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by

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1. **Introduction**.

The proof of the existence of the Standard commodity contained in Sraffa's book (section 37) has been debated recently. Lippi (2008) has argued that the algorithm in section 37 of Sraffa's book is not precisely stated and does not need to converge to the desired eigenvalue and eigenvector. The first part of the proposition has been known since the proof-reading stage of Sraffa's book when it was sustained by Alister Watson (cf. Kurz and Salvadori, 2001, p. 272-3). But the second part escaped the attention of all commentators before Lippi. Indeed, examples can be found in which an algorithm corresponding to the description provided by Sraffa converges to a vector which is not an eigenvector and it is certainly to Lippi's credit that he uncovered the problem. In Appendix A I report the example provided in a paper in which I investigated the properties that an algorithm needs to have in order to converge to the desired eigenvalue and eigenvector (cf. Salvadori, 2008). In an appendix to his paper Lippi (2008) provided a complete proof of the existence of the Standard commodity by using a very special algorithm from among all the algorithms corresponding to the description of section 37 and another special algorithm was provided, without proof, by Kurz and Salvadori (2001, p. 284). The fact that Sraffa did not choose a particular algorithm may suggest that he was convinced that any algorithm would do the job. This is wrong but, as I proved elsewhere (Salvadori, 2008), the job can actually be done by any algorithm based on a continuous function, which can start from any feasible point.

In this paper I want to shed some more light on the issue from an historical perspective. Sraffa was provided a proof of the existence of the Standard commodity by Besicovitch on 21 September 1944. This proof has not yet been discussed in the literature. In Appendix B there is a transcription of the file D3/12/39: 42 that includes it. In this paper I will show that also the proof by Besicovitch is incomplete, but it can easily be completed. Once complete, this proof concerns a family of algorithms as well, but all the algorithms in question converge to the desired eigenvalue and eigenvector. Why did Sraffa not use this proof in his book? Section 5 tries to provide an answer.

2. **Sraffa's Section 37**

Sraffa starts section 37 of his book with the following two paragraphs.

> That any actual economic system of the type we have been considering can always be transformed into a Standard system may be shown by an imaginary experiment. (The experiment involves two types of alternating steps. One type consists in changing the proportions of the industries; the other in reducing in the same ratio the quantities produced by all industries, while leaving unchanged the quantities used as means of production.)

What Sraffa calls an "imaginary experiment" is clearly what mathematicians call an algorithm: given an initial state, a definite list of well-defined instructions is given to proceed through a well-defined sequence of successive states, eventually terminating in an end-state. In order to formally reconstruct Sraffa's argument, let us introduce the square nonnegative matrix $\mathbf{A} = \left[a_{ij}\right]$ and the positive vector $\mathbf{l} = \left[l_1, l_2, ..., l_n\right]^T$ as the material input matrix and the labor input vector, on the assumption that the output matrix is the identity matrix $\mathbf{I}$. Matrix $\mathbf{A}$ is assumed to be also indecomposable, that is, all non-basic commodities are explicitly not considered. Let us continue our reading of section 37.

> We start by adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement.
>
> Let us next imagine gradually to reduce by means of successive small proportionate cuts the product of all the industries, without interfering with the quantities of labour and means of production that they employ.
>
> As soon as the cuts reduce the production of any one commodity to the minimum level required for replacement, we readjust the proportions of the industries so that there should again be a surplus of each product (while keeping constant the quantity of labour employed in the aggregate).

The initial state of the algorithm is the "actual economic system". This is able to produce a surplus, but does not need to produce a surplus consisting of all (basic) commodities, so the first step consists in determining $\mathbf{x}_0 \in \left\{\mathbf{x} > \mathbf{0} \middle| \mathbf{x}^T\mathbf{l} = \beta, \mathbf{x}^T\left[\mathbf{I} - \mathbf{A}\right] > \mathbf{0}^T\right\}$ and then building up two sequences: $\left\{\mathbf{x}_t\right\}$ and $\left\{\lambda_t\right\}$, where

$$\lambda_t = \lambda\left(\mathbf{x}_{t-1}\right) = \max_j \frac{\mathbf{x}_{t-1}^T\mathbf{A}\mathbf{e}_j}{\mathbf{x}_{t-1}^T\mathbf{e}_j}$$

so that $\mathbf{x}_{t-1}^T\left[\lambda_t\mathbf{I}-\mathbf{A}\right]\geqq\mathbf{0}^T$ and $\mathbf{x}_{t-1}^T\left[\lambda_t\mathbf{I}-\mathbf{A}\right]\ngtr\mathbf{0}^T$, and $\mathbf{x}_t$ $(t>0)$ is a vector such that $\mathbf{x}_t>\mathbf{0}$, $\mathbf{x}_t^T\mathbf{l}=\beta$ and $\mathbf{x}_t^T\left[\lambda_t\mathbf{I}-\mathbf{A}\right]>\mathbf{0}^T$. Sraffa comments "This is always feasible so long as there is a surplus of some commodities and a deficit of none." However he does not provide a proof of this sentence. As we will see, this proof is an immediate consequence of the first three Theorems provided by Besicovitch. Then Sraffa proceeds to the end-state of the algorithm.

> We continue with such an alternation of proportionate cuts with the re-establishment of a surplus for each product until we reach the point where the products have been reduced to such an extent that all-round replacement is just possible without leaving anything as surplus product.

The "imaginary experiment" concludes, in Sraffa's opinion, when $\mathbf{x}_\infty>\mathbf{0}$, $\mathbf{x}_\infty^T\mathbf{l}=\beta$ and $\mathbf{x}_\infty^T\left[\lambda_\infty\mathbf{I}-\mathbf{A}\right]=\mathbf{0}^T$. Sraffa never states that the algorithm may need an infinite number of steps, but we know indeed that this is so. Finally, we have the last paragraph of section 37.

> Since to reach this position the products of all the industries have been cut in the same proportion we are now able to restore the original conditions of production by increasing the quantity produced in each industry by a uniform rate; we do not, on the other hand, disturb the proportions to which the industries have been brought. The uniform rate which restores the original conditions of production is *R* and the proportions attained by the industries are the proportions of the Standard system.

Hence we arrive at the equation

$$\mathbf{x}_\infty^T\left[\mathbf{I}-(1+R)\mathbf{A}\right]=\mathbf{0}^T$$

where, obviously, $1+R=1/\lambda_\infty$. As Alister Watson, Kurz and Salvadori (2001) and Lippi (2008), among others, have remarked, the algorithm is not well defined since there are infinitely many ways to define $\mathbf{x}_t$. Completing the definition of the algorithm means defining a function $\phi(\mathbf{q})$ such that $\mathbf{x}_t=\phi(\mathbf{x}_{t-1})$, at each *t*. To be more precise, we introduce the sets

$$\mathsf{R}=\left\{\mathbf{q}\in\Re^n \,\middle|\, \mathbf{q}\geq\mathbf{0},\ \mathbf{q}^T\mathbf{l}=\beta,\ \mathbf{q}^T\left[\mathbf{I}-\mathbf{A}\right]\geqq\mathbf{0}^T\right\}$$

$$\mathsf{R}^*=\left\{\mathbf{q}\in\Re^n \,\middle|\, \exists\rho\geq 0\ :\ \mathbf{q}\geq\mathbf{0},\ \mathbf{q}^T\mathbf{l}=\beta,\ \mathbf{q}^T\left[\rho\mathbf{I}-\mathbf{A}\right]=\mathbf{0}^T\right\}$$

$$\mathsf{S}=\mathsf{R}-\mathsf{R}^*$$

and the set of functions

$$\mathsf{Z}(\mathsf{S}_0)=\left\{\phi\ :\ \mathsf{S}_0\rightarrow\mathsf{R} \,\middle|\, \forall\mathbf{q}\in\mathsf{S}_0\ :\ \phi(\mathbf{q})\in\mathsf{S}_0\cup\mathsf{R}^*,\lambda(\mathbf{q})\phi(\mathbf{q})-\mathbf{A}^T\phi(\mathbf{q})>\mathbf{0}\right\},$$

where S_0 is any subset of S. Each function of the set $\bigcup_{\mathsf{S}_0 \subseteq \mathsf{S}} \mathsf{Z}(\mathsf{S}_0)$ defines a different algorithm which corresponds to Sraffa's description.

If function $\phi(\mathbf{q})$ has a fixed point in S, then sequence $\{\mathbf{x}_t\}$ may converge on the fixed point of function $\phi(\mathbf{q})$. As a consequence, sequence $\{\lambda_t\}$ may converge to a number which does not even need to be close to the eigenvalue of matrix **A**. This cannot hold if function $\phi(\mathbf{q})$ has the mentioned inequality properties in the whole S, and therefore the set of functions to be considered must be

$$\mathsf{Z} = \mathsf{Z}(\mathsf{S}) = \left\{\phi \,:\, \mathsf{S} \to \mathsf{R} \middle| \, \forall \mathbf{q} \in \mathsf{S} \,:\, \phi(\mathbf{q}) \geq \mathbf{0},\, \lambda(\mathbf{q})\phi(\mathbf{q}) - \mathbf{A}^T\phi(\mathbf{q}) > \mathbf{0},\, \mathbf{l}^T\phi(\mathbf{q}) = \beta\right\}$$

and not $\bigcup_{\mathsf{S}_0 \subseteq \mathsf{S}} \mathsf{Z}(\mathsf{S}_0)$. This is the extra assumption found by Salvadori (2008). The interpretation is close at hand: the function $\phi(\mathbf{q})$ is such that $\phi(\mathbf{q}) \geq \mathbf{0},\, \lambda(\mathbf{q})\phi(\mathbf{q}) - \mathbf{A}^T\phi(\mathbf{q}) > \mathbf{0},\, \mathbf{l}^T\phi(\mathbf{q}) = \beta$, whatever is point $\mathbf{q} \in \mathsf{R}$ and not just in the support of sequence $\{\mathbf{x}_t\}$, as Sraffa's description may be interpreted. In the following two sections I will show that Besicovitch proposed a better defined algorithm and proved that the algorithm converges to the desired solution (apart from a small point to be completed).

3. Towards Besicovitch's proof

Besicovitch's proof is divided into four "Theorems". Only the last is the required proof. The first three prepare the field. In this section we discuss the first three theorems. Besicovitch does not follow the matricial notation we used above to achieve a more compact presentation.

The first Theorem of file D3/12/39: 42 reads in plain English: *With positive prices any distribution of the net outputs can be attained*. This Theorem starts from the assumption that there is a system with no profits and positive prices and a positive wage rate. The aim is to prove that industries can be operated in such a way that any proportion in which the surplus is distributed among industries is feasible. The no profit assumption is not necessary, but probably follows the exercise that Sraffa is performing. Obviously the rate of profit must be lower than the maximum one since the wage rate must be positive and this is really what is needed. In modern notation the first Theorem states:

$$\exists \mathbf{p} > \mathbf{0}, w > 0 : \mathbf{Ap} + w\mathbf{l} = \mathbf{p} \Rightarrow \exists \mathbf{x} \geq \mathbf{0} : \mathbf{x}^T = \mathbf{x}^T\mathbf{A} + \mathbf{y}^T \;\; \forall \mathbf{y} \geq \mathbf{0}\,.$$

Obviously the semipositive vector $\mathbf{y}$ is the vector of what Besicovitch calls "the Surplus outputs" (net outputs in the above). In order to obtain this result it is enough to prove that matrix $\mathbf{I} - \mathbf{A}$ is invertible and its inverse is positive, and we know that this is the case when matrix $\mathbf{A}$ is indecomposable and there is a positive vector $\mathbf{p}$ such that $[\mathbf{I} - \mathbf{A}]\mathbf{p} \geq \mathbf{0}$, because of the Perron-

Frobenius Theorem. However, Besicovitch makes no reference to the latter Theorem and indeed the proof of the existence of the Standard commodity can be interpreted as a proof of the Perron-Frobenius Theorem (see Kurz and Salvadori, 1993).

The proof provided by Besicovitch is very ingenious, but may need some explanation. Like the Gauss-Jordan elimination way to solve a linear system of equations it is based on consecutive applications of two elementary steps: (i) multiplication of an equation by a non-zero scalar, and (ii) addition to an equation of non-zero scalar multiples of other equations. Besicovitch proves that since prices are positive the *non-zero* scalar multiplications involved in both steps are indeed *positive* scalar multiplications. Let us follow step by step this recursive proof. In the first step only the last industry, n, is considered. Since

$$a_{n1}p_1 + ... + a_{nn}p_n + l_n w = p_n$$

and since $a_{n1}p_1 + ... + a_{nn-1}p_{n-1} + l_n w > 0$, then $1 - a_{nn} > 0$. Hence it is possible to find a positive λ_n such that $\lambda_n(1 - a_{nn})$ can take any positive value.

In the second step the last two industries are considered. Taking account of the equations

$$\begin{aligned} &a_{n-1,1}p_1 + ... + a_{n-1,n-1}p_{n-1} + a_{n-1,n}p_n + l_{n-1}w = p_{n-1} \\ &a_{n1}p_1 + ... + a_{nn-1}p_{n-1} + a_{nn}p_n + l_n w = p_n \end{aligned}$$

and using the first step, we can multiply the latter by a λ_n such that $\lambda_n(1 - a_{nn}) = a_{n-1,n}$ so as to obtain that the surplus of industry n equals the input of commodity n into industry $n-1$:

$$\begin{aligned} &a_{n-1,1}p_1 + ... + a_{n-1,n-1}p_{n-1} + a_{n-1,n}p_n + l_{n-1}w = p_{n-1} \\ &\frac{a_{n-1,n}}{1-a_{nn}}a_{n1}p_1 + ... + \frac{a_{n-1,n}}{1-a_{nn}}a_{nn-1}p_{n-1} + \frac{a_{n-1,n}}{1-a_{nn}}a_{nn}p_n + \frac{a_{n-1,n}}{1-a_{nn}}l_n w = \frac{a_{n-1,n}}{1-a_{nn}}p_n \end{aligned}$$

As a consequence, by summing up the two equations we obtain

$$\left(a_{n-1,1} + \frac{a_{n-1,n}}{1-a_{nn}}a_{n1}\right)p_1 + ... + \left(a_{n-1,n-1} + \frac{a_{n-1,n}}{1-a_{nn}}a_{nn-1}\right)p_{n-1} + \left(l_{n-1} + \frac{a_{n-1,n}}{1-a_{nn}}l_n\right)w = p_{n-1}$$

since

$$a_{n-1,n} + \frac{a_{n-1,n}}{1-a_{nn}}a_{nn} = \frac{a_{n-1,n}}{1-a_{nn}} .$$

Once again, since

$$\left(a_{n-1,1} + \frac{a_{n-1,n}}{1-a_{nn}}a_{n1}\right)p_1 + ... + \left(a_{n-1,n-2} + \frac{a_{n-1,n}}{1-a_{nn}}a_{nn-2}\right)p_{n-2} + \left(l_{n-1} + \frac{a_{n-1,n}}{1-a_{nn}}l_n\right)w > 0$$

then

$$1-\left(a_{n-1,n-1}+\frac{a_{n-1,n}}{1-a_{nn}}a_{n,n-1}\right)=\frac{\det\begin{bmatrix}1-a_{n-1,n-1} & -a_{n-1,n}\\ -a_{n,n-1} & 1-a_{nn}\end{bmatrix}}{1-a_{nn}}>0$$

Hence we can find two positive scalars λ_n and λ_{n-1} such that $\lambda_{n-1}-\lambda_n a_{n,n-1}-\lambda_{n-1}a_{n-1,n-1}$ can take any positive value and $\lambda_n-\lambda_n a_{nn}-\lambda_{n-1}a_{n-1,n}=0$, that is, we can proportion the two equations in such a way that the output of commodity *n* equals the sum of the inputs of commodity *n* in the last two industries and the output of commodity $n-1$ is any desired positive number. In a similar way we can proportion the two equations in such a way that the output of commodity $n-1$ equals the sum of the inputs of commodity $n-1$ in the last two industries and the output of commodity *n* is any desired positive number. Thus the two equations can be so proportioned that there is the desired surplus of the last two commodities.

The third step analyzes the last three industries. By using the second step we can proportion the last two equations in such a way that the outputs of the last two commodities equal the sum of their inputs in the last three industries.

$$\begin{gathered}a_{n-2,1}p_1+...+a_{n-2,n-1}p_{n-1}+a_{n-2,n}p_n+l_{n-2}w=p_{n-2}\\ \frac{\Delta_1}{\Delta}a_{n-1,1}p_1+...+\frac{\Delta_1}{\Delta}a_{n-1,n-1}p_{n-1}+\frac{\Delta_1}{\Delta}a_{n-1,n}p_n+\frac{\Delta_1}{\Delta}l_{n-1}w=\frac{\Delta_1}{\Delta}p_{n-1}\\ \frac{\Delta_2}{\Delta}a_{n1}p_1+...+\frac{\Delta_2}{\Delta}a_{nn-1}p_{n-1}+\frac{\Delta_2}{\Delta}a_{nn}p_n+\frac{\Delta_2}{\Delta}l_nw=\frac{\Delta_2}{\Delta}p_n\end{gathered}$$

where

$$\Delta=\det\begin{bmatrix}1-a_{n-1,n-1} & -a_{n-1,n}\\ -a_{n,n-1} & 1-a_{nn}\end{bmatrix},\Delta_1=\det\begin{bmatrix}a_{n-2,n-1} & -a_{n-1,n}\\ a_{n-2,n} & 1-a_{nn}\end{bmatrix},\Delta_2=\det\begin{bmatrix}1-a_{n-1,n-1} & a_{n-2,n-1}\\ -a_{n,n-1} & a_{n-2,n}\end{bmatrix}.$$

By adding up, we obtain

$$\left(a_{n-2,1}+\frac{\Delta_1}{\Delta}a_{n-1,1}+\frac{\Delta_2}{\Delta}a_{n1}\right)p_1+...+\left(a_{n-2,n-2}+\frac{\Delta_1}{\Delta}a_{n-1,n-2}+\frac{\Delta_2}{\Delta}a_{n,n-2}\right)p_{n-2}+l_{n-2}w=p_{n-2}$$

since

$$a_{n-2,n-1}+\frac{\Delta_1}{\Delta}a_{n-1,n-1}+\frac{\Delta_2}{\Delta}a_{n,n-1}=\frac{\Delta_1}{\Delta},\;\; a_{n-2,n}+\frac{\Delta_1}{\Delta}a_{n-1,n}+\frac{\Delta_2}{\Delta}a_{n,n}=\frac{\Delta_2}{\Delta}$$

Once again, since prices are positive, we obtain that there is a surplus of commodity $n-2$, that is,

$$1-\left(a_{n-2,n-2}+\frac{\Delta_1}{\Delta}a_{n-1,n-2}+\frac{\Delta_2}{\Delta}a_{n,n-2}\right)=\frac{\det\begin{bmatrix}1-a_{n-2,n-2} & -a_{n-2,n-1} & -a_{n-2,n}\\ -a_{n-1,n-2} & 1-a_{n-1,n-1} & -a_{n-1,n}\\ -a_{n,n-2} & -a_{n,n-1} & 1-a_{nn}\end{bmatrix}}{\Delta}>0$$

and that multipliers can be found such that the surplus of commodity $n-2$ can take any positive value, whereas the outputs of the last two commodities equal the sum of their inputs in the last three industries. This is enough to find multipliers such that there is the desired surplus of commodity $n-2$, the desired surplus of commodity $n-1$, and the desired surplus of commodity n. And so on.

The second Theorem reads in plain English*: If the wage is positive and prices are positive, then net outputs cannot be all nought and, therefore, there is a surplus of at least one commodity*. In modern notation the second Theorem states:

$$\exists \mathbf{p} > \mathbf{0}, w > 0 : \mathbf{Ap} + w\mathbf{l} = \mathbf{p} \Rightarrow \mathbf{x}^T \neq \mathbf{x}^T \mathbf{A} \ \forall \mathbf{x} \geq \mathbf{0}$$

If not, we obtain $\mathbf{x}^T \mathbf{Ap} + w\mathbf{x}^T \mathbf{l} = \mathbf{x}^T \mathbf{p} = \mathbf{x}^T \mathbf{Ap}$, and therefore $w\mathbf{x}^T \mathbf{l} = 0$, which is not possible. The proof by Besicovitch does not need a *reductio ad absurdum*. If $\mathbf{x}^T \mathbf{Ae}_i = \mathbf{x}^T \mathbf{e}_i$ each $i \neq j$, where $\mathbf{e}_i$ is the *i*-th unit vector, then

$$\mathbf{x}^T \mathbf{Ap} + w\mathbf{x}^T \mathbf{l} = \mathbf{x}^T \mathbf{Ae}_j \left(\mathbf{e}_j^T \mathbf{p}\right) + w\mathbf{x}^T \mathbf{l} + M = \mathbf{e}_j^T \mathbf{p} + M$$

where $M = \sum_{i \neq j} \mathbf{x}^T \mathbf{Ae}_i \left(\mathbf{e}_i^T \mathbf{p}\right) = \sum_{i \neq j} \mathbf{e}_i^T \mathbf{p}$ and since $w\mathbf{x}^T \mathbf{l} > 0,$ we have $\mathbf{x}^T \mathbf{Ae}_j < 1$ as required.

The third Theorem reads in plain English: *If the surplus of a commodity is positive and that of the others is nought then the prices are positive.* Note that it is always implicit that the wage rate is positive. The aim is to prove that if there is a positive surplus of at least one commodity (and a negative surplus of none), then the wage is positive and prices are positive. Also in this case it is enough to prove that matrix $\mathbf{I} - \mathbf{A}$ is invertible and its inverse is positive. In modern notation the third Theorem states:

$$\exists \mathbf{x} \geq \mathbf{0} : \mathbf{x}^T \geq \mathbf{x}^T \mathbf{A} \Rightarrow \exists \mathbf{p} > \mathbf{0} : \mathbf{Ap} + w\mathbf{l} = \mathbf{p}$$

However, in the document D3/12/39: 42 of 21 September 1944 this Theorem is not proven. What is proven is that if there is a surplus in one commodity and no surplus in all the others, then the equations can be proportioned in such a way that a surplus is obtained in every commodity (even this proof is incomplete: if the input matrix were decomposable, the statement would be false; the proof does not show why the statement holds when the input matrix is indecomposable). However, in the document D3/12/39: 42 there is a note by Sraffa saying: "Refer to blue page 1". The reference is no doubt to D3/12/39: 7, which is written on a blue piece of paper and contains a proof by

Besicovitch of the fact that if there is a surplus in every commodity, then prices are positive.[1] The transcription of this document is reported below in Appendix C.

Before arguing the proof of the third theorem I will discuss the proof in D3/12/39: 7. The statement in modern notation is:

$$\mathbf{e}^T > (1+r)\mathbf{e}^T\mathbf{A},\ (1+r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p},\ w > 0 \Rightarrow \mathbf{p} > \mathbf{0}$$

where $\mathbf{e}$ is the sum vector of the appropriate size, that is a vector of 1's. Note that the above equation always admits a solution since it is homogeneous in $(\mathbf{p},\ w)$. However, we are assuming here something more, i.e., that a solution with a positive w exists. We will deal with this problem soon. Suppose that in this solution some price (at least one) is negative or nought, and all the others (possibly none) are positive. With no loss of generality assume that the prices of the first h commodities are negative or nought, $1 \le h \le n$, and the last $n-h$ are positive. Then, with obvious meanings of symbols,

$$(1+r)\mathbf{A}_{12}\mathbf{p}_2 + w\mathbf{l}_1 = \left[\mathbf{I} - (1+r)\mathbf{A}_{11}\right]\mathbf{p}_1$$

which is impossible since $\mathbf{e}^T\left[\mathbf{I} - (1+r)\mathbf{A}_{11}\right]\mathbf{p}_1 \le 0$ whereas $(1+r)\mathbf{e}^T\mathbf{A}_{12}\mathbf{p}_2 + w\mathbf{e}^T\mathbf{l}_1 > 0$. Note that this proof holds even if matrix $\mathbf{A}$ is decomposable, and therefore some commodities are non-basic, provided that labor enters directly into the production of all commodities and, therefore, $\mathbf{l}_1 > \mathbf{0}$ (it still holds if labor enters directly or indirectly into the production of all commodities, but I will not deal with this issue here). But what happens if all solutions to the system $(1+r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p}$ have a zero w? Indeed the same proof can be slightly modified to prove that in this case $\mathbf{p} = \mathbf{0}$ and therefore the unique solution would be the trivial one. This being impossible, there are solutions with $w \ne 0$ and therefore with $w > 0$.[2]

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1 D3/12/39: 8 is also written on a blue piece of paper and contains a proof by Besicovitch, but on a different issue.

2 If $w = 0$, with no loss of generality assume that the prices of the first h commodities are positive, $1 \le h \le n$, and the last $n-h$ are either negative or zero. Then, with obvious meanings of symbols,

$$(1+r)\mathbf{A}_{12}\mathbf{p}_2 + w\mathbf{l}_1 = \left[\mathbf{I} - (1+r)\mathbf{A}_{11}\right]\mathbf{p}_1$$

which is impossible since $\mathbf{e}^T\left[\mathbf{I} - (1+r)\mathbf{A}_{11}\right]\mathbf{p}_1 > 0$ whereas $(1+r)\mathbf{e}^T\mathbf{A}_{12}\mathbf{p}_2 + w\mathbf{e}^T\mathbf{l}_1 \le 0$. Hence no price can be positive. Similarly it is proved that no price can be negative.

Now we can discuss the proof of the third theorem in the document D3/12/39: 42. With no loss of generality assume that the first h commodities have a positive surplus, $1 \leq h \leq n$, whereas the last $n-h$ have no surplus (and no loss). Therefore $\mathbf{e}^T > \mathbf{e}^T\mathbf{A}_{11} + \mathbf{e}^T\mathbf{A}_{12}$ and $\mathbf{e}^T = \mathbf{e}^T\mathbf{A}_{12} + \mathbf{e}^T\mathbf{A}_{22}$ (note that in these and in the following formulas the vectors $\mathbf{e}$ involved have different sizes). Therefore, Besicovitch maintains, if u is a real number lower than 1, but so close to 1 that $u\mathbf{e}^T > u\mathbf{e}^T\mathbf{A}_{11} + \mathbf{e}^T\mathbf{A}_{12}$ still holds, then by necessity $\mathbf{e}^T > u\mathbf{e}^T\mathbf{A}_{12} + \mathbf{e}^T\mathbf{A}_{22}$. However, this is not necessarily true. Indeed, if matrix $\mathbf{A}$ is decomposable and $\mathbf{A}_{12} = \mathbf{0}$, this is certainly false. It is reasonable to suppose that Besicovitch assumed that all commodities are basic and, therefore, matrix $\mathbf{A}$ is indecomposable. Even in this case, however, the proof is incomplete ($\mathbf{e}^T\mathbf{A}_{12}$ is semi-positive, but does not need to be positive) since we may need to iterate the process to bring home the result. In fact, if matrix $\mathbf{A}$ is indecomposable, we are sure that $\mathbf{e}^T \geq u\mathbf{e}^T\mathbf{A}_{12} + \mathbf{e}^T\mathbf{A}_{22}$ and therefore the number of commodities with a positive surplus is increased and still no commodity has a negative surplus. Further, since at any iteration of the process the number of products with a positive surplus increases, the number of iterations needed to obtain a surplus in all commodities is certainly finite since it is lower than $n-h$.

The first three theorems of file D3/12/39: 42 are intended to support two facts. First, if there is a surplus of any type, industries may be proportioned in such a way as to get the surplus anywhere it is needed. Second, there is a surplus if, and only if, prices are positive and the wage rate is positive. The relationship with section 37 of the book by Sraffa (1960) is obvious. One of the two steps of the algorithm introduced there consists exactly in "adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement". The fourth theorem concerns the existence of the Standard commodity and will be analyzed in the next section.

4. Besicovitch's proof

The fourth theorem reads in plain English: *If prices are positive, then there exist positive multipliers* $q_a, \ldots, q_k$ *such that the net output is proportional to the total of every kind of raw material.* The proof is similar to that provided by Sraffa, but is more detailed and closer to the description of an algorithm. It starts by assuming that there is a surplus with regard to all commodities. If there were a surplus only in some industries, then we could find a starting point with a surplus in all industries,

$\mathbf{x}_0 \in \left\{\mathbf{x} > \mathbf{0} \middle| \mathbf{x}^T\mathbf{l} = \beta, \mathbf{x}^T[\mathbf{I}-\mathbf{A}] > \mathbf{0}^T\right\}$, since the assumption of Theorem 1 holds. Then the second step used by Sraffa is applied. That is, it is found that

$$\lambda_1 = \lambda(\mathbf{x}_0) = \max_j \frac{\mathbf{x}_0^T\mathbf{A}\mathbf{e}_j}{\mathbf{x}_0^T\mathbf{e}_j}$$

so that $\mathbf{x}_0^T[\lambda_1\mathbf{I}-\mathbf{A}] \geqq \mathbf{0}^T$ and $\mathbf{x}_0^T[\lambda_1\mathbf{I}-\mathbf{A}] \not> \mathbf{0}^T$. Then all the equations of commodities for which there is a surplus are multiplied by a common scalar lower than 1. Besicovitch thinks this is enough to obtain that all commodities are in surplus, but this does not need to be true since input coefficients are not all positive. However, since all commodities are assumed to be basic, the input matrix $\mathbf{A}$ is indecomposable and therefore we can get the desired result by iterating the same procedure, as seen above, in the analysis of the third theorem by Besicovitch. Let us consider the point in a more formal way.

Let $\mu \in \Re$ and $\mathbf{x} \in \mathsf{S}$ be such that $\mu\mathbf{x}^T \geq \mathbf{x}^T\mathbf{A}$ and let us define the set of indices

$$\mathsf{I}_{\mu\mathbf{x}} = \left\{ i \in \{1,\ 2,\ \ldots,\ n\} \,\middle|\, \mu x_i > \sum_{j=1}^n x_j a_{ji} \right\}$$
$$\hat{\mathsf{I}}_{\mu\mathbf{x}} = \left\{ i \in \{1,\ 2,\ \ldots,\ n\} \,\middle|\, \mu x_i = \sum_{j=1}^n x_j a_{ji} \right\}$$

The aim of this step consists in finding an intensity vector $\phi(\mathbf{x})$ such that $\mathsf{I}_{\mu\phi(\mathbf{x})} = \{1,\ 2,\ \ldots,\ n\}$ and, as a consequence, $\hat{\mathsf{I}}_{\mu\phi(\mathbf{x})} = \varnothing$. Besicovitch considers that this can be obtained if $\phi(\mathbf{x})$ is the function $\mathbf{g}(\mu,\mathbf{x})$, where

$$g_i(\mu,\mathbf{x}) = \begin{cases} x_i & \text{if } i \in \hat{\mathsf{I}}_{\mu\mathbf{x}} \\ \eta x_i & \text{if } i \in \mathsf{I}_{\mu\mathbf{x}} \end{cases}$$

where η is a scalar lower than 1, but so close to 1 that

$$\mu(\eta x_i) > \sum\nolimits_{j\in\hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j a_{ji} + \eta \sum\nolimits_{j\in\mathsf{I}_{\mu\mathbf{x}}} x_j a_{ji} \qquad \text{each } i \in \mathsf{I}_{\mu\mathbf{x}}$$

That is,

$$\max_{i\in\mathsf{I}_{\mu\mathbf{x}}} \frac{\sum\nolimits_{j\in\hat{\mathsf{I}}_{\mu\mathbf{x}}} x_j a_{ji}}{\mu x_i - \sum\nolimits_{j\in\mathsf{I}_{\mu\mathbf{x}}} x_j a_{ji}} < \eta < 1\ .$$

As mentioned above, this is not enough to obtain that $\mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})} = \{1,\ 2,\ \ldots,\ n\}$ because some a_{ji} may be nought. However, by construction, $i \in \mathsf{I}_{\mu\mathbf{x}}$ implies that $i \in \mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})}$ and therefore $\mathsf{I}_{\mu\mathbf{g}(\mu,\mathbf{x})} \supseteq \mathsf{I}_{\mu\mathbf{x}}$. On the other hand, $I_{\mu\mathbf{g}(\mu,\mathbf{x})} = I_{\mu\mathbf{x}}$ if, and only if, $a_{ji} = 0$, each $i \in I_{\mu\mathbf{x}}$ and each $j \in \hat{I}_{\mu\mathbf{x}}$. But then matrix **A** would be decomposable. This being impossible, we obtain that $I_{\mu\mathbf{g}(\mu,\mathbf{x})} \supset I_{\mu\mathbf{x}}$. This is enough to say that the procedure can be iterated for a number of times lower than the (finite) number of commodities (also because if $I_{\mu\mathbf{x}} = \{1,\ 2,\ \dots,\ n\}$, then $\mathbf{g}(\mu,\mathbf{x})$ is proportional to **x**). Hence we can define:

$$\mathbf{h}_1(\mathbf{x}) = \mathbf{g}(\lambda(\mathbf{x}),\mathbf{x})$$

$$\mathbf{h}_j(\mathbf{x}) = \mathbf{g}(\lambda(\mathbf{x}),\mathbf{h}_{j-1}(\mathbf{x})) \qquad j = 2,\ \dots,n-1$$

$$\phi(\mathbf{x}) = \mathbf{h}_{n-1}(\mathbf{x})$$

There is one further aspect considered by Besicovitch. In a remark he argued that ‘we may keep one of our industries intact’ in order to avoid all multipliers becoming zero. With no loss of generality, assume that the industry in question is industry 1. Therefore function $\mathbf{g}(\mu,\mathbf{x})$ must be redefined as

$$g_i(\mu,\mathbf{x}) = \begin{cases} x_i & \text{if } i \in \hat{I}_{\mu\mathbf{x}} \text{ and } 1 \in \hat{I}_{\mu\mathbf{x}} \\ \eta x_i & \text{if } i \in I_{\mu\mathbf{x}} \text{ and } 1 \in \hat{I}_{\mu\mathbf{x}} \\ \dfrac{1}{\eta} x_i & \text{if } i \in \hat{I}_{\mu\mathbf{x}} \text{ and } 1 \notin \hat{I}_{\mu\mathbf{x}} \\ x_i & \text{if } i \in I_{\mu\mathbf{x}} \text{ and } 1 \notin \hat{I}_{\mu\mathbf{x}} \end{cases}$$

Further, this function has the property that if $I_{\mu\mathbf{x}} = \{1,\ 2,\ \dots,\ n\}$, then $\mathbf{g}(\mu,\mathbf{x}) = \mathbf{x}$. As seen in section 2, Sraffa followed a different, but equivalent, strategy to avoid all multipliers becoming zero. He kept the amount of labor fixed. If we follow this strategy, then function $\mathbf{g}(\mu,\mathbf{x})$ must be redefined as

$$g_i(\mu,\mathbf{x}) = \begin{cases} \theta x_i & \text{if } i \in \hat{I}_{\mu\mathbf{x}} \\ \theta\eta x_i & \text{if } i \in I_{\mu\mathbf{x}} \end{cases}$$

where

$$\theta = \frac{\beta}{\sum_{j \in \hat{I}_{\mu\mathbf{x}}} x_j l_j + \eta \sum_{j \in I_{\mu\mathbf{x}}} x_j l_j}$$

Also this function has the property that if $I_{\mu\mathbf{x}} = \{1,\ 2,\ \dots,\ n\}$, then $\mathbf{g}(\mu,\mathbf{x}) = \mathbf{x}$.

Also in Besicovitch's proof there is a family of potential algorithms involved. In order to have a proper algorithm we must have a way to define how η is chosen. For example, if we chose η in the middle of the range in which it can vary, we would have

$$\eta = \frac{1}{2} + \frac{1}{2} \max_{i \in I_{\mu x}} \frac{\sum_{j \in \hat{I}_{\mu x}} x_j a_{ji}}{\mu x_i - \sum_{j \in I_{\mu x}} x_j a_{ji}}$$

and in general any possible choice could be defined as a choice of $0 < \alpha < 1$ in the expression

$$\eta = \alpha + (1-\alpha) \max_{i \in I_{\mu x}} \frac{\sum_{j \in \hat{I}_{\mu x}} x_j a_{ji}}{\mu x_i - \sum_{j \in I_{\mu x}} x_j a_{ji}}$$

For each sequence $\{\alpha_i\}$, $0 < \alpha_i < 1$, we have a different algorithm; but whatever sequence $\{\alpha_i\}$ is chosen, it is easily proved that the conditions stated by Salvadori (2008) hold and therefore all the potential algorithms considered by Besicovitch converge to the desired result. In fact, for any given sequence $\{\alpha_i\}$ function $\phi(\mathbf{x})$ is continuous and can start from any point in S.

5. **Sraffa and Besicovitch**

Sraffa did not use the function $\phi(\mathbf{x})$ used by Besicovitch. He recognized that what is important is "adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement" and that at each step the desired result is closer, but he did not consider the fact that the "imaginary experiment" may work through an infinite number of steps without approaching the Standard commodity.

Why did Sraffa not use the proof available to him and provided by Besicovitch in September 1944? A simple answer could be that Sraffa thought that the exposition of the proof could be simplified and that he failed to carry out the simplification required. This is a possible interpretation. However, there are other cases in which Sraffa made no use of an available proof by Besicovitch. For instance when in the 1950s Sraffa was faced with the need to define basics and non-basics in joint production, he conceived a definition in terms of a tax on the production of single commodities (a tithe). A tax on the production of a basic commodity affects all prices and the wage rate (for a given rate of profit), whereas a tax on the production of a non-basic commodity affects only prices of some non-basic commodities (if the numeraire is fixed only in terms of basic commodities). He was convinced to use the linear dependence definition we find in the book (§ 58) by Besicovitch (the whole story is told by Kurz and Salvadori, 2004). The tax argument appears in the book (§ 65), but it is a consequence, not the definition. Nevertheless, Besicovitch proved three months after Sraffa had accepted the definition in terms of linear dependence that Sraffa's original opinion was correct and that actually the definition could be given in terms of the tax. However, the proof is extremely demanding in terms of mathematical calculations (see Kurz and Salvadori, 2004). Sraffa made no mention of this proof by Besicovitch in his book.

Both the proof of the existence of the Standard commodity and the distinction between basic and non-basic commodities recall the concluding remarks of the Preface:

> It will be only too obvious that I have not always followed the expert advice that was given to me – particularly with regard to the notation adopted, which I have insisted on retaining (although admittedly open to objection in some respects) as being easy to follow for the non-mathematical reader.

Despite his interest in the existence of a proof, Sraffa was keen to provide one only if it was "easy to follow for the non-mathematical reader". He thought that the non-mathematical reader would understand his argument in section 37. If the mathematical reader were to find it incomplete, then such a reader would also be able to find a complete proof, which Sraffa knew existed.

6. **Conclusion**

In this paper I explored the relationship between the proof of the existence of the Standard commodity contained in section 37 of Sraffa's (1960) book and the proof supplied to Sraffa by Besicovitch on 21 September 1944, and investigated the completeness and consistency of such a proof. I also postulated some reasons which led Sraffa to omit this proof in his book in favor of an incomplete argument.

Appendix A. An example

Let

$$\mathbf{A} = \begin{bmatrix} 0 & h \\ k & 0 \end{bmatrix} \qquad \mathbf{l} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \beta = 1 \qquad\qquad 0 < h < k < 1$$

It is easily calculated that the eigenvalue of maximum modulus of matrix $\mathbf{A}$ is $\sqrt{hk}$ and that the left eigenvector associated with this eigenvalue normalized by the condition $\mathbf{x}^T\mathbf{l} = \beta$ is

$$\begin{bmatrix} \dfrac{2\left(k-\sqrt{hk}\right)}{k-h} & \dfrac{2\left(\sqrt{hk}-h\right)}{k-h} \end{bmatrix}$$

It is easily recognised that

$$\mathsf{R}=\left\{\mathbf{q}\in\Re^2\,\middle|\,\frac{2k}{1+k}\le q_1\le\frac{2}{1+h},\ q_2=2-q_1\right\}.$$

Finally, let us consider the function

$$\phi(\mathbf{q})=\begin{bmatrix}1\\1-2\varepsilon\end{bmatrix}+\varepsilon\mathbf{q} \tag{6}$$

with

$$0<\varepsilon<\frac{1}{2}-\frac{1}{2}\sqrt{\frac{h}{k}} \tag{7}$$

From inequalities (7) we obtain the inequality

$$\frac{1}{1-\varepsilon}<\frac{2\left(k-\sqrt{hk}\right)}{k-h}$$

from which we obtain[3] that $\lambda(\mathbf{q})\phi(\mathbf{q})-\mathbf{A}^T\phi(\mathbf{q})>\mathbf{0}$ for each $\mathbf{q}\in\mathsf{S}_1$, whereas this property does not hold for $\mathbf{q}\in\mathsf{S}_2$, where

$$\mathsf{S}_1=\left\{\mathbf{q}\in\mathsf{S}\,\middle|\,q_1<\frac{1}{1-\varepsilon}\right\}$$

$$\mathsf{S}_2=\left\{\mathbf{q}\in\mathsf{S}\,\middle|\,\frac{1}{1-\varepsilon}\le q_1<\frac{2\left(k-\sqrt{hk}\right)}{k-h}\right\}$$

Further, it is easily verified that $\mathbf{q}\in\mathsf{S}_1\Rightarrow\phi(\mathbf{q})\in\mathsf{S}_1$. Therefore each element of any sequence defined by the conditions

$$\mathbf{q}_0\in\mathsf{S}_1,\qquad \mathbf{q}_{t+1}=\begin{cases}\mathbf{q}_t & \text{if } \lambda(\mathbf{q}_t)\mathbf{q}_t-\mathbf{A}^T\mathbf{q}_t=\mathbf{0}\\ \phi(\mathbf{q}_t) & \text{if } \lambda(\mathbf{q}_t)\mathbf{q}_t-\mathbf{A}^T\mathbf{q}_t\neq\mathbf{0}\end{cases}$$

satisfies the conditions stated by Sraffa, but

$$\lim_{i\to\infty}\lambda(\mathbf{q}_i)=(1-2\varepsilon)k>\sqrt{hk}$$

$$\lim_{i\to\infty}\mathbf{q}_i=\begin{bmatrix}\dfrac{1}{1-\varepsilon}\\[2mm]\dfrac{1-2\varepsilon}{1-\varepsilon}\end{bmatrix}\neq\begin{bmatrix}\dfrac{2\left(k-\sqrt{hk}\right)}{k-h}\\[2mm]\dfrac{2\left(\sqrt{hk}-h\right)}{k-h}\end{bmatrix}$$

---

[3] If $q_1\le 2\left(k-\sqrt{hk}\right)(k-h)^{-1}$, then $\lambda(\mathbf{q})=k(2-q_1)q_1^{-1}$. Further $\lambda(\mathbf{q})\mathbf{e}_1^T\phi(\mathbf{q})-\mathbf{e}_1^T\mathbf{A}^T\phi(\mathbf{q})>0$ if and only if $q_1<(1-\varepsilon)^{-1}$ whereas $\lambda(\mathbf{q})\mathbf{e}_2^T\phi(\mathbf{q})-\mathbf{e}_2^T\mathbf{A}^T\phi(\mathbf{q})>0$ for $q_1\le 2\left(k-\sqrt{hk}\right)(k-h)^{-1}$, provided that inequalities (7) hold.

The last limit is the unique fixed point of function (6).

Appendix B. D3/12/39: 42

Besic.: – **21.9.44**: (42) 1-4:

'Th 1 If prices are +ve, any distribution of the Surplus outputs can be attained'

Proof

(i) $A_k p_a + ... + K_k p_k + l_k w = K p_k$

obviously any desirable surplus of K can be produced since $K > K_k$ ($\because$ of +ve prices).

(ii) $\begin{cases} A_j p_a + ... + J_j p_j + K_j p_k + l_j w = J p_j \\ A_k p_a + . \qquad . \qquad . \qquad = K p_k \end{cases}$

Let the surplus of the *K*-industry be K_j. Then *J* industry produces some Surplus. Multiply{in}g both =ions {equations} by the Same factor the *J* surplus may take any assigned value. S{imilar}ly we can make *J* not to have any Surplus & *K* to have any assigned surplus. Then add{in}g the two firs{t} =ions {equations} & the two second ones we get an assigned Surplus of *J* & f{o}r *K* , a. s.o., q.e.d.

'Th 2 If the prices are +ve {positive} and the surplus of *B*, ..., *K* is 0 then the surplus of *A* is +ive {positive}.

Proof For take the surplus of *B*, ..., *K* (wrt {with respect to} *B* ... *K*) to be B_a, ... , K_a. Then the surplus f{o}r *B*, ... *K* wrt {with respect to} *A*, ... , *K* is 0. Write

$A_a p_a + ... + l_a w = A p_a$

$A_k p_a + ... + l_k w = K p_k$

& add them & drop *B*, ... , *K* terms from both sides as they are = {equal}. The result is

$(A_a + ... + A_k)\, p_a + (l_a + ... + l_k)\, w = A p_a$.

i.e. $A_a + ... + A_k < A$, q.e.d.

'Th 3 If the Surplus of *A* is +ve {positive} & of *B*, ..., *K* is 0 then the prices are +ve {positive}.

For multiplying A=ion {equation A} by u (< 1) sufficiently near 1 we shall still have the surplus of A +ve {positive} & we shall make surplus of B, ... , K +ve {positive}.

{Addition by Sraffa on bottom of page: (Refer to blue page 1)}

'Th 4 If prices are +ve {positive} there exist +ve {positive} multipliers $q_a,...,q_k$ such that the Surplus output is proportional to the total of every kind of raw materials.

Proof.

$$(1)\quad \begin{array}{l} A_a p_a + ... + l_a w = Ap_a \\ .... \\ A_k p_a + ...= Kp_k \end{array}$$

assuming the surplus for each to be +ve {positive}, i.e.

$$(2)\quad \begin{array}{l} A_a + ... + A_k < A \\ .... \\ K_a + ... + K_k < K \end{array}$$

Consider

$$(3)\quad \begin{array}{l} A_a + ... + A_k < Au \\ .... \\ K_a + ... + K_k < Ku \end{array}$$

The ≠ties {inequalities} remain true as u decreases from 1 until it reaches a certain value $u_0 > 0$ for which some of the ≠ties {inequalities} become =ties {equalities}, f.i {for instance} the first two. Then we multiply the C, ... , K =ions {equations} by $k < 1$ but near 1, so that the surplus of C, ... , K still remain positive. This will release a surplus of A & B. Then (3) will be true wrt {with respect to} the reformed system for $u = u_0$. Now we decrease u beyond u_0 a.s.o. In this way we shall reach as System

$q_a(A_k p_a + ...) = q_a Ap_a$

....

$q_k(.......................) = q_k Kp_k$

for which

$q_a A_a + ...\ q_k A_k < q_a Au$

....

$q_a K_a + ... < q_k Ku$

for $u_1 < u \leq 1$, & when $u = u_1$ all the ≠ies {inequalities} become =ies {equalities}.

Remark. All $q_a,...,q_k$ cannot become 0 since in all our adjustments we may keep one of our industries intact, f. i A, & from this it follows that $u_1 \geq A_a/A$ ($\therefore$ 1st =ion {first equation} of (3))

Appendix C. D3/12/39: 7

$$(A_a p_a + ... + K_a p_k)(1+r) + L_a w = A p_a$$
$$...........$$
$$(A_k p_a + ... + K_k p_k)(1+r) + L_k w = A p_k$$

If r is such that

$$(A_a + ... + A_k)(1+r) < A$$
$$(K_a + ... + K_k)(1+r) < K$$

then all prices are positive, assuming w > 0

Proof. Suppose not. Let $p_a < 0$, $p_b < 0$, the rest > 0. Then adding the first two equations and taking to the right A and B terms we shall have

$$\{(C_a + C_b)p_c + ... + (K_a + K_b)p_k\}(1+r) + (L_a + L_b)w$$
$$= \{A - (A_a + A_b)(1+r)\}p_a + \{B - (B_a + B_b)(1+r)\}p_b$$

which is impossible, since the expression on the left hand side is > 0, and on r. h. side < 0.

ASB

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