor input of the i$^{th}$ sector, |
| $L(t)$ | = the t$^{th}$ period total labor input, |
| $K_{ij}(t)$ | = the i$^{th}$ capital goods used in the j$^{th}$ sector in the t$^{th}$ period, |
| $\delta_i$ | = the depreciation rate of the i$^{th}$ capital goods, given as $0<\delta_i<1$, |
| $A_i(t)$ | = the t$^{th}$ period output-augmented technical-progress of the i$^{th}$ sector. |

I maintain the following standard assumptions throughout the paper.

**Assumption 1.** 1) The utility function $u(\cdot)$ is defined on $\mathbb{R}_+$ as the following:

$$u(C(t)) = \begin{cases} \frac{C^\tau}{\tau}, \text{ if } \tau \in (-\infty, 1) \\ \\ \log C(t), \quad \text{if } \tau = 0. \end{cases}$$

2) $L(t) = (1+g)^t L(0)$ where $g$ is a rate of population growth. 3) $A_i(t) = (1+\alpha_i)^t A_i(0)$ where $\alpha_i$ is a rate of output-augmented technical progress of the $i$ th sector and given as $|\alpha_i| < 1$.

2) of Assumption 1 means that the sectoral TFP is measured by the sectoral output-augmented technical progress (the Hicks-neutral technical progress), which is externally given.

**Assumption 2.** 1) All the goods are produced nonjointly with the production functions $F^i$ $(i = 1, \cdots, n)$ which are defined on $R_+^{n+1}$, homogeneous of degree one[3], strictly quasi-concave and continuously differentiable for positive inputs. 2) Any good $j$ $(j = 0, 1, \cdots, n)$ cannot be produced unless $K_{ij} > 0$ for some $i = 1, \cdots, n$. 3) Labor must be used directly in each sector. If labor input of some sector is zero, its sector's output is zero.

[3]Under the constant-returns-to-scale assumption, the Hicks-neutral technical progress implies that all the capital and labor inputs of each industry will face the same rate of technical progress. In this sense we make rather strong restrictions on each industry's production function.

Dividing all the variables by $A_i(t)L(t)$, we will transform the original model into per-capita efficiency unit model. Firstly, let us transform the $t^{th}$ sector's production function as follows; dividing both sides of the $t^{th}$ sector's production function by $A_t^i L(t)$, we have:

$$\frac{Y_i(t)}{A_i(t)L(t)} = F^i\left(\frac{K_{1i}(t)}{L(t)}, \frac{K_{2i}(t)}{L(t)}, \cdots, \frac{K_{ni}(t)}{L(t)}, \frac{L_i(t)}{L(t)}\right) \ (i = 1, \cdots, n).$$

We see that,

$$\widetilde{y}_i(t) = f^i(k_{1i}(t), k_{2i}(t), \cdots, k_{ni}(t), \ell_i(t)) \ \ (i = 1, \cdots, n)$$

where $\widetilde{y}_i(t) = \frac{Y_i(t)}{A_i(t)L(t)}$, $k_{1i}(t) = \frac{K_{1i}(t)}{L(t)}$, $k_{2i}(t) = \frac{K_{2i}(t)}{L(t)}$, $\cdots$, $k_{ni}(t) = \frac{K_{ni}(t)}{L(t)}$,and $\ell_i(t) = \frac{L_i(t)}{L(t)}$.

Applying the same transformation to the consumption sector will provide

$$\widetilde{c}(t) = f^0(k_{10}(t), k_{20}(t), \cdots, k_{n0}(t), \ell_0(t)).$$

Furthermore, we may also transform the $t^{th}$ sector's accumulation equation as follows; dividing both sides by $A_t^i L(t)$, we have

$$\frac{Y_i(t)}{A_i(t)L(t)} + (1 - \delta_i)\frac{K_i(t)}{A_i(t)L(t)} - \frac{K_i(t+1)}{A_i(t)L(t)} = 0.$$

Substituting the following relation into this equation:

$$\frac{K_i(t+1)}{A_i(t)L(t)} = \frac{(1+\alpha_i)(1+g)K_i(t+1)}{[(1+\alpha_i)A_i(t)][(1+g)L(t)]} = (1+\alpha_i)(1+g)\widetilde{k}_i(t+1)$$

where

$$\widetilde{k}_i(t+1) = \frac{K_i(t+1)}{A_i(t)}$$

, we have finally

$$\widetilde{y}_i(t) + (1-\delta_i)\widetilde{k}_i(t) - (1+\alpha_i)(1+g)\widetilde{k}_i(t+1) = 0$$

where

$$\widetilde{k}_i(t) = \frac{\sum_{j=0}^{n} k_{ji}(t)}{A_i(t)}.$$

In a vector form expression,

$$\widetilde{\mathbf{y}} + (\mathbf{I} - \Delta)\widetilde{\mathbf{k}}(t) - (1+g)\mathbf{G}\widetilde{\mathbf{k}}(t+1) = 0$$

where $G$ and $\Delta$ are the following diagonal matrices:

$$\mathbf{G} = \begin{pmatrix} (1+\alpha_i) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (1+\alpha_n) \end{pmatrix} \text{ and } \Delta = \begin{pmatrix} \delta_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \delta_n \end{pmatrix}.$$

We can also rewrite the objective function in terms of per-capita as follows: Substituting the following relation into the objective function yields:

$$\widetilde{c}(t) = \frac{C(t)}{A_t^0 L(t)} = \frac{C(t)}{(1+\alpha_i)^t (1+g)^t A_0^0 L(0)}$$

we have: assume that $A_0^0 L(0) = 1$,

$$\sum_{t=0}^{\infty}\left[\frac{(1+g)^{\tau}(1+\alpha_0)^{\tau}}{(1+r)}\right]^t\frac{\widetilde{c}^{\tau}(t)}{\tau}=\sum_{t=0}^{\infty}\rho^t u(\widetilde{c}(t)).$$

Now the original model can be rewritten as the per-capita labor efficiency unit model as defined below:

**The Per-capita Labor Efficiency Units Model**

$$\begin{aligned} Max \sum_{t=0}^{\infty}\rho^t u(\widetilde{c}(t)) \;\; where\ \rho &= \frac{(1+g)^{\tau}(1+\alpha_0)^{\tau}}{(1+r)}, \\ s.t.\ \widetilde{k}_i(0) &= \overline{k}_i \;\; (i=1,\cdots,n), \end{aligned}$$

$$\widetilde{c}(t)=f^0(k_{10}(t),k_{20}(t),\cdots,k_{n0}(t),\ell_0(t)), \tag{6}$$

$$\widetilde{y}_i(t)=f^i(k_{1i}(t),k_{2i}(t),\cdots,k_{ni}(t),\ell_i(t))\ (i=1,\cdots,n), \tag{7}$$

$$\widetilde{\mathbf{y}}+(\mathbf{I}-\Delta)\widetilde{\mathbf{k}}(t)-(1+g)\mathbf{G}\widetilde{\mathbf{k}}(t+1)=0, \tag{8}$$

$$\sum_{i=0}^{n}\ell_i(t)=1, \tag{9}$$

$$\frac{\sum_{j=0}^{n} k_{ij}(t)}{A_i(t)} = \widetilde{k}_i(t) \; (i = 1, \cdots, n). \tag{10}$$

We may add the following assumption and prove the basic property below:

**Assumptin 3.** $A_0^0 L(0) = 1$ and $0 < \rho < 1$.

**Remark 1** *The value of $\rho$ consists of four parameters; the coefficient of relative risk averse $(1-\tau)$, the rate of population growth $(g)$, the rate of subjective discount rate $(r)$ and the rate of technical progress in consumption goods sector $(\alpha_0)$. Note that the rate of population could be negative. For example, we may consider the case where $\tau = 0.5, g = -0.2, r = 0.2$ and $\alpha_0 = 0.2$.*

**Lemma 1.** Under Assumption 2, Eqs.(6)-(10) except Eq.(8) are summarized as the social production function $\widetilde{c}(t) = T(\widetilde{\mathbf{y}}(t), \widetilde{\mathbf{k}}(t))$ which is continuously differentiable in the interior of $\mathbf{R}_+^{2n}$ and concave, where $\widetilde{\mathbf{y}}(t) = (\widetilde{y}_1(t), \widetilde{y}_3(t), \cdots, \widetilde{y}_n(t))$ and $\widetilde{\mathbf{k}}(t) = (\widetilde{k}_1(t), \widetilde{k}_2(t), \cdots, \widetilde{k}_n(t))$.

**Proof.**

Solving the following problem we can derive the social production function:

$$Max\ f^0(k_{10}(t), k_{20}(t), \cdots, k_{n0}(t), \ell_0(t)),$$

$$\begin{aligned} s.t.\ \widetilde{y}_i(t) &= f^i(k_{1i}(t), k_{2i}(t), \cdots, k_{ni}(t), \ell_i(t))\ (i = 1, \cdots, n), \\ \sum_{i=0}^{n} \ell_i(t) &= 1,\ \text{and}\ \frac{\sum_{j=0}^{n} k_{ij}(t)}{A_i(t)} = \widetilde{k}_i(t)\ (i = 1, \cdots, n). \end{aligned}$$

See in detail Benhabib and Nishimura (1979). ■

If $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{z}}$ stand for initial and terminal capital stock vectors respectively, the reduced form utility function $V(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}})$ and the feasible set $D$ can be defined as follows:

$$V(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) = u(T[(1+g)\mathbf{G}\widetilde{\mathbf{z}} - (\mathbf{I} - \Delta)\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}])$$

and

$$D = \{(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) \in R^n_+ \times R^n_+ : T[(1+g)\mathbf{G}\widetilde{\mathbf{z}} - (\mathbf{I} - \Delta)\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}] \geq 0\}$$

where $\widetilde{\mathbf{x}} = (\widetilde{x}_1(t), \widetilde{x}_2(t), \cdots, \widetilde{x}_n(t))$, $\widetilde{\mathbf{z}} = (\widetilde{k}_1(t+1), \widetilde{k}_2(t+1), \cdots, \widetilde{k}_n(t+1))$ and $\mathbf{I}$ is an n-dimensional unit matrix.

Finally, the above optimization problem will be summarized as the following standard reduced form model, which is often used in turnpike theory:

**Reduced Form Model**

$$Maximize\ \textstyle\sum_{t=0}^{\infty} \rho^t V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1))$$

$$subject\ to\ (\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1)) \in D\ for\ t \geq 0\ and\ \widetilde{\mathbf{k}}(0) = \overline{\mathbf{k}}.$$

Also note that any interior optimal path must satisfy the following Euler equations; they show an intertemporal efficiency allocation:

$$\mathbf{V}_z(\widetilde{\mathbf{k}}(t-1),\widetilde{\mathbf{k}}(t))+\rho\mathbf{V}_x(\widetilde{\mathbf{k}}(t),\widetilde{\mathbf{k}}(t+1))=\mathbf{0} \text{ for all } t\geq 0 \tag{11}$$

where the partial derivative vectors mean that

$$\begin{aligned}
\mathbf{V}_x(\widetilde{\mathbf{k}}(t),\widetilde{\mathbf{k}}(t+1)) &= [\partial V(\widetilde{\mathbf{k}}(t),\widetilde{\mathbf{k}}(t+1))/\partial\widetilde{k}_1(t),\cdots,\partial V(\widetilde{\mathbf{k}}(t),\widetilde{\mathbf{k}}(t+1))/\partial\widetilde{k}_n(t)]^t,\\
\mathbf{V}_z(\widetilde{\mathbf{k}}(t-1),\widetilde{\mathbf{k}}(t)) &= [\partial V(\widetilde{\mathbf{k}}(t-1),\widetilde{\mathbf{k}}(t))/\partial\widetilde{k}_1(t),\cdots,\partial V(\widetilde{\mathbf{k}}(t),\widetilde{\mathbf{k}}(t-1))/\partial\widetilde{k}_n(t)]^t
\end{aligned}$$

,and $\mathbf{0}$ means an n dimensional zero column vector. " $t$ " implies the transposition of vectors. Note that under the differentiability assumptions, all the price vectors will satisfy the following relations:

$$\begin{aligned}
q &= \partial u(\widetilde{c})/\partial\widetilde{c}=1,\\
p_i &= -q\partial T(\widetilde{\mathbf{y}},\widetilde{\mathbf{k}})/\partial\widetilde{k}_i \quad (i=1,2,\cdots,n),\\
w_i &= q\partial T(\widetilde{\mathbf{y}},\widetilde{\mathbf{k}})/\partial\widetilde{k}_i \quad (i=1,2,\cdots,n), \text{ and}\\
w_0 &= q\widetilde{c}+\mathbf{p}\widetilde{\mathbf{y}}-\mathbf{w}\widetilde{\mathbf{k}}.
\end{aligned}$$

Using these relations, we may define the price vectors of capital goods as an $(n\times 1)$ row vector $\mathbf{p}=(p_1,p_2,\cdots,p_n)$, the output of capital goods as an $(n\times1)$ vector

$\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, \cdots, \widetilde{y}_n)^t$, the rental rate as a $(1 \times n)$ row vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ and the capital stock as an $(n \times 1)$ vector $\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2, \cdots, \widetilde{k}_n)^t$. Moreover, let us denote a wage rate as $w_0$. For simplicity, we may assume that the price of the consumption good $q$ is normalized as 1.

**Definition.** *An optimal steady state path* $\widetilde{\mathbf{k}}^\rho$ (denoted by OSS henceforth) is an optimal path which solves the above optimization problem and $\widetilde{\mathbf{k}}^\rho = \widetilde{\mathbf{k}}(t) = \widetilde{\mathbf{k}}(t+1)$ for all $t \geq 0$.

Due to the homogeneity assumption of each sector's production, it is often convenient to express a chosen technology as a technology matrix. Now, let us define the technology matrix as follows:

$$\mathbf{A} = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ a_{10} & & \\ \vdots & \overline{\mathbf{A}} & \\ a_{n0} & & \end{pmatrix} = \begin{pmatrix} a_{00} & \mathbf{a}_{0.} \\ \mathbf{a}_{.0} & \overline{\mathbf{A}} \end{pmatrix}$$

where $a_{0i} = \widetilde{\ell}_i / \widetilde{y}_i$ $(i = 0, \cdots, n)$, $a_{ij} = \widetilde{k}_{ij} / \widetilde{y}_j$ $(i = 1, \cdots, n; j = 0, 1, \cdots, n)$ and

$$\overline{\mathbf{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

It directly follows that Assumption 2 implies that for all $j = 0, 1, \cdots, n$, $a_{ij} > 0$ for some $i = 1, \cdots, n$ and $a_{0i} > 0$ for all $i$. First, we make the following assumption expressed in terms of the technology matrix to demonstrate the existence theorem.

**Assumption 4.** (*Viability*) For a given $r$ ($\geq g$), a chosen technology matrix $\overline{\mathbf{A}}$ satisfies

$$[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r]^{-1} \geq \Theta$$

where $\Theta$ is a $n \times n$ zero matrix[4].

Due to the well known equivalence theorem to the Hawkins-Simon condition and Theorem 4 of Mckenzie (1960), Assumption 4 is equivalent to the property such that the matrix $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r]$ has a dominant diagonal that is positive; there exists $\mathbf{y} \geq \mathbf{0}$ such that $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r]\mathbf{y} \geq \mathbf{0}$.

The following extra assumption will be made.

**Assumption 5.** $1 > \alpha_0 > \max_{i=1,\ldots,n} |\alpha_i|$

**Remark 2** *This assumption means that the TFP growth rate in the consumption sector is the highest one among those of sectors. Takahashi, Mashiyama and Sakagami (2004) have reported the empirical evidence such that in the postwar Japanese*

[4]Let $\mathbf{A}$ and $\Theta$ be an $n$-dimentional square matrix and an $n$-dimensional zero matrix respectively. Then $\mathbf{A} \gg \Theta$ if $a_{ij} > 0$ for all $i, j$, $\mathbf{A} > \Theta$ if $a_{ij} \geq 0$ for all $i, j$ and $a_{ij} > 0$ for some $i, j$ and $\mathbf{A} \geq \Theta$ if $a_{ij} \geq 0$ for all $i, j$.

*economy, the consumption sector has exhibited a higher per-capita output growth rate than that of the capital goods sector. If the TFP growth rate has a positive correlation with the per-capita sectoral GDP growth rate, this fact will partially support Assumption 5.*

McKenzie (1983,1984) has demonstrated the existence theorem for both an optimal and an optimal steady state paths in the reduced form model. Applying a same logic as that of McKenzie's, we can prove the following existence theorem under Assumptions 1 through 5.

**Existence Theorem:** Under Assumptions 1 through 5, there exists an optimal steady state path $\widetilde{\mathbf{k}}^{\rho}$ for $\rho \in (0, 1]$ and an optimal path $\{\widetilde{\mathbf{k}}^{\rho}(t)\}^{\infty}$ from any sufficient initial stock vector $\widetilde{\mathbf{k}}\ (0)$[5].

**Proof.** We need to show that under Assumptions 1 through 3, all the conditions[6]

[5]A capital stock x is called sufficient if there is a finite sequence (k(0),k(1),$\cdots$,k(T)) where x=k(0), (k(t),k(t+1))$\in D$ and k(T) is expansible. k(T) is expansible if there is k(T+1) such that k(T+1)$\gg$k(T) and (k(T),k(T+1))$\in D$. Note that the sufficiency will be assured by assuming "Inada-type" condition on production functions.

[6]McKenzie's conditions are followings: 1) $V(\mathbf{x}, \mathbf{z})$ are defined on a convex set $D$. 2) There is a $\eta > 0$ such that $(\mathbf{x}, \mathbf{z}) \in D$ and $|\mathbf{z}| < \xi < \infty$ implies $|\mathbf{z}| < \eta < \infty$. 3) If $(\mathbf{x}, \mathbf{z}) \in D$, then $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) \in D$ for all $\widetilde{\mathbf{x}} \geq \mathbf{x}$ and $0 \leq \widetilde{\mathbf{z}} \leq \mathbf{z}$. Moreover $V(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) \geq V(\mathbf{x}, \mathbf{z})$. 4) Ther is $\zeta > 0$ such that $|\mathbf{x}| \geq \zeta$ implies for any $(\mathbf{x}, \mathbf{z}) \in D$, $|\mathbf{z}| < \lambda\,|\mathbf{x}|$ where $0 < \lambda < 1$. 5) There is $(\overline{\mathbf{x}}, \overline{\mathbf{z}}) \in D$ such that $\rho\overline{\mathbf{z}} > \overline{\mathbf{x}}$.

in Theorem 1 of McKenzie (1983) or in the existence theorem of McKenzie (1984) are satisfied. Especially, Assumption 4 and the additional condition are needed to guarantee the non-emptiness of the interior of $D$ (Condition 5) in the footnote) as we will demonstrate as follows; from the Condition 5), there is an output vector $\mathbf{y} \geq \mathbf{0}$ such that $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r]\mathbf{y} \geq \mathbf{0}$. By a scalar multiplication of $\mathbf{y}$, we can establish $\widehat{\mathbf{x}} = \mathbf{A}^r \widehat{\mathbf{y}}$ where $\widehat{\mathbf{x}} = (1, \widetilde{\mathbf{x}})^t$ and $\widehat{\mathbf{y}} = (c, \widetilde{\mathbf{y}})$. Note that the equality of the first elements of $\widehat{\mathbf{x}}$ and $\mathbf{A}^r\widehat{\mathbf{y}}$ will provide Eq.(9): the full employment condition. Since the labor constraints are satisfied for $\widehat{\mathbf{y}}$ and that $\overline{\mathbf{A}}^r$ is a submatrix of $\mathbf{A}^r$, it follows that $\overline{\mathbf{x}} = \overline{\mathbf{A}}^r\overline{\mathbf{y}}$ holds. Now the following relation will be established:

$$
\begin{aligned}
\widetilde{\mathbf{z}} - \rho^{-1}\widetilde{\mathbf{x}} &= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I + [\mathbf{I} - \overline{\Delta} - (1+g)\mathbf{G}\rho^{-1}]\overline{\mathbf{A}}^r\right\}\widetilde{\mathbf{y}} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1} I + \mathbf{I} - \Delta - (1+g)\begin{pmatrix} (1+a_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (1+a_n) \end{pmatrix} \\
&\quad \left[\frac{(1+r)}{(1+g)(1+a_0)}\right]\Bigg]\overline{\mathbf{A}}^r\Bigg\}\widetilde{\mathbf{y}} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I + \left[\mathbf{I} - \Delta - (1+r)\begin{pmatrix} \frac{(1+a_1)}{(1+a_0)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{(1+a_n)}{(1+a_0)} \end{pmatrix}\right]\overline{\mathbf{A}}^r\right\}\widetilde{\mathbf{y}}
\end{aligned}
$$

$$\geq \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I+[\mathbf{I}-\Delta-(1+r)\mathbf{I}]\overline{\mathbf{A}}^{r}\right\}\widetilde{\mathbf{y}} \text{ due to Assumption 5,}$$
$$= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}[\mathbf{I}-(r\mathbf{I}+\overline{\Delta})]\overline{\mathbf{A}}^{r}\widetilde{\mathbf{y}} > 0 \text{ from Assumption 4,}$$

Therefore, $\overline{\mathbf{y}}$ will be chosen so that $\widetilde{\mathbf{z}} - \rho^{-1}\widetilde{\mathbf{x}} \geq \mathbf{0}$ where $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}})\varepsilon\mathbf{D}$. For the detailed discussion, see Lemma 3 through Lemma 7 in Takahashi (1985). ■

**Remark 3** *It should be noticed that since* $\widetilde{k}_i^\rho = \frac{k_i^\rho(t)}{A_i(t)L_t}$, *it follows that* $k_i^\rho(t) = \widetilde{k}_i^\rho A_i(t) = (1+\alpha_i)^t A_i(0)\widetilde{k}_i^\rho$ *for* $i = 1, \cdots, n$. *Hence the original series of the industry's optimal per-capita stock* $k_i^\rho(t)$ *is growing at the rate of its own sector's technical progress,* $(1 + a_i)$.

**Remark 4** *From now on, to avoid further complications of our notation, all the variables measured in efficiency units will be denoted without the symbol "$\sim$" unless otherwise mentioned.*

Suppose that $\mathbf{k}^\rho$ is an interior OSS in labor-efficiency units with a given $\rho$, it must satisfy the Euler equations:

$$\mathbf{V}_z(\mathbf{k}^\rho, \mathbf{k}^\rho) + \rho\mathbf{V}_x(\mathbf{k}^\rho, \mathbf{k}^\rho) = \mathbf{0}. \tag{12}$$

Due to the above definition of OSS, we will express the partial derivatives of the Euler equations in terms of price vectors:

$$\begin{aligned}\mathbf{V}_x(\mathbf{k}^\rho, \mathbf{k}^\rho) &= \mathbf{p}^\rho(\mathbf{I} - \Delta) + \mathbf{w}^\rho, \text{ and} \\ \mathbf{V}_z(\mathbf{k}^\rho, \mathbf{k}^\rho) &= -(1+g)\mathbf{G}\mathbf{p}^\rho.\end{aligned}$$

where $\mathbf{I}$ is a $n \times n$ unit matrix. Substituting these relations into the Euler equations may yield the following:

$$\rho[\mathbf{w}^\rho + \mathbf{p}^\rho(\mathbf{I} - \Delta)] - (1+g)\mathbf{G}\mathbf{p}^\rho = \mathbf{0} \tag{13}$$

and further calculation will finally yield:

$$\mathbf{p}^\rho \left[ -\mathbf{I} + \Delta + \left( \frac{1+r}{1+\alpha_0} \right) \mathbf{G} \right] = \mathbf{w}^\rho.$$

These are clearly non-arbitrage conditions among capital goods; any capital good must yield the same rate of returns as the subjective discount rate $\rho$. Hence the Euler conditions are the non-arbitrage conditions.

Because of differentiability and constant returns to scale technologies, the well-known proposition proved by Samuelson (1945) will hold: the cost function denoted by $C^i(w_0, \mathbf{w}^\rho)$ $(i = 1, \cdots, n)$ is homogeneous of degree one and $\partial C^i / \partial w_j = a_{ij}$ where $a_{ij} = k_{ij}/y_j$ $(i = 1, 2, \cdots, n;\ j = 0, 1, \cdots, n)$. Due to the cost minimization condition and this proposition, a unique technology matrix $\mathbf{A}^\rho$ is chosen on $\mathbf{k}^\rho$. Also note that due to Assumption 3, for a given $\rho \in (0,1]$, the uniquely chosen technology matrix $\overline{\mathbf{A}}^\rho$ have to satisfy

$$[\mathbf{I} - (r\mathbf{I} + \overline{\Delta})\overline{\mathbf{A}}^\rho]^{-1} \geq \Theta.$$

Moreover it follows that $a_{00}^\rho > 0$ and $\mathbf{a}_{\mathbf{0}.}^\rho \gg \mathbf{0}$ from Assumption 2. Henceforth, we use the symbol "$\rho$" to emphasize that vectors and matrices are evaluated on $\mathbf{k}^\rho$.

Combining these results, the following important property will be established:

**Lemma 2.** When $\rho \in (0,1]$ , there exists a unique $\mathbf{k}^\rho (\gg \mathbf{0})$[7] with the corresponding unique positive price vector $\mathbf{p}^\rho$ and the positive factor price vector $(w_0^\rho, \mathbf{w}^\rho)$.

**Proof.** It follows from applying the same logic as the one used in Theorem1 of Burmeister and Grahm (1975). ■

From this lemma, along the OSS with $\rho$, the nonsingular technology matrix $\mathbf{A}^\rho$ will be chosen, and the cost-minimization and full-employment conditions will be expressed as follows:

$$(1, \mathbf{p}^\rho) = (w_0^\rho, \mathbf{w}^\rho)\mathbf{A}^\rho,$$

and

$$(1, \mathbf{k}^\rho)^t = \mathbf{A}^\rho (c^\rho, \mathbf{y}^\rho)^t.$$

[7] Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional vectors. Then $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for all i, $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ for all i and at least one j, $x_i > y_i$ and $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for all i.

If $\mathbf{A}^\rho$ has an inverse matrix $\mathbf{B}^\rho$, solving these equations yields,

$$\mathbf{p}^\rho = \mathbf{w}^\rho\left(\mathbf{a}^\rho - \frac{1}{a^\rho_{00}}\mathbf{a}^\rho_{.\mathbf{0}}\mathbf{a}^\rho_{\mathbf{0}.}\right) + \frac{\mathbf{a}^\rho_{\mathbf{0}.}}{a^\rho_{00}} = \mathbf{w}^\rho(\mathbf{b}^\rho)^{-1} + \frac{\mathbf{a}^\rho_{\mathbf{0}.}}{a^\rho_{00}}$$

and

$$(\mathbf{k}^\rho)^t = \left(\mathbf{a}^\rho - \frac{1}{a^\rho_{00}}\mathbf{a}^\rho_{.\mathbf{0}}\mathbf{a}^\rho_{\mathbf{0}.}\right)(\mathbf{y}^\rho)^t + \frac{\mathbf{a}^\rho_{.\mathbf{0}}}{a^\rho_{00}} = (\mathbf{b}^\rho)^{-1}(\mathbf{y}^\rho)^t + \frac{\mathbf{a}^\rho_{.\mathbf{0}}}{a^\rho_{00}}$$

where $\mathbf{b}^\rho$ is the submatrix of $\mathbf{B}^\rho$ defined as follows:

$$\mathbf{B}^\rho = (\mathbf{A}^\rho)^{-1} = \begin{pmatrix} b^\rho_{00} & \mathbf{b}^\rho_{0\cdot} \\ \mathbf{b}^\rho_{\cdot 0} & \mathbf{b}^\rho \end{pmatrix}.$$

Note that the nonsingularity of $\mathbf{b}^\rho$ comes from the following observation: Due to the theorem in Murata (1977), it follows that $\mathbf{b}^\rho = [\mathbf{a}^\rho - (1/a^\rho_{00})\mathbf{a}^\rho_{\cdot 0}\mathbf{a}^\rho_{0\cdot}]^{-1}$. Furthermore, by Gantmacher (1960), it also follows that $det.\ \mathbf{A}^\rho = a^\rho_{00} det.[\mathbf{a}^\rho - (1/a^\rho_{00})\mathbf{a}^\rho_{\cdot 0}\mathbf{a}^\rho_{0\cdot}]$. Since $\mathbf{A}^\rho$ is non-singular, our claim holds.

From now on, we are concentrated on the OSS with $\rho = 1$ denoted by $\mathbf{k}^*$. We will also use the symbol " $*$ " to denote the elements and variables evaluated at $\mathbf{k}^*$.

**Definition.** When $\rho = 1$, the chosen technology matrix $\mathbf{A}^*$ satisfies the *Generalized Capital Intensity GCI -I* condition, if there exists a set of positive number $(d_1, \cdots, d_n)$ such that

$$d_s\left(\frac{a^*_{ss}}{a^*_{0s}} - \frac{a^*_{s0}}{a^*_{00}}\right) > \sum_{i \neq s,0}^{n} d_i \left|\frac{a^*_{si}}{a^*_{0i}} - \frac{a^*_{s0}}{a^*_{00}}\right| \quad for\ s = 1, \cdots, n.$$

Similarly, the technology matrix $\mathbf{A}^*$ satisfies the *Generalized Capital Intensity GCI -II* condition, if there exists a set of positive number $(d_1, \cdots, d_n)$ such that

$$\frac{a^*_{ss}}{a^*_{0s}} - \frac{a^*_{s0}}{a^*_{00}} < 0$$

and

$$d_s \left| \frac{a^*_{ss}}{a^*_{0s}} - \frac{a^*_{s0}}{a^*_{00}} \right| > \sum_{i \neq s,0}^{n} d_i \left| \frac{a^*_{si}}{a^*_{0i}} - \frac{a^*_{s0}}{a^*_{00}} \right| \; for \; s = 1, \cdots, n.$$

Consider a capital good sector $s$, and focus on its own capital input $s$ and its capital-labor ratio in all the other sectors. Due to the definition, the left-hand side of the GCI-I condition means the excess of the capital-labor ratio of capital input $s$ for the capital good sector $s$. The right-hand side collects the absolute values of the discrepancy between the capital-labor ratio of other sectors ( $i \neq s, 0$) to that of the consumption sector. The GCI-I condition indicates that the sum of such absolute values still fall short of the the excess of $a^*_{ss}/a^*_{0s}$ over the comparable ratio in the consumption sector, $a^*_{s0}/a^*_{00}$.We may give a similar explanation to the GCI-II condition.

The following lemma can be directly derived from the definition of both intensity conditions.

**Lemma 3.** If the technology matrix $\mathbf{A}^*$ satisfies the GCI-I (GCI- II condition) condition, its inverse matrix $\mathbf{B}^*$ has positive (negative) diagonal elements and negative (positive) off-diagonal elements.

**Proof.** From the theorem by Jones et al. (1993), under the Strong GCI-II, its inverse matrix has negative diagonal and positive off-diagonal elements. On the other hand, under the Strong CGI-I, by considering the case where one price falls with all the other prices constant in their proof, their exact logic can be applicable. Hence the first result will be established. ■

Due to Lemma 3, we may prove the following important lemma:

**Lemma 4.** Under the Strong GCI-I (the Strong GCI-II), $[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]$ has a dominant diagonal that is positive (negative) for rows[8].

**Proof.** Due to Lemma 3, under the Strong GCI-I (the Strong GCI-II), the inverse matrix $\mathbf{B}^*$ has positive (negative) diagonal elements and negative ( positive) off-diagonal elements. From the accumulation equation $\mathbf{y}^* = (1+n)\mathbf{G}\mathbf{k}^* - (\mathbf{I} - \Delta)\mathbf{k}^*$ and $\mathbf{y}^* = \mathbf{b}^*\mathbf{k}^* + \mathbf{b}^*_{.0}$, it follows that

$$[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]\mathbf{k}^* = -\mathbf{b}^*_{.0}$$

Due to Lemma 3, $-\mathbf{b}^*_{.0} < (>)\mathbf{0}$. Therefore the matrix $[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]$ has the negative (positive) dominant diagonal for rows. ■

[8]Suppose $\mathbf{A}$ is an $n \times n$ matrix and its diagonal elements are negative (positive). Let there exist a positive vector $\mathbf{h}$ such that $h_i \mid a_{ii} \mid > \sum_{j=1, j\neq i}^{n} h_j \mid a_{ij} \mid,\ i = 1, 2, \cdots, n$. Then $\mathbf{A}$ is said to have a dominant main diagonal that is negative (positive) for rows. See McKenzie (1960) and Murata (1977).

From now on, we will write the dominant diagonal that is negative as "n.d.d." and the dominant diagonal that is positive as "p.d.d." for short. From the Euler equations (12), the Jacobian $\mathbf{J}(\mathbf{k}, \rho)$ is

$$\mathbf{J}(\mathbf{k}, \rho) = \rho \mathbf{V}_{xx}(\mathbf{k}, \mathbf{k}) + \rho \mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{zx}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{zz}(\mathbf{k}, \mathbf{k}).$$

Evaluating it at $\mathbf{k}^*$ yields

$$\mathbf{J}(\mathbf{k}, 1) = \mathbf{V}_{zz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{zx}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xx}(\mathbf{k}^*, \mathbf{k}^*)$$

where all matrices are evaluated at $\mathbf{k}^*$[9]. We will show the following lemma, which is a counterpart of Lemma 2.5 of Takahashi (1992).

**Lemma 5.** Suppose that either of the GCI conditions hold. Then there exists a positive scalar $\overline{\rho}$ such that for $\rho \in [\overline{\rho}, 1]$, the OSS $\mathbf{k}^\rho$ is unique and is a continuous vector-value function of $\rho$, namely $\mathbf{k}^\rho = \mathbf{k}(\rho)$ .

**Proof.** If $\det \mathbf{J}(\mathbf{k}^*, 1) \neq 0$ holds, then due to the implicit function theorem, the result follows. To show this we will use the following fact shown in Benhabib and Nishimura (1979):

$$\mathbf{T}_1 = [\partial T / \partial \mathbf{y}] = -\mathbf{p} \text{ and } \mathbf{T}_2 = [\partial T / \partial \mathbf{k}] = \mathbf{w}$$

[9]We use the following notational convention for the partial derivative matrices:$\mathbf{V}_{xx} = [\partial^2 \mathbf{V}(\mathbf{x}, \mathbf{z}) / \partial \mathbf{x}^2]$, $\mathbf{V}_{xz} = [\partial^2 \mathbf{V}(\mathbf{x}, \mathbf{z}) / \partial \mathbf{x} \partial \mathbf{z}]$ and $\mathbf{V}_{zz} = [\partial^2 \mathbf{V}(\mathbf{x}, \mathbf{z}) / \partial \mathbf{z}^2]$. Note that each matrix is an $n \times n$ matrix.

where $\mathbf{p}$ is an output price vector. Differentiating both price vectors with respect to $\mathbf{y}$ and $\mathbf{k}$ again will yield the following second-order partial derivative matrices: $\mathbf{T}_{11} = [-\partial\mathbf{p}/\partial\mathbf{y}]$, $\mathbf{T}_{12} = [-\partial\mathbf{p}/\partial\mathbf{k}]$, $\mathbf{T}_{21} = [\partial\mathbf{w}/\partial\mathbf{y}]$ and $\mathbf{T}_{22} = [\partial\mathbf{w}/\partial\mathbf{k}]$. Note that if the matrices are evaluated at $\mathbf{k}^*$, from the previous equation, we have

$$[\partial\mathbf{p}/\partial\mathbf{w}] = (\mathbf{b}^*)^{-1}$$

,and due to the symmetry of the Hessian matrix of $c(t) = T(\mathbf{y}(t), \mathbf{k}(t))$, we have

$$[\partial\mathbf{p}/\partial\mathbf{k}] = -[\partial\mathbf{w}/\partial\mathbf{y}]^t$$

where the suffix " $t$ " means a transpose of a matrix. Utilizing these relations, all the partial derivative matrices at $\mathbf{k}^*$ will be expressed in terms of the matrices $\mathbf{b}^*$ and $\mathbf{T}_{22}$ as follows: $\mathbf{T}_{11} = (\mathbf{b}^*)^{-1}\mathbf{T}_{22}^t(\mathbf{b}^*)^{-1} = (\mathbf{b}^*)^{-1}\mathbf{T}_{22}(\mathbf{b}^*)^{-1}$, $\mathbf{T}_{12} = -(\mathbf{b}^*)^{-1}\mathbf{T}_{22}$, and $\mathbf{T}_{21} = -\mathbf{T}_{22}(\mathbf{b}^*)^{-1}$. Substituting $\mathbf{Y}_x = (g\mathbf{I} + \Delta)$ and $\mathbf{Y}_z = \mathbf{I}$ into the first terms of Eq.(3-21) through Eq.(3-23) in Takahashi (1985), the Jacobian will be finally rewritten as follows:

$$\mathbf{J}(\mathbf{k}^*, 1) = [(1+g)\mathbf{G} + \Delta - \mathbf{I}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{G} + \Delta - \mathbf{I} \\ \mathbf{I} \end{pmatrix}.$$

If the right-hand side of this is a negative definite matrix, the proof will be completed. Substituting all the relations obtained so far into the Hessian matrix of the social production function, and provided that the matrix $\mathbf{b}^*$ is nonsingular, we may yield the following equation:

$$
\begin{aligned}
& [(1+g)\mathbf{G}+\Delta-\mathbf{I},\mathbf{I}]\begin{pmatrix}\mathbf{T}_{11} & \mathbf{T}_{12}\\ \mathbf{T}_{21} & \mathbf{T}_{22}\end{pmatrix}\begin{pmatrix}(1+g)\mathbf{G}+\Delta-\mathbf{I}\\ \mathbf{I}\end{pmatrix}\\
= & ((1+g)\mathbf{G}+\Delta-\mathbf{I})\mathbf{T}_{11}((1+g)\mathbf{G}+\Delta-\mathbf{I})\\
& +((1+g)\mathbf{G}+\Delta-\mathbf{I})\mathbf{T}_{21}+\mathbf{T}_{12}((1+g)\mathbf{G}+\Delta-\mathbf{I})+\mathbf{T}_{22}\\
= & [\mathbf{b}^*-((1+g)\mathbf{G}+\Delta-\mathbf{I})]^2[(\mathbf{b}^*)^{-1}]^2\mathbf{T}_{22}.
\end{aligned}
$$

Due to Lemma 4, the matrix $[\mathbf{b}^*-((1+g)\mathbf{G}+\Delta-\mathbf{I})]$ has the negative (positive) d.d. from the GCI conditions and it must be nonsingular. $\mathbf{b}^*$ is also nonsingular as we have discussed before. $\mathbf{T}_{22}$ is negative definite and nonsingular due to the argument of Benhabib and Nishimura (1979,pp68-69). Furthermore, the first two matrices are symmetric and therefore all the elements are positive. Thus the above matrix is negative definite and the proof has been completed. ■

From this lemma, it follows that all the price vectors $\mathbf{p}^\rho$, $\mathbf{w}^\rho$, and the technology matrix $\mathbf{A}^\rho$ are continuous vecto-value functions of $\rho \in [\rho', 1]$.

## 4 The Neumann-McKenzie Facet

Now we will introduce the Neumann-McKenzie Facet (denoted by “NMF” for short), which plays an important role in stability arguments regarding neoclassical growth models as studied in Takahashi (1985) and Takahashi (1992), and has been intensively studied by L. McKenzie (see especially McKenzie (1983)). The NMF will be defined in the reduced form model as follows:

**Definition.** *The Neumann-McKenzie Facet* of an OSS, denoted by $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$, is defined as:

$$\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho) = \{(\mathbf{x}, \mathbf{z}) \in D : u(c) + \rho \mathbf{p}^\rho \mathbf{z} - \mathbf{p}^\rho \mathbf{x} = u(c^\rho) + \rho \mathbf{p}^\rho \mathbf{k}^\rho - \mathbf{p}^\rho \mathbf{k}^\rho\},$$

where $\mathbf{k}^\rho$ is the OSS and $\mathbf{p}^\rho$ is the supporting price of $\mathbf{k}^\rho$ when a subjective discount rate $r$ is given.

By the definition above, the NMF is a set of capital stock vectors $(\mathbf{x}, \mathbf{z})$ which arise from the exact same net benefit as that of the OSS when it is evaluated by the prices of the OSS. Also, the NMF is the projection of a flat segment on the surface of the utility function $V$ that is supported by the price vector $(-\mathbf{p}^\rho, \rho \mathbf{p}^\rho, 1)$ onto the $(\mathbf{x}, \mathbf{z})$-space. In Takahashi (1985), I consider a case of the objective function where n capital goods as well as pure-consumption goods are consumable. Here, the capital goods are not consumable but the discounted sum of the sequence of pure-consumption goods is directly evaluated. Due to the well-established nonsubstitution theorem, a unique technology matrix $\mathbf{A}^\rho$ defined before will be chosen on the OSS.

By exploiting this fact, we will re-characterize the NMF as a more tractable formula with an (n+1) by (n+1) matrix $\mathbf{A}^\rho$ and an (n+1)-dimensional vectors as we will demonstrate in Lemma 6 under Assumption 6:

**Assumption 6.** $u(c)$ is linear in the neighaborhood of OSS; $u(c) = c$.

**Lemma 6.** Under Assumption 6, when $\mathbf{A}^\rho$ is non-singular, $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ if and only if there exists $\widehat{\mathbf{y}} \equiv (c, \mathbf{y})' \geqq 0$ such that the following conditions hold:

$$\begin{aligned} i)\ \widehat{\mathbf{x}} &= \mathbf{A}^\rho \widehat{\mathbf{y}}, \text{ and} \\ ii)\ \widehat{\mathbf{z}} &= \left(\frac{1}{1+n}\right) \overline{\mathbf{G}}^{-1} [\widehat{\mathbf{y}} + (\mathbf{I} - \overline{\Delta})] \widehat{\mathbf{x}} \end{aligned}$$

where $\widehat{\mathbf{x}} = (1, \mathbf{x}), \widehat{\mathbf{z}} = (\mathbf{1}, \mathbf{z}), \overline{\mathbf{G}}^{-1} = \begin{pmatrix} 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \frac{1}{(1+a_1)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{1}{(1+a_n)} \end{pmatrix}$,

$\overline{\Delta} = \begin{pmatrix} 0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \delta_1 & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \delta_n \end{pmatrix}$ and $\mathbf{I}$ is an $(n+1)$-dimensional unit matrix.

**Proof.** From the definition of the NMF, we have the price-supporting relation below:

$$c + \rho \mathbf{p}^\rho \mathbf{z} - \mathbf{p}^\rho \mathbf{x} = c^\rho + \rho \mathbf{p}^\rho \mathbf{k}^\rho - \mathbf{p}^\rho \mathbf{k}^\rho$$

where $c$ and $\mathbf{y}$ correspond to $(\mathbf{x}, \mathbf{z})$. Furthermore, from the fact that $\mathbf{V}_x(\mathbf{k}^\rho, \mathbf{k}^\rho) = \mathbf{w}^\rho + \mathbf{p}^\rho(\mathbf{I} - \Delta)$ and $\mathbf{V}_z(\mathbf{k}^\rho, \mathbf{k}^\rho) = -(1+g)\mathbf{G}\mathbf{p}^\rho$, the Euler equation implies that

$\mathbf{w}^\rho = \mathbf{p}^\rho[\rho^{-1}(1+g)\mathbf{G} - (\mathbf{I} - \Delta)]$. Also note that from the accumulation relations, $\mathbf{z} = (1/(1+g))\mathbf{G}^{-1}[\mathbf{y}+(\mathbf{I}-\Delta)\mathbf{x}]$ and $\mathbf{k}^\rho = (1/(1+g))\mathbf{G}^{-1}[\mathbf{y}^\rho+(\mathbf{I}-\Delta)\mathbf{k}^\rho]$. Substituting these relations into the price-supporting equations, we finally obtain:

$$(c - c^\rho) + \rho\left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\{[\mathbf{p}^\rho\mathbf{y} - \mathbf{w}^\rho\mathbf{x}] - [\mathbf{p}^\rho\mathbf{y}^\rho - \mathbf{w}^\rho\mathbf{k}^\rho]\} = 0. \tag{14}$$

If $\{[\mathbf{p}^\rho\mathbf{y} - \mathbf{w}^\rho\mathbf{x}] - [\mathbf{p}^\rho\mathbf{y}^\rho - \mathbf{w}^\rho\mathbf{k}^\rho]\} \neq 0$, whereas the first term is scalar, the second term will be a vector. Thus the above equality never holds. Hence $\{[\mathbf{p}^\rho\mathbf{y} - \mathbf{w}^\rho\mathbf{x}] - [\mathbf{p}^\rho\mathbf{y}^\rho - \mathbf{w}^\rho\mathbf{k}^\rho]\} = 0$ as well as $c = c^\rho$ must hold. Combining the results, we finally obtain:

$$c + \mathbf{p}^\rho\mathbf{y} - \mathbf{w}^\rho\mathbf{x} = c^\rho + \mathbf{p}^\rho\mathbf{y}^\rho - \mathbf{w}^\rho\mathbf{k}^\rho.$$

This result is Condition i); $(c_0, \mathbf{y}, \mathbf{x})$ should lie on the production frontier of $T(\mathbf{y}, \mathbf{k})$ and in each sector, the chosen technology must be the same as that in the OSS. In other words, the OSS technology matrix $\mathbf{A}^\rho$ will be uniquely chosen. Therefore, on the NMF, the exact same technology matrix as that of the corresponding OSS is chosen. In other words, given an OSS technology matrix $\mathbf{A}^\rho$, the cost minimization and the full-employment conditions for labor and capital goods must be satisfied.

Hence, the following equations must hold:

$$1) q^{\rho} = w_0^{\rho} a_{00}^{\rho} + \mathbf{w}^{\rho} \mathbf{a}_{\cdot 0}^{\rho},$$

$$2)\ \mathbf{p}^{\rho} = w_0^{\rho} a_{0\cdot}^{\rho} + \mathbf{w}^{\rho} \mathbf{a}^{\rho},$$

$$3) 1 = a_{00}^{\rho} c + \mathbf{a}_{0\cdot}^{\rho} \mathbf{y},$$

$$4)\ \mathbf{x} = a_{\cdot 0}^{\rho} c + \mathbf{a}^{\rho} \mathbf{y}$$

The cost-minimization conditions 1) and 2) imply that the same technology as that of OSS is chosen. 3) and 4) means that, under the chosen technology, the full employment conditions hold. It is not difficult to see that 3) and 4) can be summarized as Condition ii). From these conditions, it follows that $c(t) > 0$ and $\mathbf{y}(t) \gg \mathbf{0}$ hold for all t. Condition ii) exhibits the (n+1)-dimensional capital accumulation equation and determines the vector $\mathbf{z}$. ■

Note that the dynamics of the NMF is expressed by the accumulation equation in Condition ii). We will rewrite it by using the element of the inverse matrix of $\mathbf{A}^{\rho}$ as follows: first note that $\mathbf{b}^{\rho} = [\mathbf{a}^{\rho} - (1/a_{00}^{\rho}) \mathbf{a}_{\cdot 0}^{\rho} \mathbf{a}_{0\cdot}^{\rho}]^{-1}$. Solving Condition ii) with respect to $\mathbf{y}$ yields:

$$\mathbf{y} = \mathbf{b}^{\rho} \mathbf{x} + \mathbf{b}_{\cdot 0.}^{\rho}$$

Substituting this into the accumulation equation ii) and solving it with respect to $\mathbf{z}$ yields:

$$\mathbf{z} = (1/(1+g))(\mathbf{b}^{\rho} + \mathbf{I} - \Delta)\mathbf{x} - ((1/(1+g))\mathbf{b}_{\cdot 0}^{\rho}.$$

Defining $\boldsymbol{\eta}(t) = (\mathbf{x} - \mathbf{k}^\rho)$ and $\boldsymbol{\eta}(t+1) = (\mathbf{z} - \mathbf{k}^\rho)$, we will finally obtain the following difference equations, which exhibits the dynamics of the NMF:

$$\boldsymbol{\eta}(t+1) = (\frac{1}{1+g})[(\mathbf{b}^\rho)^{-1} + \mathbf{I} - \Delta]\boldsymbol{\eta}(t). \tag{15}$$

It is important to note that the dimension of the NMF could be zero. The following lemma will give us an exact order of its dimension.

**Lemma 7.** dim $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho) = n$ and $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho) \subset int\ D$ .

**Proof.** Let us define $\mathbf{d} = (d_{0,}d_1, \cdots, d_n)^t \in \mathbf{R}_+^{n+1}$ such that $\sum_{i=0}^n a_{0i}d_i = 0$ holds. Because the linear constraint must be satisfied, we can exactly choose n linearly independent vectors $\mathbf{d}^h$ $(h = 1, \cdots, n)$. It is clear that $\mathbf{d}^h$ shows a reallocation of fixed labor among sectors. Moreover, let us define the followings: for $h = 1, 2, \cdots, n$ and a positive scalar $\varepsilon_h$,

$$\begin{aligned}
\widehat{\mathbf{y}}^h &\equiv \widehat{\mathbf{y}}^\rho + \varepsilon_h \mathbf{d}^h, \\
\widehat{\mathbf{x}}^h &\equiv \mathbf{A}^\rho \widehat{\mathbf{y}}^h = \mathbf{A}^\rho \widehat{\mathbf{y}}^\rho + \varepsilon_h \mathbf{A}^\rho \mathbf{d}^h, \\
&= \widehat{\mathbf{k}}^\rho + \varepsilon_h \mathbf{A}^\rho \mathbf{d}^h,
\end{aligned}$$

and

$$
\begin{aligned}
\widehat{\mathbf{z}}^h &\equiv \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}}^h + (\mathbf{I} - \overline{\Delta})\widehat{\mathbf{x}}^h], \\
&= \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}}^\rho + (\mathbf{I} - \overline{\Delta})\widehat{\mathbf{x}}^\rho] + \varepsilon_h \overline{\mathbf{G}}^{-1}[\mathbf{d}^h + (\mathbf{I} - \overline{\Delta})\mathbf{A}^\rho \mathbf{d}^h], \\
&= \widehat{\mathbf{k}}^\rho + \varepsilon_h \overline{\mathbf{G}}^{-1}[\mathbf{I} + (\mathbf{I} - \overline{\Delta})\mathbf{A}^\rho]\mathbf{d}^h.
\end{aligned}
$$

Note that the first element of the vector $\mathbf{A}^\rho \mathbf{d}^h$ is zero due to the fact that $\sum_{i=0}^n a_{0i} d_i^h = 0$ for all $h$. Since the first element of $\widehat{\mathbf{k}}^\rho$ is 1, the first element of $\widehat{\mathbf{x}}^h$ will be also 1. So the vectors $\widehat{\mathbf{x}}^h$ $(h = 1, \cdots, n)$ are well defined. Since the first element of $\widetilde{\mathbf{k}}^\rho$ is 1, $\widetilde{\mathbf{z}}^h$ is also well defined for all $h$. Due to the fact that $\widehat{\mathbf{y}}^\rho \gg \mathbf{0}$ and $\widehat{\mathbf{k}}^\rho \gg \mathbf{0}$, $\varepsilon_h$ can be chosen so that $\widehat{\mathbf{y}}^h > \mathbf{0}$,$\widehat{\mathbf{x}}^h > \mathbf{0}$ and $\widehat{\mathbf{z}}^h > \mathbf{0}$ for all $h$. From our way of construction, the vectors $\widehat{\mathbf{y}}^h$,$\widehat{\mathbf{x}}^h$ and $\widehat{\mathbf{z}}^h$ satisfy Lemma 2 and the corresponding vector $(\mathbf{x}^h, \mathbf{z}^h)$ also belongs to $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ for all $h$. This implies that there are $n$ linearly independent vectors $(\mathbf{x}^h - \mathbf{k}^\rho, \mathbf{z}^h - \mathbf{k}^\rho)$. Therefore, there are exactly n linearly independent line segments on the NMF, $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$. In other words, the NMF has an $n$-dimensional facet (flat) containing the OSS, $(\mathbf{k}^\rho, \mathbf{k}^\rho)$. This completes the proof. ■

Using Lemma 7, we will show Lemma8.

**Lemma 8.** $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ is a continuous correspondence of $\rho \in [\overline{\rho}, 1)$.

**Proof.** See the Appendix. ■

We will introduce the following definition:

**Definition.** The NMF is *stable* if there are no cyclic paths on it.

The stability of the NMF takes very important roles in demonstrating the turnpike properties as we will see soon. Under the both GCI conditions, we actually show that the NMF is stable as we will demonstrate in the next section. Due to the continuity of the NMF, if the stability of the NNF with $\rho = 1$ would have been proved, McKenzie's neighborhood turnpike theorem could be applicable as shown in Takahashi (1985) and Takahashi (1992). Finally we will demonstrate the following theorem:

**Theorem 1.** (Neighborhood Turnpike Theorem) *Provided that the NMF is stable. Then for any $\varepsilon>0$, there exists a $\overline{\rho}>0$ such that for $\rho \in [\overline{\rho}, 1)$* and the corresponding $\varepsilon(\rho)$, *any optimal path $\{\mathbf{k}_t^\rho\}^\infty$ with a sufficient initial capital stock $\mathbf{k}(0)$ eventually lies in the $\varepsilon-$neighborhood of $\mathbf{k}^\rho$*. Furthermore, as $\rho \to 1$, $\varepsilon(\rho) \to 0$.

**Proof.** See the argument of Section 4 of Takahashi (1993). ■

The neighborhood turnpike theorem means that any optimal path must be trapped in a neighborhood of the corresponding OSS and the neighborhood can be taken as small as possible by making $\rho$ sufficiently close to 1.

## 5 Turnpike Theorem

The complete turnpike theorem is described as the following theorem:

**Complete Turnpike Theorem** There is a $\overline{\rho} > 0$ close enough to 1 such that for any $\rho \in [\overline{\rho}, 1)$, an optimal path $\mathbf{k}^{\rho}(t)$ with the sufficient initial capital stock will asymptotically converge to the optimal steady state $\mathbf{k}^{\rho}$.

To show the complete turnpike theorem we need to strengthen the generalized capital intensity conditions: GCI-I and GCI-II.

**Remark 5** *The first to be noted that in the efficiency unit term, the complete turnpike implies that each sector's optimal path converges to its own optimal steady state. It follows that in original terms of series, industry's per-capita capital stock and output grow at the rate of industry's TFP. Thus our original purpose will be accomplished by demonstrating the complete turnpike theorem.*

Recall that as we have shown, under Assumption 7, the dimension of the NMF is n. Furthermore, the dynamics of the NMF is expressed by the n-dimensional linear difference equation (15). We use the following property to show the stability of the NMF.

**Lemma 9.** Let us consider the following difference equation system with the equilibrium $\mathbf{x}_e = 0$,

$$\mathbf{x}(t+1) = (\mathbf{C} + \mathbf{I})\mathbf{x}(t),$$

where $\mathbf{x}(t) \in R^n$ and $\mathbf{C}$ is an $n \times n$ matrix. If $\mathbf{C}$ has the negative d.d. for rows, $\mathbf{C} + \mathbf{I}$ is a contraction for $\mathbf{x}(t) \neq 0$ with the maximum norm $\| \cdot \|$, i.e., and the equation system is globally asymptotically stable and the Liapunov function is $\mathbf{V}(\mathbf{x}) = \|\mathbf{x}\|$, where $\|\cdot\|$ is defined as $\|\mathbf{x}\| = \max_i c_i \mid x_i \mid$ and $c_i$ is a given set of positive numbers. Furthermore, if $\mathbf{C}$ has the positive d.d. for rows, $\mathbf{C}+\mathbf{I}$ exhibits total explosiveness for $\mathbf{x}(t) \neq 0$.

**Proof.** The first part comes from the result in Neumann (1961,pp.27-29). On the contrary, if $\mathbf{C}$ has the negative d.d. for rows, $\mathbf{C}+\mathbf{I}$ has eigenvalues with their absolute values greater than one. This comes from the fact that if $\mathbf{C}$ has the positive d.d. for rows, then its eigenvalues have a positive real part. Thus the system is explosive; any path will diverge from equilibrium. ■

We may use the second property later. First, we will prove the following theorem:

**Lemma 10.** Under the negative (positive) d.d., the n-dimensional NMF: $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ where $\rho \in [\overline{\rho}, 1)$ turns out to be a linear stable (unstable) manifold.

**Proof.** Because $\mathbf{b}^\rho + \mathbf{I} - \Delta = [\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+g)\mathbf{G}] + (1+n)\mathbf{G}$, it follows that $(1/(1+g)\mathbf{G}[\mathbf{b}^\rho + \mathbf{I} - \Delta] = (1/(1+g))\mathbf{G}[\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+n)\mathbf{G}] + \mathbf{I}$. Defining $\mathbf{C} = (1/(1+g))\mathbf{G}[\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+g)\mathbf{G}]$, Eq.(8) can be rewritten as:

$$\boldsymbol{\eta}(t+1) = (\mathbf{C} + \mathbf{I})\boldsymbol{\eta}(t).$$

Note again that $\boldsymbol{\eta}(t) = (\mathbf{x} - \mathbf{k}^\rho)$ and $\boldsymbol{\eta}(t+1) = (\mathbf{z} - \mathbf{k}^\rho)$. Thus applying Lemma 4, under the negative d.d. (the positive d.d.), any path on NMF will converge to (diverge from) the OSS. ■

From this lemma, under the Strong GCI-II condition, the local stability and the stability of the NMF hold simultaneously. Furthermore, the stability of the NMF will establish the neighborhood turnpike. Thus combining both results follows the complete turnpike theorem.

**Corollary.** Under the Strong GCI-II condition, the complete turnpike theorem will be established.

**Proof.** To achieve the complete turnpike theorem, we need to combine the neighborhood turnpike theorem and the local stability of the OSS. The neighborhood turnpike theorem means that any optimal path should be trapped in the neighborhood of the OSS by choosing the discount rate properly. Therefore when the local stability also holds in the neighborhood of OSS, the optimal path must jump on the stable manifold, here the NMF itself, and will converge to the OSS. Thus the complete turnpike theorem has been established. ■

On the other hand, to show the local stability under the GCI-I condition, we need to utilize the following well-known lemma by Levhari and Liviatan (1972):

**Lemma 11.** Provided that det. $\mathbf{V}^{\rho}_{xz} \neq \mathbf{0}$, if the following characteristic equation, given by expanding the Euler equation around the

$$\left|\mathbf{V}^{\rho}_{xz}\lambda^2 + (\mathbf{V}^{\rho}_{xx} + \mathbf{V}^{\rho}_{zz})\lambda + \mathbf{V}^{\rho}_{zx}\right| = 0. \tag{16}$$

OSS, has $\lambda$ as a root, then it also has $1/(\rho\lambda)$.

**Proof.** See Levhari and Liviatan (1972). ■

**Lemma 12.** Under the GCI-I condition, the OSS satisfies the local stability.

**Proof.** All we need to show is that det. $\mathbf{V}^{\rho}_{xz} \neq \mathbf{0}$ under the GCI-I condition due to Lemma 11. From the fact that $\mathbf{V}_x(\mathbf{k}^{\rho}, \mathbf{k}^{\rho}) = \mathbf{p}^{\rho}(\mathbf{I} - \Delta) + \mathbf{w}^{\rho}$ (see Benhabib and Nishimura (1985) for a two-sector case) and Lemma 4, we may show that

$$\mathbf{V}^{\rho}_{xz} = -(\mathbf{b}^{\rho})^{-1}[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)]\mathbf{T}^{\rho}_{22}(\mathbf{b}^{\rho})^{-1}$$

where $\mathbf{T}^{\rho}_{22} = [\partial^2 T(\mathbf{y}^{\rho}, \mathbf{k}^{\rho})/\partial \mathbf{k}^2]$. Then,

$$\det .\mathbf{V}^{\rho}_{xz} = -(\det(\mathbf{b}^{\rho})^{-1})^2 \det[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)] \det \mathbf{T}^{\rho}_{22}$$

Since $\mathbf{T}^{\rho}_{22}$ is negative-definite, it is non-singular. Furthermore, $[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)]$ has a quasi-dominant diagonal that is positive under the GCI-I condition; it is non-singular too. It follows that $\mathbf{V}^{\rho}_{xz}$ is non-singular. Furthermore, the GCI-I condition implies that the NMF is explosive; there are $n$ characteristic roots with absolute value greater than one. Applying Lemma 11 will yield that there are $n$ characteristic roots with its absolute value less than one. Thus the OSS satisfies the local stability. ■

We have finally established the following theorem:

**Theorem 2** Under the both GCI conditions, the OSS $\mathbf{k}^{\rho}$ exhibits the complete turnpike property.

**Proof.** Under the GCI-II condition, the complete turnpike theorem will be established due to the above corollary. Under the GCI-I condition, from Lemma 12, the OSS will exhibit the local stability. Since any path on the NMF is totally unstable, the NMF is "stable." Hence the neighborhood turnpike theorem hold. Combining both results, the complete turnpike theorem is established too. This completes the proof. ■

## 6 Concluding Remarks

We have demonstrated turnpike property under two types of generalized capital intensity conditions. As I mentioned before, the complete turnpike property means that each industry's per capita capital stock and output converge to the industry-specific optimal steady state paths with the rate of technical progress determined by industry's TFP. It means that, the per-capita capital stock of the agriculture industry grows at its own rate of technical progress along its optimal steady state and another industry, say the manufacturing industry grows at its own rate of technical progress along its own optimal steady state. A similar explanation can be applicable to other industries. Therefore, our established theoretical results are consistent with the evidence obtained in recent empirical research.

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## APPENDIX

We will prove Lemma 8 here.

**Proof.** Since the NMF is upper-semi continuous from its definition, all I need to establish is that $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho) = \mathbf{F}(\mathbf{k}^\rho)$ is lower-semi continuous (l.s.c.) for any $\rho \in [\overline{\rho}, 1)$. Under Assumption 7, we can choose $n$ labor-redistribution vectors $\mathbf{d}^h$ as follows:

$$\mathbf{d}^h = (-1, -1, -1, \cdots, -1, \sum_{i=0}^{n} a_{0i}^\rho / a_{0h}^\rho, -1, \cdots, -1) \ (h = 0, 1, \cdots, n).$$

This means that each producing sector transfers one unit of labor to the $h^{th}$ sector. From Lemma 4, $\mathbf{d}^h$ is a continuous vector-function of $\rho$ in $\rho \in [\overline{\rho}, 1)$ and that $\sum_{i=0}^{n} a_{0i}^\rho d_i^h = 0$ for all $h$. It follows that there are $n$ linearly independent redistribution vector $\mathbf{d}^h$. Let us denote these redistribution vector as $\mathbf{d}^h$ $(h = 1, \cdots, n)$. Henceforth, we may use the notation: $\mathbf{k}(\rho), \mathbf{d}^h(\rho)$ $(h = 1, \cdots, n)$ and $\mathbf{F}(\mathbf{k}(\rho))$ for denoting OSS, redistribution vectors and the NMF respectively. Using $\mathbf{d}^h(\rho)$ and from Lemma 6, we can define the following $n$ linearly independent vectors for each $h$ $(h = 1, \cdots, n)$: $\widehat{\mathbf{x}}^h(\rho) = \mathbf{k}(\rho) + \boldsymbol{\varepsilon}_h \mathbf{A}(\rho)\mathbf{d}^h(\rho)$ and $\widehat{\mathbf{z}}^h(\rho) = \mathbf{k}(\rho) + \boldsymbol{\varepsilon}_h \overline{\mathbf{G}}^{-1}[\mathbf{I} + (\mathbf{I} - \overline{\Delta})\mathbf{A}(\rho)]\mathbf{d}^h(\rho)$ where $\boldsymbol{\varepsilon}_h$ is chosen so that $\widehat{\mathbf{x}}^h(\rho) >> \mathbf{0}$ and $\widehat{\mathbf{z}}^h(\rho) >> \mathbf{0}$ $(h = 1, \cdots, n)$. Let us arbitrarily pick up a point $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}(\rho))$. Pick up also another point $(\mathbf{x}', \mathbf{z}') \in \mathbf{F}(\mathbf{k}(\rho'))$ such that $(\mathbf{x}', \mathbf{z}') \in int.\ D$ , $(\mathbf{x}, \mathbf{z}) \neq (\mathbf{x}', \mathbf{z}')$ and choosing $\rho' \in [\overline{\rho}, 1)$ sufficiently close to $\rho$. Now let us define the plain $\mathbf{H}^\alpha$ as follows:

$$\mathbf{H}^\alpha \equiv \{(\mathbf{x}, \mathbf{z}) \in D : t_0[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho), \alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)] +$$

$$\sum_{h=1}^{n} t_h[\alpha\mathbf{x}^j(\rho') + (1-\alpha)\mathbf{x}^j(\rho), \alpha\mathbf{z}^j(\rho') + (1-\alpha)\mathbf{z}^j(\rho)]\} \ where \ \sum_{h=1}^{n} t_h = 1.$$

We can always find an intersection $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ between $\mathbf{H}^\alpha$ and the line obtained by connecting between $(\mathbf{x}, \mathbf{z})$ and $(\mathbf{x}', \mathbf{z}')$, unless $(\mathbf{x}, \mathbf{z}) = (\mathbf{x}', \mathbf{z}')$. Since $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ is on the plain $\mathbf{H}^\alpha$, it can be expressed as follows:

$$(\mathbf{x}^\alpha, \mathbf{z}^\alpha) = t_0^\alpha[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho), \alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)] +$$

$$\sum_{h=1}^{n} t_h^\alpha[\alpha\mathbf{x}^j(\rho') + (1-\alpha)\mathbf{x}^j(\rho), \alpha\mathbf{z}^j(\rho') + (1-\alpha)\mathbf{z}^j(\rho)] \ where \ \sum_{h=1}^{n} t_h^\alpha = 1$$

where $\alpha \to 0$, $[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)] \to \mathbf{k}(\rho)$ and $[\alpha\mathbf{x}^h(\rho') + (1-\alpha)\mathbf{x}^h(\rho), \alpha\mathbf{z}^h(\rho') + (1-\alpha)\mathbf{z}^h(\rho)] \to (\mathbf{x}^\rho, \mathbf{z}^\rho)$ for $h = 1, \cdots, n$. Thus, $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ converges to $(\mathbf{x}, \mathbf{z})$ as $\alpha \to 0$. Also note that $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \in int.\ D$ due to the convexity of $D$ and the fact that $(\mathbf{x}', \mathbf{z}') \in int.\ D$. Because of the continuity of $\mathbf{k}(\rho)$, $\mathbf{x}(\rho)$ and $\mathbf{z}(\rho)$ in $\rho \in [\overline{\rho}, 1)$, for any $\varepsilon^\alpha > 0$ there exists $\delta^\alpha > 0$ such that $|\rho^\alpha - \rho| < \delta^\alpha$ implies that

$$\|\mathbf{k}(\rho^\alpha) - [\alpha\mathbf{k}(\rho') + (1-\alpha)\mathbf{k}(\rho)]\| < \varepsilon^\alpha$$

and for $h = 1, \cdots, n$,

$$\left\|(\mathbf{k}(\rho^\alpha), \mathbf{k}(\rho^\alpha)) - [\alpha\mathbf{x}^h(\rho') + (1-\alpha)\mathbf{x}^h(\rho), \alpha\mathbf{z}^h(\rho') + (1-\alpha)\mathbf{z}^h(\rho)]\right\| < \varepsilon^\alpha$$

where $\rho^\alpha = \alpha\rho' + (1-\alpha)\rho$ and $\| \bullet \|$ is the Euclidean norm. Furthermore as $\alpha \to 0$, $\varepsilon^\alpha \to 0$. Now let us also define a point $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha)$ as follows:

$$(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha) = t_0^\alpha(\mathbf{k}(\rho^\alpha), \mathbf{k}(\rho^\alpha)) + \sum_{h=1}^{n} t_h^\alpha(\mathbf{x}^j(\rho^\alpha), \mathbf{z}^j(\rho^\alpha))$$

where $(t_0^\alpha, t_1^\alpha, \cdots, \cdots, t_n^\alpha)$ is used to define $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha)$. By choosing $\alpha$ sufficiently close to zero, we can make $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha)$ in the interior of D as well as sufficiently close to $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$. Therefore $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha)$ is feasible. Since it is also expressed as the linear combination of $(n-1)$ linearly independent vectors $\mathbf{x}^h$,$\mathbf{y}^h$ and $\mathbf{z}^h$, it follows that $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha) \in \mathbf{F}(\mathbf{k}(\rho^\alpha))$. Also note that due to our way of construction of $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha)$, as $\alpha \to 0$, $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha) \to (\mathbf{x}^\alpha, \mathbf{z}^\alpha)$. Now make $\alpha$ converge to zero, then $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \to (\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ and $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \to (\mathbf{x}, \mathbf{z})$. Thus $(\overline{\mathbf{x}}^\alpha, \overline{\mathbf{z}}^\alpha) \in \mathbf{F}(\mathbf{k}(\rho^\alpha))$ converges to $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}(\rho))$. Therefore, $\mathbf{F}(\mathbf{k}(\rho))$ is l.s.c. at $\rho \in [\overline{\rho}, 1)$. Apply the same arguments to any point of $\mathbf{F}(\mathbf{k}(\rho))$ and to any $\rho \in [\overline{\rho}, 1)$, it follows that $\mathbf{F}(\mathbf{k}(\rho))$ is l.s.c. for any $\rho \in [\overline{\rho}, 1)$. ■

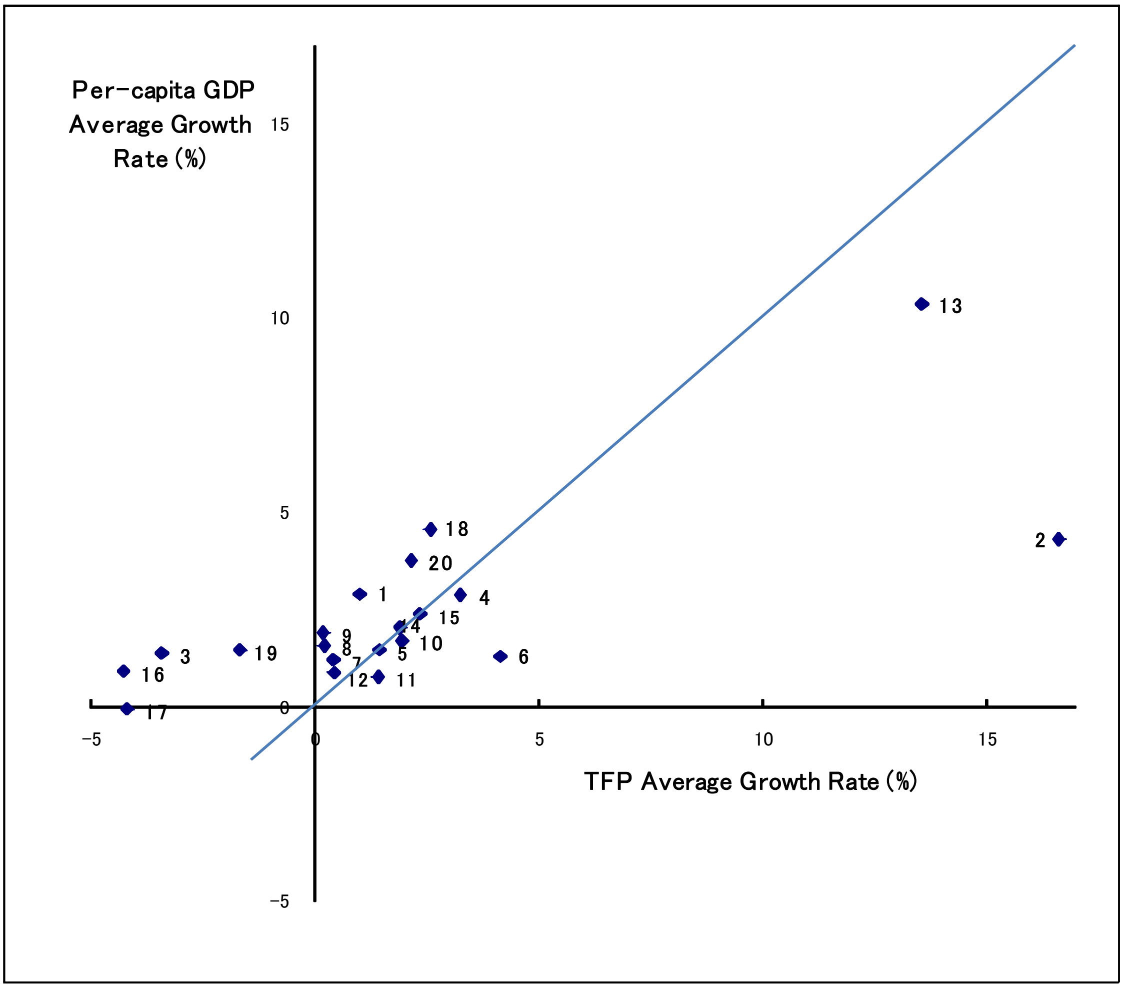


Figure 1: U.S. Economy, 1970-2005 Source: EU-KLEMS DATABASE

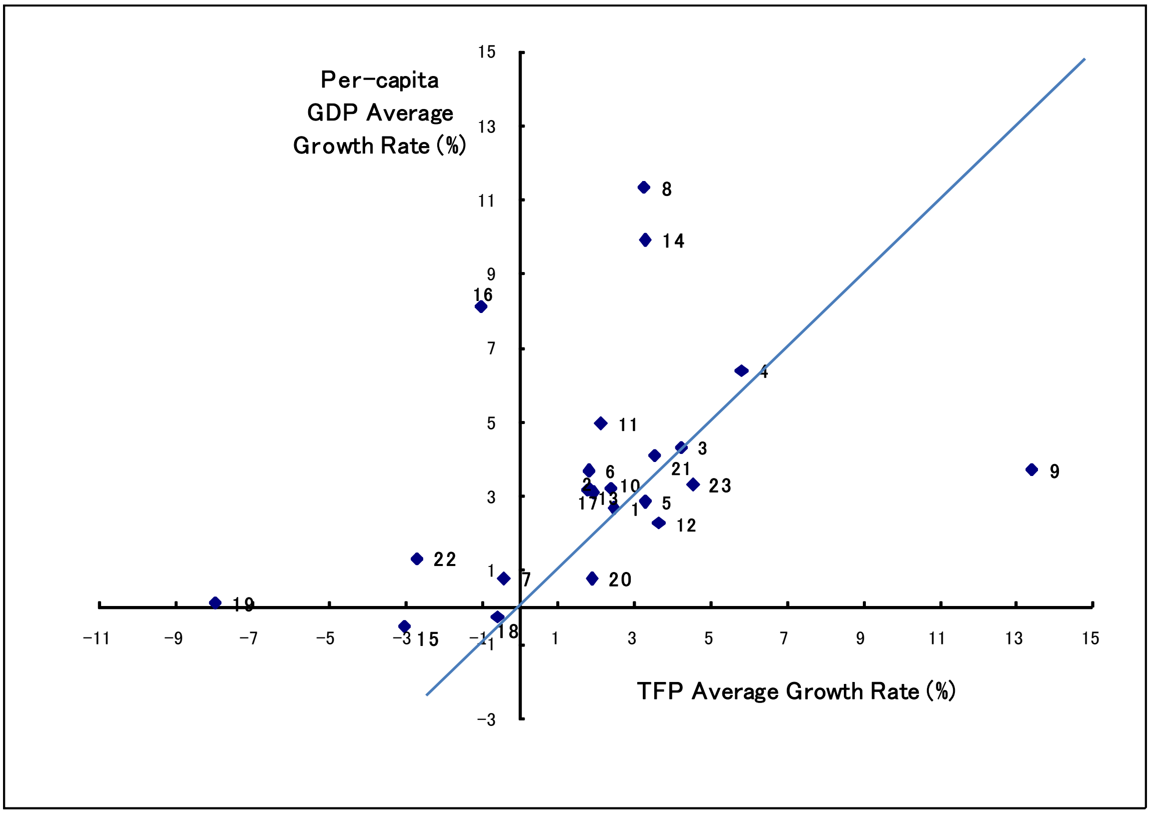


Figure 2: Japanese Economy, 1970-2005 Source: EU-KLEMS DATABASE