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Abstract

Simonson and Tversky *(Journal of Marketing Research*, 1992) demonstrated that the tendency to choose an alternative is enhanced or hindered depending on whether the tradeoffs within the set under consideration are favorable or unfavorable to that option (tradeoff contrast effect). In this paper we present an axiom that formulates this effect, and show that it yields a multiple utility representation in von-Neumann-Morgenstern framework: there is a unique convex set of vN-M utilities, such that an element is chosen if and only if it is best with respect to one of the utilities in this set. Similarly, we apply this axiom to Anscombe-Aumann framework and obtain a multiple prior representation. Finally, we study the notions of indifference, indecisiveness, and being more decisive in our models.

Key words: tradeoff contrast effect, incomplete preferences, multiple utilities, multiple priors, indecisiveness, more decisive, non-binary choice.
JEL classification: D81

1 Introduction

Choice behavior is often influenced by the set of alternatives under consideration (menu effects). Specifically, Itamar Simonson and Amos Tversky (1992) experimentally demonstrated the influence of *tradeoff contrast* effect. That is, the tendency to choose an alternative is enhanced or hindered depending on whether the tradeoffs within the set under consideration are favorable or unfavorable to that option. To quote Simonson and Tversky (1992, p. 282):

> "Consider the choice between options that vary on two attributes. If neither option dominates the other, the comparison between them involves an evaluation of differences

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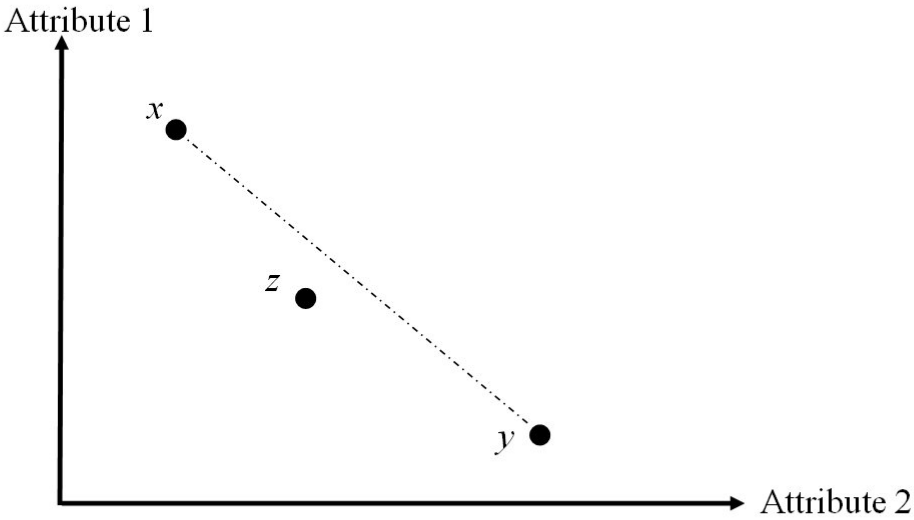
along the two attributes. Suppose that x is of higher quality and y has a better price. The decision between x and y, then, depends on whether the quality difference outweights the price difference, or equivalently on the tradeoff between price and quality implied by these options. According to the tradeoff contrast hypothesis, the choice between x and y is influenced by other implied tradeoffs in the set of options under consideration."

Consider, for example, the choice between the three objects displayed in Figure 1 (where each axis represents a positive attribute). Because the contrast between the x-y tradeoff

Figure 1. Contrast Tradeoff Detraction



and the x-z and y-z tradeoffs is unfavorable to z, z is expected to fare worse (to be chosen less often) in the triple than in the pairs. On the other hand, if z was above the x-y line, then it would be expected to fare better in the triple.

The tradeoff contrast effect can be restated as follows: an alternative fares worse in a set (compared with its choice probability in the pairs) if it is inferior to a mixture of other alternatives in this set; and it fares well if it is not inferior to any mixture in the choice set. In this paper we propose a choice theoretic formulation for this effect in a non-probabilistic framework, where "fares worse" is interpreted as not being chosen in the set. In our framework, choice behavior is described by a choice correspondence C, which selects in each closed set of alternatives, a non-empty subset of choosable alternatives. We say that alternative x is revealed *inferior* to y, if x is never chosen when y is in the choice set. The new axiom we propose, *convex axiom of revealed non-inferiority* (CARNI), requires that an alternative be chosen if and only if it is not revealed inferior to any alternative in the convex hull of the choice set. [2]

[2] See equivalent formulations for CARNI in Section 2. We note that for CARNI to induce a non-empty choice correspondence, the inferiority relation must satisfy the following requirement: if x is a mixture of other alternatives, then at least one of these alternatives is not inferior to x. A sufficient condition for this requirement, is that the inferiority relation would satisfy independence. The *maxmin expected utility* (Itzhak Gilboa and David Schmeidler, 1989) is an example for a model where this requirement is not satisfied.

Our first model uses the framework of John von-Neumann and Oscar Morgenstern (1944), where each choice set includes lotteries over a finite set of consequences.[3] In some situations, DMs may use internal randomization devices. That is, when a DM has to choose one of the elements in A, she may base her choice on a private lottery (i.e., tossing a coin), and by doing this, she can induce compound lotteries, which are equivalent to alternatives in $conv(A)$. In such situations, CARNI also has a normative appeal. If a DM would be exposed to an analysis that shows that her chosen alternative q is inferior to an element in $conv(A)$, which can be induced by a private lottery, then it seems plausible that she would like to change her choice. Thus, violating CARNI, in such situations, is irrational according to the definition of Gilboa and Schmeidler (2001, page 25): *"An action, or a sequence of actions is rational for a DM if, when the DM is confronted with an analysis of the decisions involved, but with no additional information, she does not regret her choices."*

Our first result shows that satisfying three standard axioms (non-triviality, continuity, and independence) and CARNI is equivalent to the following multiple utility representation: There exists a unique (up to linear transformations) convex and compact set U of vN-M utility functions, such that for every choice set A and every lottery q, q is chosen in A if and only if it is best with respect to one of the utilities in U.[4] This representation is interpreted as follows: The DM has several ways to evaluate alternatives, each with a different justification (rationale). Additional payoff-irrelevant information that is observable or available during the choice process determines which justification is used. Each justification triggers the DM to base her evaluation on a specific anchoring utility.[5]

Kfir Eliaz and Efe A. Ok (2006) presented a similar axiomatic model. Their key axiom, weak axiom of revealed non-inferiority (WARNI), requires that an alternative be chosen if and only if it is not revealed inferior to any alternative in the choice set. This yields the following representation: there exists a convex set U of vN-M utility functions, such that for every choice set A and every lottery q, q is chosen in A if and only if, for every lottery r in the choice set, there is a utility u_r in the set, such that q is better than r with respect to u_r. Observe, that unlike our representation, in Eliaz and Ok's representation, lottery q can be chosen in the triple $\{q, r, r'\}$ if it is better than r according to one utility, and better than r' according to a different utility, even though q does not maximize any utility in U.

Recently, Paola Manzini and Marco Mariotti (2010) experimentally tested how people violate the weak axiom of revealed preference (WARP) in their choices. Specifically, they divide the possible violations of WARP into two groups: 1) *pairwise inconsistency* - choices over the couples are not transitive; and 2) *menu effects* - choices over the couples do not induce choices over larger sets. Manzini and Mariotti show that menu effects are largely responsible for failures of WARP, and they conclude that on the basis of their data, "*any*

[3] Simonson and Tversky (1992, 1993) experimentally demonstrated the tradeoff contrast effect for choices between multi-attribute products. In Heller (2010) we present some experimental evidence that this effect also exists in choices between lotteries (von-Neumann-Morgenstern framework).

[4] A similar representation was presented non-axiomatically in Isaac Levi (1974).

[5] In this paper, we do not explicitly model the process in which payoff-irrelevant information determines the justification. Some examples for such processes are: framing effect (Tversky and Daniel Kahneman, 1981), availability heuristics, and anchoring (Tversky and Kahneman, 1974).

procedure that fails to account for menu effects will not make a significant improvement of the standard maximization model". WARNI implies that choices over couples induce choices over larger sets,[6] and thus it cannot account for menu effects. Their result has motivated us to modify Eliaz and Ok's model in a way that can account for some menu effects (specifically, for the tradeoff contrast effect), while retaining a normative appeal.

As discussed earlier, in Eliaz and Ok's representation an alternative can be chosen based on the simultaneous use of conflicting rationales (different utilities). Our second motivation is to model choice behavior in which in each choice problem the DM uses a single coherent rationale to evaluate all alternatives ("multiple-selves" approach). This choice behavior is more natural in many choice situations. One example for such a situation, which is described in Ehud Lehrer and Roee Teper (2010) is choices in large-scale organizations, where responsibility for different choices is delegated to different employees, each employee has a different rationale, and all rationales are consistent with the organization's common information and policy. Another example for such a situation is given in Example 1.

Example 1 There are four consequences: *bn*="beef near", *bf*="beef far", *cn*="chicken near", *cf*="chicken far". Let q be a 50:50 lottery with prizes bf and cf. Assume that the DM may like either chicken or beef (two justifications) and also dislikes eating too far from home. Then q may beat bn based on the "chicken" justification (that is, $\{bn, q\} = C(\{bn, q\})$); similarly, q may beat cn based on the "beef" rationale ($\{cn, q\} = C(\{cn, q\})$). But intuitively, if both bn and cn are available, q should not be chosen ($\{bn, cn\} = C(\{bn, cn, q\})$): the DM can get her favorite meal at a nearby restaurant, regardless of whether she wants beef or chicken. Observe, that this choice behavior is consistent with CARNI (q is not chosen in the triple because it is inferior to the mixture of bn and cn), and is inconsistent with WARNI (as WARNI implies: $q \in C(\{bn, q\})$ and $q \in C(\{cn, q\}) \Rightarrow q \in C(\{bn, cn, q\})$)

In our second model we apply CARNI to the Anscombe-Aumann framework, where each alternative is an *act* – a function that assigns a lottery in each state of nature. Our second result shows that satisfying four standard axioms (non-triviality, monotonicity, continuity, and independence) and CARNI is equivalent to the following representation: There exists a unique (up to linear transformations) vN-M utility u, and a unique convex and closed set P of priors (probability distributions over the state of nature), such that for every set A and every act f, f is chosen from A if and only if it is a best act with respect to one of the priors in the set.[7]

In Section 3 we show how observable choice data can be used to identify when the DM is indifferent between two alternatives, and when she is indecisive between them, and we characterize these notions in terms of our representations (adapting the methods presented in Eliaz and Ok, 2006). Specifically, we define a DM to be *indecisive* between q and r,

[6] That is, element q is chosen in A if and only if it is chosen in any couple $\{q, r\}$ for each r in A.
[7] Our representation is similar to the binary choice correspondence that is derived from Knightian preference (Truman F. Bewely, 2002; see also justifiable preference of Lehrer and Teper, 2010): There is a unique vN-M utility, and a unique convex set of priors, such that act f is chosen if for every act in the choice set, g, there is a prior p_g such that f is better than g with respect to the utility and the prior p_g. As in the vN-M framework, our representation differs in the requirement that a chosen act would be better than all other acts in the set with respect to the same prior.

if her preference between the two alternatives depends on the menu: there is a menu where both elements are available but only q is chosen, and there is a menu where both elements are available but only r is chosen. We define Alice to be *more decisive* than Bob, if whenever Alice is indecisive between q and r, so is Bob. Our last result characterizes the more decisive notion in terms of our representations. In the multiple utility representation Alice is more decisive than Bob if: 1) Alice has a single utility, or 2) Alice's set of utilities is included in Bob's set of utilities, or 3) Alice's set of utilities is included in the opposite of Bob's set of utilities.[8] Similarly, in the multiple prior representation Alice is more decisive than Bob if: 1) Alice has a single prior, or 2) Alice's set of priors is included in Bob's set of priors, and in addition they share the same utility, or have exactly opposite utilities. This characterization can be applied to other models of incomplete preferences (Bewley, 2002; Juan Dubra, Maccheroni and Ok, 2004; and Eliaz and Ok, 2006; see Section 3).

The paper is organized as follows. Section 2 presents the models and the results. In Section 3 we investigate the notions of indecisiveness and indifference in our model. Different aspects of our model, and its relations to the existing literature are discussed in Section 4. Section 5 includes the proofs.

2 Models and Results

2.1 Risk (von-Neumann-Morgenstern Framework)

2.1.1 Preliminaries

Let X be a finite set of consequences (certain prizes).[9] Let $Y = \triangle(X)$ be the set of lotteries over X. The mixture (convex combination) of two lotteries is defined as follows: $(\alpha q + (1-\alpha) r)(x) = \alpha q(x) + (1-\alpha) r(x)$ (where $\alpha \in [0,1]$, $q, r \in Y$ and $x \in X$). Similarly, given $A \subseteq Y$, let $\alpha q + (1-\alpha) A$ denote the set of lotteries that include all convex combinations of q with lottery r in A, with weights α and $1-\alpha$ respectively: $(\alpha q + (1-\alpha) A) = \{\alpha q + (1-\alpha) r | r \in A\}$.

The primitive of the model is a choice correspondence C over Y. The domain of C is all the non-empty closed sets in Y.[10] For each such set $A \subseteq Y$, $C(A)$ is a non-empty subset of A. The interpretation of C is the following: when a DM faces a choice from the elements in A, she chooses one of the alternatives in $C(A)$, and any alternative in $C(A)$ may be chosen. That is, the DM considers all the elements in $C(A)$, and only them, as choosable alternatives. The choice of a specific element in $C(A)$ is not explicitly modeled.[11] When

[8] That is, for each utility u_A of Alice there is utility u_B of Bob such that $u_A = a \cdot u_B + b$ where $a < 0$ and $b \in \mathbb{R}$.

[9] We define X to be finite for simplicity of presentation. Both models can be extended to a compact metric space of outcomes by adapting the proofs as in Ozgur Evren (2010) and Gilboa et al. (2010).

[10] We define C only on closed sets because in non-closed sets the Pareto frontier might be an empty set. Our results remain the same if C is defined only on finite (non-empty) sets.

[11] In the model's interpretation, the choice of a specific act in $C(A)$ depends on the payoff-irrelevant information that is observable during the choice process. In subsection 4.2 we present an alternative stochastic interpretation.

$q \in C(A)$ we say that q is choosable in A, or that the DM sometimes chooses q in A; similarly, when $q \notin C(A)$ we say that q is not choosable in A, or that the DM does not choose q in A. Given $A \subseteq Y$, $conv(A)$ denotes the convex hull of A (the smallest convex set that contains A).

The following three standard axioms (assumptions) are imposed on C:

A1 *Non-triviality.* $\exists A \subseteq Y$ and $\exists q \in A$, such that $q \notin C(A)$.
A2 *Continuity.* For any lottery $q \in Y$, the set $\{r \in Y | r \in C(\{q, r\})\}$ is closed, and the set $\{r \in Y | \{r\} = C(\{q, r\})\}$ is open.
A3 *Independence.* Let $q \in A \subseteq Y$, $r \in Y$ and $\alpha \in (0, 1)$. $q \in C(A) \Leftrightarrow \alpha r + (1 - \alpha) q \in C(\alpha r + (1 - \alpha) A)$.

Axioms A1-A3 are standard. Axiom A1 requires that C be non-trivial (there is a choice set with at least one unchoosable act). Axiom A2 (continuity) is equivalent to the requirement that for any lottery $q \in A$, the sets $\{r | r \succeq q\}$ and $\{r | r \preceq q\}$ are closed, where $\succeq$ is the revealed preference relation: $r \succeq q \Leftrightarrow r \in C(q, r)$.[12]

Assume that the DM is going to choose lottery q in A, when she finds out that there is some probability that event E occurs, and in that case she will have to take lottery r. Axiom A3 (independence) requires the DM to choose the mixture of q and r in the new choice problem (the mixture of A and r). That is, to choose lottery q if E does not occur. Observe that violating independence is time-inconsistent.

2.1.2 Convex Axiom of Revealed Non-Inferiority (CARNI)

Von-Neumann and Morgenstern (1944) assume that the choice correspondence satisfies the following axiom:

WARP *(Weak Axiom of Revealed Preference)* - Let $q, r \in A \cap B \subseteq Y$. $q \in C(A)$ and $r \in C(B)$ implies $q \in C(B)$.

That is, if q, r are in the intersection of two sets, q is chosen in the first set, and r is chosen in the second set, then both alternatives are chosen in both sets. Von-Neumann and Morgenstern show that Axioms A1-A3 and WARP are equivalent to expected utility representation: There exists a unique vN-M utility function u, such that the chosen lotteries are best according to u. That is, for every closed set $A \subseteq L$ and every lottery $q \in A$: $q \in C(A) \Leftrightarrow u(q) \geq u(r)\ \ \forall r \in A$.

Most models that generalize expected utility (such as, Mark J. Machina, 1982) and subjective expected utility (such as, Gilboa and Schmeidler, 1989; Schmeidler, 1989; Paolo Ghirardato, Maccheroni and Marinacci 2004, Maccheroni, Marinacci, and Aldo Rustichini, 2006), choose to weaken the independence axiom, and keep WARP. Some support for the independence axiom is found in Howard Raiffa (1961)'s results: most people that violate the independence axiom in Ellsberg's paradox, change their choices when presented with an analysis that shows that their original choices counter the independence axiom. R. Duncan Luce and Detlof von Winterfeldt (1994) discuss the experimental violations of

[12] Alternatively, A2 is equivalent to the requirement that sets $\{r | r \succ q\}$ and $\{r | r \prec q\}$ are open, where $\succ$ is the revealed strict preference relation ($r \succ q \Leftrightarrow \{r\} = C(q, r)$, which is equivalent to the psychological preferences of the DM as discussed in Subsection 3).

the independence axiom in the literature, and show that they are mostly caused by the violation of the assumption of reduction of compound lotteries to normal form, which is implicitly assumed in the frameworks of von Newman and Morgenstern, and Francis J. Anscombe and Robert J. Aumann (1963), which are used in the models in this paper, and in all the existing models mentioned above. It seems less likely to assume that people follow the reduction to normal form, but violate independence.

In this paper we weaken WARP and keep the independence axiom. With an eye to this relaxation we formulate WARP slightly differently:

WARP *(an alternative equivalent formulation)* - Let $q \in A \subseteq Y$. If there exists $r \in C(A)$ and $B \subseteq Y$ such that $q \in C(B)$ and $r \in B$, then $q \in C(A)$.

WARP is appropriate when the psychological preferences of the DM are complete. In such cases, $q \in C(B)$ and $r \in B$ imply that q is revealed to be weakly-superior to r (i.e., q is as good as r). Thus if r is chosen in A so is q.

When the psychological preferences are incomplete, there is a difference between something being superior and it being non-inferior for a DM. Eliaz and Ok (2006) propose the following axiom to deal with choice that is induced from incomplete preferences:

WARNI (Weak Axiom of Revealed Non-Inferiority) - Let $q \in A \subseteq Y$. If for every $r \in C(A)$ there exists a set $B \subseteq Y$ such that $q \in C(B)$ and $r \in B$, then $q \in C(A)$.[13]

When the psychological preferences are incomplete, $q \in C(B)$ and $r \in B$ only imply that q is revealed non-inferior to r, but it is not necessary that q is weakly-superior to r. WARNI requires that if q is revealed non-inferior to all the chosen alternatives in A, then it must be chosen from A as well. Following Eliaz and Ok (2006) one can show that axioms A1-A3 and WARNI are equivalent to the following multiple utility representation: There exists a convex and compact set U of vN-M utility functions (unique up to linear transformations), such that for every closed set $A \subseteq L$ and every lottery $q \in A$:[14]

$$q \in C(A) \Leftrightarrow \forall r \in A, \exists u_r \in U, \text{ s.t. } u_r(q) \geq u_r(r). \tag{1}$$

As discussed in the introduction, in some choice situations, it seems more appropriate to require a convex variation of WARNI, where a chosen act has to be non-inferior to all acts in the convex hull of A. This requirement is captured by CARNI:

A4 *Convex Axiom of Revealed Non-Inferiority (CARNI).* Let $q \in A \subseteq Y$. If $\forall r \in conv(C(A))$ there exists a set $B \subseteq Y$ with $q \in C(B)$ and $r \in conv(B)$, then $q \in C(A)$.

CARNI requires that if q is revealed non-inferior to all the alternatives in $conv(C(A))$, then it must be chosen in A as well. In order to have a better understanding of CARNI, we present in the following lemma equivalent formulations for CARNI.

Lemma 2 *Let C be a choice correspondence over Y. The following are equivalent:*

[13] Observe that it is immediate that $q \in C(A) \Rightarrow \exists B \subseteq Y$ s.t. $q \in C(B)$ and $r \in B$. Therefore, WARNI can be equivalently stated with an if and only if formulation: Let $q \in A \subseteq Y$. $q \in C(A) \Leftrightarrow (\forall r \in C(A), \exists B \subseteq Y \text{ s.t. } q \in C(B) \text{ and } r \in B)$.

[14] Eliaz and Ok ([?])'s representation is somewhat different than (1) due to their different continuity requirements. Their representation is as follows: $q \in C(A) \Leftrightarrow \forall r \in A$, $(\exists u_r \in U, \text{ s.t. } u_r(q) > u_r(r) \text{ or } \forall u \in U\, u(q) = u(r))$.

CARNI-1 *Let $q \in A \subseteq Y$. If $\forall r \in conv\,(C(A))$ there exists a set $B \subseteq Y$ with $q \in C\,(B)$ and $r \in conv\,(B)$, then $q \in C(A)$.*

CARNI-2: *Let $q \in A \subseteq Y$. $q \in C(A)$ if and only if $\forall r \in conv\,(C(A))$ $q \in C\,(\{q, r\})$.*

CARNI-3: *Let $q \in A \subseteq Y$. $q \in C(A)$ if and only if $\forall r \in conv\,(C(A))$ there exists a set $B \subseteq Y$ with $q \in C\,(B)$ and $r \in B$.*

The second formulation (CARNI-2) shows that satisfying CARNI is equivalent to a convex binariness property of C: an element is chosen in a set if and only if it is chosen in any couple in the convex hull of the set. The last formulation is the one motivated in the introduction: alternative q is revealed inferior to alternative r if q is never chosen when r is in the choice set; CARNI-3 requires that an alternative is chosen in a choice set if and only if it is not inferior to any other alternative in the convex hull of the set. In Subsection 4.3 we discuss the logical implications between WARP, WARNI and CARNI.

2.1.3 Representation Theorem

Replacing WARP/WARNI with CARNI yields the following multiple utility representation.

Theorem 3 *Let C be a choice correspondence over Y. The following are equivalent:*

(1) C satisfies axioms A1-A4 (non-triviality, continuity, independence and CARNI).

(2) There exists a convex and compact set U of linear (vN-M) utility functions, such that for every closed set $A \subseteq Y$ and every lottery $q \in A$:

$$q \in C(A) \Leftrightarrow \exists u \in U,\ s.t. \forall r \in A,\ u\,(q) \geq u\,(r)\,. \tag{2}$$

That is, a lottery is chosen if and only if it is best with respect to one of the utilities in U. Moreover:

(a) U is unique up to positive linear transformations. That is, if both U and V are convex compact sets that represent the same choice correspondence then $\forall u \in U$, $\exists v \in V$ such that $u = a \cdot v + b$ where $a > 0$ and $b \in R$.

(b) There are two consequences $\underline{x}, \overline{x} \in X$ such that $\forall u \in U$, $u\,(\underline{x}) < u\,(\overline{x})$.

Remark 4 Observe the difference in the orders of the quantifiers between Eliaz and Ok's representation (1) and our representation (2). In (1), each comparison of a chosen lottery q with some lottery $r \in A$ may be based on a different utility $u_r \in U$, while in (2) all comparisons are based on the same utility function $u \in U$. This change in the order of the quantifiers is implied by the extra convexity of CARNI (with respect to WARNI), which allows us to apply a minimax theorem in the proof.

2.2 Uncertainty (Anscombe-Aumann Framework)

2.2.1 Model

In this model we follow the framework of Anscombe-Aumann ([**?**], as reformulated in Peter C. Fishburn, 1970). Similar to the first model, X is a finite set of outcomes and $Y = \triangle\,(X)$ is the set of lotteries. Let S be a finite set of states of nature, and, abusing notation, let $S = |S|$. Let $L = Y^S$ be the set of all functions from states of nature to lotteries. Such functions are referred to as acts. Endow this set with the product topology, where the topology on Y is the relative topology inherited from $[0,1]^X$. Abusing notation, for an act $f \in L$ and a state $s \in S$, we denote by $f(s)$ the constant (unambiguous) act that assigns the lottery $f(s)$ to every state of nature. Similarly for a set $A \subseteq L$ and a state $s \in S$, let $A(s)$ denote the act-wise set of constant acts: $A(s) = \{f(s) | f \in A\}$.

Mixtures (convex combinations) of acts are performed point-wise. In particular if $f, g \in L$ and $\alpha \in [0,1]$, then $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$ for every $s \in S$. Similarly, let $(\alpha f + (1-\alpha)A)$ denote the set where each $g \in A$ is replaced by $\alpha f + (1-\alpha)g$: $(\alpha f + (1-\alpha)A) = \{\alpha f + (1-\alpha)g | g \in A\}$. As in the former model, the primitive is a choice correspondence C over L. The domain of C is all the non-empty closed sets in L. For each such set $A \subseteq L$, $C(A)$ is a non-empty subset of A.

The following five axioms are imposed on the choice correspondence:

B0 *Monotonicity.* Let $f \in A \subseteq L$ and $g \in B \subseteq L$. If $\forall s \in S$, $f(s) \in C(f(s), g(s))$ then: (i) $g \in C(B) \Rightarrow f \in C(B \cup \{f\})$, and (ii) $C(A) \subseteq C(A \cup \{g\})$.
B1 *Non-triviality.* There is an act $f \in A \subseteq L$ such that $f \notin C(A)$.
B2 *Continuity.* For any act $f \in L$, the set $\{g \in L | g \in C(f, g)\}$ is closed, and the set $\{g \in L | \{g\} = C(\{f, g\})\}$ is open.
B3 *Independence.* Let $f \in A \subseteq L$, $h \in L$ and $\alpha \in (0,1)$. $f \in C(A) \iff \alpha h + (1-\alpha) f \in C(\alpha h + (1-\alpha) A)$.
B4 *Convex Axiom of Revealed Non-Inferiority (CARNI).* Let $f \in A \subseteq L$. If $\forall g \in conv(C(A))$ there exists a set $B \subseteq L$ with $f \in C(B)$ and $g \in conv(B)$, then $f \in C(A)$.

We say that an act f (weakly) dominates an act g if for every state of nature $s \in S$ $\{f(s)\} \in C(\{f(s), g(s)\})$. That is, for every state of nature s, if the DM knows s, act f would be chosen in the pair $\{f, g\}$. Thus, f is better than g in all states of nature. Axiom B0 (monotonicity) requires that if f dominates g, then: (i) f is chosen whenever it is added to a set where g was a choosable alternative, and (ii) any alternative that is chosen in a set that includes f is also chosen after adding g to this set. Axioms B1-B3 are standard and are analogous to axioms A1-A3, which were discussed in the first model. Axiom B4 (CARNI) was discussed in the introduction.

Axioms B0-B3 and WARP [15] are equivalent to the subjective expected utility representation (Anscombe-Aumann, 1963; see also Leonard Jimmie Savage, 1954): There exists a unique vN-M utility function u, and a unique probability distribution p over S (prior), such that for every closed set $A \subseteq L$ and every act $f \in A$: $f \in C(A) \iff E_p(u(f)) \geq E_p(u(g))$ $\forall g \in A$.

That is, f is a best act according to the prior p (and the utility u).

Axioms B0-B3 and WARNI [16] are equivalent to the following representation: There exists a unique non-degenerate vN-M utility function u, and a unique set $P \subseteq \triangle(S)$ of priors, such that for every closed set $A \subseteq L$ and every act $f \in A$:

$$f \in C(A) \iff \forall g \in A, \exists p_g \in P \text{ s.t. } E_{p_g}(u(f)) \geq E_{p_g}(u(g)). \tag{3}$$

[15] In the Anscombe-Aumann framework WARP is formulated as follows: Let $f, g \in A \cap B \subseteq L$. $f \in C(A)$ and $g \in C(B)$ implies $f \in C(B)$.
[16] In the Anscombe-Aumann framework WARNI is formulated as follows: Let $f \in A \subseteq L$. If for every $g \in C(A)$ there exists a set $B \subseteq L$ with $f \in C(B)$ and $g \in B$, then $f \in C(A)$.

This representation is equivalent to the binary choice correspondence that is induced from Knightian preferences (Bewley, 2002) and from justifiable preferences (Lehrer and Teper, 2010).

2.2.2 Representation Theorem

Replacing WARNI with CARNI yields the following multiple prior representation:

Theorem 5 *Let C be a choice correspondence over L. The following are equivalent:*

(1) C satisfies axioms B0-B4.
(2) There exists a unique non-degenerate linear (vN-M) utility function u (up to positive linear transformations), and a unique convex and closed set $P \subseteq \triangle(S)$ of probability distributions over S (priors), such that for every closed set $A \subseteq L$ and every act $f \in A$:

$$f \in C(A) \Leftrightarrow \exists p \in P \text{ s.t. } \forall g \in A,\ E_p(u(f)) \geq E_p(u(g)). \tag{4}$$

That is, an act is chosen if and only it is best according to one of the priors in P (and the utility u).

Remark 6 As in the previous model, the extra convexity of CARNI allows us to change the order of the quantifiers in the representation. Specifically, in (3), each comparison of a chosen act f with some act $g \in A$ may be based on a different prior $p_g \in P$, while in (4), all comparisons are based on the same prior $p \in P$.

3 Indecisiveness and Indifference

As argued by Aumann (1962), Bewely (2002), Dubra, Maccheroni and Ok (2004), and Michael Mandler (2005), among others, rationality does not imply completeness of preferences. Incomplete preferences allows a DM to exhibit both indifference and indecisiveness. In this section we show (by adapting the methods presented by Eliaz and Ok, 2006) how choice data that satisfies CARNI allows us to fully characterize the psychological preferences of the DM, and to distinguish between indecisiveness and indifference. Finally, we define the notion of one DM being more decisive than another DM, and characterize it in terms of our representations.

3.1 Psychological Preferences

The DM's revealed psychological preference relation $\succeq^*$ is defined for each $q, r \in Y$ as follows:

$$q \succeq^* r \Leftrightarrow (\forall A \subseteq Y, \text{s.t.}\, q, r \in A,\ r \in C(A) \Rightarrow q \in C(A))$$

That is, alternative q is revealed to be as good as r, if q is chosen whenever r is chosen in a set that includes both alternatives. We also define a strict psychological preference relation $\succ$ for each $q \neq r \in Y$ as follows:

$$q \succ r \Leftrightarrow (\forall A \subseteq Y, \text{ s.t. } q \in A \Rightarrow r \notin C(A))$$

That is, alternative q is revealed to be strictly better than r if r is never chosen whenever q is included in the choice set. Observe that $q \succ r \Rightarrow q \succ^* r$ (where $\succ^*$ is the asymmetric part of $\succeq^*$), but the opposite is not necessary true.

The following lemma shows that the psychological preferences are transitive, and that they can be fully derived from choices over the couples.

Lemma 7 *Let C be a choice correspondence over Y that satisfies CARNI. Then:*

(1) For each $q, r \in Y$: $q \succeq^* r \Leftrightarrow (\forall p \in Y, r \in C(\{p, r\}) \Rightarrow q \in C(\{p, q\}))$.
(2) For each $q, r \in Y$: $q \succ r \Leftrightarrow (\{q\} = C(\{q, r\}))$
(3) The relations $\succeq^*$ and $\succ$ are transitive.

Next we define indifference and indecisiveness. For each $q, r \in Y$, we say that the DM is *indifferent* between q and r if $q \sim^* r$ (where $\sim^*$ is the symmetric part of $\succeq^*$). That is, the DM is indifferent between two alternatives if whenever both elements are available in the menu (choice set), then one of them is chosen if and only if the other one is chosen. We say that the DM is *indecisive* between q and r and we denote it by $q \bowtie r$ if $\neg q \succeq^* r$ and $\neg r \succeq^* q$. That is, the DM is indecisive between q and r if her choice between them depends on the menu: There is a menu where both elements are available but only q is chosen, and there is a menu where both elements are available but only r is chosen. Observe that $\bowtie$ does not have to be transitive.

Given two DMs, Alice and Bob, we say that Alice is *more decisive* than Bob if $q \bowtie_{Alice} r \Rightarrow q \bowtie_{Bob} r$. That is, whenever Alice is indecisive between two alternatives, so is Bob. Observe that when Alice and Bob are both decisive between q and r, their preferences might be different: it might be that $q \succeq^*_{Alice} r$ and $r \succeq^*_{Bob} q$.

3.2 Multiple Utility Characterization

Intuitively, a DM with a multiple utility representation is indifferent between q and r if all of her utilities assign both alternatives the same value ($u(q) = u(r)$ for every utility $u \in U$), and she is indecisive between the two alternatives if one of her utilities assigns a better value for q and another utility assigns a better value for r ($\exists u_1, u_2 \in U$ such that $u_1(q) > u_1(r)$ and $u_2(q) < u_2(r)$). In order to prove the equivalence between the multiple utility criteria for indifference and indecisiveness and the choice-derived definitions presented earlier, we have to assume that there is a best element in X: a choice correspondence C has a *best element* if there exists $x^b \in X$ such that for every $x \in X$, $\left\{x^b\right\} = C\left(\left\{x^b, x\right\}\right)$. Under this assumption, the following lemma shows this equivalence.[17]

Lemma 8 *Let C be a choice correspondence over Y that satisfies axioms A2-A4 and that has a best element . Let U be the multiple utility representation. Then for each $q, r \in Y$:*

[17] The following example shows why the best element assumption is required. Let X=$\{x, 0, 2\}$. Let $u_i(0) = 0$ and $u_i(2) = 2$ for $i = 1, 2$, $u_1(x) = 3$ and $u_2(x) = 1$. Let $U = conv(\{u_1, u_2\})$ be the set of utilities in the multiple utility representation. Then, x is chosen in all sets (as it maximizes u_1), and thus it is as good as any other element according to the choice-derived definition. Specifically, $x \succeq^* 2$ despite the fact that $u_2(x) < u_2(2)$.

(1) $r \succeq^* q \Leftrightarrow \forall u \in U,\ u(r) \geq u(q)$.
(2) $r \sim^* q \Leftrightarrow \forall u \in U,\ u(r) = u(q)$.
(3) $r \succ q \Leftrightarrow \forall u \in U,\ u(r) > u(q)$.
(4) $r \bowtie q \Leftrightarrow \exists u_1, u_2 \in U,\ u_1(r) > u_1(q)$ *and* $u_2(r) < u_2(q)$.

Finally, the following proposition characterizes when Alice is more decisive then Bob in terms of multiple utility representation . It shows that Alice is more decisive than Bob if: 1) Alice has a single utility, or 2) Alice's set of utilities is included in Bob's set of utilities., or 3) Alice's set of utilities is included in Bob's set of opposite utilities. The proposition assumes that both choice correspondences have a best element and a worse element. Choice correspondence C has a *worse element* if there exists $x^w \in X$ such that for every $x \neq x^w \in X$ $\{x\} = C(\{x^w, x\})$.

Proposition 9 *Let Alice and Bob be two DMs with respective choice correspondences* (C_A, C_B) *over* Y *that satisfy axioms A2-A4 and that have respective best elements* $\left(x_A^b, x_B^b\right)$ *and respective worse elements* (x_A^w, x_B^w) *with respective to multiple utility representations* (U_A, U_B)*. Then Alice is more decisive than Bob if and only if at least one of the following holds:*

(1) U_A *is a singleton (up to positive linear transformations). That is, each* $u_1, u_2 \in U_A$ *satisfy* $u_1 = a \cdot u_2 + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*).*
(2) $U_A \subseteq U_B$ *(up to positive linear transformations). That is, for each* $u_A \in U_A$ *there exist* $u_B \in U_B$*,* $a > 0$*, and* $b \in \mathbb{R}$ *such that* $u_B = a \cdot u_A + b$*.*
(3) $U_A \subseteq -U_B$ *(up to positive linear transformations). That is, for each* $u_A \in U_A$ *there exist* $u_B \in U_B$*,* $a < 0$*, and* $b \in \mathbb{R}$ *such that* $u_B = a \cdot u_A + b$*.*

Remark 10 *Proposition 9 can be applied to other models of incomplete preferences:*

(1) It would remain valid if one replaces axiom A4 (CARNI) with Eliaz and Ok (2006)'s WARNI.
(2) It induces a similar characterization for Dubra, Maccheroni, and Ok (2004)'s multiple utility preferences: let Alice and Bob be DMs with incomplete transitive preferences $(\succeq_A, \succeq_B)$ *on* Y*. Assume that each preference satisfies continuity and independence and has best and worse elements; let* U_{Alice}, U_{Bob} *be their respective multiple utility representations; then Alice is more decisive than Bob if and only if at least one of the following holds: 1)* U_A *is a singleton, 2)* $U_A \subseteq U_B$*, or 3)* $U_A \subseteq -U_B$*.*

3.3 Multiple Prior Characterization

Intuitively, a DM with a multiple prior representation is indifferent between f and g if all of her priors assign both alternatives the same value, and she is indecisive between the two alternatives if one of her priors assigns a better value for f and another prior assigns a better value for g. In order to prove the equivalence between the multiple prior criteria for indifference and indecisiveness and the choice-derived definitions presented earlier, we have to strengthen the monotonicity requirement (B0) and require *strong monotonicity*: Let $f, g \in L$. If $\forall s \in S,\ f(s) \in C(f(s), g(s))$ and there is $s^* \in S$ such that $\{f(s^*)\} = C(f(s^*), g(s^*))$ then $\{f\} = C(\{f, g\})$.[18] Under this assumption, the following lemma

[18] Strong monotonicity implies that every prior in the representation would have full support.

shows this equivalence. [19]

Lemma 11 *Let C be a choice correspondence over Y that satisfies axioms B0-B4 and strong monotonicity . Let u be the utility and P the set of priors in the multiple prior representation. Then for each $f, g \in L$:*

(1) $f \succeq^* g \Leftrightarrow \forall p \in P,\ E_p(u(f)) \geq E_p(u(g))$.
(2) $f \sim^* g \Leftrightarrow \forall p \in P,\ E_p(u(f)) = E_p(u(g))$.
(3) $f \succ g \Leftrightarrow \forall p \in P,\ E_p(u(f)) > E_p(u(g))$.
(4) $f \bowtie g \Leftrightarrow \exists p_1, p_2 \in P,\ E_{p_1}(u(f)) > E_{p_1}(u(g))$ *and* $E_{p_2}(u(f)) < E_{p_2}(u(g))$.

Finally, the following proposition characterizes the condition on the multiple prior representation that is equivalent to the choice-theoretic definition of more decisive that was given earlier. The lemma shows that Alice is more decisive than Bob if: 1) Alice has a single prior, or 2) Alice's set of priors is included in Bob's set of priors, and in addition Alice's utility is equal to Bob's utility or exactly the opposite of Bob's utility.

Proposition 12 *Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over L that satisfy axioms B0-B4 and strong monotonicity with respective multiple prior representations $((u_A, P_A), (u_B, P_B))$. Then Alice is more decisive than Bob if and only if at least one of the following holds:*

(1) P_A is a singleton (includes a single prior).
(2) $P_A \subseteq P_B$ and $u_A = u_B$ (up to positive linear transformations; that is, there exist $a > 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$).
(3) $P_A \subseteq P_B$ and $u_A = -u_B$ (up to positive linear transformations; that is, there exist $a < 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$).

Remark 13 *Proposition 9 can be applied to other models of incomplete preferences:*

(1) It would remain valid if one replaces axiom A4 (CARNI) with WARNI.
(2) It induces a similar characterization for Bewley's (2002) Knightian preferences: let Alice and Bob be DMs with incomplete transitive preferences $(\succeq_A, \succeq_B)$ on L. Assume that each preference satisfies non-triviality, completeness on the constant acts, strong monotonicity, continuity and independence; let $(u_{Alice}, P_{Alice}), (u_{Bob}, P_{Bob})$ be their respective multiple prior representations; then Alice is more decisive than Bob if and only if at least one of the following holds: 1) P_A is a singleton, 2) $P_A \subseteq P_B$ and $u_A = \pm u_B$.

[19] The following example shows why the strong monotonicity assumption is required. Let $X = \{0, 1\}$ and $S = \{s_1, s_2\}$. Let the utility of the DM satisfy $u(0) = 0$ and $u(1) = 1$, and let her set of priors P be the set of all priors. Let $p_1 \in P$ the prior that assigns probability one to state s_1. Finally, let $f = (0, 1)$ and $g = (1, 1)$. Then, f is chosen in all sets (as the prior p_1 assigns it a maximal value of 1), and thus it is as good as any other act according to the choice-derived definition. Specifically, $f \succeq^* g$ though g is better than f according to any prior in P except p_1.

4 Discussion

4.1 Applying CARNI in other frameworks

One can use CARNI to extend other axiomatizations of binary preferences to axiomatizations of non-binary justifiable choice correspondences. Ok, Pietro Ortoleva and Gil Riella (2010) present an axiomatization for a preference that is represented by either multiple priors or multiple utilities, and a few axiomatizations of multiple state-dependent utilities. Teddy Seidenfeld, Mark J. Scharvish and Joseph B. Kadane (1995) present an axiomatization for a preference that is represented by a set of pairs of state-dependent utilities/priors. It is possible to add CARNI to each of these axiomatic models and get the appropriate justifiable choice representation.

CARNI can also be used to axiomatize choice in multi-criteria problems. Assume that each alternative (e.g, a laptop) is characterized by n attributes (e.g., price, processor's speed, memory's size, weight, etc.), and the choices of the DM are derived from a choice correspondence C over $\mathbb{R}^n$ (vectors of attributes). Similarly to the proof of Theorem 3, one can show that C satisfies monotonicity ($\forall i \; x^i > y^i$ implies that $\{x\} = C(\{x, y\})$), continuity, independence to linear transformations ($\forall y \in \mathbb{R}^n$, $0 < \alpha \quad x \in C(A) \Leftrightarrow y + x \in C(y + A) \Leftrightarrow \alpha x \in C(\alpha A)$) and CARNI if and only if it has a multiple weight representation: There is a convex set of weights (unit vectors); each such weight is a linear evaluation of the tradeoff between the different attributes (weight w evaluates vector of attributes x as the scalar multiplication $w \cdot x$); an element is chosen if and only if it is best with respect to one of these weights.

4.2 Random Expected Utility

In this subsection we discuss the relations between Theorem 3 and Faruk Gul and Wolfgang Pesendorfer's model of random expected utility (2006). They consider choice data that consists of the frequency with which a DM chooses each of the elements in each finite choice set. A *random choice rule* is a function ρ that associates each finite choice set A with a probability distribution over the elements in A. That is, $\rho(A)(q)$ is the probability that q is chosen in the choice set A. Define a *random utility function* as a probability measure μ over the set of vN-M utilities, and say that it is regular if, in every choice set with probability 1, the realized utility function has a unique maximizer. Say that ρ maximizes μ if the probability that an element q is chosen according to ρ in A is equal to the probability of choosing a utility function that is maximized in A at q.[20] Gul-Pesendorfer's result provides necessary and sufficient conditions under which a random choice rule maximizes a regular random utility function. Specifically, they show that ρ satisfies (1) monotonicity, (2) continuity, (3) independence (linearity), and (4) extremeness (with probability 1, the chosen lottery is an extreme point of the choice set), if and only if it maximizes some regular random utility function.

Consider a situation where the choice data only consists the support of ρ. That is, the data

[20] If ρ is not regular, assume that each of the maximal elements of the chosen utility is chosen with some positive probability.

consists of a choice correspondence C with the following interpretation: $C(A)$ includes the elements that are chosen with positive probability. In this setup, our result is interpreted as providing necessary and sufficient conditions under which a choice correspondence can be explained as the support of a random choice rule that maximizes a (not necessarily regular) random utility function with a convex support.

To simplify the presentation of the result, we limit the characterization to the case where there is a best element in X. Specifically, say that a random utility function μ has a *best element* if there is $x^b \in X$ such that $u\left(x^b\right) > u(x)$ for every $x \in X$ and $u \in supp(\mu)$. Recall that a choice correspondence $\overline{C}(A)$ has a best element if there is $x^b \in X$ such that $\left\{x^b\right\} = \overline{C}\left(\left\{x, x^b\right\}\right)$ for every $x \in X$. Define the closure, $\overline{C}(A)$, of $C(A)$ as follows: $q \in \overline{C}(A)$ if for each $\epsilon > 0$ there exists a lottery q_ϵ in an ϵ-neighborhood of q such that $q_\epsilon \in C(A \cup \{q_\epsilon\})$. That is, $q \in \overline{C}(A)$ if it is chosen in A, or if it may become a chosen element by ϵ-perturbing it.

The following representation theorem is implied by Theorem 3: A choice correspondence $\overline{C}$ satisfies axioms A2-A4 (continuity, independence and CARNI) and has a best-element if and only if the following conditions hold: (1) $\overline{C}$ is a closure of some choice correspondence C; (2) C is the support of some random choice rule ρ; and (3) ρ maximizes some random utility function with a convex compact support and a best element.

4.3 Logical Implications between WARP, WARNI, CARNI and Binariness

A choice correspondence satisfies binariness if $q \in C(A) \Leftrightarrow \forall r \in A$, $q \in C(\{q, r\})$. That is, the revealed preference relation that describes choices over the couples, $q \succeq r \Leftrightarrow q \in C(\{q, r\})$, induces choices over larger sets: an alternative is chosen if and only if it is maximal with respect to $\succeq$.

Figure 1 describes the logical implications between different properties of choice correspondence: WARP, WARNI, independence, CARNI and binariness.

Figure 2. Logical Implications Between Different Properties of a Choice Correspondence

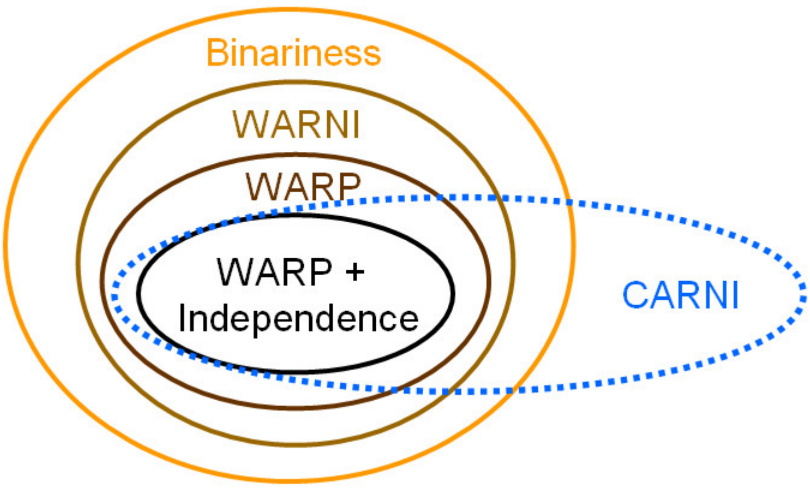


Figure 1 shows that:

- WARP implies WARNI, and WARNI implies binariness.

- WARP together with independence imply CARNI.[21]
- CARNI does not imply and is not implied by any of the other properties. Example 1 demonstrates a choice correspondence that satisfies CARNI and independence, and violates WARNI (and binariness). Modifying the choice in that example (having $\{bn, cn, q\} = C(\{bn, cn, q\})$) would give a choice correspondence that satisfies WARNI and independence, and violates CARNI (given that q is inferior to $0.5bn + 0.5cn$).

In most existing literature, the primitive of the model of rational choice under risk and uncertainty is a preference order. This implicitly assumes that the choice correspondence of a rational DM satisfies binariness. Recently Manzini and Mariotti (2007) showed in an experimental setting, that people often violate WARP, and that most violations are due to not satisfying binariness (menu effects). As discussed in the introduction, CARNI (unlike WARNI) allows us to axiomatize non-binary choice correspondence, and to capture the contrast tradeoff menu effect (Simonson and Tversky, 1992). Examples in the literature for other models with non-binary choice include the social choice models of Raveendran N. Batra and Prasanta K. Pattanaik (1972) and Rajat Deb (1983), and Klaus Nehring (1997)'s model for preference relation between an act and a set of acts. The choice correspondence in our models has a *global* binariness property that is not shared by the existing models mentioned above: the choices of the DM over all the couples in the global set L (or at least over all the couples in $conv(A)$) determine her choices in A.

4.4 Properties of CARNI

Empirical content: In some models (see, e.g., Gil Kalai, Ariel Rubinstein and Ran Spiegler, 2002), one can "rationalize" any choice data with enough justifications. This is not the case in our model. CARNI restricts the set of rationalizable choice correspondences by requiring that, when an element is not chosen from some set, then no justification can rank it as the top element in this set. Specifically, CARNI implies two standard properties with empirical content: contraction property (Sen's property α, $A \subseteq B$, $q \in A, q \in C(B) \Rightarrow q \in C(A)$), and irrelevant acts invariance (Aizerman's property, $A \subseteq B$, $C(B) \subseteq A \Rightarrow C(A) \subseteq C(B)$).[22]

Status-quo justification: In a dynamic environment in which at each stage the DM faces a new choice problem, violating the weak axiom (by following CARNI) may make the DM vulnerable to *money pumps*. This can be avoided if the choices from the choosable alternatives at each stage are based on a *status-quo justification*: The DM is triggered to evaluate alternatives according to utilities (or priors) that are consistent with her past choices. This kind of behavior has strong empirical support in the psychological literature. A closely related formal model is found in Bewley (2002).

Non-convexity of the chosen elements: Luce and Raiffa (1957, Chapter 13.3) present a list of 9 reasonable axioms for a rational choice correspondence under uncertainty. Satisfying all of them is equivalent to the subjective expected utility model. Our second

[21] This is evident from the following equivalent formulation of WARP (given Independence): Let $q \in A$. If $\exists r \in conv(C(A))$ and $B \subseteq Y$ such that $q \in C(B)$ and $r \in conv(B)$, then $q \in C(A)$.

[22] CARNI is equivalent to the combination of four independent properties: contraction property, irrelevant acts invariance, convex expansion ($\cup A_n$ is convex, $q \in \cap A_n$, and $\forall n \; q \in C(A_n) \Rightarrow q \in C(\cup A_n)$), and invariance to mixtures - $q \in C(A) \Rightarrow q \in C(conv(A))$.

model satisfies all of these axioms except the convexity of the set of chosen acts: if both acts f and g are chosen in A, and $\alpha f + (1-\alpha) g$ is an element of A, then $\alpha f + (1-\alpha) g$ is chosen in A. The following example demonstrates why this violation is plausible. Let $|S| = 2$, and $f, g, h \in L$ three acts with the following vN-M utilities: $u(f) = (1,0)$, $u(g) = (0,1)$ and $u(h) = (0.6, 0.6)$. Assume that the DM considers all priors to be possible. Let $A = \{f, g, h, 0.5f + 0.5g\}$. It is plausible that both $f, g \in C(A)$, as the DM believes that the probability of either state of nature may be high, and there are justifications for choosing either act. However, it is not rational to choose $0.5f + 0.5g$ because it has utility $(0.5, 0.5)$, which is strictly dominated by h.

Attitude to uncertainty: Consider the following example: $|S| = 2$, $X = \{\underline{x}, \overline{x}\}$, $\overline{x} = C(\underline{x}, \overline{x})$, $g = (\underline{x}, \overline{x})$ and $f = (0.5\underline{x} + 0.5\overline{x}, 0.5\underline{x} + 0.5\overline{x})$. Act f gives unambiguous probability 0.5 of obtaining the better outcome $\overline{x}$, while g gives $\overline{x}$ with the ambiguous probability that state 2 occurs. Assume that P, the set of possible priors, includes $(0.5, 0.5)$. Gilboa and Schmeidler (1989)'s model predicts that people would strictly prefer f over g, i.e., people are uncertainty averse, as experimentally observed in Ellsberg's paradox ([**?**]). Our model predicts that both acts are choosable, and that the attitude to uncertainty depends on the relevant justification. An experimental support for this prediction is found in Chip Heath and Tversky (1991), where it is shown that people may be uncertainty averse or uncertainty seekers, and that it depends on payoff-irrelevant observable information. Specifically, people prefer ambiguous events over equiprobable chance events when they consider themselves knowledgeable in the area that is the source of the uncertainty, and they prefer chance events when they consider themselves ignorant or uninformed.

4.5 Related Literature

In our models, choices are derived from multiple justifications (rationales) with the following properties: (1) Each justification is represented by an ordering. (2) The chosen acts are best with respect to one of the justifications. (3) Each justification may be used in all choice problems. Some related models for choice with multiple justifications are:

- Kalai, Rubinstein and Spiegler (2002) - The DM has several justifications, and each of them is used in a disjoint subset of choice problems.
- Manzini and Mariotti (2010) - The DM has several justifications that are used sequentially in a fixed order. Each justification is represented by incomplete preferences.
- Rubinstein and Yuval Salant (2008) - The DM has a set of justifications, and she uses one of the justifications according to how the choice problem is framed (for example, the order in which the acts are presented).
- Vadim Cherpanov, Timothy Feddersen and Alvaro Sandroni (2009) - The DM has several justifications, but only one preference relation. The chosen alternative is the most preferred among all the justifiable alternatives.

Unlike these models, we work with a more structured framework and this allows us to impose more structure on the justifications: the set of justifications is convex and closed, and each justification is a linear ordering.

Recently, Seidenfeld, Schervish and Kadane (2010) presented an axiomatic model for choice under uncertainty. They require three axioms, which are implied by CARNI: contraction (Sen's property α), irrelevant acts invariant (Aizerman's property), and invariance to mixtures - $f \in C(A) \Rightarrow f \in C(conv(A))$, and five standard axioms (non-triviality, continuity, independence, monotonicity and domination), and get a representation where the set of justifications (pairs of state dependent utilities/priors) is non-convex. Replacing these three axioms with CARNI would give a convex set of justifications.

5 Proofs

5.1 Equivalent formulations

In this subsection we prove Lemma 2:

Lemma 2: Let C be a choice correspondence over Y. The following are equivalent:

CARNI-1: Let $q \in A \subseteq Y$. If $\forall r \in conv(C(A))$ there exists set $B \subseteq Y$ with $q \in C(B)$ and $r \in conv(B)$, then $q \in C(A)$.

CARNI-2: Let $q \in A \subseteq Y$. $q \in C(A)$ if and only if $\forall r \in conv(C(A))$ $q \in C(\{q,r\})$.

CARNI-3: Let $q \in A \subseteq Y$. $q \in C(A)$ if and only if $\forall r \in conv(A)$ there exists set $B \subseteq Y$ with $q \in C(B)$ and $r \in B$.

PROOF. We show that each formulation implies the following formulation:

- $1 \Rightarrow 2$:
  - Let $q \in C(A)$ and $r \in conv(A)$. Assume to the contrary that $\{r\} = C(\{q,r\})$. This implies (by CARNI-1) that there is $r' \in conv(\{q,r\}) \subseteq conv(A)$ such that for every $B \subseteq Y$ with $r' \in conv(B)$, $q \notin C(B)$. Specifically (as $r' \in conv(A)$) $q \notin C(A)$ (a contradiction).
  - Let $q \in A \backslash C(A)$. This implies (by CARNI-1) that there is $r \in conv(C(A)) \subseteq conv(A)$ such that for every $B \subseteq Y$ with $r \in conv(B)$, $q \notin C(B)$. Specifically, $q \notin C(\{q,r\}) \Rightarrow \{r\} = C(\{q,r\})$.
- $2 \Rightarrow 3$: We show that given CARNI-2, $\forall q,r \in Y$ $q \in C(\{q,r\}) \Leftrightarrow \exists B \subseteq Y$ such that $q \in C(B)$ and $r \in B$. The $\Rightarrow$ part is immediate ($B = \{q,r\}$). The $\Leftarrow$ part is proved by observing that $q \in C(B)$, $r \in B$ and CARNI-2 imply that $q \in C(\{q,r\})$.
- $3 \Rightarrow 1$: Assume that $\forall r \in conv(C(A))$ there exists a set $B_r \subseteq Y$ with $q \in C(B_r)$ and $r \in conv(B_r)$. By CARNI-3, $\forall r' \in conv(B_r)$ there exists a set $B_{r'} \subseteq Y$ with $q \in C(B_{r'})$ and $r' \in B_{r'}$. Using CARNI-3 again, this implies that $q \in C(conv(B_r))$. $\forall r \in conv(C(A))$, $q \in C(conv(B_r))$ and $r \in conv(B_r)$. By CARNI-3 this implies that $q \in C(A)$.

5.2 Risk (von-Neumann-Morgenstern framework)

In this subsection we prove Theorem 3. We have to show that axioms A1-A4 (non-triviality, continuity, independence and CARNI) are sufficient for the multiple utility representation. The other direction is immediate. Let $\succ$ denote the *revealed strict preference relation* that is induced from C: $q \succ r \Leftrightarrow \{q\} = C(\{q,r\})$ $(q \neq r)$.

The following lemma shows that $\succ$ satisfies transitivity, non-triviality, continuity and independence.

Lemma 14 *Let C be a choice correspondence that satisfies axioms A1-A4, and let $\succ$ be the revealed strict preference. Then $\succ$ satisfies the following properties:*

C1 *Non-triviality* - There are $q, r \in Y$ such that $q \succ r$.
C2 *Continuity* - For each $q \in Y$ the sets $\{q|q \succ r\}$ and $\{q|q \prec r\}$ are open.
C3 *Independence* - For any $p, q, r \in Y$ and any $\alpha \in (0,1)$, $q \succ r \Leftrightarrow \alpha p + (1-\alpha) q \succ \alpha p + (1-\alpha) r$
C4 *Transitivity* - For any $p, q, r \in Y$, $p \succ q$ and $q \succ r$ implies that $p \succ r$.

PROOF. Axioms C1-C3 are immediately implied from the analogous properties of C (A1-A3). C4 (transitivity) is proved as follows. Let $p \succ q$ *and* $q \succ r$. By CARNI-2 (Lemma 2) $q, r \notin C(\{p, q, r\})$. This implies $\{p\} = C(\{p, q, r\})$. Assume to the contrary that $r \in C(\{p, r\})$. CARNI implies that $r \in C(\{p, q, r\})$ and we get a contradiction.

The following proposition (Theorem 1 in Evren, 2010) shows that $\succ$ has a unique multiple utility representation.[23]

Proposition 15 *(Evren, 2010, Theorem 1) Let $\succ$ be a strict binary relation over Y. The following are equivalent:*

(1) $\succ$ satisfies axioms C1-C4 (transitivity, non-triviality, continuity and independence).
(2) There exists a nonempty convex compact set U of linear (vN-M) utility functions, such that for every two lotteries $q, r \in Y$, $q \succ r \Leftrightarrow \forall u \in U$, $u(q) > u(r)$.
Moreover:
  (a) U is unique up to positive linear transformations. That is if both U and V are convex compact sets that represent the same choice correspondence then $\forall u \in U$, $\exists v \in V$ such that $u = a \cdot v + b$ where $a > 0$ and $b \in R$.
  (b) There are two outcomes $\underline{q}, \overline{q} \in X$ such that $\forall u \in U$, $u\left(\underline{q}\right) < u(\overline{q})$

We use Prop. 15 to finish Theorem 3's proof, by showing that axioms A1-A4 are sufficient for the multiple prior representation. Let C be a choice correspondence that satisfies these axioms, and let $\succ$ be the revealed strict preference. Let U be the unique (up to linear transformations) convex and compact set of utilities of Prop. 15. We have to show for each $q \in A \subseteq Y$, $q \in C(A) \Leftrightarrow \exists u \in U$, s.t. $u(q) \geq u(r)$ $\forall r \in A$. This is done as follows:

$$\begin{aligned}
q \in C(A) &\Longleftrightarrow \neg\exists r \in conv(A)\, s.t.\, r \succ q && (5)\\
&\Longleftrightarrow \forall r \in conv(A)\ \exists u \in U \text{ such that } u(q) \geq u(r) && (6)\\
&\Longleftrightarrow \min_{r \in conv(A)} \max_{u \in U} (u(q) - u(r)) \geq 0 &&\\
&\Longleftrightarrow \max_{u \in U} \min_{r \in conv(A)} (u(q) - u(r)) \geq 0 && (7)\\
&\Longleftrightarrow \exists u \in U \text{ such that } \forall r \in conv(A),\ u(q) \geq u(r) && (8)\\
&\Longleftrightarrow \exists u \in U \text{ such that } \forall r \in A,\ u(q) \geq u(r) &&
\end{aligned}$$

[23] Earlier versions of this paper included a different proof for Prop. 8. Recently, Evren ([**?**]) has independently proved a more general result that applies for a compact metric space X (and not only for a finite X). For brevity, we omit our original proof and rely on Evren's result.

where (5) is implied by CARNI-2 (Lemma 2), (6) is due to Proposition 15, (7) is implied by the minimax theorem (von-Neumann and Morgenstern, 1944) using the convexity of the sets U and $conv(A)$ and the linearity of each utility $u \in U$, and (8) is implied by the linearity of u.

5.3 Uncertainty (Anscombe-Aumann Framework)

In this subsection we prove Theorem 5. We have to show that axioms B0-B4 are sufficient for the multiple prior representation. The other direction is immediate. Let $\succeq$ denote the *revealed (weak) preference relation* that is induced from C: $q \succeq r \Leftrightarrow q \in C(\{q, r\})$, and let $\succ$ be its strict part (which is defined as in the previous subsection: $q \succ r \Leftrightarrow \{q\} = C(\{q, r\})$. Observe that CARNI-2 (Lemma 2) implies the *contraction* (alpha) *property*: $f \in B \subseteq A$ and $f \in C(A)$ imply that $f \in C(B)$.

The following proposition shows that $\succeq$ satisfies unambiguous transitivity, non-triviality, continuity, independence, completeness and favorable mixing.

Proposition 16 *Let C be a choice correspondence that satisfies axioms B0-B4, and let $\succeq$ be the revealed preferences . Then $\succeq$ satisfies the following properties:*

D0 *Unambiguous Transitivity.* Let $f, g, h \in L$ such that $\forall s \in S$ $f(s) \succeq g(s)$. Then, (i) $h \succeq f \Rightarrow h \succeq g$, and (ii) $g \succeq h \Rightarrow f \succeq h$.
D1 *Non-triviality.* There are acts $f, g \in L$ s.t. $f \succ g$.
D2 *Continuity.* For any $f \in L$, the sets $\{g|g \succeq f\}$ and $\{g|g \preceq f\}$ are closed.
D3 *Independence.* Let $f, g \in L$. $f \succeq g$ if and only if $\alpha h + (1-\alpha) f \succeq \alpha h + (1-\alpha) g$ for every $h \in L$ and $\alpha \in (0, 1)$.
D4 *Completeness and reflexivity.* For any $f, g \in L$, $f \succeq g$ or $g \succeq f$, and $f \sim f$.
D5 *Favorable mixing.* For every $f, g, h \in L$ *and* $\alpha \in (0, 1)$, *if* $g \succ f$ *and* $\alpha f + (1-\alpha) h \succeq g$, *then* $\lambda f + (1-\lambda) h \succeq g$, *for every* $0 < \lambda \leq \alpha$.

PROOF. Property D0 is implied by property A0 (monotonicity) and by the contraction property described above. Axioms D1-D3 are immediately implied by the analogous properties B1-B3. Axiom D4 is immediately implied from the definition of $\succeq$ as a revealed preference relation. D5 is proved as follows. Let $h' = \lambda f + (1-\lambda) h$ where $0 < \lambda \leq \alpha$. Assume to the contrary that $h' \prec g$. Observe that there exists $\beta \in (0, 1)$ such that $\alpha f + (1-\alpha) h = \beta f + (1-\beta) h'$. Independence (D3) implies that $h' \prec g$ $\Rightarrow \beta g + (1-\beta) h' \prec \beta g + (1-\beta) g = g$, and $f \prec g \Rightarrow \beta f + (1-\beta) h' \prec \beta g + (1-\beta) h'$. The transitivity of the strict preference $\succ$ (which is proved as in the previous subsection) implies that $6\alpha f + (1-\alpha) h = \beta f + (1-\beta) h' \prec g$, which contradicts the fact that $\alpha f + (1-\alpha) h \succeq g$.

The following proposition (Lehrer and Teper, 2010, Theorem 1) shows that $\succeq$ has a unique multiple prior representation.

Proposition 17 *(Lehrer and Teper, 2010, Theorem 1). Let $\succeq$ be a binary relation over L. The following are equivalent:*

(1) $\succeq$ satisfies axioms D0-D5.

(2) *There exists a unique (up to positive linear transformations) non-degenerate vN-M utility u, and a unique convex and closed set P of priors over the state of nature, such that for every two acts $f, g \in L$: $f \succeq g \Leftrightarrow \exists p \in P,\ E_p(u(f)) \geq E_p(u(g))$.*

Observe that Proposition 17 immediately implies that the strict relation $\succ$ has Knightian representation (Bewely, 2002): $f \succ g \Leftrightarrow \forall p \in P,\ E_p(u(f)) > E_p(u(g))$. We use Proposition 17 to finish Theorem 5's proof, by showing that axioms B0-B4 are sufficient for the multiple prior representation. Let C be a choice correspondence that satisfies these axioms, and let $\succ$ be the revealed strict preference. Let u be the unique (up to linear transformations) utility, and let P be the unique convex and closed set of priors of Proposition 17. We have to show, for each $f \in A \subseteq L$, $f \in C(A) \Leftrightarrow \exists p \in P$, s.t. $E_p(u(f)) \geq E_p(u(g))\ \forall g \in A$. This is done as follows:

$$
\begin{aligned}
f \in C(A) &\Longleftrightarrow \neg\exists g \in conv(A)\ s.t.\ g \succ f && (9)\\
&\Longleftrightarrow \forall g \in conv(A)\ \exists p \in P \,\text{such that}\, p\cdot u(f) \geq p \cdot u(g) && (10)\\
&\Longleftrightarrow \min_{g\in conv(A)} \max_{p\in P} \left(p\cdot u(f) - p\cdot u(g)\right) \geq 0 &&\\
&\Longleftrightarrow \max_{p\in P} \min_{g\in conv(A)} \left(p\cdot u(f) - p\cdot u(g)\right) \geq 0 && (11)\\
&\Longleftrightarrow \exists p \in P \,\text{such that}\, \forall g \in conv(A),\ p\cdot u(f) \geq p\cdot u(g) && (12)\\
&\Longleftrightarrow \exists p \in P \,\text{such that}\, \forall g \in A,\ p\cdot u(f) \geq p\cdot u(g) &&
\end{aligned}
$$

where (9) is implied by CARNI-2 (Lemma 2), (10) is due to Proposition 17, (11) is implied by the minimax theorem using the convexity of the sets P and $conv(A)$ and the linearity of each utility $u \in U$, and (12) is implied by the linearity of u.

5.4 Indecisiveness and indifference

In this subsection we prove the results of Section 3. The following lemma shows that the psychological preferences are transitive, and that they can be fully derived from choices over the couples.

Lemma 7 Let C be a choice correspondence over Y that satisfies CARNI. Then:

(1) For each $q, r \in Y$: $q \succeq^* r \Leftrightarrow (\forall p \in Y,\ r \in C(\{p,r\}) \Rightarrow q \in C(\{p,q\}))$.
(2) For each $q, r \in Y$: $q \succ r \Leftrightarrow (\{q\} = C(\{q,r\}))$
(3) The relations $\succeq^*$ and $\succ$ are transitive.

PROOF. Recall that $q \succeq^* r \Leftrightarrow (\forall A \subseteq Y, \text{s.t.}\ q, r \in A,\ r \in C(A) \Rightarrow q \in C(A))$.

(1)

- " $\Rightarrow$ ": Let $q \succeq^* r$. Assume to the contrary that there exists $p \in Y$ such that $r \in C(\{p,r\})$ and $q \notin C(\{p,q\})$. CARNI-2 implies that: $q \notin C(\{p,q\}) \Rightarrow q \notin C(\{p,q,r\})$. CARNI-1 implies that $r \in C(\{p,q,r\})$ (because if $\{p\} = C(\{p,q,r\})$ then $r \notin C(\{p,q,r\})$ contradicts CARNI-1). Thus there exists $A = \{p,q,r\}$ such that $q, r \in A$, $r \in C(A)$ and $q \notin C(A)$ (a contradiction).
- " $\Leftarrow$ ": Let $q, r \in Y$ satisfy $\forall p \in Y,\ r \in C(\{p,r\}) \Rightarrow q \in C(\{p,q\})$. Assume to the contrary that $q \not\succeq^* r$. Let $A \subseteq Y$ be such that $q, r \in A$, $r \in C(A)$ and $q \notin C(A)$.

By CARNI-2 there exists $p \in conv(A)$ such that $q \notin C(\{p,q\})$ and $r \in C(\{p,r\})$ (a contradiction).

(2) This follows immediately from Lemma 2.

(3) The transitivity of $\succ$ was proved in Lemma 14. We now prove the transitivity of $\succeq^*$. Assume to the contrary that $\exists p,q,r \in Y$ such that $p \succeq^* q$, $q \succeq^* r$ and $p \not\succeq^* r$. Let $t \in Y$ such that $r \in C(\{r,t\})$ and $p \notin C(\{p,t\})$. $\{t\} = C(\{p,q,r,t\})$ contradicts CARNI-1 (as it would imply that $r \in C(\{p,q,r,t\})$). $\{p,q,r\} \cap C(\{p,q,r,t\}) \neq \emptyset$ implies that $p \in C(\{p,q,r,t\})$ (because $p \succeq^* q$ and $q \succeq^* r$) and this contradicts $p \notin C(\{p,t\})$ (by CARNI-2).

5.4.1 Multiple Utility Representation

The following lemma characterizes the conditions on the multiple utility representation that are equivalent to the choice data definitions given in Section 3.

Lemma 8 Let C be a choice correspondence over Y that satisfies axioms A2-A4 and has a best element $x^b \in X$. Let U be the multiple utility representation. then:

(1) $r \succeq^* q \Leftrightarrow \forall u \in U,\ u(r) \geq u(q)$.

(2) $r \sim^* q \Leftrightarrow \forall u \in U,\ u(r) = u(q)$.

(3) $r \succ q \Leftrightarrow \forall u \in U,\ u(r) > u(q)$.

(4) $r \bowtie q \Leftrightarrow \exists u_1, u_2 \in U,\ u_1(r) > u_1(q)$ and $u_2(r) < u_2(q)$.

PROOF.

(1) " $\Leftarrow$ " is immediate. " $\Rightarrow$ " is proved as follows. Let $q, r \in Y$ be such that $r \succeq^* q$. Assume to the contrary that there exists $u^* \in U$ such that $u^*(q) > u^*(r)$. Observe that the fact that x_b is a best element implies that $u(p) < u\left(x^b\right)$ for every $u \in U$ and $p \neq x^b \in Y$. This implies that for every $u \in U$ and $0 < \alpha < 1$, $u(r) < u\left(\alpha r + (1-\alpha) x^b\right)$. Let $0 < \alpha < 1$ be large enough such that $u^*(q) > u^*(\alpha r + (1-\alpha) x_b)$. This implies that $r \notin C\left(\left\{r, \alpha r + (1-\alpha) x^b\right\}\right)$ and $q \in C\left(\left\{q, \alpha r + (1-\alpha) x^b\right\}\right)$. This contradicts the fact that $r \succeq^* q$.

Conditions (2) and (4) are immediate from condition (1). Condition (3) is immediate from the representation theorem.

Recall that Alice is more decisive than Bob, if whenever Alice is indecisive between q and r, so is Bob. The following proposition characterizes the condition on the multiple utility representation that is equivalent to being more decisive.

Proposition 9 Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over Y that satisfy axioms A2-A4 and that have respective best elements $\left(x_A^b, x_B^b\right)$ and respective worse elements (x_A^w, x_B^w) with respective multiple utility representations (U_A, U_B). Alice is more decisive than Bob if and only if at least one of the following holds:

(1) U_A is a singleton (up to positive linear transformations). That is, each $u_1, u_2 \in U_A$ satisfy $u_1 = a \cdot u_2 + b$ for some $a > 0$ and $b \in \mathbb{R}$).

(2) $U_A \subseteq U_B$ (up to positive linear transformations). That is, for each $u_A \in U_A$ there exist $u_B \in U_B$, $a > 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$.

(3) $U_A \subseteq -U_B$ (up to positive linear transformations). That is, for each $u_A \in U_A$ there exist $u_B \in U_B$, $a < 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$.

PROOF. It is immediate to see that either (1), (2) or (3) implies that Alice is more decisive than Bob. We now prove the opposite. We normalize each utility in U_A (U_B) such that $u_A\left(x_A^b\right) = 1$ and $u_A\left(x_A^w\right) = 0$ for each $u_A \in U_A$ ($u_B\left(x_B^b\right) = 1$ and $u_B\left(x_B^w\right) = 0$ for each $u_B \in U_B$). Assume that Alice is more decisive than Bob and that U_A is not a singleton.

We first show that $x_B^b \in \left\{x_A^b, x_A^w\right\}$. Assume to the contrary that there is a utility $u_A^* \in U_A$ such that $0 < u_A^*\left(x_B^b\right) < 1$. Let $\alpha = u_A^*\left(x_B^b\right)$. Assume first that there exists $u_A' \in U_A$ such that $u_A'\left(x_B^b\right) = \beta \neq \alpha$. Then Alice is indecisive between x_B^b and $\frac{\alpha+\beta}{2}x_A^b + \left(1 - \frac{\alpha+\beta}{2}\right)x_A^w$ due to Lemma 8 (if $\alpha > \beta$ x_B^b is better according to u^* while $\frac{\alpha+\beta}{2}x_A^b + \left(1 - \frac{\alpha+\beta}{2}\right)x_A^w$ is better according to u', and if $\alpha < \beta$ the opposite holds). This contradicts the fact that Bob is decisive between them because x_B^b is his best element. We are left with the case that $u_A\left(x_B^b\right) = \alpha$ for every $u_A \in U_A$. As U_A is not a singleton, there exists $q \in Y$ and $u_A^1, u_A^2 \in U_A$ such that $u_A^1(q) > u_A^2(q)$. By mixing q with x_A^b or x_A^w one can have q' such that $u_A^1(q') > \alpha > u_A^2(q')$. This implies that Alice is indecisive between x_B^b and q' while Bob is not (a contradiction). Similarly, one can show that $x_B^w \in \left\{x_A^b, x_A^w\right\}$. So only two cases are possible: 1) Alice and Bob has the same best element and the same worse element, or 2) Alice's best (worse) element is Bob's worse (best) element.

Case 1: $x_A^b = x_B^b = x^b$ and $x_A^w = x_B^w = x^w$. Let $u_A \in U_A$. Assume to the contrary that $u_A \not\subseteq U_B$. By a standard separation theorem (using the convexity and the compactness of U_B) there are $q, r \in Y$ such that $1 > \alpha = u_A(r) - u_A(q) > u_B(r) - u_B(q)$ for each $u_B \in U_B$.[24] Let $max_{u_B \in U_B}\left(u_B(r) - u_B(q)\right) = \beta$. Assume first that there is $u_A' \in U(A)$ such that $\gamma = u_A'(r) - u_A'(q) \neq \alpha$. By the convexity of U_A one can assume that $\beta < \gamma$. This implies that Alice is indecisive between

$$\frac{1}{1+\frac{\alpha+\gamma}{2}}r + \left(1 - \frac{1}{1+\frac{\alpha+\gamma}{2}}\right)x^w \text{ and } \frac{1}{1+\frac{\alpha+\gamma}{2}}q + \left(1 - \frac{1}{1+\frac{\alpha+\gamma}{2}}\right)x^b$$

(because if $\gamma > \alpha$ then the former is better according to u_A' and the latter is better according to u_A, and if $\gamma < \alpha$ the opposite holds), while Bob is decisive (the latter is better according to all of Bob's utilities) - a contradiction. So we are left with the case that $u_A'(r) - u_A'(q) = \alpha$ for every $u_A' \in U_A$. As U_A is not a singleton, there is $p \in Y$ and $u_A^1, u_A^2 \in U_A$ such that $u_A^1(p) > u_A^2(p)$. For sufficiently small $\delta > 0$, $r' = (1-\delta)r + \delta p$ and $q' = (1-\delta)q + \delta p$ satisfy: 1) $u_A^1(r') - u_A^1(q'), u_A^2(r') - u_A^2(q') > u_B(r') - u_B(q')$ for each $u_B \in U_B$, 2) $u_A^2(r') - u_A^2(q') \neq u_A^1(r') - u_A^1(q')$. By the previous argument, this leads to a contradiction. Thus, we have proved that in this case $U_A \subseteq U_B$.

[24] Extending each utility u from $\triangle(X)$ to $\mathbb{R}^{|X|}$, the standard separation theorem yields a signed unit vector v (possibly with negative values) such that $u_A(v) > u_B(v)$ for each $u_B \in U_B$. This vector v induces the two lotteries $q, r \in \triangle(X)$ as follows: $q = \frac{v^+}{\|v^+\|}$ ($q = x_w$ if $v^+ = \vec{0}$) and $r = \frac{v^-}{\|v^-\|}$ ($r = x_w$ if $v^- = \vec{0}$), where $v_i^+ = \max(v_i, 0)$ and $v_i^- =$ -min $(v_i, 0)$.

Case 2: Let Charlie be a DM with the exact opposite multiple utility representation with respect to Bob ($U_C = -U_B$). This implies that: 1) Charlie and Alice share the same best element and the same worse element, 2) The fact that Alice is more decisive than Bob implies that Alice is more decisive than Charlie (because Charlie is as decisive as Bob). By the proof of case 1 $U_A \subseteq U_C = -U_B$, which completes the proof.

5.4.2 Multiple Prior Representation

The following lemma characterizes the conditions on the multiple prior representation that are equivalent to the choice data definitions given in Section 3.

Lemma 11 Let C be a choice correspondence over Y that satisfies axioms B0-B4 and strong monotonicity . Let u be the utility and P the set of priors in the multiple prior representation. Then for each $f, g \in L$:
(1) $f \succeq^* g \Leftrightarrow \forall p \in P,\ E_p(u(f)) \geq E_p(u(g))$.
(2) $f \sim^* g \Leftrightarrow \forall p \in P,\ E_p(u(f)) = E_p(u(g))$.
(3) $f \succ g \Leftrightarrow \forall p \in P,\ E_p(u(f)) > E_p(u(g))$.
(4) $f \bowtie g \Leftrightarrow \exists p_1, p_2 \in P,\ E_{p_1}(u(f)) > E_{p_1}(u(g))$ and $E_{p_2}(u(f)) < E_{p_2}(u(g))$.

PROOF.

(1) " $\Leftarrow$ " is immediate. " $\Rightarrow$ " is proved as follows. Let $f, g \in L$ be such that $f \succeq^* g$. Assume to the contrary that there exist $p^* \in P$ such that $E_{p^*}(u(f)) < E_{p^*}(u(g))$. Let x_b be a weak best element in X ($u(x_B) \geq u(x)$ for each $x \in X$). By the strong monotonicity for every $p \in P$ and $0 < \alpha < 1$, $E_p(u(f)) < E_p\left(u\left(\alpha f + (1-\alpha)x^b\right)\right)$. Let $0 < \alpha < 1$ be large enough such that $E_p(u(g)) > E_p\left(u\left(\alpha f + (1-\alpha)x^b\right)\right)$. This implies that: $f \notin C\left(\left\{f, \alpha f + (1-\alpha)x^b\right\}\right)$ and $g \in C\left(\left\{g, \alpha f + (1-\alpha)x^b\right\}\right)$. This contradicts the fact that $f \succeq^* g$.

Conditions (2) and (4) are immediate from condition (1). Condition (3) is immediate from the representation theorem.

The following proposition characterizes the condition on the multiple prior representation that is equivalent to being more decisive.

Proposition 12 Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over L that satisfy axioms B0-B4 and strong monotonicity with respective multiple prior representations $((u_A, P_A), (u_B, P_B))$. Then Alice is more decisive than Bob if and only if at least one of the following holds:
(1) P_A is a singleton (includes a single prior).
(2) $P_A \subseteq P_B$ and $u_A = u_B$ (up to positive linear transformations; that is, there exist $a > 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$).
(3) $P_A \subseteq P_B$ and $u_A = -u_B$ (up to positive linear transformations; that is, there exist $a < 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$).

PROOF. It is immediate to see that either (1), (2) or (3) implies that Alice is more decisive than Bob. We now prove the opposite. Let $x_A^b \in X$ ($x_B^b \in X$) be a weak best element of Alice (Bob), and let $x_A^b \in X$ ($x_B^b \in X$) be a weak worse element of Alice (Bob).

We normalize u_A and u_B such that such that $u_A\left(x_A^b\right) = u_A\left(x_B^b\right) = 1$ and $u_A\left(x_A^w\right) = u_A\left(x_B^w\right) = 0$. Assume that Alice is more decisive than Bob and that P_A is not a singleton.

We first show that $u_B\left(x_A^b\right) \in \{0,1\}$. Assume to the contrary that $0 < u_B\left(x_A^b\right) < 1$. Then there are $0 < \alpha_1, \alpha_2$ satisfying: 1) $u_B\left(x_A^b\right) = u_B\left(\alpha_1 x_B^w + \alpha_2 x_B^b + (1-\alpha_1-\alpha_2)\, x_A^w\right)$, and 2) $1 > \alpha_1 + \alpha_2$. As P_A is not a singleton, there are priors $p, p' \in P_A$ and state $s, s' \in S$ such that $p'(s') > p(s')$ and $p'(s') < p(s)$. Let

$$f = \begin{cases} \alpha_1 x_B^w + \alpha_2 x_B^b + (1-\alpha_1-\alpha_2)\, x_A^w & s \\ x_A^b & \text{all other states} \end{cases},$$

and

$$f' = \begin{cases} \alpha_1 x_B^w + \alpha_2 x_B^b + (1-\alpha_1-\alpha_2)\, x_A^w & s' \\ x_A^b & \text{all other states} \end{cases}.$$

Then Bob is decisive between f and f' (he is indifferent between them), while Alice is indecisive between them (f is better according to p and f' is better according to p') - a contradiction. Similarly, one can show that $u_B\left(x_A^w\right) \in \{0,1\}$. So only two cases are possible: 1) Alice and Bob share a weak best element and a weak worse element: $u_B\left(x_A^w\right) = u_A\left(x_A^w\right) = 0$ and $u_B\left(x_A^b\right) = u_A\left(x_A^b\right) = 1$, or 2) one of Alice's (Bob's) weak best elements is one of Bob's (Alice's) weak worse elements: $u_B\left(x_A^b\right) = u_A\left(x_A^w\right) = 0$ and $u_B\left(x_A^w\right) = u_A\left(x_A^b\right) = 1$.

Case 1: Assume W.L.O.G. that $x_A^b = x_B^b = x^b$ and $x_A^w = x_B^w = x^w$. We now show that $u_A = u_B$. Assume to the contrary that there is $q \in Y$ such that $u_A(q) \neq u_B(q) = \beta$. As P_A is not a singleton, there are priors $p, p' \in P_A$ and state $s, s' \in S$ such that $p'(s') > p(s')$ and $p'(s) < p(s)$. Let

$$f = \begin{cases} \beta x^b + (1-\beta)\, x^w & s \\ q & \text{all other states} \end{cases}, \quad f' = \begin{cases} \beta x^b + (1-\beta)\, x^w & s' \\ q & \text{all other states} \end{cases}.$$

Then Bob is decisive (indifferent) between f and f' while Alice is indecisive between them - a contradiction.

Let $u = u_A = u_B$. We are left with proving that $P_A \subseteq P_B$. Assume to the contrary that $p_A \nsubseteq P_B$. Let $p_A \in P_A \backslash P_B$. By a standard separation theorem (using the convexity and the compactness of P_B, see footnote 24) there are $f, g \in L$ such that $1 > \alpha = E_{p_A}(u(f) - u(g)) > E_{p_B}(u(f) - u(g))$ for each $p_B \in P_B$. Let $max_{p_B \in P_B} E_{p_B}(u(r) - u(q)) = \beta$. Assume first that there is $p'_A \in P_A$ such that $\gamma = E_{p'_A}(u(f) - u(g)) \neq \alpha$. By the convexity of P_A we can assume that $\beta < \gamma$. This implies that Alice is indecisive between

$$\frac{1}{1+\frac{\alpha+\gamma}{2}} f + \left(1 - \frac{1}{1+\frac{\alpha+\gamma}{2}}\right) x^w \text{ and } \frac{1}{1+\frac{\alpha+\gamma}{2}} g + \left(1 - \frac{1}{1+\frac{\alpha+\gamma}{2}}\right) x^b$$

(because if $\gamma > \alpha$ then the former is better according to p'_A and the latter is better according to p_A and if $\gamma < \alpha$ the opposite holds), while Bob is decisive (the latter is better according to all of Bob's utilities) - a contradiction. So we are left with the case that $E_{p'_A}(u(f) - u(g)) = \alpha$ for every $p'_A \in P_A$. As P_A is not a singleton, there are $h \in L$

and $p_A^1, p_A^2 \in P_A$ such that $E_{p_A^1}(u(h)) > E_{p_A^2}(u(h))$. For sufficiently small $\delta > 0$, $f' = (1-\delta)f + \delta h$ and $g' = (1-\delta)g + \delta h$ satisfy: 1) $E_{p_A^1}(u(f') - u(g'))$, $E_{p_A^2}(u(f') - u(g')) > E_{p_B}(u(f') - u(g'))$ for each $p_B \in P_B$, 2) $E_{p_A^1}(u(f') - u(g')) \neq E_{p_A^2}(u(f') - u(g'))$. By the previous argument, this leads to a contradiction. Thus, we have proved that in this case $P_A \subseteq P_B$.

Case 2: Let Charlie be a DM with the opposite of Bob's utility ($u_C = -u_B$) and the same set of priors as Bob ($P_C = P_B$). This implies that: 1) Alice and Charlie share a best element and a worse element:, 2) The fact that Alice is more decisive than Bob, implies that Alice is more decisive than Charlie. By the proof of case 1 $u_A = u_C = -u_B$ and $P_A \subseteq P_C = P_B$, which completes the proof.

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