Significance of the characteristic roots of linearized econometric models

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1. Introduction

The stability of a dynamic econometric model depends on the roots of the associated characteristic equation.

As the structural estimated coefficients of the model are subject to sampling errors, also the characteristic roots are affected by an error.

The estimation of the asymptotic standard errors of the characteristic roots in linear econometric models has been dealt with by Theil and Boot (1962) (whose numerical results for the Klein-I model have been recently revised by Bianchi, Calzolari and Corsi, 1980), Neudecker and van de Panne (1966), Oberhofer and Kmenta (1973) and Schmidt (1974). By means of these asymptotic standard errors, it is possible to test the model for

stability (however, the power of the test is open to question, because the null hypothesis must be always stability, rather than instability, as well pointed out in Oberhofer and Kmenta, 1973, fn.5 and Gustafson, 1978). Unfortunately, all the procedures proposed to derive the asymptotic standard errors of the characteristic roots seem to have some drawbacks, which can be summarized as follows.

1) They are confined to linear models.
2) To operate "analytically" as long as possible, they generally involve the use of large scale (sparse) matrices even for small-medium size models.
3) In case of nonlinear models, an explicit linearization must be preliminarily performed and explicit values must be obtained for the variances-covariances of the linearized model's parameters, which are time-varying.

These difficulties, together with the consideration that all the "analytical" methods must in any case resort to numerical techniques to solve the characteristic equation, suggested to implement a purely numerical (simulation) method, whose description is the main subject of this paper.

1) The method works in the same way both on linear and nonlinear models; model's linearization, in the neighborhood of a given point, is performed only implicitly and reference is made only to the estimated parameters of the structural form.
2) The procedure does not involve the use of large sparse matrices.
3) No special input format is requested for the model; it can be

simply written in the Gauss-Seidel format usually adopted by most of the builders of medium-large scale models. The possibility of computational errors is, therefore, drastically reduced.

4) The experiments performed on several models show that the computation is very fast for small models and reasonably fast for medium size models; with higher cost, the computation is still feasible for moderately large models (for which, even if linear, the analytical methods would probably lead to untreatable dimensions).

In section 2 the problem of the characteristic roots and associated asymptotic standard errors in linear models is briefly recalled. In section 3 the method to deal with local linearizations of nonlinear models is briefly described; an outline of a flow-chart is given in section 4. Finally, in section 5, numerical results are displayed for some models of different sizes.

2. Linear dynamic models

Let a linear dynamic econometric model be represented, in its structural form, as

$$Ay_t + Bx_t + Cy_{t-1} = u_t \qquad t=1,2,\dots,T$$

where y_t is the $(m \times 1)$ vector of the endogenous variables at time t, x_t is the $(n \times 1)$ vector of the exogenous variables, y_{t-1} is the vector of the endogenous variables lagged one period, A, B and C are the $(m \times m)$, $(m \times n)$ and $(m \times m)$ matrices of the

structural coefficients (including restrictions) and u_t is the $(m \times 1)$ vector of the structural random disturbances (including zeroes for the nonstochastic definitional equations). Defining

$$\Pi_0 = -A^{-1}C, \quad \Pi_1 = -A^{-1}B, \quad v_t = A^{-1}u_t$$

the restricted reduced form is

$$y_t = \Pi_1 x_t + \Pi_0 y_{t-1} + v_t .$$

Models with endogenous variables lagged more than one period can be reconducted to the above scheme simply by adding definitional (nonstochastic) equations (a simple matrix formulation can be found in Dhrymes, 1978, p.128).

The dynamic behavior (and stability) of the model depends on the characteristic roots of the $(m \times m)$ matrix Π_0. Let a consistent and asymptotically normally distributed estimate of the subset of reduced form coefficients which form Π_0 be available, so that

$$\sqrt{T}\,(\mathrm{vec}\,\hat{\Pi}_0 - \mathrm{vec}\,\Pi_0) \xrightarrow[T \to \infty]{} N\,(0,\Omega).$$

Let also a consistent estimate of Ω be available (for example, with the method by Goldberger, Nagar and Odeh, 1961). Then the asymptotic variance of a characteristic root can be obtained via computation of the partial derivatives of such a root with respect to the elements of Π_0. This computation is, in principle, rather simple, each derivative being equal to the product of elements of the eigenvector corresponding to the given root (this is basically the method by Theil and Boot, 1962 and by Neudecker and van de Panne, 1966).

A difficulty, however, arises in practice even for small-medium size models, as the dimensions of the matrix Ω may be extremely large; its dimensions are, in fact, $(m^2 \times m^2)$, where m is the number of endogenous variables of the "first order" system derived from the model, that is the one obtained by adding equations due to possible presence of some endogenous variables lagged more than one period.

The dimensional problems involved in the method by Oberhofer and Kmenta (1973) seem to be similar, as this method deals with the asymptotic covariance matrix of

$$\sqrt{T}\begin{bmatrix}\text{vec } \hat{A} - \text{vec } A \\ \text{vec } \hat{C} - \text{vec } C\end{bmatrix}$$

whose dimensions are $(2m^2 \times 2m^2)$.

However, most of the elements of $\hat{A}$ and $\hat{C}$ are generally restricted a-priori (for example zeroes and ones) so that the estimated coefficients can be collected into a vector $\hat{a}$ whose dimension is generally much smaller than $2m^2$. If the model is linear, it can be possible to maintain the correspondence between the elements of $\hat{a}$ and the corresponding sparse elements of $\hat{A}$ and $\hat{C}$, so that the dimensions of the problem can be reconducted to those of the asymptotic covariance matrix of $\sqrt{T}(\hat{a}-a)$.

If the model is nonlinear and has been preliminarily linearized, it seems difficult to treat in a simple way the correspondence between the elements of the vector $\hat{a}$ of the estimated nonlinear structural form and the elements of $\hat{A}$ and $\hat{C}$ in the linearized model (due to nonlinearities, $\hat{A}$ and $\hat{C}$ may be time varying and may involve nonlinear transformations of the elements of $\hat{a}$).

The method described in section 4 is close to the methods by Theil and Boot (1962) and by Neudecker and van de Panne (1966), but it avoids to explicitly compute (and store) the large covariance matrix of the elements of $\hat{\Pi}_0$.

3. Nonlinear dynamic models

Let us represent a structural nonlinear econometric model by

$$y_t = f(y_t, x_t, y_{t-1}, a) + u_t \qquad t = 1,2,\ldots,T$$

where:

f is a column vector of functional operators (f_i, $i=1,2,\ldots,m$), continuous and differentiable with respect to the elements of y_t, x_t, y_{t-1} and a, with continuous derivatives up to the second order;

y_t, x_t and y_{t-1} are the column vectors of endogenous, exogenous and lagged endogenous variables at time $t = 1,2,\ldots,T$ ($y_{i,t}$, $i=1,2,\ldots,m$; $x_{j,t}$, $j=1,2,\ldots,n$)

a is the column vector (a_k, $k=1,2,\ldots,s$) of structural coefficients to be estimated (the other known coefficients of the model being included in the functional operators);

u_t is the column vector of structural stochastic disturbances at time t ($u_{i,t}$, $i=1,2,\ldots,m$).

We assume existence of a vector $\hat{a}$ of consistent estimates of a, the asymptotic normality of $\sqrt{T}(\hat{a}-a) \sim N(0,\Psi)$ and the availability of a consistent estimate of Ψ, $\hat{\Psi}$.

The restricted reduced form of the model will be indicated as

$$y_t = g(x_t, y_{t-1}, a, u_t),$$

where the vector g of functional operators is, of course, generally unknown if the model is nonlinear.

The dynamic behavior (and stability) of a local linearization of this model at time t is determined by the characteristic roots of the (m x m) matrix $\Pi_{0.t}$ ($\hat{\Pi}_{0.t}$ is its estimate) of partial derivatives, in the neighborhood of the solution point at time t, of y_t with respect to y_{t-1}.

Let λ_t be a real characteristic root of $\Pi_{0.t}$ and $\hat{\lambda}_t$ the corresponding characteristic root of $\hat{\Pi}_{0.t}$. Under the assumptions of continuity and differentiability of the functions involved in the structural form, since $\sqrt{T}(\hat{a}-a)$ is asymptotically normally distributed as $N(0,\Psi)$, then $\sqrt{T}(\hat{\lambda}_t-\lambda_t)$ is asymptotically normally distributed as $N(0,j_t'\Psi j_t)$, where j_t is the vector of partial derivatives of λ_t with respect to the elements of a (Rao, 1965,p.321). If the computation is performed for $\hat{\lambda}_t$, through the matrix $\hat{\Pi}_{0.t}$, we get a consistent estimate $\hat{j}_t$ of j_t and the square root of $(\hat{j}_t'\hat{\Psi}\hat{j}_t)/T$ is the estimated asymptotic standard error of the given root.

If λ_t is complex, the above results hold both for the modulus and for the argument (or for the corresponding period); in this case we have to compute two vectors $\hat{p}_t$ and $\hat{q}_t$, partial derivatives of the modulus and of the argument, respectively.

4. The adopted simulation procedure

The $\hat{j}_t$ (or $\hat{p}_t$ and $\hat{q}_t$) vectors for all the characteristic roots can be computed using a simulation method, which makes no difference between linear and nonlinear models; the method is based on numerical differentiation of the characteristic roots of the matrix $\hat{\Pi}_{0.t}$, with respect to the structural estimated coefficients, where the matrix $\hat{\Pi}_{0.t}$ is repeatedly constructed using numerical differentiation of the endogenous variables with respect to the lagged endogenous. The procedure can be summarized as follows.

1) A scan is preliminarily performed on the model to determine which are the lagged endogenous variables actually present and the maximum lag of each of them.
2) The model is numerically solved at time t.
3) An increment is given to the value of a lagged endogenous variable, the model is again solved at time t and the partial derivatives of y_t with respect to the given lagged endogenous are numerically computed. This step is repeated for all the lagged endogenous variables and the computed derivatives are stored into a matrix ($\hat{\Pi}_{0.t}$).
4) The characteristic roots of the matrix are computed and stored; if they are complex, we compute and store moduli and arguments (or periods).
5) An increment is given to one structural estimated coefficient.
6) The process is repeated from step 2 to step 5 as many times as the number of structural estimated coefficients (a_k, k=1,2,...,s).
7) The partial derivatives of each root, with respect to the structural coefficients, are then computed, stored into the

vector $\hat{j}_1$ (or the two vectors $\hat{p}_1$ and $\hat{q}_1$, if the root is complex) and the asymptotic standard error of the root is finally obtained.

Some care must be taken of the choice of the increments to be given to the lagged endogenous variables and to the coefficients to compute the derivatives. In all the models on which these experiments have been performed, relative increments in the range 0.001-0.000001 have always led to results equal at least in the first 2-3 significant digits (quite enough for a standard error). Of course, to appreciate these small increments, the tolerance at convergence in the iterative Gauss-Seidel method had to be much smaller than the one usually adopted by econometricians for the solution of simultaneous systems; 10^{-13} has been used for these experiments.

5. Results on some models

As already mentioned in the introduction, the purpose of this paper is not to give any judgement on the stability of the various models analyzed nor on the significance of their characteristic roots, but rather to describe in some details a simulation method and its computational performances.

Therefore, no comments will be made on the results displayed in this section, like unreasonably long periods of complex roots, or differences among periods associated with characteristic roots obtained for alternative estimation methods, or characteristic roots greater than 1.

5.1. The Klein-1 model

The model consists of three stochastic plus three definitiona equations; 12 are the estimated coefficients, 4 for eac equation. 3 are the lagged endogenous variables, all with la 1. One of the three nonzero roots is real, the other two ar conjugate complex; the table below displays, for severa estimation methods (some of these results have been presented i Bianchi, Calzolari and Corsi, 1980), the real root and th modulus and period corresponding to the conjugate couple with, i parentheses, their standard errors. For each estimation method the computation has required approximately two seconds of C time on a computer IBM/370 model 158.

Estimate	Real root	Complex pair	
		Modulus	Peri
2SLS	.297 (.075)	.846 (.063)	14. (2.0
3SLS	.344 (.060)	.871 (.057)	13. (1.6
LIVE	.318 (.072)	.847 (.063)	15. (2.4
FIVE	.367 (.063)	.870 (.069)	14. (2.1
FID	.334 (.068)	.838 (.059)	14. (1.7
LIVE iterat.	.317 (.072)	.847 (.060)	15. (2.2
FIVE iterat.	.423 (.062)	.761 (.078)	34. (26.
FIML	.423 (.069)	.761 (.127)	34. (66.

5.2. The Klein-Goldberger model

The model on which the experiments have been performed is the revised version described in Klein (1969). It is nonlinear and consists of sixteen stochastic and four definitional equations. It includes 54 estimated coefficient and 18 lagged endogenous variables with maximum lag of two years. Estimation has been performed by means of 2SLS with 4 principal components, as in Klein (1969). The derivatives with respect to the lagged endogenous variables are computed in the solution point at 1964 (last year of the sample period).

For brevity's sake, the table below displays only the two largest real roots and the two largest (in modulus) couples of complex roots with standard errors.

The computation has required approximately one minute of CPU time on a computer IBM/370 model 158.

Estimate	Real roots	Complex pairs Modulus	Complex pairs Period
2SLS - 4 PC	1.15 (.043)	.389 (.392)	87.8 (428.)
	1.01 (.020)	.226 (.689)	6.13 (8.91)

5.3. The ISPE model of the Italian economy

The nonlinear model analyzed in this section is an annual model of the Italian economy developed by a team led by ISPE (Istituto Studi Programmazione Economica). The model, originally described in Sartori (1978), has been reestimated for the period 1955-1976 using 2SLS with principal components, according to method 4 by Kloek and Mennes (1960). It consists of 19 stochastic plus 15 definitional equations; 75 are the estimated coefficients and 18 are the lagged endogenous variables, with a maximum lag of 2 years. The derivatives with respect to the lagged endogenous variables are computed in the solution point at 1976 (last year of the sample period).

The table below displays the three largest real roots and the three largest pairs of complex roots with standard errors in parentheses.

The computation has required 6 minutes of CPU time on a computer IBM/370 model 158.

Estimate	Real roots	Complex pairs Modulus	Complex pairs Period
2SLS - PC	1.21 (.012)	.932 (.116)	28.5 (20.4)
	1.20 (.014)	.411 (.079)	4.42 (.352)
	.751 (.164)	.100 (.073)	4.96 (.262)

5.4. The IBM model of United Kingdom

The model analyzed in this section is the model of United Kingdom developed by the IBM CHQ Economics. It is a quarterly model, with 120 equations, 21 of which stochastic and 32 exogenous variables; the structural estimated coefficients are 68 and the lagged endogenous variables are 33, with a maximum lag of 4 quarters. The sample period starts from 1956/2 to 1969/1 and always ends at 1975/4. Estimation has been performed by means of the Limited information Instrumental Variables Efficient method (LIVE), described in Brundy and Jorgenson (1971). The derivatives with respect to the lagged endogenous variables are computed in the solution point at 1975/4 (last quarter of the sample period).

The table below displays the five largest real roots and the five largest pairs of complex roots with standard errors in parentheses.

The computation has required approximately one hour of CPU time on a computer IBM/370 model 158.

Estimate	Real roots	Complex pairs	
		Modulus	Period
LIVE	1.06 (.007)	1.04 (.003)	3602. (7156.)
	1.05 (.003)	.808 (.147)	414.2 (6213.)
	1.04 (.002)	.762 (.078)	25.3 (8.91)
	1.03 (.004)	.659 (.002)	4.30 (.004)
	1.01 (.001)	.657 (.001)	4.30 (.002)

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