Fixed point theorems for multivalued contractive operators on generalized metric spaces

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Abstract

In this paper we give a fixed point results for multivalued operators on generalized metric spaces endowed with a generalized w-distance. Then we study the data dependence for this new result.

Key words and phrases: multivalued weakly Picard operator, w-distance, fixed point, multivalued operator.

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1 Introduction

It is well known that Caristis fixed point theorem [2] is equivalent to Ekland variational principle [4], which is nowadays is an important tool in nonlinear analysis. Most recently, many authors studied and generalized Caristis fixed point theorem to various directions. Using the concept of Hausdorff metric, Nadler Jr. [13] has proved multivalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space, has a fixed point.

Recently, Bae [1] introduced a notion of multivalued weakly contractive maps and applying generalized Caristis fixed point theorems he proved several fixed point results for such maps in the setting of metric and Banach spaces. Many authors have been using the Hausdorff metric to obtain fixed point results for multivalued maps on metric spaces, but, in fact for most cases the existence part of the results can be proved without using the concept of Hausdorff metric.

Recently, using the concept of w-distance [9], Suzuki and Takahashi [20] introduced a notion of multivalued weakly contractive in short, w-contractive maps and improved the Nadlers fixed point result without using the concept of Hausdorff metric. Most recently, Latif [10] generalized the fixed point result of Suzuki and Takahashi [[20], Theorem 1].

The concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence phenomenon for the fixed point set of multivalued operators on complete metric space, by I. A. Rus, A. Petruşel and A. Sântămărian, see [19]. In [17] is presented the theory of multivalued weakly Picard operators in L-spaces.

In 1966 A. I. Perov was introduced the concept of generalized metric space and obtained a generalization of the Banach principle for contractive operators on spaces endowed with vector-valued metrics, see [16].

The purpose of this paper is to recall the notion of generalized w-distance in a generalized metric space. Also, we present some generalizations of some fixed point results obtained in [5] with respect to a generalized w-distance and we give a data dependence result for the new theorem of fixed point.

2 Preliminaries

Let (X, d) be a metric space. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;

$\mathcal{P}(X) = P(X) \bigcup \emptyset$

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_b(X)$ - *the set of all nonempty bounded subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of* X;

For two subsets $A, B \in P_b(X)$ we recall the following functionals.

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A,B) := sup\{d(a,b)|x \in A, b \in B\}$ - *the diameter functional*;

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A,B) := max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}$ - *the Pompeiu-Hausdorff functional*;

If $T : X \to P(X)$ is a multivalued operator, then we denote by $FixT$ the fixed point set of T, i.e. $FixT := \{x \in X | x \in T(x)\}$.

First we define the concept of **L-space**.

Definition 2.1 *Let* X *be a nonempty set and* $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X, n \in \mathbb{N}\}$. *Let* $c(X) \subset s(X)$ *a subset of* $s(X)$ *and* $Lim : c(X) \to X$ *an operator. By definition the triple* $(X, c(X), Lim)$ *is called an* **L-space** *if the following conditions are satisfied:*

(i) If $x_n = x$, *for all* $n \in \mathbb{N}$, *then* $(x_n)_{n\in\mathbb{N}} \in c(X)$ *and* $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ *and* $Lim(x_n)_{n\in\mathbb{N}} = x$, *then for all subsequences,* $(x_{n_i})_{i\in\mathbb{N}}$, *of* $(x_n)_{n\in\mathbb{N}}$ *we have that* $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ *and* $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By the definition an element of $c(X)$ is convergent and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we can write $x_n \to x$ as $n \to \infty$.

We will denote an L-space by $(X, \to)$.

Let us give some examples of L-spaces, see [17].

Example 2.1 *(L-structures on Banach spaces) Let* X *be a Banach space. We denote by* $\to$ *the strong convergence in* X *and by* $\rightharpoonup$ *the weak convergence in* X. *Then* $(X, \to)$, $(X, \rightharpoonup)$ *are L-spaces.*

Example 2.2 *(L-structures on function spaces) let* X *and* Y *be two metric spaces. Let* $\mathbb{M}(X, Y)$ *the set of all operators from* X *to* Y. *We denote by* $\xrightarrow{p}$

the point convergence on $\mathbb{M}(X,Y)$, by $\overset{unif}{\to}$ *the uniform convergence and by* $\overset{cont}{\to}$ *the convergence with continuity. Then* $(\mathbb{M}(X,Y),\overset{p}{\to})$, $(\mathbb{M}(X,Y),\overset{unif}{\to})$ *and* $(\mathbb{M}(X,Y),\overset{cont}{\to})$ *are L-spaces.*

Definition 2.2 *Let $(X,\to)$ be an L-space. Then $T: X \to P(X)$ is a* **multi-valued weakly Picard operator***(briefly MWP operator)if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:*

(i)$x_0 = x$, $x_1 = y$;

(ii)$x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

(iii)the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Let us give some examples of MWP operators, see [17],[19].

Example 2.3 *Let (X,d) be a complete metric space and $T: X \to P_{cl}(X)$ be a Reich type multivalued operator, i.e. there exists $\alpha,\beta,\gamma \in \mathbb{R}_+$ with $\alpha+\beta+\gamma < 1$ such that*

$$H(T(x),T(y)) \leq \alpha d(x,y) + \beta D(x,T(x)) + \gamma D(y,T(y)),$$

for all $x,y \in X$. Then T is a MWP operator.

Example 2.4 *Let (X,d) be a complete metric space and $T: X \to P_{cl}(X)$ be a closed multifunction for which there exists $\alpha,\beta \in \mathbb{R}_+$ with $\alpha+\beta < 1$ such that $H(T(x),T(y)) \leq \alpha d(x,y) + \beta D(y,T(y))$, for every $x \in X$ and every $y \in T(x)$. Then T is a MWP operator.*

Example 2.5 *Let (X,d) be a complete metric space and $T_1,T_2: X \to P_{cl}(X)$ for which there exists $\alpha \in]0,\frac{1}{2}[$ such that*

$$H(T_1(x),T_2(y)) \leq \alpha[D(x,T_1(x)) + D(y,T_2(y))],$$

for each $x,y \in X$. Then T_1 and T_2 are a MWP operators.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[9]) as follows:

Let (X,d) be a metric space. A functional $w: X \times X \to [0,\infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x,z) \leq w(x,y) + w(y,z)$, for any $x, y, z \in X$;
2. for any $x \in X : w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z,x) \leq \delta$ and $w(z,y) \leq \delta$ implies $d(x,y) \leq \varepsilon$.

Some examples of w-distance are given in [9].

Example 2.6 *Let (X,d) be a metric space. Then the metric "d" is a w-distance on X.*

Example 2.7 *Let X be a normed linear space with norm $|| \cdot ||$. Then the function $w : X \times X \to [0, \infty)$ defined by $w(x,y) = ||x|| + ||y||$ for every $x, y \in X$ is a w-distance.*

Example 2.8 *Let (X,d) be a metric space and let $g : X \to X$ a continuous mapping. Then the function $w : X \times Y \to [0, \infty)$ defined by:*

$$w(x,y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ is a w-distance.

Let us recall a crucial lemma for w-distance (see[20] for more details).

Lemma 2.1 *Let (X,d) be a metric space, and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following statements hold:*

1. *If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*
4. *If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

For the rest of the paper, if $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then by $v \leq r$ means $v_i \leq r_i$, for each $i \in \{1, 2, \cdots, m\}$, while $v < r$ means $v_i < r_i$, for each $i \in \{1, 2, \cdots, m\}$.

Also, $|v| := (|v_1|, |v_2|, \cdots, |v_m|)$ and, if $c \in \mathbb{R}$ then $v \leq c$ means $v_i \leq c_i$, for each $i \in \{1, 2, \cdots, m\}$.

If $x_0 \in X$ and $r \in \mathbb{R}^m_+$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$ we will denote by $B(x_0; r) := \{x \in X|\ d(x_0, x) < r\}$ the open ball centered in x_0 with radius $r := (r_1, r_2, \cdots, r_m)$ and by $\widetilde{B}(x_0; r) := \{x \in X|\ d(x_0, x) \leq r\}$ the closed ball centered in x_0 with radius r.

In [8] we can find the notion of generalized w-distance as follows.

Definition 2.3 *Let (X, d) a generalized metric space. The mapping $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ defined by $\widetilde{w}(x, y) = (v_1(x, y), v_2(x, y), ..., v_m(x, y))$ is said to be a generalized w-distance if it satisfies the following conditions:*

(w_1) $\widetilde{w}(x, y) \leq \widetilde{w}(x, z) + \widetilde{w}(z, y)$, for every $x, y, z \in X$;

(w_2) $v_i : X \times X \to \mathbb{R}_+$ is lower semicontinuous, for $i \in \{1, 2, \ldots, m\}$;

(w_3) For any $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) > 0$, for $m \in \mathbb{N}$, there exists $\delta := (\delta_1, \delta_2, ..., \delta_m) > 0$ such that $\widetilde{w}(z, x) \leq \delta$ and $\widetilde{w}(z, y) \leq \delta$ implies $\widetilde{d}(x, y) \leq \varepsilon$.

The notion of generalized w-distance with his properties was discussed in [8].

Let us present now an important lemma for w-distances into the terms of generalized w-distances.

Lemma 2.2 *Let $(X, \widetilde{d})$ be a generalized metric space, and let $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let (x_n) and (y_n) be two sequences in X, let $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)}, ..., \alpha_n^{(m)}) \in \mathbb{R}_+$ and $\beta_n = (\beta_n^{(1)}, \beta_n^{(2)}, ..., \beta_n^{(m)}) \in \mathbb{R}_+$ be two sequences such that $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ converge to zero for each $i \in \{1, 2, \ldots, m\}$. Let $x, y, z \in X$. Then the following assertions hold:*

1. *If $\widetilde{w}(x_n, y) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*

2. If $\widetilde{w}(x_n, y_n) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.

3. If $\widetilde{w}(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.

4. If $\widetilde{w}(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

Throughout this paper we will denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by Θ the zero $m \times m$ matrix, by I the identity $m \times m$ matrix and by U the unity $m \times m$ matrix. If $A \in M_{m,m}(\mathbb{R}_+)$, then the symbol A^τ stands for the transpose matrix of A.

Recall that a matrix A is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

For the proof of the main result we need the following theorem, see [16].

Theorem 2.1 *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) A is a matrix convergent to zero;

(i) $A^n \to 0$ as $n \to \infty$;

(ii)The eigen-values of A are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;

(iii) The matrix $I - A$ is non-singular and

$$(I - A)^{-1} = I + A + ... + A^n + ...;$$

(iv) The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnenegative elements.

(v) $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

3 Main results

Throughout this section (X, d) is a generalized metric space in Perov's sense and w is a generalized w-distance on the generalized metric space.

Let $x_0 \in X$ and $r := (r_i)_{i=1}^n$ for each $i = \{1, 2, ..., m\}$. Let us define:

$\widetilde{B}_w(x_0; r) := \{x \in X | \widetilde{w}(x_0, x) < r\}$ the open ball centered at x_0 with radius r with respect to $\widetilde{w}$;

$\widetilde{B}_w(x_0; r) := \{x \in X | \widetilde{w}(x_0, x) \leq r\}$ the closed ball centered at x_0 with radius r with respect to $\widetilde{w}$;

$\widetilde{B}_w^d(x_0; r)$- the closure in (X, d) of the set $\widetilde{B}_w(x_0; r)$.

Theorem 3.2 *Let (X, d) be a complete generalized metric space, $x_0 \in X$, $r := (r_i)_{i=1}^n$ for each $i = \{1, 2, ..., m\}$, $\widetilde{w} : X \times X \to [0, \infty)$ a generalized w-distance on X and let $T : \widetilde{B}_w(x_0; r) \to P(X)$ be a multivalued operator with the property that there exists $A = (a_{i,j})_{i,j \in \{1,2,...,m\}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ a matrix convergent to zero and $B = (b_{i,j})_{i,j \in \{1,2,...,m\}} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \setminus \{U\}$ such that, for every $x, y \in X$ and each $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\widetilde{w}(u, v) \leq A\widetilde{w}(x, y) + BD_{\widetilde{w}}(y, T(y)),$$

where $D_{\widetilde{w}}(x, T(x)) := inf\{\widetilde{w}(x, y) : y \in T(x)\}$.

(This means, that for each $x, y \in Y$ and each $u \in T(x)$, there exists $v \in T(y)$ such that

$$\begin{pmatrix} w_1(u,v) \\ \cdots \\ w_m(u,v) \end{pmatrix} \leq \begin{pmatrix} a_{11} & \cdots & a_{1,m} \\ \cdots & & \\ a_{m1} & \cdots & a_{m,m} \end{pmatrix} \cdot \begin{pmatrix} w_1(x,y) \\ \cdots \\ w_m(x,y) \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1,m} \\ \cdots & & \\ b_{m1} & \cdots & b_{m,m} \end{pmatrix} \cdot \begin{pmatrix} D_{w_1}(x,y) \\ \cdots \\ D_{w_m}(x,y) \end{pmatrix}$$
)

Suppose that:

1. $\inf\{\widetilde{w}(x, y) + D_{\widetilde{w}}(x, T(x))\} > 0$, *for every $x, y \in X$ and $y \notin T(y)$.*
2. *There exists $x_1 \in T(x_0)$ such that $\widetilde{w}(x_0, x_1)(I - A)^{-1} \leq r$.*
3. *If $u \in \mathbb{R}_+^m$ is such that $u(I - A)^{-1} \leq (I - A)^{-1}r$, then $u \leq r$.*

Then $FixT \neq \emptyset$.

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$ such that

$$\widetilde{w}(x_0, x_1)(I - A)^{-1} \leq r \leq (I - A)^{-1} \cdot r$$

Then, by (2), $x_1 \in \widetilde{B}_w(x_0;r)$. For $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$\widetilde{w}(x_1,x_2) \leq A\widetilde{w}(x_0,x_1) + BD_{\widetilde{w}}(x_1,T(x_1))$$

$$\leq A\widetilde{w}(x_0,x_1) + B\widetilde{w}(x_1,x_2)$$

Thus $\widetilde{w}(x_1,x_2) \leq \frac{A}{U-B}\widetilde{w}(x_0,x_1)$

We denote $C := \frac{A}{U-B}$ and we observe that the matrix $C \in \mathcal{M}_{m,m}(\mathbb{R})$ is a matrix convergent to zero and satisfy the following inequalities

- $I + \frac{A}{U-B} \leq I + A + A^2 + \cdots + A^n + \cdots$, therefore $I + C \leq (I-A)^{-1}$
- $(I-C)^{-1} \leq (I-A)^{-1}$

Thus $\widetilde{w}(x_1,x_2)(I-A)^{-1} \leq \frac{A}{U-B}w(x_0,x_1)(I-A)^{-1} \leq Cr$. Notice that $x_2 \in \widetilde{B}_w(x_0;r)$.

Indeed, since $\widetilde{w}(x_0,x_2) \leq \widetilde{w}(x_0,x_1) + \widetilde{w}(x_1,x_2)$ we get that $w(x_0,x_2)(I-A)^{-1} \leq \widetilde{w}(x_0,x_1)(I-A)^{-1} + \widetilde{w}(x_1,x_2)(I-A)^{-1} \leq Ir + Cr \leq (I-A)^{-1}r$, which immediately implies (by hypothesis (2)) that $\widetilde{w}(x_0,x_2) \leq r$.

By induction, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{B_w}(x_0;r)$ having the properties:

(a) $x_{n+1} \in T(x_n), n \in \mathbb{N}$;

(b) $\widetilde{w}(x_0,x_n)(I-A)^{-1} \leq (I-A)^{-1}r$, for each $n \in \mathbb{N}^*$, that means $\widetilde{w}(x_0,x_n) \leq r$;

(c) $\widetilde{w}(x_n,x_{n+1})(I-A)^{-1} \leq C^n r$, for each $n \in \mathbb{N}$.

By (c), for every $m,n \in \mathbb{N}$, with $m > n$, we get that

$$\widetilde{w}(x_n,x_m)(I-A)^{-1} \leq C^n(I-C)^{-1}r \leq C^n(I-A)^{-1}r.$$

By Lemma 2.2(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in the complete metric space. Denote by x^* its limit in $\widetilde{B}^d_w(x_0;r)$.

Assume that $x^* \notin T(x^*)$. Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}}$ is a sequence in $\widetilde{B}_w(x_0;s)$ which converge to x^* and $\widetilde{w}(x_n,\cdot)$ is lower semicontinuous we have

$$\widetilde{w}(x_n,x^*) \leq \lim_{m\to\infty}\inf \widetilde{w}(x_n,x_m) \leq C^n r, \text{ for every } n \in \mathbb{N}.$$

Therefore, by hypothesis (1) and using above inequality we have

$$0 \leq \inf\{\widetilde{w}(x, x^*) + D_{\widetilde{w}}(x, T(x)) : x \in X\}$$
$$\leq \inf\{\widetilde{w}(x_n, x^*) + \widetilde{w}(x_n, x_{n+1}) : n \in \mathbb{N}\}$$
$$\leq \inf\{2C^n r\} = 0.$$

Which is a contradiction. Thus conclude that $x^* \in T(x^*)$.

A global version of the previous theorem is the following result.

Theorem 3.3 *Let (X, d) be a complete generalized metric space, $x_0 \in X$, $r := (r_i)_{i=1}^n$ for each $i = \{1, 2, ..., m\}$, $\widetilde{w} : X \times X \to [0, \infty)$ a generalized w-distance on X and let $T : X \to P(X)$ be a multivalued operator with the property that there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ a matrix convergent to zero and $B \in \mathcal{M}_{m,m}(\mathbb{R}_+) \setminus \{U\}$ such that, for every $x, y \in X$ and each $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\widetilde{w}(u, v) \leq A\widetilde{w}(x, y) + BD_{\widetilde{w}}(y, T(y)),$$

where $D_{\widetilde{w}}(x, T(x)) := inf\{\widetilde{w}(x, y) : y \in T(x)\}$.

Suppose that $\inf\{\widetilde{w}(x, y) + D_{\widetilde{w}}(x, T(x))\} > 0$, for every $x, y \in X$ and $y \notin T(y)$ then

1. *$FixT \neq \emptyset$.*
2. *The sequence $(x_n)_{n\in\mathbb{N}} \in X$ given by relation $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$, is convergent and its limit is a fixed point of T.*
3. *One has the estimation $\widetilde{w}(x_n, x^*) \leq C^n\widetilde{w}(x_0, x_1)$ where $C \in \mathcal{M}_{m,m}(\mathbb{R})$, $C := \frac{A}{U-B}$, and $x^* \in FixT$.*

Remark 3.1 *In the condition of the previous theorem we observe that T is a MWP operator.*

4 Data dependence theorem for weakly contractive type operators in generalized metric spaces

The main result of this section is the following data dependence theorem with respect to the Theorem 3.3.

Theorem 4.4 *Let (X,d) be a complete generalized metric space, $x_0 \in X$, $r := (r_i)_{i=1}^n$ for each $i = \{1,2,...,m\}$, $\widetilde{w} : X \times X \to [0,\infty)$ a generalized w-distance on X and let $T_1, T_2 : X \to P(X)$ be a multivalued operator with the property that there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ a matrix convergent to zero and $B \in \mathcal{M}_{m,m}(\mathbb{R}_+) \setminus \{U\}$ such that, for every $x, y \in X$ and each $u \in T_j(x)$, for every $j \in \{1,2\}$, there exists $v \in T_j(y)$ such that*

$$\widetilde{w}(u,v) \leq A\widetilde{w}(x,y) + BD_{\widetilde{w}}(y, T_j(y)),$$

where $D_{\widetilde{w}}(x, T_j(x)) := inf\{\widetilde{w}(x,y) : y \in T_j(x)\}$.

Suppose that the following are true:

1. *$FixT_1 \neq \emptyset \neq FixT_2$.*

2. *We suppose that there exists $\eta := (\eta_i)_{i=1}^n$, for each $i = \{1,2,...,m\}$, with $\eta > 0$, such that for every $u \in T_1(x)$ there exists $v \in T_2(x)$ such that $\widetilde{w}(u,v) \leq \eta$, (respectively for every $v \in T_2(x)$ there exists $u \in T_1(x)$ such that $\widetilde{w}(v,u) \leq \eta$).*

3. *$\inf\{\widetilde{w}(x,y) + D_{\widetilde{w}}(x, T_j(x))\} > 0$ for each $j \in \{1,2\}$, for every $x, y \in X$ and $y \notin T_j(y)$.*

Then for every $u^* \in FixT_1$ there exists $v^* \in FixT_2$ such that

$$\widetilde{w}(u^*, v^*) \leq U(1-C)^{-1}\eta, \text{ where } C \in \mathcal{M}_{m,m}(\mathbb{R}),\ C := \tfrac{A}{U-B};$$

(respectively for every $v^* \in FixT_2$ there exists $u^* \in FixT_1$ such that

$$\widetilde{w}(v^*, u^*) \leq U(1-C)^{-1}\eta, \text{ where } C \in \mathcal{M}_{m,m}(\mathbb{R}),\ C := \tfrac{A}{U-B}).$$

Proof. Let $u_0 \in FixT_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis (2) we have that there exists $u_1 \in T_2(u_0)$ such that $\widetilde{w}(u_0, u_1) \leq \eta$.

We have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that $\widetilde{w}(u_1, u_2) \leq A\widetilde{w}(u_0, u_1) + BD_{\widetilde{w}}(u_1, T_2(u_1)) \leq A\widetilde{w}(u_0, u_1) + B\widetilde{w}(u_1, u_2)$.

Thus $\widetilde{w}(u_1, u_2) \leq \frac{A}{U-B}\widetilde{w}(u_0, u_1)$

We denote $C := \frac{A}{U-B}$ and we observe that the matrix $C \in \mathcal{M}_{m,m}(\mathbb{R})$ is a matrix convergent to zero.

Thus $\widetilde{w}(u_1, u_2) \leq C\widetilde{w}(u_0, u_1)$.

By induction we obtain a sequence $(u_n)_{n\in\mathbb{N}} \in X$ such that

(1) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(2) $\widetilde{w}(u_n, u_{n+1}) \leq C^n\widetilde{w}(u_0, u_1)$

For $n, p \in \mathbb{N}$ we have the inequality

$$\widetilde{w}(u_n, u_{n+p}) \leq C^n(I - C)^{-1}\widetilde{w}(u_0, u_1).$$

By the Lemma 2.2(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.

Assume that $v^* \notin T_2(v^*)$. Fix $n \in \mathbb{N}$. By the lower semicontinuity of $\widetilde{w}(x, \cdot) : X \to [0, \infty)$ we have

$$\widetilde{w}(u_n, v^*) \leq \lim_{p\to\infty} \inf \widetilde{w}(u_n, u_{n+p}) \leq C^n(I - C)^{-1}\widetilde{w}(u_0, u_1) \tag{4.1}$$

Therefore, by hypothesis (3) and using the relation 4.1 we have the inequality:

$$\begin{aligned} 0 &< \inf\{\widetilde{w}(u, v^*) + D_{\widetilde{w}}(u, T_2(u)) : x \in X\} \\ &\leq \inf\{\widetilde{w}(u_n, v^*) + \widetilde{w}(u_n, u_{n+1}) : n \in \mathbb{N}\} \\ &\leq \inf\{C^n(I - C)^{-1}\widetilde{w}(u_0, u_1) + C^n\widetilde{w}(u_0, u_1) : n \in \mathbb{N}\} = 0. \end{aligned}$$

Which is a contradiction. Thus we conclude that $v^* \in T_2(v^*)$.

Then, by $\widetilde{w}(u_n, v^*) \leq C^n(I - C)^{-1}\widetilde{w}(u_0, u_1)$, with $n \in \mathbb{N}$, for $n = 0$ we obtain $\widetilde{w}(u_0, v^*) \leq U(I - C)^{-1}\widetilde{w}(u_0, u_1) \leq U(I - C)^{-1}\eta$, which complete the proof. □

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