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# Lévy Subordinator Model : A Two Parameter Model of Default Dependency

B. S. BALAKRISHNA\*

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*“There is no “right” model. The best you can do is pick a model that mimics the most important behavior of the underlyer in your market. Then add perturbations if necessary.”*

*– Emanuel Derman in “Modeling the Volatility Smile”.*

## Abstract

Subordinators are Lévy processes with non-decreasing sample paths. They are natural processes to model default dependency. They help ensure that the loss process is non-decreasing leading to a promising class of dynamic models. The simplest subordinator is the Lévy subordinator, a maximally skewed stable process with index of stability  $1/2$ . Interestingly, this simplest subordinator turns out to be the appropriate choice as the basic process in modeling default dependency. It involves just two parameters to assess dependency risk, a measure of correlation and that of the likelihood of a catastrophe. Its attractive feature is that it admits a closed form expression for its distribution function. This helps in automatic calibration to individual hazard rate curves and efficient pricing with Fast Fourier Transform techniques. It is structured similar to the one-factor Gaussian copula model and can easily be implemented within the framework of the existing computational infrastructure. As it turns out, the Gaussian copula model can itself be recast into this framework highlighting its limitations. The model can also be investigated numerically with a Monte Carlo simulation algorithm. As is now well appreciated, random recovery is helpful in better pricing of the senior tranches and the model admits a tractable framework of random recovery. The model is investigated numerically and the implied base correlations are presented over a wide range of its parameters. The investigation also demonstrates its ability to generate reasonable hedge ratios.

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Default dependency has remained a challenging issue from the modeling perspective. An understanding of its implications is needed for pricing derivative instruments referencing a collection of credit names. Modeling dependent events and their correlations is also of interest outside the realm of credit derivatives. Many models have been developed for pricing the correlation products, but the market standard has remained the Gaussian copula model in spite of all the criticisms for its alleged role in the recent credit crisis. As has been emphasized by many, it has been well-known that the Gaussian copula model has serious limitations and is inadequate as a model of default dependency.

Major attraction of the Gaussian copula model is its simplicity and tractability. It can easily be calibrated to individual hazard rate curves. It can be formulated in closed form providing a semi-analytical framework for pricing. It admits efficient pricing with recursive methods or Fast Fourier Transform techniques. As it turns out, there exists another simple and tractable model similar in architecture that also enjoys these properties. Unlike the Gaussian copula model, it is a dynamical two-parameter model capable of offering a reasonable explanation of the correlation smile. The two parameters provide the two measures necessary to assess dependency risk, a measure of correlation and that of the likelihood of a catastrophe. The model is based on the Lévy subordinator, an  $\alpha = 1/2$  stable process maximally skewed to the right, whose distribution function is expressible in closed form and is known as the Lévy distribution. Though it is inevitable that, with a model of such few parameters, there is bound to exist a residual smile, the ability to capture the smile characteristics will be helpful in sensitivity analysis and stress testing.

Issues with the Gaussian copula model have been addressed before. Brigo, Pallavicini and Torresetti [2010] provide a discussion of its limitations and an account of the developments in this field. They also note that, since the start of the credit crisis, the probability mass associated to a catastrophic or armageddon event, i.e. the default of the entire pool of credit references, has increased dramatically. The need for such a catastrophic scenario while pricing the super-senior tranche was noted earlier by many authors, see for instance Balakrishna [2009]. However, in many of the models, such a scenario needs to be enforced, somewhat artificially. An attractive feature of the subordinator models discussed here is that such a scenario arises naturally as a consequence of a drift term that is well-known to be a natural component of the dynamics of subordinators.

Pricing models are helpful in computing hedges. Delta-hedges are sensitivities to the underlyings and are relatively more important. When the underlyings are credit default swaps as in our case, sensitivities are determined with respect to the individual hazard rates, assuming an environment wherein the parameters governing default dependency are relatively stable. Hazard rates are intensity-like variables having dimension of inverse time. Many of the pricing models have other intensity-like variables in them. It is not obvious while computing delta-hedges whether such variables need to be bumped along with the hazard rates. An important virtue of the Gaussian copula model is that it has no such additional intensity-like variables. As we will see in this article, the subordinator models discussed here share this virtue as they too have no additional intensity-like variables.

Models of the volatility smile have taught us that an explanation of the smile alone is not a guarantee for obtaining satisfactory hedge ratios. Local volatility models, though capable of providing a perfect fit to the smile, are criticized for giving rise to hedge corrections inconsistent with typical market behavior. Models respecting some concept of stationarity

have been pursued to obtain better hedge ratios. Within the context of the correlation smile, subordinator models discussed here attempt to achieve a similar goal. Numerical investigation of the Lévy subordinator model demonstrates its ability to generate reasonable hedge ratios under a wide range of its parameters.

In Balakrishna [2007] and some of the literature in the field, it is found that the modeled loss distribution displays one or more bumps along its tail. Even if such a distribution is able to reproduce the market prices providing an explanation of the correlation smile, it is not immediately obvious whether the bumps are a realistic feature of the distribution or an artifact of the model. Models capable of reproducing the market prices without such bumps, even if less accurate, can potentially give rise to better behaved prices and sensitivities. As it turns out, the Lévy subordinator model presented here exhibits no such bumps along the tail of its default probability distribution.

The article is organized as follows. To start with, a brief review of subordinators is presented in section 1. In section 2, a one-factor framework of default dependency is formulated starting with the de Finetti theorem from probability theory. Two classes of models arise naturally that are termed type-I and type-II. These models formulated on an infinitely large homogeneous collection to start with are further extended in section 3 to be applicable to finite heterogeneous collections and general hazard rate curves. Type-I models are well-known that include many of the reduced form models that have a systemic component to their dynamics. Type-II models are formulated as new that include the subordinator models introduced here but it turns out, as discussed in section 4, that the popular copula models can also be recast into this framework. Section 5 discusses type-II models governed by stable subordinators, in particular the Lévy subordinator model based on the  $\alpha = 1/2$  stable subordinator called the Lévy subordinator. Section 6 discusses some niceties of the  $\alpha = 1/2$  choice found numerically as appropriate. An attractive feature of the Lévy subordinator model is the possibility of semi-analytical pricing that is discussed in section 7. Section 8 discusses the large homogeneous pool approximation. Section 9 is a short presentation of an efficient pricing technique based on the Fast Fourier Transform. The model can also be investigated numerically with a Monte Carlo simulation algorithm as discussed in section 10. As is now well appreciated, random recovery is helpful in better pricing of the senior tranches, and section 11 presents a tractable framework. The effect of initial conditions is analyzed in section 12. Section 13 discusses default contagion within the present framework. Some possible extensions such as intensity based modeling are discussed in section 14. Section 15 concludes with a discussion and a brief summary. Figures 1-10 present the results of a numerical investigation into the model's implications.

## 1 A Brief on Subordinators

Lévy processes, and hence subordinators, is a well researched branch of mathematics. For the sake of completeness, the following gives a brief review of subordinators.

A stochastic process is a collection of random variables. A continuous-time stochastic process is such a collection over continuous time. A Lévy process is a continuous-time stochastic process starting as zero that has independent and stationary increments, and is stochastically continuous. Independence is a statement that increments over disjoint time-

intervals are independent random variables. Stationarity is a statement that increment over any time-interval is distributed with its time-dependence only on the length of the time-interval. Stochastic continuity means that jumps are random and rare, that the probability of a jump occurring at a given time is zero. A realization or a sample path of a stochastic process is an assignment of a sample of the random variable to each time of the time series. Subordinators are real-valued Lévy processes with non-decreasing sample paths.

Given a subordinator  $X(t)$ , its Laplace transform, or equivalently its Laplace exponent  $\eta(u)$ , is given by

$$e^{-t\eta(u)} = \mathbb{E} \{ e^{-uX(t)} \}, \quad u \geq 0, \quad (1)$$

where  $\mathbb{E} \{ \}$  denotes expectation value. The specific time-dependence assumed for the Laplace transform above is a consequence of the properties of the subordinator as a Lévy process. It follows that the Laplace exponent of a sum of two independent subordinators is the sum of their Laplace exponents. An important result for Lévy Processes is the Lévy-Khintchine formula. In the case of subordinators, it gives for the Laplace exponent

$$\eta(u) = bu + \int_0^\infty \lambda(dy) (1 - e^{-uy}). \quad (2)$$

$b \geq 0$  is called the drift coefficient that contributes a non-negative drift  $bt$  to  $X(t)$  so that  $X(t) \geq bt$  for all  $t$ .  $\lambda(dy)$  is called the Lévy measure that is required to satisfy  $\int_0^\infty \lambda(dy) \min(y, 1) < \infty$ . It is also true that any function of the above form is the Laplace exponent of a subordinator.

An important subclass of subordinators are stable subordinators. They are stable processes having index of stability  $\alpha \in (0, 1)$  and maximally skewed to the right, that is, with their skew parameter set to one. Their Laplace exponent is  $\eta(u) = bu + au^\alpha$  for some constant  $a$  and  $\alpha \in (0, 1)$ , obtainable from the Lévy measure  $a\alpha [\Gamma(1 - \alpha)]^{-1} y^{-1-\alpha} dy$ . It follows that stable subordinators (more generally stable processes) feature an additive property, that is if  $X(t)$  and  $Y(t)$  are two independent stable subordinators with index of stability  $\alpha$  (and parameters  $b_X, a_X$  and  $b_Y, a_Y$ ), then  $Z(t) = pX(t) + qY(t)$  is also a stable subordinator with index of stability  $\alpha$  (having  $b_Z = pb_X + qb_Y$  and  $a_Z = p^\alpha a_X + q^\alpha a_Y$ ).

Inversion of the Laplace transform gives us the probability density function of the random variable  $X(t)$  at time  $t$ , or equivalently its cumulative distribution function  $g_t(x)$  given by

$$g_t(x) = \mathbb{E} \{ 1_{X(t) \leq x} \}, \quad (3)$$

where  $1_{\{\dots\}}$  is the indicator function. No closed form expression is available for  $g_t(x)$  in general for the stable subordinators, except for the  $\alpha = 1/2$  stable subordinator called the Lévy subordinator<sup>1</sup>. In the case of the Lévy subordinator that has  $\eta(u) = bu + a\sqrt{u}$ , the distribution is known as the Lévy distribution and is given by

$$\begin{aligned} g_t(x) &= 2N \left( -at / \sqrt{2(x - bt)} \right), \\ \partial_x g_t(x) &= \frac{1}{2\sqrt{\pi}} at(x - bt)^{-3/2} e^{-\frac{1}{4}(at)^2/(x - bt)}, \end{aligned} \quad (4)$$

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<sup>1</sup>In the literature, one sometimes finds the term “Lévy subordinator” used for all subordinators. As in Applebaum [2005], it is used here just for the  $\alpha = 1/2$  stable subordinator. Similarly, the term “Lévy distribution” is used here just for its distribution.

where  $N()$  is the cumulative standard normal distribution function. This includes a non-negative drift component  $bt$  discussed above so that  $g_t(x)$  and  $\partial_x g_t(x)$  can be taken to be zero for  $x < bt$ .

Stable subordinators also feature a scaling property such that  $(at)^{-1/\alpha}(X(t) - bt)$  is independent of  $t$  in distribution, that is  $g_t(x)$  is a function of the combination  $(at)^{-1/\alpha}(x - bt)$ . This scaling property is evident in the behavior of the tail of their distributions. The long tail of the distribution of a stable subordinator  $X(t)$  obeys a power-law decay with

$$g_t(x + bt) \rightarrow 1 - at [\Gamma(1 - \alpha)]^{-1} x^{-\alpha}, \quad \text{for large } x. \quad (5)$$

At the short end, the log-distribution exhibits a power-law behavior with

$$-\ln g_t(x + bt) \rightarrow (1 - \alpha) (\alpha(at)^{1/\alpha})^{\alpha/(1-\alpha)} x^{-\alpha/(1-\alpha)}, \quad \text{for small } x. \quad (6)$$

A consequence of power-law decay for large  $x$  is that stable subordinators have both infinite mean and infinite variance.

Lévy distribution (4) is also the first passage time distribution of a Brownian motion over time variable  $x \geq bt$  with the barrier set at  $at/\sqrt{2}$ . The first passage time distribution of a Brownian motion with drift rate  $c\sqrt{2}$ , that is of a Gaussian process, is also available in closed form and is known as the inverse Gaussian distribution. The associated subordinator is the inverse Gaussian subordinator that has  $\eta(u) = bu + a(\sqrt{u + c^2} - c)$  and

$$\begin{aligned} g_t(x) &= N(-at/z + cz) + e^{2act} N(-at/z - cz), \\ \partial_x g_t(x) &= \frac{2}{\sqrt{2\pi}} a t e^{act} z^{-3} e^{-\frac{1}{2}((at)^2 z^{-2} + c^2 z^2)}, \quad z = \sqrt{2(x - bt)}. \end{aligned} \quad (7)$$

Though not a stable subordinator, inverse Gaussian subordinator is useful as the natural extension of the Lévy subordinator. Its Lévy measure is that of the Lévy subordinator damped exponentially with  $e^{-c^2 y}$ . Other stable subordinators are also generalized in this way with an exponential damping called tempering of the Lévy measure.

Simplest example of a subordinator is the Poisson process having  $\eta(u) = \lambda(1 - e^{-u})$ . It has unit jump sizes occurring at intensity  $\lambda$ . The form of the Laplace exponent in (2) indicates that a subordinator should be constructible from Poisson processes with varying jump sizes. Lévy-Itô decomposition theorem applied to subordinators provides us with such a construction that reads, as a differential equation,

$$dX(t) = bdt + \int_{y=0}^{\infty} y dN(\lambda(dy), t). \quad (8)$$

Here  $N(\lambda(dy), t)$  is a Poisson process of intensity  $\lambda(dy)$  associated with the interval  $(y, y + dy)$ . Increments  $dN(\lambda(dy), t)$  at disjoint  $t$  and  $y$  intervals are independent random variables. If  $N(\lambda(dy), t)$  jumps up by one at time  $t$ ,  $dN(\lambda(dy), t)$  causes  $X(t)$  to jump up by  $y$  at time  $t$ . Over an infinitesimal time interval  $(t, t + dt)$ , we have

$$E \{ \exp[-uy dN(\lambda(dy), t)] \} = \exp[-dt\lambda(dy) (1 - e^{-uy})]. \quad (9)$$

This follows simply on noting that  $dN(\lambda(dy), t)$  takes values zero and one with probabilities  $1 - dt\lambda(dy)$  and  $dt\lambda(dy)$  respectively, irrespective of the value of  $N(\lambda(dy), t)$ . It is now straightforward to obtain the Laplace exponent (2) starting from the solution  $X(t)$  of the differential equation.

## 2 One Factor Formulation

It is instructive to proceed formulating the model starting with an infinitely large homogeneous collection of credit names. This offers an intuitive insight into its structure that has a basis in probability theory due to a theorem attributed to de Finetti. The model then evolves as a natural extension of this formulation.

Given an infinite homogeneous collection of credit names, consider a configuration of its defaulted and undefaulted states at some future time  $t$ . Let us say our interest is not in the actual assignment of states among the names, but only on the fraction of names in the defaulted states. The collection being infinite, this fraction  $\nu$  can take any value from zero to one. The configuration is thus characterized by just one common risk factor that is identifiable with the fraction  $\nu$ .

Given such a configuration, the assignments of fraction  $\nu$  names defaulted and the rest undefaulted are all equally possible. The probability of finding a given name defaulted is  $\nu$ . Because the collection is infinite, the probability of finding a second name defaulted given that the first one has been is also  $\nu$ , so that the probability of finding both the names defaulted is  $\nu^2$ . More generally, the probability of finding a set of  $j$  names defaulted and  $k$  names undefaulted is  $\nu^j(1 - \nu)^k$ . In other words, the states can be treated as independent variables. If the configuration under consideration itself has a probability density function  $\partial_\nu F_t(\nu)$ ,  $F_t(\nu)$  being the cumulative default distribution function, the probability of finding  $j$  names defaulted and  $k$  names undefaulted can hence be written as

$$P_{[j,k]}(t) = \int_0^1 d\nu \partial_\nu F_t(\nu) \nu^j (1 - \nu)^k. \quad (10)$$

This is a one-factor formulation since, given a value of just one variable  $\nu$ , defaults get treated as independent random variables. This intuitive result has a basis in probability theory due to a theorem attributed to de Finetti. It is sometimes helpful to express  $F_t(\nu)$  in terms of a random variable  $\mathcal{V}(t)$  taking values in  $[0, 1]$  as

$$F_t(\nu) = \mathbb{E} \{ 1_{\mathcal{V}(t) \leq \nu} \}. \quad (11)$$

Random variables  $\mathcal{V}(t)$  for all  $t > 0$  with  $\mathcal{V}(0) = 0$  can be viewed together as defining a stochastic process that we may denote for simplicity as  $\mathcal{V}(t)$  itself.

Some general characteristics of  $F_t(\nu)$  can be inferred to start with. It is of course a non-decreasing function of  $\nu$ . When there are firm-specific contributions to defaults, as is expected to be the case in a realistic model, one expects a minimum value  $\nu_{\min}(t)$  for  $\nu$  below which  $F_t(\nu) = 0$ . This is because firm-specific contributions are independent of each other and hence, in an infinitely large homogeneous collection, one expects at least a fraction of names equal to the firm-specific default probability to have defaulted. Further, it has been usual to consider  $F_t(\nu) \rightarrow 1$  as  $\nu \rightarrow 1$ . But, as the recent credit crisis has indicated, there can be a non-zero probability for all the names in the collection to have defaulted by time  $t$  so that  $F_t(\nu)$  should be allowed to tend to some  $F_{\max}(t) < 1$  as  $\nu \rightarrow 1$ . For  $\nu$  in-between,  $F_t(\nu)$  is expected to be a decreasing function of  $t$ . Note that  $\mathcal{V}(t)$  can not be decreasing as a function of  $t$  since we do not allow for recovery of defaulted names. Given two times  $t_1$  and  $t_2$ ,  $t_1 < t_2$ , we have

$$F_{t_1}(\nu) - F_{t_2}(\nu) = \mathbb{E} \{ 1_{\mathcal{V}(t_1) \leq \nu} - 1_{\mathcal{V}(t_2) \leq \nu} \} = \mathbb{E} \{ 1_{\mathcal{V}(t_1) \leq \nu, \mathcal{V}(t_2) > \nu} \} - \mathbb{E} \{ 1_{\mathcal{V}(t_1) > \nu, \mathcal{V}(t_2) \leq \nu} \}. \quad (12)$$

For a non-decreasing stochastic process  $\mathcal{V}(t)$ , the last expectation above is zero. Assuming that there are nonzero contributions to the first term, as is usually the case, we thus have  $F_{t_1}(\nu) > F_{t_2}(\nu)$  for  $\nu$  in-between.

Formulation (10) has all the complexities of the model bundled into one common function  $F_t(\nu)$ . We may proceed with it by modeling  $\mathcal{V}(t)$ , but there is an interesting and more flexible alternate formulation that emphasizes individual behavior. Let us rewrite (10) as

$$P_{[j,k]}(t) = \int_0^1 dF [p_t(F)]^j [1 - p_t(F)]^k, \quad (13)$$

where  $p_t(F)$  is the inverse of  $F_t(\nu)$  defined by  $p_t(F_t(\nu)) = \nu$ ,  $\nu \geq \nu_{\min}(t)$ . Because  $F_t(\nu)$  is now an integration variable, both of its  $t$  and  $\nu$  dependences have been conveniently dropped.  $F$  can be viewed as a uniformly distributed random variable. Note that all the functional intricacies are now bundled into the conditional individual default probability  $p_t(F)$ .

Generic characteristics of  $F_t(\nu)$  discussed above imply similar ones for  $p_t(F)$ . Equivalently, they can be inferred from  $F$  viewed as an indicator of economic developments, with higher  $F$  corresponding to less favorable circumstances. Conditional individual survival probability  $q_t(F) = 1 - p_t(F)$  is a non-increasing function of  $F$  for all  $t > 0$ . With  $F = 1$  corresponding to the worst case scenario, that of total collapse with all the names defaulting, we have  $q_t(1) = 0$ . A non-zero probability of such a scenario implies that  $q_t(F) = 0$  for some  $F \geq F_{\max}(t)$ . At the  $F = 0$  end, the common variables are ineffective in causing defaults so that  $q_t(0)$  is firm-specific. As noted earlier,  $F_t(\nu)$  for  $\nu$  in-between is a decreasing function of  $t$ . Consequently, for  $F$  in-between,  $q_t(F)$  is a decreasing as a function of  $t$ , starting at one and ending up at zero as  $t$  runs from zero to infinity.

The characteristics of  $q_t(F)$  suggest that  $1 - q_t(F)/q_t(0)$  can be viewed as the cumulative distribution function of a random variable  $\Phi_i(t)$  taking values in  $[0, 1]$ . In other words,

$$q_t(F) = q_t(0) \mathbb{E} \left\{ 1_{\Phi_i(t) \geq F} \right\}. \quad (14)$$

Random variables  $\Phi_i(t)$  for all  $t > 0$  can be viewed together as defining a stochastic process, denoted for simplicity as  $\Phi_i(t)$  itself. It is a non-increasing process with  $\Phi_i(0) = 1$  and  $\Phi_i(\infty) = 0$ . There is one such independent stochastic process for each name in the collection, hence the name-subscript. In the homogeneous collection under discussion here, they are identically distributed. For the unconditional individual default probability  $P(t)$ , or equivalently its survival counterpart  $Q(t) = 1 - P(t)$ , we then have

$$Q(t) = \int_0^1 dF q_t(F) = q_t(0) \mathbb{E} \left\{ \int_0^1 dF 1_{\Phi_i(t) \geq F} \right\} = q_t(0) \mathbb{E} \{ \Phi_i(t) \}. \quad (15)$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves.

Models based on formulation (10) are referred to here as type-I models. Note that there is only one stochastic process  $\mathcal{V}(t)$  in its one-factor formulation modeling the common factor that governs the collective behavior. Many reduced form models belong to this class. Intensity based models that have a systemic component to their stochastic default intensities are type-I models. Because  $\mathcal{V}(t)$  is directly related to the loss process (for uniform recovery rates), many loss process models can also be viewed as type-I models.



In contrast, formulation (13) appears new. Here the collection, though homogeneous, has one independent stochastic process  $\Phi_i(t)$  for each of its names. A model of these  $\Phi_i(t)$ s is referred to here as a type-II model. These models are in some sense like structural models. It turns out that the popular copula models can be reformulated as belonging to this class. The reformulation highlights their unnaturalness modeling  $\Phi_i(t)$ s as static objects. Interestingly, as we will discover, there also exist new models, perhaps more promising, that model  $\Phi_i(t)$ s as genuinely dynamic stochastic processes.

### 3 Finite Heterogeneous Collection

The previous discussion was confined to an infinite homogeneous collection leading to two formulations of default dependency. The formulations can be considered to be applicable as such to a finite homogeneous collection under the assumption that the latter can be extended to an infinite one. The two formulations result in different models depending on the choices made for  $\mathcal{V}(t)$  or  $\Phi_i(t)$ s. The following introduces further extensions of the two homogeneous versions to heterogeneous collections.

Consider type-I formulation (10) to start with. It is convenient to work with  $\Lambda(t) = -\ln[1 - \mathcal{V}(t)]$ , a non-decreasing process taking values in  $[0, \infty]$  with  $\Lambda(0) = 0$  and  $\Lambda(\infty) = \infty$ . The joint survival probability  $Q_\Omega(t)$  for a list of names in  $\Omega$  can then be expressed as

$$Q_\Omega(t) = \mathbb{E} \left\{ e^{-\sum_{i \in \Omega} \Lambda_i(t)} \right\}. \quad (16)$$

A name-subscript  $i$  has been attached to  $\Lambda(t)$  to make it applicable to a heterogeneous collection. Since we had only one process  $\mathcal{V}(t)$  to start with, expectation  $E\{\}$  is taken with respect to a common process  $X(t)$ . All  $\Lambda_i(t)$ s are considered to be driven by  $X(t)$ , for instance as  $\Lambda_i(t) = a_i X(t) + b_i t$  for some parameters  $a_i$ s and  $b_i$ s. Expression (16) indicates that defaults are independent given a realization of  $X(t)$ . For the unconditional individual survival probability  $Q_i(t)$ , we now have

$$Q_i(t) = \mathbb{E} \left\{ e^{-\Lambda_i(t)} \right\}. \quad (17)$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves.

Type-II formulation (13) can be similarly generalized to a finite heterogeneous collection. Here too, it is convenient to work with  $\Lambda_i(t) = -\ln \Phi_i(t)$  for each name, a non-decreasing stochastic process taking values in  $[0, \infty]$  with  $\Lambda_i(0) = 0$  and  $\Lambda_i(\infty) = \infty$ . The conditional default and survival probabilities  $p_i(F)$  and  $q_i(F) = 1 - p_i(F)$  are now denoted with name-subscripts as  $p_i(F, t)$  and  $q_i(F, t)$  respectively. Integration variable  $F$  may now be viewed simply as a uniformly distributed common factor. In terms of  $\Lambda_i(t)$ ,  $q_i(F, t)$  reads

$$q_i(F, t) = q_i(0, t) \mathbb{E} \left\{ 1_{\Lambda_i(t) \leq -\ln F} \right\}. \quad (18)$$

For the unconditional individual survival probability  $Q_i(t)$  we then have

$$Q_i(t) = q_i(0, t) \mathbb{E} \left\{ e^{-\Lambda_i(t)} \right\}. \quad (19)$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves. The joint survival probability  $Q_\Omega(t)$  for a list of names in  $\Omega$  can be expressed as

$$Q_\Omega(t) = \int_0^1 dF \prod_{i \in \Omega} q_i(F, t) = \left[ \prod_{i \in \Omega} q_i(0, t) \right] \mathbb{E} \left\{ e^{-\text{Max}_{i \in \Omega} \Lambda_i(t)} \right\}, \quad (20)$$

where  $\text{Max}_{i \in \Omega}$  picks up the largest  $\Lambda_i(t)$  in  $\Omega$ . This follows from the fact that  $\Lambda_i(t)$ s are independent stochastic processes. It is interesting to note that this defines the model with no reference to the common factor that has been integrated away. Dependency is built into the combination  $\text{Max}_{i \in \Omega} \Lambda_i(t)$ .

The two apparently equivalent formulations have resulted in very different models of default dependency, especially at the heterogeneous level. Expression (16) emphasizes conditional independence of defaults. In contrast, expression (20) indicates maximum dependency, a supplier-consumer kind of dependency. To understand it better, consider a realization of the  $\Lambda_i(t)$  processes. Within the context of the realization, (20) suggests that the joint survival probability of a list of names in  $\Omega$  is  $e^{-\text{Max}_{i \in \Omega} \Lambda_i(t)}$  (here and below, it is assumed that they have survived their firm-specific risk factors). Consider further just two names in  $\Omega$  say 1 and 2, and a realization with  $\Lambda_1(t) < \Lambda_2(t)$ . The probability of both the names having survived during  $(0, t)$  is  $e^{-\Lambda_2(t)}$ . Since the survival probability of name 2 irrespective of the state of name 1 is also  $e^{-\Lambda_2(t)}$ , this implies that it is not possible to have name 1 defaulted and name 2 survived. If name 1 has defaulted, then name 2 should have defaulted as well. More generally, without loss of generality, consider a realization with the ordering  $\Lambda_1(t) \leq \Lambda_2(t) \leq \dots$ . In this ordering, if name  $i$  is known to have defaulted during  $(0, t)$ , all names labeled  $j > i$  should have defaulted as well. That is, if a common crisis has resulted in name  $i$  defaulting, then all names known to be more vulnerable to the crisis in a realization should have defaulted as well. Stated this way, maximum dependency appears to be more realistic than the conditional independence formulation of expression (16). Besides, some contagion effects appear to be already in place.

The  $F$ -formulation can be viewed as realizing the  $\Lambda_i(t)$  processes further to obtain an assignment of defaulted and undefaulted states. This follows after re-introducing the  $F$ -integral as

$$e^{-\text{Max}_{i \in \Omega} \Lambda_i(t)} = \int_0^1 dF \prod_{i \in \Omega} 1_{\Lambda_i(t) \leq -\ln F}. \quad (21)$$

$F$  can be sampled from a uniform distribution. Given a value for  $F$ , a name is in defaulted or undefaulted state at time  $t$  depending on whether  $\Lambda_i(t)$  is above  $-\ln F$  or not. Thus, in some sense,  $\Lambda_i(t)$  can be viewed as the amount of impact a common crisis has on a name and  $-\ln F$  as the minimum amount of impact needed to lead to a default. In an infinite homogeneous collection, because  $\Lambda_i(t)$ s are independent and identically distributed, the fraction of  $\Lambda_i(t)$ s below  $-\ln F$ , and hence the fraction  $1 - \nu$  of names in undefaulted states at time  $t$  (conditional on surviving their firm-specific risk factors) is expected to agree with the cumulative distribution function of  $\Lambda_i(t)$ , namely  $\mathbb{E} \left\{ 1_{\Lambda_i(t) \leq -\ln F} \right\}$ . This is consistent with our earlier discussion in section 2 and realizes  $\mathcal{V}(t)$  as  $p_t(F)$ . Note that  $\Lambda_i(t)$  is assumed here to keep evolving even after default, but does not get counted below  $-\ln F$  after default (consider  $-\ln F$  time-independent as in the following).

In the case of type-II models, a straightforward extension of (20) to joint distribution of default times is

$$\begin{aligned} \text{Prob}(\tau_i > t_i, \tau_j > t_j, \dots) &= \int_0^1 dF [q_i(F, t_i) q_j(F, t_j) \dots] \\ &= [q_i(0, t_i) q_j(0, t_j) \dots] \mathbb{E} \{ e^{-\text{Max}(\Lambda_i(t_i), \Lambda_j(t_j), \dots)} \}, \end{aligned} \quad (22)$$

where  $\tau_i$ s are random default times. The resulting model can be formulated as a first passage model with the crossing of barrier  $-\ln F$  by the non-decreasing  $\Lambda_i(t)$  triggering default of the  $i^{\text{th}}$  credit name, conditional on surviving firm-specific risk factors.  $F$  is then a random variable uniformly distributed and a possible interpretation is that of  $\Lambda_i(t)$  as an intrinsic age process and  $-\ln F$  as a common age limit or, as noted above,  $\Lambda_i(t)$  as the amount of impact the common crisis has on a name and  $-\ln F$  as the minimum amount of impact needed to lead to a default.

A further generalization to a multi-factor joint distribution of default times is

$$\begin{aligned} \text{Prob}(\tau_i > t_i, \tau_j > t_j, \dots) &= \int \mathcal{D}C(F_1, F_2, \dots) [q_i(F_i, t_i) q_j(F_j, t_j) \dots] \\ &= [q_i(0, t_i) q_j(0, t_j) \dots] \mathbb{E} \{ C(\Phi_i(t_i), \Phi_j(t_j), \dots) \}, \end{aligned} \quad (23)$$

where  $C()$  is a copula and  $\mathcal{D}$  is a short notation for the  $n$ -dimensional differential,  $n$  being the number of factors. Similar distribution has been discussed in Schönbucher and Schubert [2001] within the context of an intensity model, but in the absence of firm-specific risk factors. Recall that copula is a joint distribution of uniformly distributed random variables. Since  $F_i, F_j, \dots$  are uniformly distributed, their joint distribution is given by a copula. To keep things simple, we have kept the copula as time-independent. For simplicity of presentation, we have also considered maximum number of factors, that is as many  $F$ s as there are names in the collection. Fewer factors can be handled with suitable restrictions on the copula. The one-factor case is recovered with the maximally dependent copula  $C(F_1, F_2, \dots) = \text{Min}(F_1, F_2, \dots)$ .

Like the one-factor formulation, the above multi-factor version, when all the  $t_i$ s are identical to say  $t$ , has a basis in probability theory. Consider again an infinitely large collection as in section 2, but now heterogeneous having finitely many, say  $n$ , types of names. A configuration of its defaulted and undefaulted states at time  $t$  is now characterized by  $n$  kinds of fractions,  $\nu_i, i = 1, \dots, n$ . Arguing as before, we can express the joint survival probability  $Q_{ij\dots}(t)$  for a list of names in  $\{i, j, \dots\}$  as

$$Q_{ij\dots}(t) = \int \mathcal{D}F_t(\nu_1, \nu_2, \dots) [(1 - \nu_i)(1 - \nu_j) \dots], \quad (24)$$

where  $F_t(\nu_1, \nu_2, \dots)$  is the joint cumulative distribution function of the  $\nu_i$ s. Next, as usual in copula based dependency modeling, introduce individual survival probabilities  $q_i(F_i, t)$  satisfying  $q_i(F_t^{(i)}(\nu_i), t) = 1 - \nu_i$  where  $F_t^{(i)}(\nu_i)$  is the  $i^{\text{th}}$  marginal cumulative distribution function. Changing the integration variables to  $F_i = F_t^{(i)}(\nu_i)$ , we then obtain the multi-factor formulation (23) for identical  $t_i$ s, though under a scenario where the names are assumed immersed in an infinitely extended pool. The assumption of extendibility to an infinite pool may not always hold, but note that the type-II formulation is independent of the pool characteristics making its applicability more natural as compared to a type-I formulation.

## 4 Copula Models

Interestingly, the popular copula models can also be recast into the present formulation. This is instructive as it highlights their unnaturalness, in particular their static nature.

In the present setting, a copula model is just a type-II model that appears as an attempt at constructing  $\Phi_i(t)$ s with the right properties. Stochasticity of  $\Phi_i(t)$  arises from a single random variable, one for each name in the collection. Time-dependence of  $\Phi_i(t)$  results from a time-dependent mean of the random variable. To recover the standard results, let us express  $\Phi_i(t)$  as

$$\Phi_i(t) = 1 - N_Y \left( \frac{1}{\sqrt{\rho}} \left( K_i(t) - \sqrt{1-\rho} Z_i \right) \right). \quad (25)$$

$Z_i$ s, one for each name in the collection, are independent random variables.  $N_Y()$  is the cumulative distribution function of some random variable  $Y$  independent of all the  $Z_i$ s.  $K_i(t)$  is a function of time to be determined.  $\rho$  is a correlation parameter. Assuming that there are no firm-specific risk factors, and expressing  $F$  in terms of a random variable  $Y$  as  $F = 1 - N_Y(Y)$ , we have

$$\begin{aligned} p_i(F, t) &= \mathbb{E} \{ 1_{\Phi_i(t) \leq F} \} = \mathbb{E} \left\{ 1_{\frac{1}{\sqrt{\rho}} (K_i(t) - \sqrt{1-\rho} Z_i) \geq Y} \right\} \\ &= \mathbb{E} \left\{ 1_{Z_i \leq \frac{1}{\sqrt{1-\rho}} (K_i(t) - \sqrt{\rho} Y)} \right\} = N_{Z_i} \left( \frac{1}{\sqrt{1-\rho}} (K_i(t) - \sqrt{\rho} Y) \right). \end{aligned} \quad (26)$$

$N_{Z_i}()$  is the cumulative distribution function of  $Z_i$ . The unconditional individual default probability  $P_i(t)$  can now be expressed as

$$P_i(t) = \mathbb{E} \{ 1 - \Phi_i(t) \} = \mathbb{E} \left\{ 1_{Y \leq \frac{1}{\sqrt{\rho}} (K_i(t) - \sqrt{1-\rho} Z_i)} \right\} = \mathbb{E} \{ 1_{X_i \leq K_i(t)} \} = N_{X_i}(K_i(t)), \quad (27)$$

where  $X_i = \sqrt{1-\rho} Z_i + \sqrt{\rho} Y$ .  $N_{X_i}()$  is the cumulative distribution function of  $X_i$ . This determines  $K_i(t)$  to be  $N_{X_i}^{-1}(P_i(t))$ . It is an increasing function of time resulting in a decreasing time-dependence for  $\Phi_i(t)$ .

These are the standard results known for one-factor copula models. Copula results from expression (22) for joint distribution of default times.  $Z_i$ s, and hence  $X_i$ s, are usually taken to be identically distributed. Then  $N_{Z_i}()$ s, and similarly  $N_{X_i}()$ s, are the same for all the names in the collection.  $Y$  and  $Z_i$ s, and hence  $X_i$ s, are usually normalized to have zero mean and unit variance.  $\rho$  is then the correlation between two different  $X_i$ s. In the case of the Gaussian copula model,  $Y, Z_i$  and  $X_i$  are all standard normal random variables, and hence  $N_Y, N_{Z_i}$  and  $N_{X_i}$  are all identical to the cumulative standard normal distribution function. In other copula models,  $N_Y$  and  $N_{Z_i}$  may be known by choice, but then  $N_{X_i}$  is not guaranteed to be easily computable.

It is also possible to recover multi-factor copula models in the present setting. As discussed earlier, a multi-factor type-II model can be extended to introduce joint distribution of default times by expression (23). The  $\Phi_i$ -construct is as before, but in terms of name-subscripted  $Y$ s and  $\rho$ s. For simplicity of presentation, this considers maximum number of factors, that is as many  $Y$ s as there are names in the collection. Assuming no firm-specific risk factors, that is  $q_i(0, t_i) = 1$ , the joint survival probability (23) can be rewritten as

$$\text{Prob}(\tau_i > t_i, \tau_j > t_j, \dots) = \mathbb{E} \{ C(\Phi_i(t_i), \Phi_j(t_j), \dots) \} = \mathbb{E} \{ 1_{U_i \leq \Phi_i(t_i)} 1_{U_j \leq \Phi_j(t_j)} \dots \}. \quad (28)$$

In the last step, uniformly distributed random variables  $U_i, U_j, \dots$  are introduced to express the copula  $C()$  as their joint distribution function. Now, expressing  $U_i$ s in terms of random variables  $Y_i$ s as  $U_i = 1 - N_{Y_i}(Y_i)$ , we have

$$\text{Prob}(\tau_i > t_i, \tau_j > t_j, \dots) = \text{E} \{ 1_{X_i \geq K_i(t_i)} 1_{X_j \geq K_j(t_j)} \dots \}, \quad (29)$$

where  $X_i = \sqrt{1 - \rho_i} Z_i + \sqrt{\rho_i} Y_i$ . Restricting to one name, we find  $K_i(t) = N_{X_i}^{-1}(P_i(t))$ . Writing  $X_i \geq K_i(t_i)$  in terms of a uniform random variable  $N_{X_i}(X_i)$  as  $N_{X_i}(X_i) \geq P_i(t_i)$ , we note that the above is in fact a survival copula of default times.

It is well-known that copula models are static models. That they lack dynamics is evident from the  $\Phi_i(t)$ -construct that has just one random variable  $Z_i$  generating the whole process  $\Phi_i(t)$ . A sample path of  $\Phi_i(t)$  is fully determined given a sample of  $Z_i$ . Its time-dependence is due to  $K_i(t)$  alone. Besides, these models have no support for firm-specific risk. Better models can be constructed with the help of subordinators that we next turn to.

## 5 Stable Subordinator Models

To proceed, it is helpful to invoke some concept of stationarity, that the model looks similar at some future time given conditions similar to as it exists today. This is easier to do when the names involved have flat hazard rate curves. It is better done at individual levels since, at the collective level, there can be dependency information available from defaults that needs to be incorporated. For the unconditional individual survival probability  $Q_i(t)$ , a flat hazard rate curve implies  $Q_i(t_2) = Q_i(t_1)Q_i(t_2 - t_1)$  (assumed similarly for any firm-specific component) given two times  $t_1$  and  $t_2$  such that  $0 < t_1 < t_2$ . Writing

$$\text{E} \{ e^{-\Lambda_i(t_2)} \} = \text{E} \{ e^{-\Lambda_i(t_1)} \text{E} \{ e^{-(\Lambda_i(t_2) - \Lambda_i(t_1))} | \Lambda_i(t_1) \} \}, \quad (30)$$

we note that a stationarity concept can be naturally accommodated with a  $\Lambda_i(t)$  that has independent and stationary increments. This is as expected since it is well-known that the most natural choice for  $\Lambda_i(t)$  incorporating stationary is a Lévy process.  $\Lambda_i(t)$  is further expected to be a subordinator, that is a Lévy process involving only non-decreasing sample paths. Realizations of  $\Lambda_i(t)$  then contribute a non-increasing factor  $e^{-\Lambda_i(t)}$  to unconditional individual survival probability.

Though type-II models are more promising and are the main focus of the article, a little discussion of type-I models is helpful to put them in perspective. Type-I models were formulated in a heterogeneous setting in section 3. Consider the individual processes  $\Lambda_i(t)$  modeled as  $\Lambda_i(t) = a_i X(t) + b_i t$  for some constants  $a_i$ s and  $b_i$ s in terms of a common process  $X(t)$ , a subordinator having Laplace exponent  $\eta(u)$ . This gives for the unconditional individual survival probability

$$Q_i(t) = \text{E} \{ e^{-\Lambda_i(t)} \} = e^{-t(\eta(a_i) + b_i)}. \quad (31)$$

This generates a constant hazard rate  $\eta(a_i) + b_i$ , leading to a heterogeneous model with flat hazard rate curves. Drift term  $b_i t$  generates the firm-specific component. However, to generate a nonzero probability of a total collapse, a nonzero probability of  $X(t) = \infty$  is needed. Besides, though time-dependent  $a_i$ s and  $b_i$ s can be helpful, it is not straightforward

to extend the model to allow for general hazard rate curves while retaining some concept of stationarity. Replacing  $X(t)$  by a time-changed subordinator  $X(\phi(t))$  given some non-decreasing function of time  $\phi(t)$  can be helpful. But, allowing for  $\phi(t)$  to be name-dependent leads to both conceptual and practical issues.

Type-II models are better suited for the purpose since they are formulated independently at the individual levels to start with. If  $\Lambda_i(t)$  has Laplace exponent  $\eta_i(u)$  and the firm-specific hazard rate is  $b_i$  so that  $q_i(0, t) = e^{-tb_i}$ , we get for the unconditional individual survival probability

$$Q_i(t) = q_i(0, t) \mathbb{E} \{ e^{-\Lambda_i(t)} \} = e^{-t(b_i + \eta_i(1))}. \quad (32)$$

This generates a constant hazard rate  $b_i + \eta_i(1)$  leading to a heterogeneous model with flat hazard rate curves. Unlike in type-I models, firm-specific component here arises from  $q_i(0, t)$  and as we will see later, drift component of  $\Lambda_i(t)$  has a different role to play, to generate a nonzero probability of a total collapse. As for extending this to a general hazard rate curve, there can be various approaches, some of them possibly offering an explanation of its time-dependence. If our interest is just accommodating the curve, the easiest thing to do is to make some of the parameters time-dependent. This can be done independently at the individual level making it applicable to heterogeneous collections. Note that all model intricacies here are built into  $q_i(F, t)$ , that is  $q_i(0, t)$  and  $\Lambda_i(t)$ . We may thus simply consider some of the parameters of  $\Lambda_i(t)$  as time-dependent. An equivalent but more convenient approach is to work with a time-changed subordinator, time-changed on an individual basis with regard to an intrinsic time-like variable  $\theta_i(t)$  given by

$$\theta_i(t) = -\ln Q_i(t). \quad (33)$$

Since the individual survival probability  $Q_i(t)$  is a non-increasing function of time,  $\theta_i(t)$  is appropriate to play the role of time. This approach provides a simple and neat extension of the stationarity concept to general hazard rate curves.

Specifically, consider subordinators  $X_i(t)$ , independent ones for each of the names in the collection, whose Laplace exponent is  $\eta(u)$ . Let us set

$$q_i(0, t) = e^{-(1-\sigma\eta(1))\theta_i(t)}, \quad \Lambda_i(t) = X_i(\sigma\theta_i(t)). \quad (34)$$

This introduces  $\sigma$  as one of the parameters of the model. Now, the hazard rate curve is automatically calibrated to, since

$$Q_i(t) = q_i(0, t) \mathbb{E} \{ e^{-\Lambda_i(t)} \} = e^{-(1-\sigma\eta(1))\theta_i(t)} \mathbb{E} \{ e^{-X_i(\sigma\theta_i(t))} \} = e^{-\theta_i(t)}. \quad (35)$$

As for the conditional survival probability  $q_i(F, t)$ , we have

$$q_i(F, t) = q_i(0, t) \mathbb{E} \{ 1_{X_i(\sigma\theta_i(t)) \leq -\ln F} \} = e^{-(1-\sigma\eta(1))\theta_i(t)} g_{\sigma\theta_i(t)}(-\ln F), \quad (36)$$

where  $g_t(x)$  is the cumulative distribution function of  $X_i(t)$ .

Note that we require  $\sigma\eta(1) \leq 1$  so that the firm-specific survival probability  $q_i(0, t)$  is non-increasing as a function of  $t$ . Further note that, due to the introduction of parameter  $\sigma$ , an overall scale for the Laplace exponent  $\eta(u)$  can be conveniently chosen. Also,  $\sigma$  can be both name and time dependent. Time-dependence, or more conveniently  $\theta_i(t)$ -dependence,

is subject to  $q_i(F, t)$  non-increasing with respect to  $t$ . In other words, suppressing name and time dependence for simplicity,  $\sigma\theta$  and  $(1 - \sigma\eta(1))\theta$  should both be non-decreasing with respect to  $t$ . Assuming constant values for any other parameters so that  $\eta(1)$  is time-independent, this requires

$$0 \leq \frac{d(\rho\theta)}{d\theta} \leq 1, \quad \text{where } \rho = \sigma\eta(1). \quad (37)$$

Satisfying this is sufficient to ensure  $0 \leq \rho \leq 1$ .  $d(\rho\theta)/d\theta$  is interpretable as local- $\rho$  with  $\rho$  as its  $\theta$ -average. It is hence possible to have a consistent  $\rho$  term-structure, for instance with piecewise constant  $d(\rho\theta)/d\theta$ .

In addition to  $\sigma$  or  $\rho$ , the model is expected to have at least one more parameter  $\mu$ . This is because a subordinator  $X_i(t)$  can have a non-negative drift component  $\mu t$  so that distribution  $g_t(x)$  gets set to zero for  $x < \mu t$ . This introduces a positive drift  $\mu\sigma\theta_i(t)$  to  $\Lambda_i(t)$  that forces  $q_i(F, t)$  to zero for  $F > e^{-\mu\sigma\theta_i(t)}$ . In the infinite homogeneous pool, this implies that  $F_t(\nu)$  tends to  $F_{\max}(t) = e^{-\mu\sigma\theta(t)}$  as  $\nu \rightarrow 1$ . This realizes the possibility envisioned earlier of a probability mass  $1 - F_{\max}(t)$  at  $\nu = 1$ . It implies a non-zero probability  $1 - F_{\max}(t)$  of all the names in the pool defaulting by time  $t$ ,  $\mu$  controlling the likelihood of such a catastrophe. Recall also that  $F_t(\nu) = 0$  for  $\nu$  below  $\nu_{\min} = 1 - q_t(0)$ . Both the probability mass and the minimum  $\nu$  increase with  $t$  starting from zero.

Let us now confine ourselves to modeling with subordinators whose distributions have power-law tails. In particular, we will be dealing with stable subordinators reviewed briefly in section 1. A rationale for this choice can be noted from expression (20) for the joint survival probability. It contains the combination  $\text{Max}_{i \in \Omega} \Lambda_i(t)$ , the maximum of independent, and say identically distributed,  $\Lambda_i$ -processes. This is also the combination that is central to extreme value theory. Distributions having power-law tails give rise naturally to extreme value distributions in an appropriate limit. Note that the extreme events we are concerned with here occur for large  $\Lambda_i$ s, necessitated by large  $-\ln F$  or small  $F$  which is the short-end behavior of  $F_t(\nu)$ . A dependent default when its likelihood  $\nu$  is small (more precisely when  $(\nu - \nu_{\min})/(1 - \nu_{\min})$  is small) is in fact an extreme event.

It turns out that type-II models using stable subordinators can be calibrated reasonably well to market data on CDOs (see Balakrishna [2010]). Interestingly, applicable index of stability  $\alpha$  turns out to be just  $1/2$ . The  $\alpha = 1/2$  stable subordinator is called the Lévy subordinator. Its Laplace exponent is  $\eta(u) = \mu u + \sqrt{u}$ . Its cumulative distribution function is known as the Lévy distribution and is available in closed form,

$$g_t(x) = 2N \left( -t/\sqrt{2(x - \mu t)} \right), \quad (38)$$

where  $N()$  is the cumulative standard normal distribution function. This includes a non-negative drift component  $\mu t$ . With the Lévy subordinator chosen for  $X_i(t)$ , the above distribution gives us for the conditional survival probability

$$q_i(F, t) = 2e^{-(1-\sigma(1+\mu))\theta_i(t)} N \left( -\sigma\theta_i(t)/\sqrt{-2(\ln F + \mu\sigma\theta_i(t))} \right). \quad (39)$$

Note that, though this result is obtained with a time-changed Lévy subordinator, it can be viewed as arising from an ordinary Lévy subordinator, but time-inhomogeneous with

time-dependent parameters  $\sigma\theta_i(t)/t$  and  $\mu\sigma\theta_i(t)/t$ , or, if a local description preferred, with time-dependent local parameters  $d(\sigma\theta_i(t))/dt$  and  $d(\mu\sigma\theta_i(t))/dt$ .

Consistency requirement  $\sigma\eta(1) \leq 1$  now reads  $\sigma(1 + \mu) \leq 1$ . As earlier indicated, it may be sometimes preferable to work with  $\rho = \sigma(1 + \mu)$  in place of  $\sigma$ . If we further introduce  $\kappa = \mu/(1 + \mu)$ ,  $\sigma$  and  $\mu$  are given in terms of  $\rho$  and  $\kappa$  by

$$\sigma = \rho(1 - \kappa), \quad \mu = \frac{\kappa}{1 - \kappa}. \quad (40)$$

Both  $\rho$  and  $\kappa$  are allowed take values in the range  $[0, 1]$ . Note that  $\rho$  is the fraction of hazard rate that is attributable to systemic factors. Of this fraction, a further fraction  $\kappa$  is attributed to catastrophic scenarios.

$F_t(\nu)$  in the infinite homogeneous pool can be obtained by solving  $q_t(F) = 1 - \nu$  for  $F = F_t(\nu)$ . In the Lévy subordinator model, when  $\sigma(1 + \mu) \leq 1$  as is required for consistency, the minimum  $\nu$  below which  $F_t(\nu) = 0$  is

$$\nu_{\min}(t) = 1 - e^{-(1-\sigma(1+\mu))\theta(t)}. \quad (41)$$

This increases with  $t$  starting from zero at  $t = 0$ . For  $\nu$  above  $\nu_{\min}(t)$ ,  $F_t(\nu)$  is

$$F_t(\nu) = \exp \left\{ -\mu\sigma\theta(t) - \frac{1}{2}(\sigma\theta(t))^2 \left[ N^{-1} \left( \frac{1}{2}(1 - \nu)e^{(1-\sigma(1+\mu))\theta(t)} \right) \right]^{-2} \right\}, \quad \nu \geq \nu_{\min}(t). \quad (42)$$

Note that  $\int_0^1 d\nu F_t(\nu) = 1 - \int_0^1 dF_t(\nu)\nu = e^{-\theta(t)}$  as expected. Further,  $F_t(\nu) \rightarrow e^{-\mu\sigma\theta(t)}$  as  $\nu \rightarrow 1$  so that there is a probability mass at  $\nu = 1$  as observed earlier, suggesting a finite probability  $1 - e^{-\mu\sigma\theta(t)}$  of a total collapse.

## 6 Why $\alpha = 1/2$

An intriguing aspect of the model is the choice  $\alpha = 1/2$  for the index of stability of the stable subordinator. Calibration results indicate this as appropriate so that the Lévy subordinator can be considered to be the basic model of  $\Lambda_i(t)$  processes. This is somewhat analogous to the Brownian motion being the choice as the basic model of many economic processes. Perhaps this has something to do with fact that Lévy distribution is dual to the half-normal distribution or that it is the first passage time distribution of the Brownian motion. Though the following discussion does not answer why  $\alpha = 1/2$  is appropriate, it may become clearer from the discussion that the question is better phrased as to why  $1/\alpha = 2$ , a choice that has some obvious niceties and appealness associated with it. Note that  $p_t(F)$  and  $F_t(\nu)$  are inverses of each other and hence an index in the range  $[1/2, 1]$  applicable to  $p_t(F)$  has an effective applicability to  $F_t(\nu)$  in the range  $[1, 2]$ .

The long tail of the distribution of a stable subordinator  $X(t)$  obeys a power-law decay as in (5). Thus for  $F$  small (dropping the name-subscripts), we have

$$\begin{aligned} q_t(F) &= q_t(0)g_{\sigma\theta(t)}(-\ln F) \\ &\rightarrow q_t(0) - q_t(0)\sigma\theta(t) [\Gamma(1 - \alpha)]^{-1} (-\ln F)^{-\alpha}. \end{aligned} \quad (43)$$



Recall that  $q_t(0) = e^{-(1-\sigma\eta(1))\theta(t)} = 1 - \nu_{\min}(t)$ . This suggests that for  $\nu > \nu_{\min}(t)$  but close to  $\nu_{\min}(t)$ , that is for small positive  $(\nu - \nu_{\min}(t))/(1 - \nu_{\min}(t))$ , we have

$$-\ln F_t(\nu) \rightarrow (\sigma\theta(t))^{1/\alpha} [\Gamma(1 - \alpha)]^{-1/\alpha} [(\nu - \nu_{\min}(t))/(1 - \nu_{\min}(t))]^{-1/\alpha}. \quad (44)$$

This is the expected form arising from extreme value theory considerations with its relevant index  $1/\alpha$ . For  $\alpha = 1/2$ , the index is two. Similar behavior is obtained for small  $\theta(t)$  since  $g_{\sigma\theta(t)}(-\ln F)$  is a function of the combination  $(\sigma\theta(t))^{-1/\alpha}(-\ln F - \mu\sigma\theta(t))$ ,

$$-\ln F_t(\nu) \rightarrow \mu\sigma\theta(t) + (\sigma\theta(t))^{1/\alpha} [\Gamma(1 - \alpha)]^{-1/\alpha} \nu^{-1/\alpha}. \quad (45)$$

Again for  $\alpha = 1/2$ , the index is two and  $F_t(\nu)$  is regular in  $\theta(t)$  as  $\theta(t) \rightarrow 0$ , consistent with the closed form solution (42).

The log-distribution of a stable subordinator  $X(t)$  exhibits power-law at the short end as in (6). This suggests that for  $\nu$  close to one, we have

$$-\ln F_t(\nu) \rightarrow \mu\sigma\theta(t) + \alpha(\sigma\theta(t))^{1/\alpha}(1 - \alpha)^{(1-\alpha)/\alpha} [-\ln(1 - \nu)]^{-(1-\alpha)/\alpha}. \quad (46)$$

This gives a fat long tail as a function of  $-\ln(1 - \nu)$  with a tail-index of  $1/\alpha - 1$ . For  $\alpha = 1/2$ , this index is one. Thus,  $-\ln F_t(\nu)$  as a function of  $-\ln[(1 - \nu)/(1 - \nu_{\min}(t))]$  features a power-law at the long and short ends with indices of one and two respectively. Attempt to model  $F_t(\nu)$  directly as arising from a stable process is expected to come close with a tail-index in-between, but requiring a cutoff to prevent negative moves. This is what is borne out by a study in Balakrishna [2009].

Let us next look at default correlation over period  $(0, t)$  for small  $\theta(t)$ . We will assume here that  $\alpha \in (1/2, 1)$ . The two-point survival probability over period  $(0, t)$  in the homogeneous case can be expressed as

$$\int_0^1 dF[q_t(F)]^2 = [q_t(0)]^2 e^{-\mu\sigma\theta(t)} \int_0^\infty dx x^{-\alpha-1} f(x) e^{-(\sigma\theta(t))^{1/\alpha} x}, \quad (47)$$

where  $f(x) = x^{\alpha+1} d(h(x))^2/dx$  and  $h(x) = g_{\sigma\theta(t)}((\sigma\theta(t))^{1/\alpha} x + \mu\sigma\theta(t))$  (independent of  $t$  for a stable subordinator). For small  $\theta(t)$ , we have (see Balakrishna [2009])

$$\int_0^1 dF[q_t(F)]^2 \simeq 1 + (\mu\sigma - 2)\theta(t) + c(\sigma\theta(t))^{1/\alpha}, \quad (48)$$

with some constant  $c$ . This leads to a two-point default probability  $\simeq \mu\sigma\theta(t) + c(\sigma\theta(t))^{1/\alpha}$  and a default correlation  $\simeq \mu\sigma + c\sigma(\sigma\theta(t))^{(1-\alpha)/\alpha}$ . For  $\alpha \in (1/2, 1)$ , the exponent of the last term above is  $1/\alpha \in (1, 2)$ . As  $\alpha \rightarrow 1/2$ , this exponent tends to two and the next correction term in the expansion becomes significant. It can be shown that in this limit, that is in the Lévy subordinator model for which  $h(x) = 2N(-1/\sqrt{2x})$ , the two-point default probability  $\simeq \mu\sigma\theta(t) - (2\sigma^2/\pi)(\theta(t))^2 \ln \theta(t)$  and default correlation  $\simeq \mu\sigma - (2\sigma^2/\pi)\theta(t) \ln \theta(t)$ . The default correlation remains nonzero at  $\mu\sigma$  as  $\theta(t) \rightarrow 0$  resulting in what may be called the instantaneous default correlation. It leads to a tail dependence for the loss distribution that has become necessary recently for better pricing of the senior tranches.

## 7 Semi-Analytical Pricing

An attractive feature of the Lévy subordinator model is that it admits a closed form expression for  $q_i(F, t)$ . This provides an analytical expression for the expected loss making it possible to price CDOs semi-analytically. The procedure is identical to that used in the Gaussian copula model. The following is a brief review of the steps involved.

Consider the loss process  $L(t)$ . Its contribution at time  $t$  to tranche  $[a, b]$  with attachment point  $a$  and detachment point  $b$  can be expressed as a call-spread, or equivalently as  $b - a$  minus the put-spread so that the loss per tranche size is

$$L(t)_{[a,b]} = 1 - \frac{1}{b-a} [(b - L(t), 0)_+ - (a - L(t), 0)_+]. \quad (49)$$

Subscript  $+$  denotes picking up the positive term. If the cumulative distribution function of  $L(t)$  is  $H_t(x)$  at time  $t$ , the expected loss per tranche size is

$$\bar{L}(t)_{[a,b]} = 1 - \frac{1}{b-a} \int_0^b dx (dH_t(x)/dx) [(b-x, 0)_+ - (a-x, 0)_+]. \quad (50)$$

After a partial integration, this can be written as

$$\bar{L}(t)_{[a,b]} = 1 - \frac{1}{b-a} \int_a^b dx H_t(x). \quad (51)$$

Loss distribution is usually taken to be a discrete distribution, in which case more care can be exercised in deriving this. If discrete,  $H_t(x)$  is flat in-between successive points giving appropriate interpolations at  $x = a$  and  $x = b$ .

Given the expected loss per tranche size, the default leg of a tranche can be priced per tranche size as

$$DL_{[a,b]} = \int_0^T D(t) d\bar{L}(t)_{[a,b]}, \quad (52)$$

where  $T$  is the maturity and  $D(t)$  is the discount factor for the time period  $(0, t)$ . Similarly the premium leg at unit spread can be priced per tranche size as

$$PL_{[a,b]} = \sum_{i=1}^{N_\delta} \delta_i(t_i) D(t_i) [1 - \bar{L}(t_i)_{[a,b]}] + PL'_{[a,b]}, \quad (53)$$

where  $\delta_i(t_i)$  is the accrual factor for the period  $(t_{i-1}, t_i)$ ,  $t_{N_\delta} = T$  and  $N_\delta$  is the number of periods.  $PL'_{[a,b]}$  is the contribution from accrued interest payments made upon default,

$$PL'_{[a,b]} = \sum_{i=1}^{N_\delta} \int_{t_{i-1}}^{t_i} \delta_i(t) D(t) d\bar{L}(t)_{[a,b]}, \quad (54)$$

where  $\delta_i(t)$  is the accrual factor for the partial period covering  $(t_{i-1}, t)$ . Given the leg values, fair spread can be obtained by dividing the default leg by the premium leg, after taking care of any upfront payments.

Integrations can be performed over a sufficiently fine time-grid. Time-steps making up the grid can be as wide as the periods themselves for efficient pricing, and hence the factors multiplying the increments  $d\bar{L}(t)_{[a,b]}$  are evaluated at mid-points of the time-steps. The super senior tranche can be priced like an ordinary tranche along with a part of the notional that is a fraction  $R$  of the total notional of the underlying credit default swaps outstanding, or, if recovery rates are nonuniform, sum of fractions  $R_i$  of the individual notionals.

## 8 Large Homogeneous Pool

Because the Lévy subordinator model is structured similar to the Gaussian copula model, efficient pricing techniques of the latter can be directly employed in the present case. One of them is the large homogeneous pool approximation that can be a useful tool since it admits an explicit expression for the loss distribution.

We have already discussed an infinitely large homogeneous pool and its cumulative distribution function  $F_t(\nu)$ . It can also be approached starting from a finite collection of  $n$  names. The joint default probability that  $k$  or less number of names are in the defaulted state at time  $t$  and the rest are not is given by

$$P_{\{k\}}(t) = \sum_{j=0}^k \binom{n}{j} \int_0^1 dF [p_t(F)]^j [1 - p_t(F)]^{n-j}. \quad (55)$$

For an infinitely large homogeneous pool of names, that is as  $n \rightarrow \infty$ , it is well-known that, by the law of large numbers, the above simplifies to

$$F_t(\nu) = P_{\{\nu n\}}(t) = \int_0^1 dF 1_{p_t(F) \leq \nu}, \quad (56)$$

where  $\nu = k/n$  is the fraction of names in the defaulted state at time  $t$ . This indicates that  $F_t(\nu)$  can be obtained by summing up the region of  $F$  over which  $p_t(F) \leq \nu$ . We have considered  $p_t(F)$  to be an increasing function of  $F$ . Hence,  $F_t(\nu)$  can be obtained by solving  $p_t(F) = \nu$ , or equivalently  $q_t(F) = 1 - \nu$ , for  $F = F_t(\nu)$ . This is in agreement with our earlier result (42) in the Lévy subordinator model.

The expected loss per tranche size for a tranche  $[a, b]$  can be computed as

$$\bar{L}(t)_{[a,b]} = 1 - \frac{1}{\nu_b - \nu_a} \int_{\nu_a}^{\nu_b} d\nu F_t(\nu), \quad (57)$$

where  $\nu_a = a/(1 - R)$ ,  $\nu_b = b/(1 - R)$  and  $R$  is the uniform recovery rate. Because  $F_t(\nu) = 0$  for  $\nu \leq \nu_{\min}(t)$ , the expected loss becomes 100% of the tranche size once  $\nu_{\min}(t)$  crosses  $\nu_b$ , if  $b$  is small enough for this to occur within the maturity of the trade. This leads to overpricing of the equity tranches. Finite  $n$  and heterogeneity is expected to offer better pricing by smoothening out the small  $\nu$  behavior.

## 9 Finite Pool with FFT

Large homogeneous pool approximation yields fast results, but at the expense of accuracy. As is well-known, many of the factor models admit efficient pricing for finite  $n$  with recursive

methods or Fast Fourier Transform (FFT) techniques. Being structured similar to the Gaussian copula model, the present model can be handled analogously. The following outlines the steps involved in computing with FFT.

To obtain the loss distribution for finite  $n$ , consider the loss variable at time  $t$  conditional on  $F$  given by

$$\mathcal{L}(F, t) = \sum_{i=1}^n L_i \xi_i(F, t), \quad (58)$$

where  $\xi_i(F, t)$  is the conditional default indicator at time  $t$  and  $L_i = (1 - R_i)w_i$ ,  $R_i$  being the recovery rate and  $w_i$  the fraction of the total pool notional associated with the  $i^{\text{th}}$  name. Though not explicitly shown,  $L_i$  can be dependent on both  $F$  and  $t$  (as in section 11 on random recovery rates). Default indicators being independent conditional on  $F$ , the above has the characteristic

$$\mathbb{E} \{ e^{iu\mathcal{L}(F,t)} \} = \prod_{m=1}^n [q_m(F, t) + p_m(F, t)e^{iuL_m}], \quad (59)$$

where  $i = \sqrt{-1}$ . This characteristic is the Fourier transform of the density function of the loss distribution (conditional on  $F$  unless mentioned otherwise). Hence, the loss distribution can be obtained by inverting it using FFT techniques. The result can be used to compute the expected loss per tranche size for a tranche with attachment point  $a$  and detachment point  $b$  according to

$$\bar{L}(F, t)_{[a,b]} = 1 - \frac{1}{b-a} \int_a^b dx H_t(F, x), \quad (60)$$

where  $H_t(F, \cdot)$  is the cumulative loss distribution function.

FFT requires discretization of  $u$ . Discretization is straightforward if  $L_i$ 's are uniform at  $L$  across the collection ( $L = (1 - R)/n$  if  $R_i$ s are uniform). Inversion then yields the loss distribution at loss-points  $j = 0, \dots, n$  in units of  $L$ . This gives the default probability density  $P_{[j]}(F, t)$ , the sum of products of various combinations of  $j$  of the  $p_i(F, t)$ s and  $n - j$  of the  $q_i(F, t)$ s. Consider it extended up to  $j = N - 1 \geq n$  by padding with zeros where  $N$  is a power of 2, as is usually done for an efficient FFT. In this case, (59) reads

$$\sum_{j=0}^{N-1} P_{[j]}(F, t) e^{i\omega j k} = \prod_{m=1}^n [q_m(F, t) + p_m(F, t) e^{i\omega k L_m}], \quad k = 0, \dots, N - 1, \quad (61)$$

where  $\omega = 2\pi/N$ . This can easily be computed and inverted using FFT techniques to obtain  $P_{[j]}(F, t)$ ,  $j = 0, \dots, n$ , and hence its cumulative counterpart  $G_t(F, \nu)$  (that corresponds to  $H_t(F, jL)$ ) where  $\nu = j/n$  is the fraction of names in the defaulted state. Expected loss per tranche size is then

$$\bar{L}(F, t)_{[a,b]} = 1 - \frac{1}{\nu_b - \nu_a} \int_{\nu_a}^{\nu_b} d\nu G_t(F, \nu), \quad (62)$$

where  $\nu_a = a/(nL)$ ,  $\nu_b = b/(nL)$ , and  $G_t(F, \nu)$  is flat in-between successive  $\nu$ -points. Integration of  $\bar{L}(F, t)_{[a,b]}$  over  $F$  gives  $\bar{L}(t)_{[a,b]}$ , the unconditional expected loss per tranche size. This result can be used to price the CDO tranches as discussed in section 7. Integration over  $F$  can be performed numerically using the Gauss-Legendre quadrature formula. It is efficient to perform the integration after the prices are computed conditional on  $F$ .

## 10 Monte Carlo Pricing

Though the Lévy subordinator model can be handled semi-analytically as detailed above, a Monte Carlo simulation algorithm can be a useful tool to price non-standard products. It can also be useful for pricing standard tranches as it is found to be efficient, accurate and easily implementable, and does not involve discretization of time. The following algorithm can be viewed as simulating the model defined by expression (22) or simply as a method of computing the integrals involved in the semi-analytical pricing. Efficiency of the algorithm can be improved substantially by using quasi random sequences such as Sobol sequences to generate each of the independent uniform random numbers.

The algorithm reads as follows.

1. Draw a uniformly distributed random number  $F$  and  $n$  independent uniformly distributed random numbers  $u_i, i = 1, \dots, n$ .
2. For each credit name  $i$ , first determine whether it defaults before the time horizon  $T$  by checking if  $q_i(F, T) < u_i$  where  $q_i(F, \cdot)$  is given in equation (39). If so, solve the equation  $q_i(F, t_i) = u_i$  for  $\theta_i(t_i)$ . Determine default time  $t_i$  of credit name  $i$  by a table look up into its hazard rate curve.
3. Given the default times before the time horizon, price the instrument. For the next scenario, go to step 1.
4. Average all the prices thus obtained to get a price for the instrument.

Given a scenario of default times, it is straightforward to price the CDOs. One proceeds processing the defaults one by one, starting from the first up to maturity, picking up payments by the default leg, switching to the next tranche whenever a tranche gets wiped out, at the same time computing the premium legs per unit spread for all the surviving tranches. Whenever a default leg pays out the loss amount, the notional of that tranche gets reduced by the same amount, and the notional of the super senior tranche gets reduced by the recovery amount (when the super senior is the only survivor, it gets treated like a default swap). The leg values can be added across tranches to obtain those for the index default swap. Fair spreads can be computed given the leg values at the end of the simulation.

## 11 Random Recovery Rates

It has become apparent, especially during the recent crisis, that random recovery rate is helpful in better pricing of the senior tranches. Random recovery rates have been discussed by Andersen and Sidenius [2004], and recently by Amraoui and Hitier [2008] and Krekel [2008] in response to the recent crisis within the context of the Gaussian copula model. Here, let us consider a similar approach with an emphasis on tractability, and randomness of recovery rates arising from a decreasing dependence on  $F$ . Because we are concerned with just one individual name in this section, name-subscripts are omitted. Further, because time-dependence of  $q_t(F)$  is only through its dependence on  $\theta(t)$ , all time-dependences in this section can be expressed as a dependence on  $\theta(t)$  if desired.

Consider a possibly time-dependent recovery rate  $R(t)$  used in building the hazard rate curve.  $R(t)$  can be time-dependent when modeled in line with some of the empirical findings supporting an inverse relationship between recovery rates and hazard rates. The expected

loss contribution during an infinitesimal time interval  $(t, t + dt)$  is  $(1 - R(t))dP(t)$  where  $P(t) = 1 - e^{-\theta(t)}$  is as before the default probability over the period  $(0, t)$ . If  $R(t)$  were constant, this accumulates to an expected loss at time  $t$  of  $(1 - R)P(t)$ . If  $R(t)$  is time-dependent and one prefers working with the expected loss, it is convenient to introduce a recovery rate  $\bar{R}(t)$  such that the expected loss at time  $t$  is  $(1 - \bar{R}(t))P(t)$ . The two are related by  $(1 - R(t))dP(t) = d[(1 - \bar{R}(t))P(t)]$  that solves to

$$\bar{R}(t) = \frac{1}{1 - e^{-\theta(t)}} \int_0^{\theta(t)} d\phi(s) e^{-\phi(s)} R(s). \quad (63)$$

Hence  $\bar{R}(t)$  is the expected recovery rate conditional on default during  $(0, t)$ . We may refer to  $R(t)$  as the instantaneous recovery rate and  $\bar{R}(t)$  as the period recovery rate.

Similar results can be written down when conditioned by  $F$  in a model of the kind implied by expression (22). Let the conditional instantaneous recovery rate be  $R(F, t)$  and the conditional period recovery rate be  $\bar{R}(F, t)$ . The conditional expected loss should integrate to the unconditional expected loss so that

$$\int_0^1 dF \bar{R}(F, t) p(F, t) = \bar{R}(t) (1 - e^{-\theta(t)}). \quad (64)$$

The conditional equivalent of  $R(t)dP(t) = d(\bar{R}(t)P(t))$  is

$$R(F, t) = \frac{\partial_t (\bar{R}(F, t) p(F, t))}{\partial_t (p(F, t))}. \quad (65)$$

$\bar{R}(F, t)$  is the recovery rate to be used in the semi-analytical pricing and  $R(F, t)$  is the recovery rate to be used in the Monte-Carlo algorithm presented earlier.

A tractable choice for  $\bar{R}(F, t)$  that has decreasing  $F$ -dependence is, for some  $\chi > 0$ ,

$$\bar{R}(F, t) = R_0(t) - (R_0(t) - R_1) F^\chi. \quad (66)$$

This decreases from  $R_0(t)$  to  $R_1$  (assuming  $R_0(t) > R_1$ ) as  $F$  runs from zero to one. For simplicity,  $R_1$  is considered to be  $t$ -independent.  $R_0(t)$  gets related to  $\bar{R}(t)$  (and hence  $R(t)$ ) by relation (64) that can be evaluated to obtain

$$R_0(t) = R_1 + (\bar{R}(t) - R_1) \left[ 1 - \frac{1 - e^{-c\theta(t)}}{(1 + \chi)(1 - e^{-\theta(t)})} \right]^{-1},$$

where  $c = 1 + \sigma(\eta(1 + \chi) - \eta(1)) = 1 + \sigma(\sqrt{1 + \chi} + \mu\chi - 1)$ . (67)

Recall that  $\eta(u)$  is the Laplace exponent of the subordinator ( $\mu u + \sqrt{u}$  for the Lévy Subordinator). Decreasing  $F$ -dependence of  $\bar{R}(F, t)$  implies  $R_0(t) > \bar{R}(t) > R_1$ . For constant or decreasing  $\bar{R}(t)$  (and constant  $\chi$ ),  $R_0(t)$  is a decreasing function of  $t$ . Requirement  $\bar{R}(F, t) \leq 1$  is thus satisfied by ensuring  $R_0(0) \leq 1$ . Instantaneous random recovery rate  $R(F, t)$  can be obtained from  $\bar{R}(F, t)$  using relation (65). No simple criterion is available to ensure that  $R(F, t)$  is within bounds.

## 12 Initial Conditions

We have assumed that all the  $\Lambda_i(t)$  processes start off as  $\Lambda_i(0) = 0$ . In one possible interpretation of  $\Lambda_i(t)$  as the amount of negative economic impact on the name, this implies that conditions prior to time zero are assumed to be completely favorable. In this section, let us explore the consequences of relaxing this assumption.

The joint survival probability  $Q_\Omega(t)$  over the period  $(t_0, t)$  for a list of names in  $\Omega$  is given in (20) for zero initial conditions, that is  $\Lambda_i(t_0) = 0$  for all  $i$ ,  $t_0 = 0$  being today. Let assume that it holds good for some earlier time  $t_0 < 0$  given a realization of the  $\Lambda_i(t)$  processes from  $t_0$  to today. Now, the joint survival probability over the period  $(0, t)$  conditional on survival up to today, denoted for simplicity as  $Q_\Omega(t)$  itself, can be written down as

$$Q_\Omega(t) = e^{\Lambda_\Omega(0)} \left[ \prod_{i \in \Omega} q_i(0, t) \right] \mathbb{E} \left\{ e^{-\text{Max}_{i \in \Omega}(\Lambda_i(t))} \right\} = e^{\Lambda_\Omega(0)} \int_0^1 dF \prod_{i \in \Omega} q_i(F e^{\Lambda_i(0)}, t), \quad (68)$$

where  $\Lambda_\Omega(0) = \text{Max}_{i \in \Omega}(\Lambda_i(0))$  is the largest  $\Lambda_i(0)$  and

$$q_i(F, t) = q_i(0, t) \mathbb{E} \left\{ 1_{\Lambda_i(t) - \Lambda_i(0) \leq -\ln F} \right\}. \quad (69)$$

As before,  $q_i(0, t)$  and  $q_i(F, t)$  are functions defined from today onwards with  $q_i(0, 0) = q_i(F, 0) = 1$ . Note that  $Q_\Omega(t)$  depends only on the differences  $\Lambda_\Omega(0) - \Lambda_i(0)$ . To see this, note that  $q_i(F e^{\Lambda_i(0)}, t) = 0$  for  $F > e^{-\Lambda_\Omega(0)}$  for the name with  $\Lambda_i(0) = \Lambda_\Omega(0)$ . The integrand is hence zero for  $F > e^{-\Lambda_\Omega(0)}$  so that the integration variable can be changed to  $F' = F e^{\Lambda_\Omega(0)}$  that ranges from zero to one. Because  $Q_\Omega(t)$  depends only on the differences, an overall impact does not contribute to it; only the relative values are relevant. This is as it should be since the names are all known to have survived up to today. The presence of initial conditions with relative values leads to higher joint survival probabilities. However, calibration to individual hazard rate curves remains the same as before since, when there is only one name in  $\Omega$ ,  $Q_\Omega(t)$  is independent of the initial condition.

Though the above is expressed as an integral over  $F$ , it is not a conditionally independent representation since the factor  $e^{\Lambda_\Omega(0)}$  multiplying the integral is  $\Omega$ -dependent. One way to render it conditionally independent is to express the factor as

$$e^{\Lambda_\Omega(0)} = e^{\Lambda_M(0)} - \int_1^{e^{\Lambda_M(0)}} dG \prod_{i \in \Omega} 1_{\Lambda_i(0) \leq \ln G}, \quad (70)$$

where  $\Lambda_M(0)$  is the largest  $\Lambda_i(0)$  in the collection. Using this in (68) we obtain

$$Q_\Omega(t) = e^{\Lambda_M(0)} \int_0^1 dF \prod_{i \in \Omega} q_i(F e^{\Lambda_i(0)}, t) - \int_0^1 dF \int_1^{e^{\Lambda_M(0)}} dG \prod_{i \in \Omega} q_i(F e^{\Lambda_i(0)}, t) 1_{\Lambda_i(0) \leq \ln G}. \quad (71)$$

This is a weighted difference of two conditionally independent representations that can be used for semi-analytical pricing in the presence of non-trivial initial conditions, but the presence of a double integral can make it computationally inefficient.

### 13 Default Contagion

We have modeled  $\Lambda_i(t)$ s as a set of independent stochastic processes evolving from time zero onwards. As our zero of time passes by, one would expect  $\Lambda_i(t)$ s to get realized. However, it is not obvious how this information can be extracted from the names. Let us assume in this section that the only information available from the names is their defaulted or undefaulted status. In such a scenario, one expects the hazard rate of a given name to jump up on every default information as it arrives.

A simple analysis in our one-factor formulation helps us to infer that such a contagion tendency exists at the probability level itself. Arguments are similar to those discussed in Balakrishna [2007]. Consider  $\pi_n^{(\nu)}(t)$ , the conditional probability of default during  $(0, t)$  of say name  $n$  given the information that names  $1, \dots, \nu$  have defaulted,

$$\pi_n^{(\nu)}(t) = \frac{\int_0^1 dF [p_1(F, t) \cdots p_\nu(F, t)] p_n(F, t)}{\int_0^1 dF [p_1(F, t) \cdots p_\nu(F, t)]}. \quad (72)$$

This can be compared to  $\pi_n^{(\nu-1)}(t)$ , conditional probability of default without the default information about name  $\nu$ . If  $\pi_n^{(\nu)}(t) > \pi_n^{(\nu-1)}(t)$ , it suggests that the likelihood of name  $n$  having defaulted increases with the number of names known to have defaulted. This check amounts to verifying

$$\int_0^1 dF \int_0^F dG w_{\nu-1}(F, t) w_{\nu-1}(G, t) [p_\nu(F, t) - p_\nu(G, t)] [p_n(F, t) - p_n(G, t)] > 0, \quad (73)$$

where  $w_{\nu-1}(F, t) = p_1(F, t) \cdots p_{\nu-1}(F, t)$ . This always holds since  $p_i(F, t)$ s are monotonic with respect to  $F$  for all the names.

Now coming to jumps in the hazard rate due to default contagion, again for name  $n$ , consider the joint survival probability

$$Q(t_-) = \text{Prob}(\tau_1 > t_1, \dots, \tau_n > t_n), \quad (74)$$

where  $\tau_i$ s are random default times,  $t_1 < \dots < t_n$ ,  $t_-$  denotes dependence on  $t_1, \dots, t_n$  and for convenience the names are labeled according to the same order. The probability density (or intensity) that names  $1, \dots, \nu$  have defaulted respectively at times  $t_1, \dots, t_\nu$  and the rest have survived up to times  $t_{\nu+1}, \dots, t_n$  is then given by

$$Q_{(\nu)}(t_-) = (-1)^\nu \frac{\partial^\nu Q(t_-)}{\partial t_1 \cdots \partial t_\nu}. \quad (75)$$

We may allow  $\nu = 0$  as well in which case  $Q_{(0)}(t_-) = Q(t_-)$ . Given the default and the survival information defining it, the hazard rate for name  $n$  can now be written as

$$h_n^{(\nu)}(t_-) = -\partial_n \ln Q_{(\nu)}(t_-), \quad (76)$$

where  $\partial_n = \partial/\partial t_n$ . The jump in the hazard rate  $h_n^{(\nu-1)}(t_-)$  due to name  $\nu$  defaulting is, dropping  $t$ -arguments for simplicity,

$$\Delta_\nu h_n^{(\nu-1)} = h_n^{(\nu)} - h_n^{(\nu-1)} = -\partial_n \ln \left( \frac{Q_{(\nu)}}{Q_{(\nu-1)}} \right) = -\partial_n \ln (-\partial_\nu \ln Q_{(\nu-1)}). \quad (77)$$



If positive, this indicates that defaults are contagious, that for every credit name defaulting, the hazard rate of the name  $n$  being observed jumps up. To know this jump for any time  $t_n$ , given all the survival and default information up to time  $t \leq t_n$ , all of  $t_{\nu+1}, \dots, t_{n-1}$  should be set to  $t$ . If interested in this jump just after the last credit name  $\nu$  has defaulted,  $t$  should be set to  $t_\nu$  as well.

Hazard rate jumps are, as one would expect, proportional to default correlation. This can be seen by rewriting the jump as

$$\Delta_\nu h_n^{(\nu-1)} = \frac{Q_{(\nu-1)}}{Q_{(\nu)}} \left[ \frac{\partial_n \partial_\nu Q_{(\nu-1)}}{Q_{(\nu-1)}} - \frac{(-\partial_n Q_{(\nu-1)})}{Q_{(\nu-1)}} \frac{(-\partial_\nu Q_{(\nu-1)})}{Q_{(\nu-1)}} \right] = \sqrt{\frac{-\partial_n Q_{(\nu-1)}}{Q_{(\nu)}}} \rho_{n\nu}^{(\nu-1)}, \quad (78)$$

where  $\rho_{n\nu}^{(\nu-1)} \sqrt{dt_n dt_\nu}$  is the default correlation between credit names  $n$  and  $\nu$  for defaults during infinitesimal intervals  $(t_n, t_n + dt_n)$  and  $(t_\nu, t_\nu + dt_\nu)$  respectively, given the conditions implicit in  $Q_{(\nu-1)}$ . For  $\nu = 1$ , this relates to default correlation in the absence of contagion and for  $\nu > 1$  to default correlation in the middle of a contagion.

Consider now the extended one-factor model defined by (22) that gives for the joint survival probability,

$$Q(t_-) = \int_0^1 dF [q_1(F, t_1) \cdots q_n(F, t_n)]. \quad (79)$$

In this model, the probability density  $Q_\nu(t_-)$  is

$$Q_\nu(t_-) = \int_0^1 dF w(F, t_-) [r_1(F, t_1) \cdots r_\nu(F, t_\nu)], \quad (80)$$

where  $w(F, t_-) = q_1(F, t_1) \cdots q_n(F, t_n)$  and  $r_i(F, t_i) = -\partial_i(\ln q_i(F, t_i))$ . This gives for the hazard rate

$$h_n^{(\nu)}(t_-) = \frac{\int_0^1 dF w(F, t_-) [r_1(F, t_1) \cdots r_\nu(F, t_\nu)] r_n(F, t_n)}{\int_0^1 dF w(F, t_-) [r_1(F, t_1) \cdots r_\nu(F, t_\nu)]}. \quad (81)$$

This is again of the form  $\pi_n^{(\nu)}(t)$  given in expression (72). Similar steps let us conclude that  $h_n^{(\nu)}(t_-)$  will jump up on every default if it can be shown that  $r_i(F, t_i)$  is monotonic in  $F$  for all the names. That this is the case can easily be verified in the Lévy subordinator model for  $\mu = 0$ , and appears to hold good for nonzero  $\mu$  as well.

## 14 Intensity Modeling

We modeled the individual process  $\Lambda_i(t)$  as a time-changed Lévy subordinator. An alternate approach to modeling it as a non-decreasing process is to express it as the time-integral of a non-negative stochastic process that in some sense can be interpreted as stochastic default intensity. However, as we see below, when driven by a stable subordinator, the resulting model is effectively a time-inhomogeneous stable subordinator model. Hence, our Lévy subordinator model can be considered to be sufficiently generic.

In the following, we are concerned with one credit name and hence the name-subscript is dropped from the results. Consider  $\lambda(t)$  obeying the stochastic differential equation

$$d\lambda(t) = -m\lambda(t)dt + dS(t), \quad (82)$$

where  $S(t)$  is a subordinator and  $m$  is the mean reversion rate (better referred to as the decay rate since the mean need not be finite). This can be solved to obtain

$$\begin{aligned}\lambda(t) &= \lambda(0)e^{-mt} + \int_0^t e^{-m(t-s)}dS(s), \\ \Lambda(t) &= \int_0^t \lambda(s)ds = \lambda(0)b(t) + \int_0^t b(t-s)dS(s),\end{aligned}\tag{83}$$

where  $b(t) = (1 - e^{-mt})/m$ . For simplicity, we have not considered time-changing the process here. Given above, one can determine the Laplace transform

$$e^{-\psi_t(u)} = \mathbb{E} \left\{ e^{-u\Lambda(t)} \right\} = \exp \left\{ -\lambda(0)b(t)u - \int_0^t \eta(ub(s))ds \right\},\tag{84}$$

where  $\eta(u)$  is the Laplace exponent of  $S(t)$ . If  $S(t)$  is chosen to be the Lévy subordinator with  $\eta(u) = \mu\sigma u + \sigma\sqrt{u}$ , we have

$$\psi_t(u) = \left\{ \lambda(0)b(t) + \mu\sigma \int_0^t b(s)ds \right\} u + \left\{ \sigma \int_0^t \sqrt{b(s)}ds \right\} \sqrt{u}.\tag{85}$$

Note that this reduces to  $t(\mu'\sigma'u + \sigma'\sqrt{u})$  when the limit  $m \rightarrow \infty$  is appropriately taken, that is with  $\mu/\sqrt{m} \rightarrow \mu'$  and  $\sigma/\sqrt{m} \rightarrow \sigma'$  as  $m \rightarrow \infty$ . Hence, our earlier Lévy subordinator model is the  $m \rightarrow \infty$  limit of this intensity model. Besides,  $\psi_t(u)$  itself is of the form  $p(t)u + q(t)\sqrt{u}$  for some parameters  $p(t)$  and  $q(t)$  so that the Laplace transform is that of a Lévy distribution. It can be generated by a time-inhomogeneous Lévy subordinator. Drift  $p(t)$  is now stochastic since, when viewed as a dynamical model, it has a dependence on the present value of the stochastic process  $\lambda(t)$ .

There can be some reservations about intensity modeling in this type-II framework. Note that  $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$  is a predictive process and for  $t$  infinitesimally small can be written as  $\approx \lambda_i(0)t$ . Consider  $\lambda_i(0)$  to be a fraction  $\rho$  of the initial hazard rate  $h_i(0)$  inferred from the hazard rate curve, fraction  $1 - \rho$  being the firm-specific default intensity  $\bar{\lambda}_i(0)$ . Fixing  $\Lambda_i(t)$  deterministically may itself be questionable, but leaving this aside, consider the joint survival probability  $Q_{12}(t)$  for two names 1 and 2. Assuming  $\lambda_1(0) \leq \lambda_2(0)$ , we have  $Q_{12}(t) \approx 1 - \lambda_2(0)t - \bar{\lambda}_1(0)t - \bar{\lambda}_2(0)t$ . This leads to maximum dependability, a restrictive feature of the model. Instantaneous default correlation at time zero can be computed to be  $\rho\sqrt{h_1(0)/h_2(0)}$ . This can turn out to be significant even under normal economic conditions giving rise to an unacceptable number of simultaneous defaults. These observations can also be made directly from our result above that involves  $\lambda_i(0)$  contributing to the drift component that, as we have already noted, accounts for catastrophic scenarios.

Instead, it may be more appropriate to make  $\theta_i(t)$  dynamic, say as the time-integral of some non-negative stochastic process  $\lambda_i(t)$ . Our earlier Lévy subordinator model can now be defined for a realization of  $\lambda_i(t)$ . Time-changing the subordinator is still with  $\theta_i(t)$ , but the latter is no longer given by (33). Instead, it leads to the individual survival probability  $Q_i(t) = \mathbb{E} \left\{ e^{-\theta_i(t)} \right\}$ . A possible choice for the  $\lambda_i(t)$  process is square-root diffusion that provides analytical results to compute  $Q_i(t)$ . A closed form expression is also available for the distribution of  $\lambda_i(t)$ , but computing joint default probabilities involves distribution of its time-integral making it harder pricing semi-analytically. Nevertheless, this approach is appealing as it accounts for stochasticity of hazard rates.

## 15 Discussions and Conclusions

*“Mathematics is important, and the more mathematics you know the better off you’re going to be. But don’t fall too much in love with it.”*

– Emanuel Derman in *“Introduction to the Smile; The Principles of Valuation”*.

Modeling the dependency structure of defaults or other such extreme events is a problem of multi-dimensional mathematical complexity. But the need for a simple and tractable solution is evident from the popularity of the Gaussian copula model. The model has remained the market standard for pricing correlation products in spite of all its limitations, very well-known in the field. It is the purpose of the article to demonstrate that there do exist other models, equally simple and tractable, that have better loss process dynamics and explanatory power. The Lévy subordinator model presented here is one such model offering a reasonable explanation of the correlation smile.

The Lévy subordinator model is a one-factor model driven by the Lévy subordinator, an  $\alpha = 1/2$  stable process maximally skewed to the right. The distribution function of the Lévy subordinator is known in closed form as the Lévy distribution. The model shares many of the attractive features of the Gaussian copula model. It gets automatically calibrated to individual hazard rate curves. It can be used for pricing both semi-analytically by employing recursive methods or Fast Fourier Transform techniques, and via a Monte Carlo algorithm. Being structured similar to the Gaussian copula model, it can easily be implemented within the framework of the existing computational infrastructure. In fact, the only modifications needed are to use (39) for the conditional survival probability in place of a similar one of the Gaussian copula model and to integrate the conditional results over a uniform distributed common factor instead of a normally distributed one.

The model has just two parameters  $\sigma$  and  $\mu$ , or equivalently  $\rho = \sigma(1 + \mu)$  and  $\kappa = \mu/(1 + \mu)$ . Both  $\rho$  and  $\kappa$  are permitted to range from zero to one.  $\rho$  is the fraction of hazard rate that is attributable to systemic factors. Of this fraction, a further fraction  $\kappa$  is attributed to catastrophic scenarios. When deep in crisis,  $\rho$  may well tend to its upper limit of one. When  $\rho = 1$ , the model still has a freedom of one parameter, namely  $\kappa$ . This can be helpful in, at least qualitatively, accounting for in-crisis correlation smiles.

Figures 1-10 present the results of a numerical investigation into the model’s implications. Results are for a homogeneous collection with flat hazard rates. Figures 1-4 plot the base correlations for various ranges of parameters under fixed recovery rates. Base correlations are implied by the Gaussian copula model for the Lévy subordinator model prices. Model prices and hedge ratios are computed for tranches [0,3]%, [3,7]%, [7,10]%, [10,15]% and [15,30]%. As can be seen from the figures, despite having only two parameters at its disposal, the model is capable of generating correlation smiles of various slope characteristics. Figures 5-8 present the hedge ratios under similar conditions. The hedge ratios appear reasonable, at least qualitatively, with the right dependencies on the model parameters. Figures 9 and 10 show the effect of random recovery rates on base correlations and hedge ratios, in particular that of the parameter  $\chi$  introduced in equation (66).

We have considered the model parameters as uniform across the collection for simplicity. As already noted, they can be name-dependent (and also time-dependent). An attractive feature of the model is that the pool specifics are completely encoded into the names so that

it becomes easier to combine pools to construct larger ones. For example, if we have two pools calibrated separately in the one-factor setup giving rise to two sets of parameters, we can price trades on a pool constructed out of names picked up from the two pools assuming applicability of the one-factor formulation. This is of course the first step since in general one would need the multi-factor formulation (23) with at least two factors and a copula describing their joint distribution.

We modeled the individual process  $\Lambda_i(t)$  as a time-changed Lévy subordinator. Other subordinators can also be attempted that admit closed form solutions to their distributions such as the inverse Gaussian subordinator that is a natural extension of the Lévy subordinator. Alternately, the conditional survival probability  $q_i(F, t)$  can be modeled directly, for instance as a mixture of Lévy distributions. Though these extensions were not found to be helpful in improving the fit to market data on CDOs in a preliminary study in Balakrishna [2010], they may be helpful under different market conditions.

Though the model has been developed with an application to CDOs in mind, it could be useful in other disciplines that involve modeling a dependent set of events. Simplicity and tractability with its large homogeneous pool approximation, an efficient semi-analytical framework and a Monte Carlo algorithm makes the model an attractive choice.

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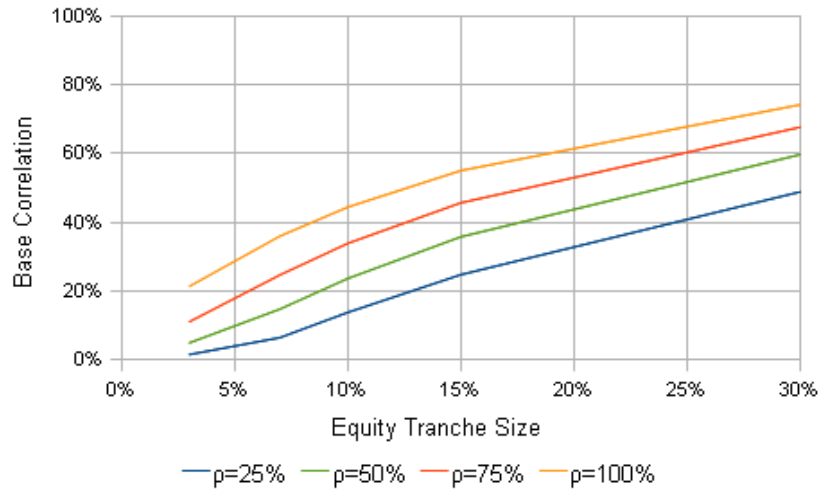


Figure 1: Base correlations for  $\rho = 25\%, 50\%, 75\%$  and  $100\%$  with fixed  $\kappa = 25\%$  and recovery rate  $R = 40\%$ . Maturity is 5 years and reference spread is at 100bp.

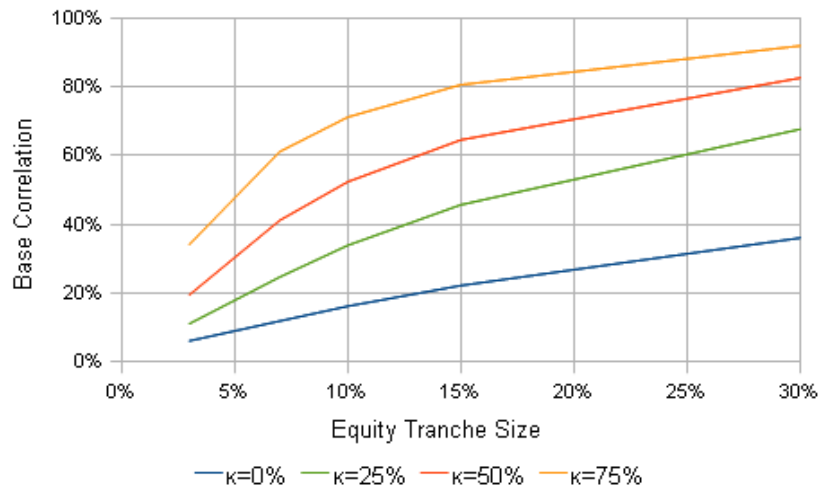


Figure 2: Base correlations for  $\kappa = 0\%, 25\%, 50\%$  and  $75\%$  with fixed  $\rho = 75\%$  and recovery rate  $R = 40\%$ . Maturity is 5 years and reference spread is at 100bp.

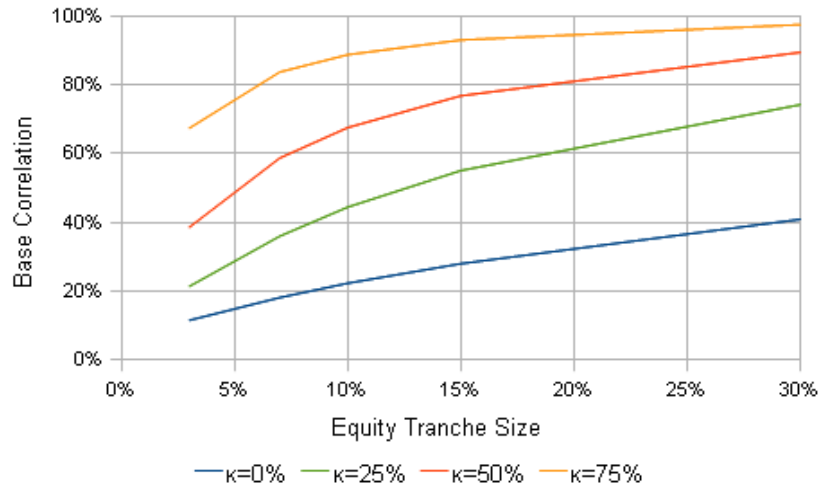


Figure 3: Base correlations for  $\kappa = 0\%, 25\%, 50\%$  and  $75\%$  with fixed  $\rho = 100\%$  and recovery rate  $R = 40\%$ . Maturity is 5 years and reference spread is at 100bp.

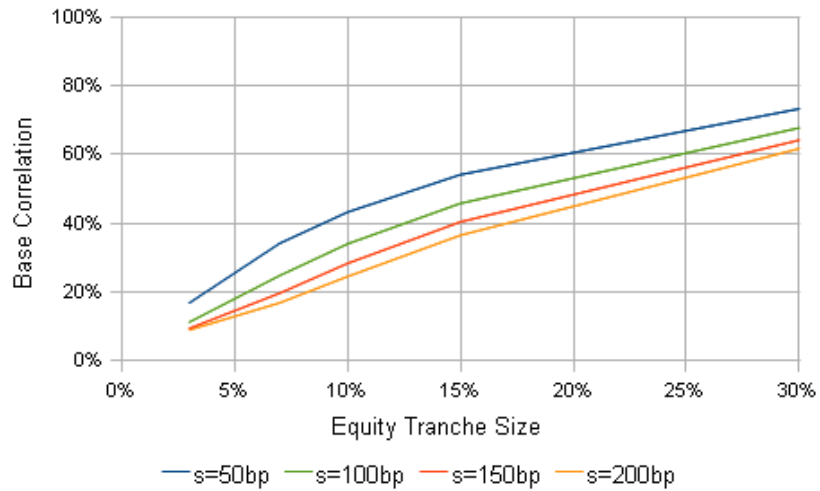


Figure 4: Base correlations for reference spreads  $s = 50, 100, 150$  and  $200\text{bp}$  with fixed  $\rho = 75\%$  and  $\kappa = 25\%$ . Recovery rate  $R = 40\%$  and maturity is 5 years.

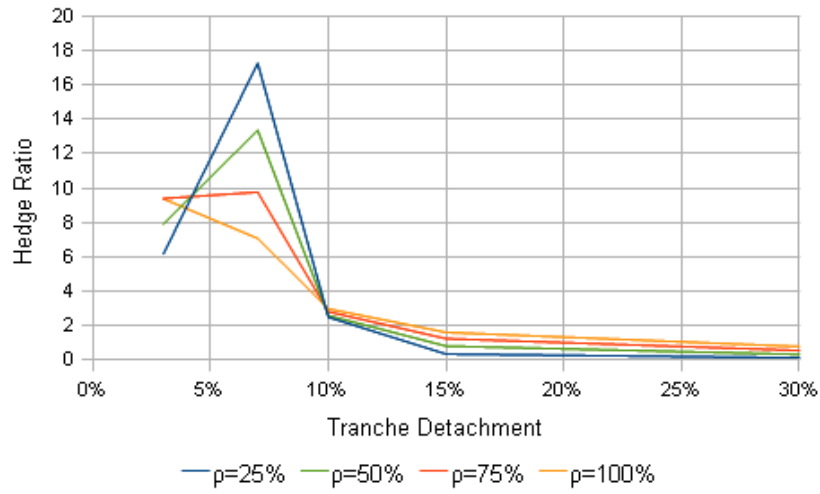


Figure 5: Hedge ratios for  $\rho = 25\%, 50\%, 75\%$  and  $100\%$  with fixed  $\kappa = 25\%$  and recovery rate  $R = 40\%$ . Maturity is 5 years and reference spread is at 100bp.

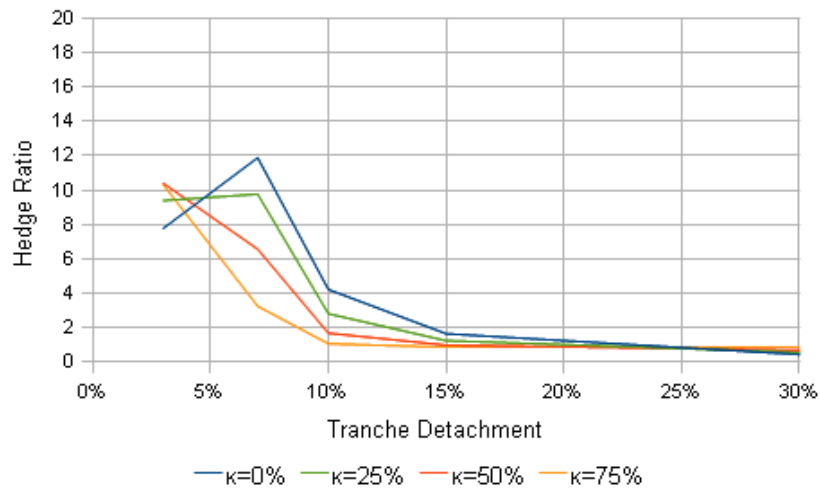


Figure 6: Hedge ratios for  $\kappa = 0\%, 25\%, 50\%$  and  $75\%$  with fixed  $\rho = 75\%$  and recovery rate  $R = 40\%$ . Maturity is 5 years and reference spread is at 100bp.



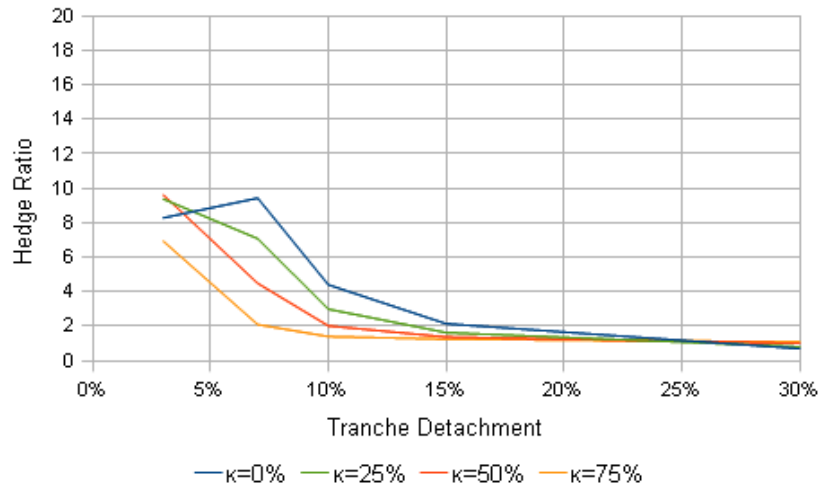


Figure 7: Hedge ratios for  $\kappa = 0\%, 25\%, 50\%$  and  $75\%$  with fixed  $\rho = 100\%$  and recovery rate  $R = 40\%$ . Maturity is 5 years and reference spread is at 100bp.

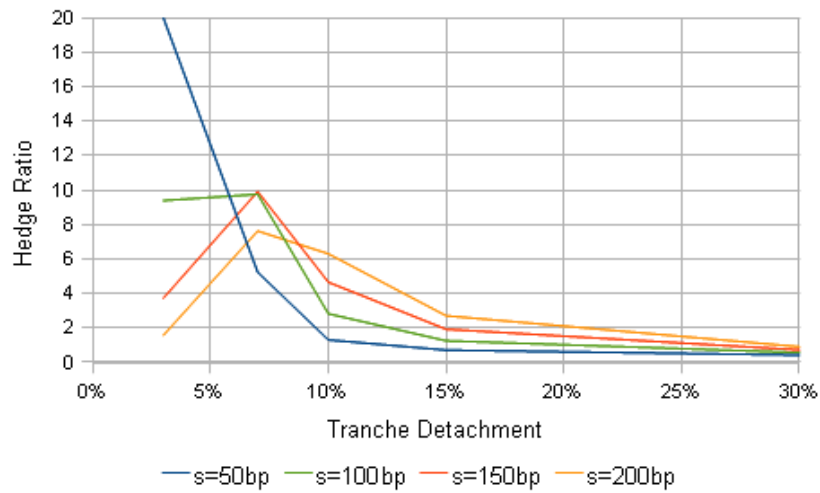


Figure 8: Hedge ratios for reference spreads  $s = 50, 100, 150$  and  $200\text{bp}$  with fixed  $\rho = 75\%$  and  $\kappa = 25\%$ . Recovery rate  $R = 40\%$  and maturity is 5 years.

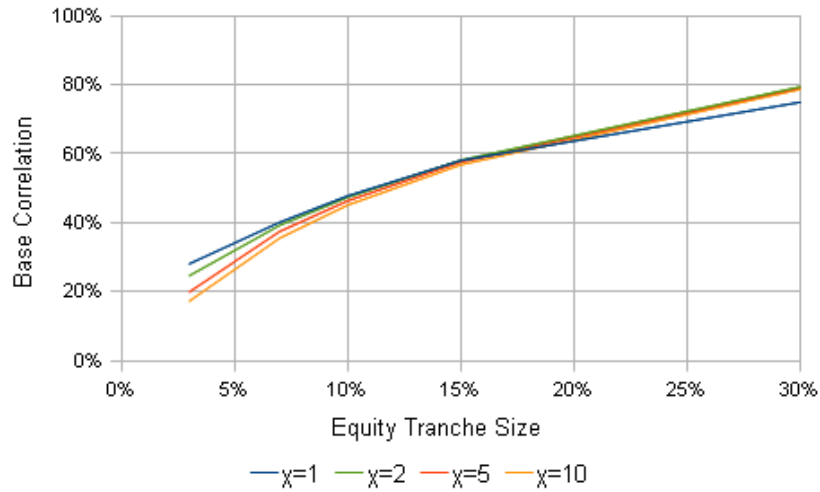


Figure 9: Base correlations for  $\chi = 1, 2, 5$  and  $10$  with fixed  $\rho = 75\%$  and  $\kappa = 25\%$ . Recovery rate is random with mean  $R = 40\%$  and minimum  $R_1 = 20\%$ . Maturity is 5 years and reference spread is at 100bp.

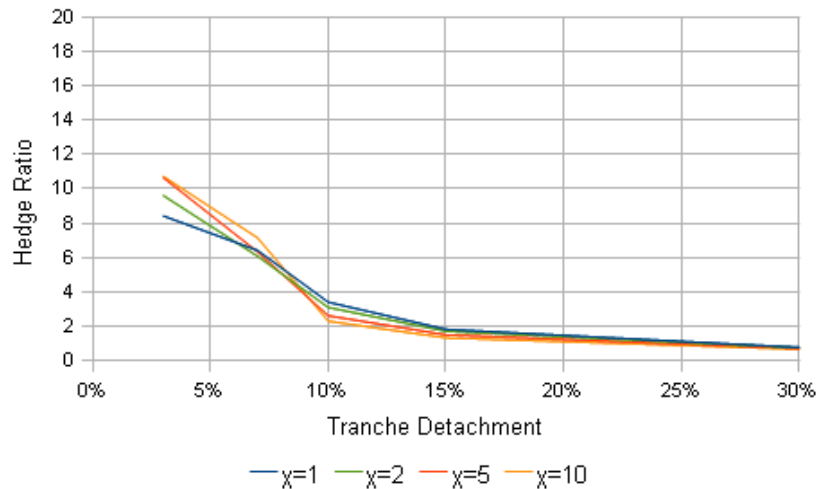


Figure 10: Hedge ratios for  $\chi = 1, 2, 5$  and  $10$  with fixed  $\rho = 75\%$  and  $\kappa = 25\%$ . Recovery rate is random with mean  $R = 40\%$  and minimum  $R_1 = 20\%$ . Maturity is 5 years and reference spread is at 100bp.