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4 October 2010

Online at <https://mpra.ub.uni-muenchen.de/25645/>  
MPRA Paper No. 25645, posted 05 Oct 2010 00:08 UTC

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## Vanilla Option Pricing on Stochastic Volatility market models

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### **Abstract**

We want to discuss the option pricing on stochastic volatility market models, in which we are going to consider a generic function  $\beta(\nu_t)$  for the drift of volatility process. It is our intention choose any equivalent martingale measure, so that the drift of volatility process, respect at the new measure, is zero. This technique is possible when the Girsanov theorem is satisfied, since the stochastic volatility models are uncomplete markets, thus one has to choice an arbitrary risk price of volatility. In all this cases we are able to compute the price of Vanilla options in a closed form. To name a few, we can think to the popular Heston's model, in which the solution is known in literature, unless of an inverse Fourier transform.

# 1 Introduction

The Black-Scholes model rests upon a number of assumptions that are, to some extent, strategic. Among these there are continuity of stock-price process (it does not jump), the ability to hedge continuously without transaction costs, independent Gaussian returns, and constant volatility. We are going to focus here on relaxing the last assumption by allowing volatility to vary randomly, for the following reason, a well-known discrepancy between the Black-Scholes predicted European option prices and market-traded options prices, the **smile curve**, can be accounted for by stochastic volatility models. Modeling volatility as a stochastic process is motivated a priori by empirical studies of stock-price returns in which estimated volatility is observed to exhibit random characteristics. Additionally, the effects of transaction costs show up under many models, as uncertainty in the volatility; fat-tailed returns distributions can be simulated by stochastic volatility. The assumption of constant volatility is not reasonable, since we require different values for the volatility parameter for different strikes and different expiries to match market prices. The volatility parameter that is required in the Black-Scholes formula to reproduce market prices is called the implied volatility. This is a critical internal inconsistency, since the implied volatility of the underlying should not be dependent on the specifications of the contract. Thus to obtain market prices of options maturing at a certain date, volatility needs to be a function of the strike. This function is the so called volatility skew or smile. Furthermore for a fixed strike we also need different volatility parameters to match the market prices of options maturing on different dates written on the same underlying, hence volatility is a function of both the strike and the expiry date of the derivative security. This bivariate function is called the volatility surface. There are two prominent ways of working around this problem, namely, local volatility models and stochastic volatility models. For local volatility models the assumption of constant volatility made in Black and Scholes [1973] is relaxed. The underlying risk-neutral stochastic process becomes

$$dS_t = r(t)S_t dt + \sigma(t, S_t)S_t d\tilde{W}_t$$

where  $r(t)$  is the instantaneous forward rate of maturity  $t$  implied by the yield curve and the function  $\sigma(S_t, t)$  is chosen (calibrated) such that the model is consistent with market data, see Dupire [1994], Derman and Kani [1994] and [Wilmott, 2000, x25.6]. It is claimed in Hagan et al. [2002] that local volatility models predict that the smile shifts to higher prices (resp. lower prices) when the price of the underlying decreases (resp. increases). This is in contrast to the market behavior where the smile shifts to higher prices (resp. lower prices) when the price of the underlying increases (resp. decreases). Another way of working around the inconsistency introduced by constant volatility is by introducing a stochastic process for the volatility itself; such models are called stochastic volatility models. The major advances in stochastic volatility models are Hull and White [1987], Heston [1993] and Hagan et al. [2002].

## 2 Market Model

Suppose to be given two assets. One is a riskless asset (bond) with price  $B_t$  at time  $t \in [0, T]$ , described by ordinary differential equation  $dB_t = rB_t dt$ , where  $r$  is a nonnegative constant, is the instantaneous interest rate for lending or borrowing money. The price  $S_t$  of the other asset, the risky stock or stock index, evolves according to the stochastic differential equation  $dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^{(1)}$ , where  $\mu$  is a constant mean return rate, and  $\nu_t$  is the variance (we want to remember that the variance is equal to volatility square:  $\nu_t = \sigma_t^2$ ); we suppose in what

follows that  $\nu_t$  will be a random process too,  $d\nu_t = \beta(\nu_t)dt + \alpha\sqrt{\nu_t}dW_t^{(2)}$ . By Girsanov theorem, we change the natural probability measure  $\mathbb{P}$  into a equivalent martingale risk-free measure  $\mathbb{Q}$ , by which our market model becomes the following:

$$\begin{aligned} dS_t &= rS_tdt + \sqrt{\nu_t}S_td\tilde{W}_t^{(1)}, \\ d\nu_t &= \alpha\sqrt{\nu_t}d\tilde{W}_t^{(2)}, \quad \alpha \in \mathbb{R}^+ \\ d\tilde{W}_t^{(1)}d\tilde{W}_t^{(2)} &= \rho dt, \quad \rho \in (-1, +1) \\ dB_t &= rB_tdt. \end{aligned} \tag{1}$$

From here, we are able to say that the risky stock price  $S_t$  increases, if the correlation factor increases too. Major is the market liquidity lower is the correlation  $\rho$ , therefore we are going to use the correlation factor as index of market liquidity.

## 2.1 PDE Approach

Generally speaking, stochastic volatility models are not complete, and thus a typical contingent claim (such as a european option) cannot be priced by arbitrage. In Other words, the standard replication arguments cannot longer be applied to most contingent claims. For this reason, the issue of valuation of derivative securities under market incompleteness has attracted considerable attention in recent years, and various alternative approaches to this problem were subsequently developed. Seen form a different perspective, the incompleteness of a generic stochastic volatility model is reflected by the fact that the class of all martingale measure for the process  $S_t/B_t$  comprises more than one probability measure, and thus the necessity of specifying a single pricing probability arises. Since under (2), we deal with a two-dimensional diffusion process, it is possible to derive, under mild additional assumptions, the partial differential equation satisfied by the value function of a European contingent claim. For this purpose, one needs first to specify the market price of volatility risk  $\lambda(\nu, t)$ . Mathematically speaking, the market price for the risk is associated with the Girsanov transformation of the underlying probability measure leading to a particular martingale measure. Let us observe that pricing of contingent claims using the market price of volatility risk is not preferences-free, in general (typically, one assumes that the representative investor is risk-averse and has a constant relative risk-aversion utility function). To illustrate the PDE approach mentioned above, assume that the dynamic of two dimensional diffusion process  $(S, \nu)$ , under a martingale measure, is given by (2), with Brownian motions  $\tilde{W}_t^{(1)}$  and  $\tilde{W}_t^{(2)}$  such that  $d\tilde{W}_t^{(1)}d\tilde{W}_t^{(2)} = \rho dt$  for some constant  $\rho \in (-1, +1)$ . Suppose also that both processes  $S$  and  $\nu$ , are nonnegative. Then the price function  $f = f(t, S, \nu)$  of a European contingent claim is well known to satisfy a specific PDE. Let us stress once again that we do not claim here that  $\mathbb{Q}$  is a unique martingale measure for a given model. Hence unless volatility-based derivatives are assumed to be among primary assets, the market price of volatility risk needs to be exogenously specified. For some specifications of stochastic volatility dynamics and the market price of volatility risk, a closed-form expression for the option's price is available. In other cases, suitable numerical procedures need to be employed. The calculation based on the discretization of the partial differential equation satisfied by the pricing function appear excessively time-consuming. An alternative Monte Carlo approach for stochastic volatility models was examined by Fournié (1997).

### 3 Derivatives Pricing

When the volatility is a Markov Itô processes, we have a pricing function for European derivatives of the form  $f(t, S, \nu)$  from no-arbitrage arguments, as in the Black-Scholes case, the function  $f(t, S, \nu)$  satisfies a partial differential equation with two space dimensions ( $S$  and  $\nu$ ); the price of the derivative depends on the value of the process  $\nu$ , which is not directly observable. We now write the partial differential equation for pricing, assuming that (2) is the market model:

$$\begin{aligned}
 dS_t &= rS_t dt + \sqrt{\nu_t} S_t d\tilde{W}_t^{(1)}, \\
 d\nu_t &= \alpha \sqrt{\nu_t} d\tilde{W}_t^{(2)}, \quad \alpha \in \mathbb{R}^+ \\
 d\tilde{W}_t^{(1)} d\tilde{W}_t^{(2)} &= \rho dt, \quad \rho \in (-1, +1) \\
 dB_t &= rB_t dt. \\
 f(T, S_T, \nu_T) &= \Phi(S_T)
 \end{aligned} \tag{2}$$

and  $f(T, S_T, \nu_T)$  is a derivative contract. Thus we can write by Feynman-Kač formula the pricing PDE:

$$\begin{aligned}
 \frac{\partial f}{\partial t} + \frac{1}{2} \nu \left( S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\alpha S \frac{\partial^2 f}{\partial S \partial \nu} + \alpha^2 \frac{\partial^2 f}{\partial \nu^2} \right) + rS \frac{\partial f}{\partial S} - rf &= 0 \\
 f(T, S, \nu) &= \Phi(S_T)^+, \quad \rho \in (-1, +1), \quad \alpha \in \mathbb{R}^+ \\
 S \in [0, +\infty) \quad \nu \in [0, +\infty) \quad t \in [0, T]
 \end{aligned} \tag{3}$$

where  $\phi(S_T)$  is a general payoff function for a derivative security; in what follows we consider only Vanilla options. In order to manage a simpler PDE, we make some coordinate transformations:

$$1^a \text{ transformation } \begin{cases} x = \ln S, & x \in (-\infty, +\infty) \\ \tilde{\nu} = \nu/\alpha, & \tilde{\nu} \in [0, +\infty) \\ f(t, S, \nu) = f_1(t, x, \tilde{\nu}) e^{-r(T-t)} \end{cases}$$

$$2^a \text{ transformation } \begin{cases} \xi = x - \rho\tilde{\nu} & \xi \in (-\infty, +\infty) \\ \eta = -\tilde{\nu}\sqrt{1-\rho^2} & \eta \in (-\infty, 0] \\ f_1(t, x, \tilde{\nu}) = f_2(t, \xi, \eta) \end{cases}$$

$$3^a \text{ transformation } \begin{cases} \gamma = \xi + r(T-t) & \gamma \in (-\infty, +\infty) \\ \delta = -\eta & \delta \in [0, +\infty) \\ \tau = -\frac{\alpha\eta}{2\sqrt{1-\rho^2}}(T-t) & \tau \in [0, +\infty) \\ f_2(t, \xi, \eta) = f_3(\tau, \gamma, \delta) \end{cases}$$

$$4^a \text{ transformation } \begin{cases} \phi = \tau + \epsilon\delta & \phi \in [0, +\infty) \\ \psi = \delta & \psi \in [0, +\infty) \\ f_3(\tau, \gamma, \delta) = f_4(\phi, \gamma, \psi) \end{cases}$$

The PDE that out comes from the above indicated transformations, is the following:

$$\left[ \frac{\partial f_4}{\partial \phi} - (1 - \rho^2) \left( \frac{\partial^2 f_4}{\partial \gamma^2} + \frac{\partial^2 f_4}{\partial \psi^2} \right) - \frac{\partial f_4}{\partial \gamma} \right] = 0$$

$$f_4(0, \gamma, \psi) = (e^{\gamma + \rho\psi / \sqrt{1 - \rho^2}} - E)^+, \quad \rho \in (-1, +1), \quad \alpha \in \mathbb{R}^+;$$

$$\gamma(-\infty, +\infty) \quad \psi \in [0, +\infty) \quad \phi \in [0, +\infty);$$
(4)

The new PDE (4) is simpler than (3) and we now are able to find its solution in closed form, and in order to obtain this, impose that:

$$f_4(\phi, \gamma, \psi) = e^{\frac{1}{4(1-\rho^2)}\phi + \frac{1}{2(1-\rho^2)}\gamma} f_5(\phi, \gamma, \psi),$$
(5)

thus we have:

$$\frac{\partial f_5}{\partial \phi} = (1 - \rho^2) \left( \frac{\partial^2 f_5}{\partial \gamma^2} + \frac{\partial^2 f_5}{\partial \psi^2} \right)$$

$$f_5(0, \gamma, \psi) = e^{-\frac{\gamma}{2(1-\rho^2)}} \left( e^{\gamma + \rho\psi / \sqrt{1 - \rho^2}} - E \right)^+$$

$$\phi \in [0, +\infty) \quad \psi \in [0, +\infty) \quad \gamma \in (-\infty, +\infty)$$
(6)

The new variables are linked to old variables as follows:

$$\begin{cases} \phi = \nu(T - t) \\ \psi = \frac{\nu}{\alpha} \sqrt{1 - \rho^2} \\ \gamma = \ln S - \rho \frac{\nu}{\alpha} + r(T - t) \end{cases}$$
(7)

$$f(t, S, \nu) = e^{-r(T-t) + \frac{\phi}{4(1-\rho^2)} + \frac{\gamma}{2(1-\rho^2)}} f_5(\phi, \gamma, \psi);$$
(8)

The solutions of PDE (6) is known in literature, and it can be written as integral, whose kernel  $G(0, \gamma', \psi' | \phi, \gamma, \psi)$  is a bivariate gaussian function:

$$f_5(\phi, \gamma, \psi) = \int_0^{+\infty} d\psi' \int_{-\infty}^{+\infty} d\gamma' f_5(0, \gamma', \psi') G(0, \gamma', \psi' | \phi, \gamma, \psi)$$

$$+ \int_0^{\phi} d\phi' \int_{-\infty}^{+\infty} d\gamma' f_5(\phi', \gamma', 0) \frac{\partial G(0, \gamma', \psi' | \phi, \gamma, \psi)}{\partial \psi'} \Big|_{\psi'=0}$$

where the second term is zero, because for  $\psi = 0$ , also  $\phi = 0$  at any time, see eq. (7), thus we can write as follows:

$$\begin{aligned}
f_5(\phi, \gamma, \psi) &= \int_0^{+\infty} d\psi' \int_{-\infty}^{+\infty} d\gamma' f_5(0, \gamma', \psi') G(0, \gamma', \psi' | \phi, \gamma, \psi) \\
&= \int_0^{+\infty} d\psi' \int_{-\infty}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left( e^{\gamma' + \rho\psi' / \sqrt{1-\rho^2}} - E \right)^+ G(0, \gamma', \psi' | \phi, \gamma, \psi) \\
&= \int_0^{+\infty} d\psi' \int_{\ln E - \rho\psi' / \sqrt{1-\rho^2}}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left( e^{\gamma' + \rho\psi' / \sqrt{1-\rho^2}} - E \right) G(0, \gamma', \psi' | \phi, \gamma, \psi)
\end{aligned} \tag{9}$$

the kernel  $G(0, \gamma', \psi' | \phi, \gamma, \psi)$  is given by:

$$G(0, \gamma', \psi' | \phi, \gamma, \psi) = \frac{1}{4\phi(1-\rho^2)} \left[ e^{-\frac{(\gamma'-\gamma)^2 + (\psi'-\psi)^2}{4\phi(1-\rho^2)}} - e^{-\frac{(\gamma'-\gamma)^2 + (\psi'+\psi)^2}{4\phi(1-\rho^2)}} \right]$$

Therefore we can rewrite:

$$\begin{aligned}
f_5(\phi, \gamma, \psi) &= \int_0^{+\infty} d\psi' \int_{\ln E - \rho\psi' / \sqrt{1-\rho^2}}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left( e^{\gamma' + \rho\psi' / \sqrt{1-\rho^2}} - E \right) \times \\
&\quad \frac{1}{4\phi(1-\rho^2)} \left[ e^{-\frac{(\gamma'-\gamma)^2 + (\psi'-\psi)^2}{4\phi(1-\rho^2)}} - e^{-\frac{(\gamma'-\gamma)^2 + (\psi'+\psi)^2}{4\phi(1-\rho^2)}} \right].
\end{aligned}$$

At this point we are able to write the fair price of a Call option in European style as follows:

$$\begin{aligned}
f(t, S, \nu) &= e^{-r(T-t) + \frac{\phi}{4(1-\rho^2)} + \frac{\gamma}{2(1-\rho^2)}} \int_0^{+\infty} d\psi' \int_{\ln E - \rho\psi' / \sqrt{1-\rho^2}}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left( e^{\gamma' + \rho\psi' / \sqrt{1-\rho^2}} - E \right) \times \\
&\quad \frac{1}{4\phi(1-\rho^2)} \left[ e^{-\frac{(\gamma'-\gamma)^2 + (\psi'-\psi)^2}{4\phi(1-\rho^2)}} - e^{-\frac{(\gamma'-\gamma)^2 + (\psi'+\psi)^2}{4\phi(1-\rho^2)}} \right];
\end{aligned}$$

Thus, if we indicate with  $C(t, S_t, \nu_t)$  the price at any time  $t$  of a Call option, we have:

$$\begin{aligned}
C(t, S_t, \nu_t) &= e^{\frac{\nu_t(T-t)}{2(1-\rho^2)}} S_t \left[ N(d_1, a_{0,1} \sqrt{1-\rho^2}) - e^{-(2\frac{\rho}{\alpha}\nu_t)} N(d_2, a_{0,2} \sqrt{1-\rho^2}) \right] \\
&\quad - e^{\frac{\nu_t(T-t)}{2(1-\rho^2)}} E e^{-r(T-t)} \left[ N(\tilde{d}_1, \tilde{a}_{0,1} \sqrt{1-\rho^2}) - N(\tilde{d}_2, \tilde{a}_{0,2} \sqrt{1-\rho^2}) \right]. \tag{10}
\end{aligned}$$

At difference of Black-Scholes market model with deterministic volatility in which we have one dimensional normal distribution functions, in the Black-Scholes with stochastic volatility, one has bivariate normal distribution functions, in which the arguments are:

$$\left\{ \begin{array}{l} d_1 = \frac{\nu_t/\alpha + 2\rho\nu_t(T-t)}{\sqrt{2\nu_t(T-t)}} \\ d_2 = \frac{-\nu_t/\alpha + 2\rho\nu_t(T-t)}{\sqrt{2\nu_t(T-t)}} \\ a_{0,1} = \frac{\ln(S_t/E) + (r+\nu_t)(T-t)}{\sqrt{2(1-\rho^2)\nu_t(T-t)}} \\ a_{0,2} = \frac{\ln(S_t/E) + (r+\nu_t)(T-t) - 2\rho\nu_t/\alpha}{\sqrt{2(1-\rho^2)\nu_t(T-t)}} \end{array} \right. \quad \left\{ \begin{array}{l} \tilde{d}_1 = \frac{\nu_t/\alpha}{\sqrt{2\nu_t(T-t)}} \\ \tilde{d}_2 = \frac{-\nu_t/\alpha}{\sqrt{2\nu_t(T-t)}} \\ \tilde{a}_{0,1} = \frac{\ln(S_t/E) + (r-\nu_t)(T-t)}{\sqrt{2(1-\rho^2)\nu_t(T-t)}} \\ \tilde{a}_{0,2} = \frac{\ln(S_t/E) + (r-\nu_t)(T-t) - 2\rho\nu_t/\alpha}{\sqrt{2(1-\rho^2)\nu_t(T-t)}} \end{array} \right. \tag{11}$$

It is worth noting that for  $\rho$  equal to zero we have:

$$d_2 = -d_1, \quad a_{0,1} = a_{0,2},$$

$$\tilde{d}_2 = -\tilde{d}_1, \quad \tilde{a}_{0,1} = \tilde{a}_{0,2}.$$

Similarly, we are able to write the fair price of a Put option in european style as follows:

$$\begin{aligned} P(t, S_t, \nu_t) = & e^{\frac{\nu_t(T-t)}{2(1-\rho^2)}} E e^{-r(T-t)} \left[ N\left(\tilde{d}_1, -\tilde{a}_{0,1}\sqrt{1-\rho^2}\right) - N\left(\tilde{d}_2, -\tilde{a}_{0,2}\sqrt{1-\rho^2}\right) \right] \\ & - S_t e^{\frac{\nu_t(T-t)}{2(1-\rho^2)}} \left[ N\left(d_1, -a_{0,1}\sqrt{1-\rho^2}\right) - e^{-2\frac{\rho}{\alpha}\nu_t} N\left(d_2, -a_{0,2}\sqrt{1-\rho^2}\right) \right]. \end{aligned} \quad (12)$$

### 3.1 Greeks and Put-Call-parity

The way to reduce the sensitivity of a portfolio to the movement of something by taking opposite positions in different financial instruments is called hedging. Hedging is a basic concept in finance. For the stochastic volatility market models used in literature, is not possible to write the Greeks in closed form. Contrariwise by our model we are able to compute the Greeks in closed form; and in order to verify the **Put-Call parity** relation, we compute only  $\Delta$  and  $\Gamma$  as follows:

$$\begin{aligned} \Delta_{call} = & \frac{\partial C(t, S, \nu)}{\partial S} \\ = & e^{\frac{\nu_t(T-t)}{2(1-\rho^2)}} \left[ N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right) - e^{-2\frac{\rho}{\alpha}\nu} N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right) \right] \\ + & E \frac{e^{\frac{\nu_t(T-t)}{2(1-\rho^2)}}}{\sqrt{2(1-\rho^2)}\nu(T-t)} \left[ \frac{\partial N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}} - e^{-2\frac{\rho}{\alpha}\nu} \frac{\partial N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}} \right] \\ - & \left( \frac{E^2}{S} \right) \frac{e^{-\left(r - \frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{\sqrt{2(1-\rho^2)}\nu(T-t)} \left[ \frac{\partial N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}} - \frac{\partial N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}} \right]. \end{aligned} \quad (13)$$



$$\begin{aligned}
\Delta_{put} &= \frac{\partial P(t, S, \nu)}{\partial S} \\
&= \left( \frac{E^2}{S} \right) \frac{e^{-\left(r - \frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[ \frac{\partial N\left(\tilde{d}_1, -\tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}} - \frac{\partial N\left(\tilde{d}_2, -\tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}} \right] \\
&\quad - e^{\frac{\nu t(T-t)}{2(1-\rho^2)}} \left[ N\left(d_1, -a_{0,1}\sqrt{1-\rho^2}\right) - e^{-\frac{2\rho}{\alpha}\nu} N\left(d_2, -a_{0,2}\sqrt{1-\rho^2}\right) \right] \\
&\quad - E \frac{e^{\frac{\nu t(T-t)}{2(1-\rho^2)}}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[ \frac{\partial N\left(d_1, -a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}} - e^{-\frac{2\rho}{\alpha}\nu} \frac{\partial N\left(d_2, -a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}} \right].
\end{aligned} \tag{14}$$

Thus we have:

$$\begin{aligned}
\Gamma_{call} &= \Gamma_{put} \\
&= \left( \frac{E}{S} \right) \frac{e^{\frac{\nu t(T-t)}{2(1-\rho^2)}}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[ \frac{\partial N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}} - e^{-\frac{2\rho}{\alpha}\nu} \frac{\partial N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}} \right] \\
&\quad + \left( \frac{E^2}{S} \right) \frac{e^{\frac{\nu t(T-t)}{2(1-\rho^2)}}}{2(1-\rho^2)\nu(T-t)} \left[ \frac{\partial^2 N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}^2} - e^{-\frac{2\rho}{\alpha}\nu} \frac{\partial^2 N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}^2} \right] \\
&\quad - \left( \frac{E^2}{S^2} \right) \frac{e^{-\left(r - \frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{2(1-\rho^2)\nu(T-t)} \left[ \frac{\partial^2 N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}^2} - \frac{\partial^2 N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}^2} \right] \\
&\quad + \left( \frac{E^2}{S^2} \right) \frac{e^{-\left(r - \frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[ \frac{\partial N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}} - \frac{\partial N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}} \right],
\end{aligned}$$

and we can conclude that our model verifies the **Put-Call parity** relation, and it is consistency with no-arbitrage hypothesis, as well as it is necessary in a perfect market.

## 4 Numerical Experiments

The price of a Call option is shown hereafter; but it is straightforward compute the value of a Digital option, or for example also of a cover warrant contract. We can observe that our model verifies the law of monotony according to volatility behaviour; in fact, if the volatility increases, the price increases too. Analogously, when the maturity date  $T$  increases also the price increases; again, if the interest rate increases, we obtain that the option price is more high than before. This behaviour is corrected and it can be observed in a true market, making trading. It is worth noting that we obtain the same price for negative and positive sign of correlation factor. We want to observe that in our model (2) the volatility oscillates around the initial value of volatility  $\sqrt{\nu_0}$ , in fact we have:

$$\sqrt{\nu_t} = \sqrt{\nu_0} + \frac{\alpha}{2} (\tilde{W}_t - \tilde{W}_0) \quad (15)$$

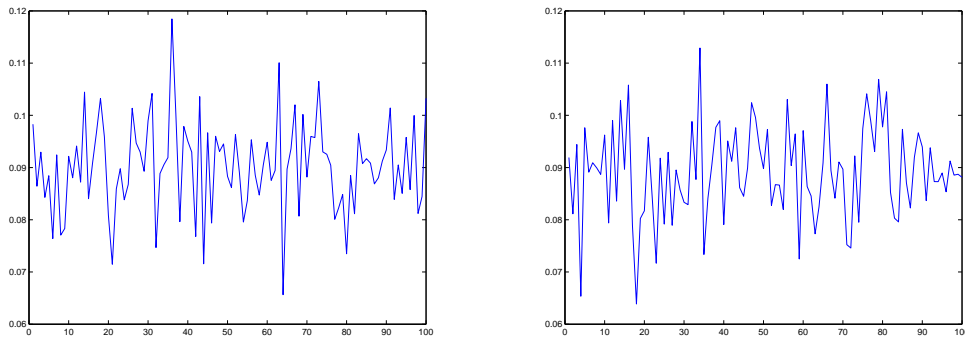


Figure 1: We have considered, for the simulation of the variance process  $\nu_t$ , the variance at the time zero, equal to 0.09,  $\alpha = 0.1$  and  $\rho = 0.6$ , where the time life or maturity date  $T$  in the first case is 1 year and in the second case is 10 years (our code is written in MatLab).

Table 1: At the money Call price for  $S_t = 100, E = 100, \alpha = 0.1, \rho = 0.6$

r	$\nu_t$	T-t	Value
0.03	0.01	6/12	4.3956
0.03	0.04	6/12	8.4123
0.03	0.09	6/12	11.8321
0.03	0.16	6/12	15.2633
0.03	0.25	6/12	18.6649

Table 2: In the money Call price for  $S_t = 100, E = 90, \alpha = 0.1, \rho = 0.6$

r	$\nu_t$	T-t	Value
0.03	0.01	6/12	9.8676
0.03	0.04	6/12	14.1145
0.03	0.09	6/12	16.9518
0.03	0.16	6/12	19.9909
0.03	0.25	6/12	23.1117

Table 3: Out the money Call price for  $S_t = 100, E = 110, \alpha = 0.1, \rho = 0.6$

r	$\nu_t$	T-t	Value
0.03	0.01	6/12	1.2788
0.03	0.04	6/12	4.6110
0.03	0.09	6/12	8.0435
0.03	0.16	6/12	11.5509
0.03	0.25	6/12	15.0435

Table 4: At the money Call price for  $S_t = 100, E = 100 \alpha = 0.1, \rho = 0.6$

r	$\nu_t$	T-t	Value
0.07	0.01	12/12	7.1764
0.07	0.04	12/12	13.9619
0.07	0.09	12/12	18.6758
0.07	0.16	12/12	23.1458
0.07	0.25	12/12	27.7939

Table 5: In the money Call price for  $S_t = 100, E = 90 \alpha = 0.1, \rho = 0.6$

r	$\nu_t$	T-t	Value
0.07	0.01	12/12	11.4257
0.07	0.04	12/12	19.2650
0.07	0.09	12/12	23.5779
0.07	0.16	12/12	27.6163
0.07	0.25	12/12	31.9621

Table 6: Out the money Call price for  $S_t = 100, E = 110 \alpha = 0.1, \rho = 0.6$

r	$\nu_t$	T-t	Value
0.07	0.01	12/12	3.8234
0.07	0.04	12/12	9.002
0.07	0.09	12/12	14.6888
0.07	0.16	12/12	19.3951
0.07	0.25	12/12	24.2216

## 5 Conclusions

The model proposed here is a general stochastic volatility model (2), and by using the latter we are able to write the option price in closed form. The advantages are a lot of. On one hand, it is cheaper from computational point of view, we have not the problems which plague the numerical methods. For example, one can consider the inverse Fourier transform method, in which we have to compute an integral between zero and infinity. In this case in fact, there is always some problem in order to define the correct domain of integration; or equivalently, considering also the finite difference method, in which we have to define a suitable grid, in other words we have some problems about the choice of the grid's meshes. Thus we suggest that our model might be easier, from the algorithmic point of view, and on other hand, it is easier to manage a portfolio, because through this is possible to compute in an explicit way the greeks. This is particularly interesting when we want to use the VaR technique in Risk-Management. In this case we have to know the sensitivity of first and second order, with respect to the variables  $S, \nu, t, r$  to evaluate how different our distribution is compared to Normal-distribution of yields, and by using the proposed model we are able to accomplish this. Another important consideration that we can make is due to correlation factor, by which we can accord our model at the liquidity of several markets. All this makes became that our model is a suitable and ductile market model

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