Interim Partially Correlated Rationalizability

Tang, Qianfeng

The University of Texas at Austin

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Qianfeng Tang*

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Abstract

In game theory, there is a basic methodological dichotomy between Harsanyi's "game-theoretic" view and Aumann's "Bayesian decision-theoretic" view of the world. We follow the game-theoretic view, propose and study interim partially correlated rationalizability for games with incomplete information. We argue that the distinction between this solution concept and the interim correlated rationalizability studied by Dekel, Fudenberg and Morris (2007) is fundamental, in that the latter implicitly follows Aumann's Bayesian view.

Our main result shows that two types provide the same prediction in interim partially correlated rationalizability if and only if they have the same infinite hierarchy of beliefs over conditional beliefs. We also establish an equivalence result between this solution concept and the Bayesian solution–a notion of correlated equilibrium proposed by Forges (1993).

*Department of Economics, The University of Texas at Austin. Email: tangqianfeng198@mail.utexas.edu.
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1 Introduction

In complete information games, rationalizability is an important solution concept. It was first introduced independently by Bernheim (1984) and Pearce (1984). Intuitively, a rationalizable action is one that a player may play given the minimal assumption of common knowledge of rationality among players. We join the effort in extending rationalizability to games with incomplete information. In particular, we study interim rationalizable actions: actions that are rationalizable to a player after she receives her private information. Harsanyi type spaces (Harsanyi, 1967-1968), which model players' private information as their (private) types and parameters of payoff functions as states of nature, are the basic tool for studying games with incomplete information. With this tool, the problem transforms into studying rationalizable actions for any given type of a player.

Similar to rationalizable actions in complete information games, interim rationalizable actions can also be defined using the procedure of iterative elimination of never best response actions. In this procedure, actions that are not a best response to any conjectures are eliminated step by step, and the actions that survive to the end are called rationalizable. In games with incomplete information, players need to conjecture on both the others' actions and states of nature. If we fix a type space, how should we define a player's belief over both the others' actions and states of nature?

There are generally two approaches to model such beliefs: Harsanyi's game-theoretic view (Harsanyi, 1967-1968), or principle, and Aumann's Bayesian (decision-theoretic) view (Aumann, 1987)[1]. Harsanyi's principle distinguishes states of nature as independent variables and actions as type-contingent variables, and insists that subjective probabilities should be assigned only to independent variables. Instead, Aumann's Bayesian view holds that

[1]This distinction between Aumann's Bayesian view and Harsanyi's principle is also adopted by Forges (1993) in defining correlated equilibria for games with incomplete information. In her terminologies, the two viewpoints are named the universal Bayesian approach and the partial Bayesian approach, respectively.

subjective probabilities are assignable to anything unknown, including the others' actions.

We use an example taken from Ely and Pęski (2006) to illustrate the effects of these different approaches.

Example 1. *This is a two-player game with incomplete information, with states of nature parameterized by* $\Theta = \{\theta_1, \theta_2\}$. *Each player has three actions,* $A_i = \{a_i, b_i, c_i\}, i = 1, 2,$ *and players' payoffs are given by*

θ_1	a_2	b_2	c_2
a_1	$1, 1$	$-10, -10$	$-10, 0$
b_1	$-10, -10$	$1, 1$	$-10, 0$
c_1	$0, -10$	$0, -10$	$0, 0$

θ_2	a_2	b_2	c_2
a_1	$-10, -10$	$1, 1$	$-10, 0$
b_1	$1, 1$	$-10, -10$	$-10, 0$
c_1	$0, -10$	$0, -10$	$0, 0$

Figure 1.

Given the payoffs, players would like to match, on a *or* b, *in state* θ_1 *and mismatch in state* θ_2. *Players can also play action* c, *which is a safe action and always pays* 0.

Consider first a trivial type space T *in which each player has just one type:* $T_1 = T_2 = \{*\}$. *Assume it is common knowledge between players that* θ_1 *and* θ_2 *happen with equal probability. Since players are symmetric, we concentrate on player* 1.

With Harsanyi's principle, players' actions must be type contingent. Since player 2 *has only one type, player* 1 *expect player* 2 *to play the same strategies (pure or mixed) in states* θ_1 *and* θ_2. *Given any strategy of player* 2, *actions* a_1 *and* b_1 *give player* 1 *strictly negative expected payoffs and thus are strictly dominated by* c_1. *As a result,* c_1 *is the only rationalizable action for player* 1.

If instead we follow Aumann's Bayesian view, player 1 *could legitimately conjecture that player* 2 *plays* a_2 *at state* θ_1 *and* b_2 *at state* θ_2*. Given this conjecture, it is a unique best response for player* 1 *to play* a_1*. We can similarly check that the product set* $\{a_1, b_1\} \times \{a_2, b_2\}$ *is a best reply set, and thus a subset of rationalizable action profiles.*

Previously, Dekel, Fudenberg and Morris (2007) proposed a notion of interim correlated rationalizability. Their approach implicitly fits with Aumann's Bayesian view; they assume that a player's conjecture over the others' types, states of nature and the others' actions could be an arbitrary probability measure over the product space, as long as it is consistent with her belief in the type space. The type space that models incomplete information about states of nature, in their view, is the marginal of an epistemic type space that models incomplete information about both states of nature and the others' actions.

We, instead, adopt Harsanyi's principle and define interim partially correlated rationalizability. We assume that actions are type-contingent variables, and that a player's conjecture over the others' actions and states of nature are induced by her belief in the type space together with a type-correlated strategy of the others'. A type-correlated strategy of the others' maps each profile of their types to a probability measure on their action profiles. If we take the agent-normal-form view of a type space, i.e., if we view each type as an agent, the correlation is exactly the same as that in correlated rationalizability in complete information games. In other words, the correlation we permit can be viewed as *interim* correlation, while that permitted by Dekel et al. can be viewed as *ex post* correlation.

Although interim partially correlated rationalizability may seem to be a refinement of interim correlated rationalizability at the first sight, the distinction between them is purely methodological and therefore more fundamental. A type space is an artificially constructed object used to model incomplete information. In order to define the "right" solution concept on it, we need to know beforehand what information is incorporated into the types; more precisely, we need to know whether types contain enough information to tell if the others' actions are type-contingent or not. Conventional construction of types (Mertens and Zamir, 1986) relies on eliciting players' beliefs and higher-order beliefs about states of nature. These type spaces, although sufficient for Aumann's Bayesian view of modeling games, are insufficient for Harsanyi's principle. Indeed, a player's hierarchy of beliefs about states of nature does not contain any information about whether there is direct correlation between the others' actions and states of nature. This can be illustrated with a simple type space presented in Ely and Pęski (2006).

Example 2. *Fix the type space T in Example 1; we describe a type space $\hat{T}$ that has the same set of hierarchies of beliefs about states of nature. Let $\hat{T}_1 = \hat{T}_2 = \{+1, -1\}$, and assume there is a common prior on $\hat{T}_1 \times \hat{T}_2 \times \Theta$:*

$$\theta_1 : \begin{array}{c|c|c|} t_1 \backslash t_2 & +1 & -1 \\ \hline +1 & \frac{1}{4} & 0 \\ \hline -1 & 0 & \frac{1}{4} \\ \hline \end{array} \qquad \theta_2 : \begin{array}{c|c|c|} t_1 \backslash t_2 & +1 & -1 \\ \hline +1 & 0 & \frac{1}{4} \\ \hline -1 & \frac{1}{4} & 0 \\ \hline \end{array}$$

Figure 2.

Given the prior, two players have the same type if and only if the state is θ_1 and two players have different types if and only if the state is θ_2. At both $+1$ and -1 in $\hat{T}$, each player has the same hierarchy of beliefs about states of nature, i.e., common knowledge that θ_1 and θ_2 happen with equal probability, which is the same as that at type $*$ in T. Thus $\hat{T}$ is redundant with respect to conventional hierarchies of beliefs[2]. The information we elicited from players is insufficient for us to tell which of T and $\hat{T}$ models the actual game environment.

We return to the game in Example 1. If player 1 believes that the distribution on $\Theta \times A_2$

[2] See Liu (2005) for a general study on the redundancy of hierarchies of beliefs in type spaces and the state-dependent correlating mechanism that characterizes it.

is $\frac{1}{2}(\theta_1, a_2)$ and $\frac{1}{2}(\theta_2, b_2)$, in T she must conjecture that player 2's action directly depends on states of nature; however, in $\hat{T}$, at her type $+1$ for example, the belief can be justified by the conjecture that player 2 plays a type-contingent strategy: a_2 at $+1$ and b_2 at -1. Because from a player's conventional hierarchy of beliefs we cannot tell apart T and $\hat{T}$, we cannot tell from it whether the others' actions are type-contingent or not.

Since Harsanyi's principle is almost always implicitly assumed in applications, it is important to know that in order for a type space to satisfy the principle, what additional information needs to be gathered to incorporate into it? The other side of the same question, which is more straightforward, is to study how we represent such information, in some form of hierarchies of beliefs, after the construction of the type space. Example 2 suggests that the representation must be sensitive to correlated signals that directly depend on states of nature. The hierarchy of beliefs constructed in following way is called Δ-hierarchy of beliefs, and was first introduced by Ely and Pęski (2006): if we fix a type of a player, then, conditional on each profile of types of the others, the player will have a conditional belief about states of nature, and her belief about the others' types induces sequentially her belief and higher-order beliefs on the set of conditional beliefs.

Our main result shows that two types have the same interim partially correlated rationalizable behavior if and only if they have the same Δ-hierarchy of beliefs. Not only does this result identify the information that characterizes rationalizable behavior, but also, it provides us with the representation of information necessary for Harsanyi's principle. The sufficiency part of this result can be contrasted with Proposition 1 in Dekel et al. (2007). They show that the identification of interim correlated rationalizability requires only infinite hierarchies of beliefs over states of nature. The distinction between the two identifications explicitly describes the distinction between the methodological viewpoints behind the two solution concepts.

This paper directly extends Ely and Pęski (2006). Ely and Pęski study interim independent rationalizability in two-player games, and introduce Δ-hierarchies of beliefs for its identification. There are multiple extensions of their definition to games with more than two players, due to the existence of multiple ways to formulate correlations; our definition is exactly the one that retains the full implication of Δ-hierarchies of beliefs. The key difference is that we study interim "correlated" rationalizability, instead of interim independent rationalizablity. Naturally, the proof to our main result can be readily extended from Ely and Pęski's work. Nevertheless, we adopt approaches from theirs and make our proofs to both the necessity part and sufficiency part of the main result more direct and accessible.

To justify interim partially correlated rationalizability, we also establish an equivalence result between it and the Bayesian solution–a notion of correlated equilibrium proposed by Forges (1993). The Bayesian solution is defined obeying the partial Bayesian approach, which is equivalent to Harsanyi's principle. We show that type-correlated strategies of the others' can be justified by the Bayesian solution; this result describes explicitly how correlations in the others' actions can be achieved. Brandenburger and Dekel (1987) show, for complete information games, the payoff equivalence between correlated rationalizability and *a posteriori* equilibrium. As an analogue of their result, we show the payoff equivalence between interim partially correlated rationalizability and the Bayesian solution.

Some other research are also related to this paper. Liu (2005) and Liu (2009) study type spaces with the same set of conventional hierarchies of beliefs and Liu (2005) characterize the redundancy with state-dependent correlating mechanisms. The type space $\hat{T}$ in Example 2 can be explained as one such mechanism. Tang (2010) further characterizes the correlation embedded in type spaces with the same set of Δ-hierarchies of beliefs, and studies its implication for the Bayesian solution. These characterizations make more explicit the connections between interim correlated rationalizability and interim partially correlated rationalizability[3]. Using garblings instead of correlating devices, Lehrer, Rosenberg and Shmaya (2006) examine the connections between type spaces that are payoff equivalent in all Bayesian games, for various notions of correlated equilibrium, including the Bayesian solution. The non-communicating garblings they use are inherently equivalent to information mappings that preserve conditional beliefs.

We organize the paper as follows. We introduce notations and models and define solution concepts in Section 2. Examples are also given to distinguish different solutions. We describe the constructions of hierarchies of beliefs in Section 3, and present our main results and results on the connections between solution concepts in Section 4. Section 5 studies the equivalence between the Bayesian solution and our solution. Section 6 concludes.

2 Model

2.1 Set up

We begin with some notations. For any metric space X, let ΔX denote the space of probability measures on the Borel σ-algebra of X endowed with the weak*-topology. Let the product of two metric spaces be endowed with the product Borel σ-algebra. Let $\operatorname{supp}\mu$ be the support of a probability measure μ, i.e., the smallest closed set with probability 1 under μ. For any measure $\mu \in \Delta(X \times Y)$, denote $\operatorname{marg}_X \mu$ the marginal distribution of μ on X. For any measure $\mu \in \Delta X$ and integrable function $f : X \to R$, denote $\mu[f]$ the expectation of f under μ.

We study games with incomplete information with n players. The set of players is $N = \{1, 2, ..., n\}$. For each $i \in N$, let $-i$ denote the set of i's opponents. Players play a game in which the payoffs are uncertain and parameterized by a finite set Θ. Each element $\theta \in \Theta$

[3] And also the connections between the universal Bayesian solution (Forges, 1993) and the Bayesian solution.

is called a state of nature. For each $i \in N$, denote A_i the set of actions for player i, and $A \equiv \times_{i \in N} A_i$ the set of action profiles. A (strategic form) *game* is a profile $G = (g_i, A_i)_{i \in N}$. For each $i \in N$, we assume the payoff function is bounded: $g_i : A \times \Theta \to [-M, M]$, for some positive real number M. The set of finite bounded games is denoted by $\mathcal{G}$.

A *type space* over Θ is defined as $T = (T_i, \pi_i)_{i \in N}$, where for each i, T_i is a compact metric space of types for player i and $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$ is a measurable mapping that describes player i's belief over the others' types and states of nature for any type of player i. A strategy of player i is a mapping $\sigma_i : T_i \to \Delta A_i$. Let $\sigma = (\sigma_i)_{i \in N}$ be a strategy profile, and with a little abuse of notation, let $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ be a type-correlated strategy of the others. The intuition behind type-correlated strategies is provided in the next section.

Throughout, given arbitrary $x \in X$ and $y \in Y$, we use the notation $\pi_i(x)[y]$ to denote player i's belief about y conditional on x. More precisely, the object in the round bracket always denotes the object that player i conditions on, and the object in the square bracket always denotes the object that player i assigns probability to.

2.2 Solution concepts

We propose and study interim partially correlated rationalizability, or IPCR, for games with incomplete information. Previously, Dekel, Fudenberg and Morris (2007, DFM, hereafter) propose both interim correlated rationalizability (ICR) and interim independent rationalizability (IIR); and for two-player games, Ely and Pęski (2006) independently define IIR in a formulation equivalent to DFM's. In this section, we first define our new solution concept and then compare it with the other two. Examples are given at the end of the section.

2.2.1 Interim partially correlated rationalizability

Rationalizability can be defined in many equivalent approaches; we start with the iterative elimination of never best response actions procedure. Player i's (joint) *conjecture* on the others' types, states of nature and the others' actions is a joint distribution $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$. Let $m^v[(\theta, a_{-i})] \equiv \int_{T_{-i}} v[(dt_{-i}, \theta, a_{-i})]$ denote the marginal probability of v at (θ, a_{-i}), i.e., $m^v = \mathrm{marg}_{\Theta \times A_{-i}} v$. An action $a_i \in A_i$ is *a best response to a conjecture* v if

$$a_i \in \arg\max_{a_i' \in A_i} \sum_{\theta, a_{-i}} g_i((a_i', a_{-i}), \theta) m^v(\theta, a_{-i}).$$

Without referring to specific constraints on conjectures, interim rationalizability can in general be defined as follows: for each player $i \in N$, the first round of elimination eliminates actions in A_i that are not a best response to any conjectures about the others' play. In the $k+1$-th round, a level-k conjecture assigns positive probability only to actions of the others' that are level-$(k-1)$ rationalizable, and actions that are not a best response to any level-k conjectures are eliminated. The elimination procedure stops in finite rounds. Actions that survive k rounds of elimination are called *level-k rationalzable actions* and actions that survive to the end are called *rationalizable actions.* Different notions of interim rationalizability may be defined using the same procedure. We first define interim partially correlated rationalizability.

Definition 1. *Fix a game G and a type space T. For all $t_i \in T_i$, $R^T_{i,0}(t_i|G) \equiv A_i$. An action is level-$k$ rationalizable at t_i, i.e., $a_i \in R^T_{i,k}(t_i|G)$, if there exists $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ such that*

1. $(t_{-i}, \theta, a_{-i}) \in \mathrm{supp}\, v \Rightarrow a_{-i} \in R^T_{-i,(k-1)}(t_{-i})$, *where* $R^T_{-i,(k-1)}(t_{-i}) \equiv (R^T_{j,(k-1)}(t_j)|G)_{j \neq i}$;
2. $a_i \in \arg\max_{a_i' \in A_i} \sum_{\theta, a_{-i}} g_i((a_i', a_{-i}), \theta) m^v[(\theta, a_{-i})]$;

3. *(constraint on conjectures) There exists a type-correlated strategy* $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ *such that*

$$m^v[(\theta, a_{-i})] = \int_{T_{-i}} \sigma_{-i}(t_{-i})[a_{-i}] \cdot \pi_i(t_i)[(dt_{-i}, \theta)]. \tag{2.1}$$

Let $R_i^T(t_i|G) = \bigcap_{k=1}^{\infty} R_{i,k}^T(t_i|G)$. *Actions in* $R_i^T(t_i|G)$ *are said to be interim partially correlated rationalizable at type* t_i.

By definition, $R_i^T(t_i|G)$ is always non-empty. Hereafter, we suppress the notation G in $R_i^T(t_i|G)$ unless it is necessary for clarity.

In the definition of IPCR, each joint conjecture $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ is induced by player i's belief $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$ in the type space and a type-correlated strategy $\sigma_{-i}(t_{-i}) \in \Delta A_{-i}$ of the others'. When type spaces are finite, item 2.1 can be simplified as

$$v[(t_{-i}, \theta, a_{-i})] = \pi_i(t_i)[(t_{-i}, \theta)] \cdot \sigma_{-i}(t_{-i})[a_{-i}].$$

By adopting this *constraint on conjectures*, we are following *Harsanyi's principle* on modeling games with incomplete information. Harsanyi models actions as variables dependent on types. This expression also connects interim partially correlated rationalizability with Forges's *partial Bayesian approach* (Forges, 1993): players form subjective beliefs about the others' types and states of nature, but their beliefs over the others' actions are not subjectively formed. See subsubsection 4.2.1 for more discussions.

The type-correlated strategy $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ also deserves some clarification. We are not assuming that the others are sharing information with each other and playing in a coordinated fashion; instead, we take the view that the correlation may come from possibly correlated type-contingent extraneous signals that other players receive (see Section 5), or from player i's ignorance over the others' beliefs about each other's action (Aumann, 1987, section 6).

2.2.2 Interim correlated rationalizability and interim independent rationalizability

To promote understanding, we present the definitions of ICR and IIR proposed by DFM. Since the definitions differ only in *constraint on conjectures* (item 3 in Definition 1), it suffices for us to present the respective variations of item 3.

Definition 2. *Fix a game G and a type space T. We can define the set of interim correlated rationalizability actions at t_i, denoted as $ICR_i^T(t_i|G)$, and the set of interim independent rationalizability at t_i, denoted as $IIR_i^T(t_i|G)$, by replacing item 3 in Definition 1, respectively,*

1. *ICR (constraint on conjectures)* $\mathrm{marg}_{T_{-i}\times\Theta}\, v = \pi_i(t_i)$.
2. *IIR (constraint on conjectures) There exist independent strategies $\sigma_j : T_j \to \Delta A_j, j \neq i$, such that*

$$m^v = \int_{T_{-i}} \prod_{j\neq i} \sigma_j(t_j)[a_j] \cdot \pi_i(t_i)[dt_{-i}, \theta]. \tag{2.2}$$

In the definition of ICR, the constraint requires only that the conjecture $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ be consistent with player i's belief $\pi_i(t_i)$ over $T_{-i} \times \Theta$ in the type space. DFM follow Aumann's Bayesian view and treat every player as a Bayesian decision maker who faces three uncertainties: states of nature, the others' types and their actions. Conjectures are explained as players' subjective beliefs over these uncertainties; actions are not treated as type-contingent variables anymore. In Forges's terminology, this approach is called the *universal Bayesian approach*, as in contrast with the partial Bayesian approach.

In the definition of IIR, the constraint is that player i believes that the others are playing independently. Correlations among the others' actions, if there is any, are characterized by the correlations among the types of the others', which have already been incorporated in

$\pi_i(t_i)$. When type spaces are finite, item 2.2 can be simplified as

$$v[(t_{-i}, \theta, a_{-i})] = \pi_i(t_i)[(t_{-i}, \theta)] \cdot \prod_{j \neq i} \sigma_j(t_j)[a_j].$$

By definition, IIR and IPCR coincide in two-player games.

2.3 Examples

We now show in examples how distinct notions of rationalizability differ in predictions. The distinction between IPCR and ICR has been illustrated in Example 1 in the introduction. For player 1, the set of interim partially correlated rationalizable actions at the type $t_1 = *$ is $\{c_1\}$, while the set of interim correlated rationalizable actions at that type is $\{a_1, b_1, c_1\}$. Now we illustrate with an example the distinction between IPCR and IIR. To do that, we need a game with at least three players

Example 3. *Consider a three-player game with no payoff uncertainty,* $\Theta = \{*\}$. *The action sets are* $A_1 = \{a_1, b_1\}, A_2 = \{a_2, b_2\}, A_3 = \{a_3, b_3, c_3\}$, *and the payoffs are given by*

a_3:

	a_2	b_2
a_1	$1,1,2$	$0,0,2$
b_1	$0,0,2$	$0,0,0$

b_3:

	a_2	b_2
a_1	$0,0,0$	$0,0,2$
b_1	$0,0,2$	$1,1,2$

c_3:

	a_2	b_2
a_1	$1,1,1$	$0,0,0$
b_1	$0,0,0$	$1,1,1$

Figure 3.

The type space is also trivial: $T_1 = T_2 = T_3 = \{*\}$. *In fact, this is a complete information game. As both strategy profiles* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *are Bayesian Nash equilibria,* $\{a_1, b_1\} \times \{a_2, b_2\} \times \{a_3, b_3\}$ *is a subset of rationalizable action profiles (for any notion of rationalizability).*

With IIR, for player 3, actions a_3 and b_3 strictly dominate c_3; because for any product conjecture on player 1 and player 2's actions, the maximal payoff of player 3 from playing a_3 and b_3 is at least $\frac{3}{2}$, while playing c_3 pays at most 1. As a result, c_3 is never a best response, and hence is not rationalizable for player 3.

With IPCR, c_3 is rationalizable. Player 3 may conjecture that player 1 and 2 play the following correlated strategy: each of (a_1, a_2) and (b_1, b_2) is played with probability half. Given this correlated strategy, the payoff for player 3 is 1, no matter which strategy in ΔA_3 she takes. In other words, c_3 also becomes rationalizable.

3 Hierarchies of beliefs

We first present Mertens and Zamir's conventional formulation of hierarchies of beliefs (see also Brandenburger and Dekel (1993)), and based on that present Ely and Pęski's construction of Δ-hierarchies of beliefs.

3.1 Mertens-Zamir's formulation of hierarchies of beliefs

Type spaces are objects artificially constructed by the modeler to overcome the difficulty of working with players' infinite hierarchies of beliefs. An infinite hierarchy of beliefs describes a player's belief and higher-order beliefs about states of nature. For any type space, the following definition recovers for us the hierarchy of beliefs that each type t_i of player i represents.

Let $X_0 = \Theta$, and for $k \geq 1, X_k = X_{k-1} \times \times_{j \neq i} \Delta(X_{k-1})$. Let $h^1(t_i) = \mathrm{marg}_\Theta\, \pi_i(t_i)$, which is player i's belief over Θ at type t_i. For each $k \geq 1$, let $h^k(t_i)[S] = \pi_i(t_i)[\{(\theta, t_{-i}) : (\theta, (h^l(t_{-i}))_{1 \leq l \leq k-1}) \in S\}]$, for any measurable subset $S \subseteq X_k$. In the construction, $h^k(t_i) \in \Delta(X_{k-1})$ represents player i's k-th order belief at t_i. The profile $h(t_i) = (h^1(t_i), ..., h^k(t_i), ...) \in \times_{k=0}^{\infty} \Delta X_k$ is called player i's hierarchy of beliefs at type t_i. Mertens and Zamir show the existence of a universal type space[4] into which all other belief-closed subspaces[5] can be embedded through a belief preserving mapping.

The main result from DFM sets up a connection between conventional hierarchies of beliefs and interim correlated rationalizability:

Proposition 1 (Dekel, Fudenberg and Morris, 2007)**.** *If $t_i \in T$, $t'_i \in T'$, and $h(t_i) = h(t'_i)$, then $ICR_i^T(t_i|G) = ICR_i^T(t'_i|G), \forall G \in \mathcal{G}$.*

Thus if two types induce the same conventional hierarchy of beliefs, no matter which type spaces they belong to, an action that is interim correlated rationalizable at one must also be interim correlated rationalizable at another.

3.2 Δ-hierarchy of beliefs

A Δ-hierarchy of beliefs describes a player's belief and higher-order beliefs about conditional beliefs on states of nature. The concept was introduced by Ely and Pęski (2006) in their study of interim independent rationalizability. Ely and Pęski observe that conditional beliefs over the states of nature play a key role in identifying the information that is necessary and sufficient for the behavioral prediction of IIR, and that hierarchy of beliefs over conditional beliefs fully identifies such information.

We begin with defining conditional beliefs. Given a belief $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$, the conditional belief[6] of type t_i over Θ, conditioning on the others' types being t_{-i}, is $\pi_i(t_i)(t_{-i}) \in \Delta\Theta$, also written as $\pi_i(t_i, t_{-i})$. For any type t_i in a type space T, denote the set of all possible conditional beliefs at t_i as $B_i(t_i) = \{\pi_i(t_i, t_{-i}) \in \Delta\Theta : t_{-i} \in T_{-i}\}$. Type t_i's belief over T_{-i}

[4] Throughout, we do not actually work on the universal type space, and thus explicit construction of it is omitted.

[5] A subspace $(T_i, \pi_i)_{i \in N}$ is belief-closed if $\forall i \in N$, each type $t_i \in T_i, \pi_i(t_i)[T_{-i}] = 1$.

[6] Since $\Delta(T_{-i} \times \Theta)$ is a complete metric space, there always exists a version of regular conditional probability (cf., e.g., Durrett (2004)).

then induces a belief over $\Delta\Theta$: for any measurable subset $S \subseteq \Delta\Theta, \pi_i(t_i)[S] = \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) \in S\}]$.

Now we define Δ-hierarchy of beliefs at t_i by treating the set of possible conditional beliefs, i.e., $\Delta\Theta$, as the set of basic uncertainty. Let the first-order belief be player i's belief over the set of conditional beliefs, second-order belief be player i's belief over the others' beliefs over the set of conditional beliefs, and so on.

Formally, fix any type space $T = (T_i, \pi_i)_{i \in N}$ on Θ, we transform it into a type space $T^\Delta = (T_i, \pi_i^\Delta)_{i \in N}$ on $\Delta\Theta$. In the new type space, players' type sets are unchanged, and $\pi_i^\Delta(t_i) \in \Delta(T_{-i} \times \Delta\Theta)$ is given by

$$\pi_i^\Delta(t_i)[S] = \pi_i(t_i)[\{t_{-i} : (t_{-i}, \pi_i(t_i, t_{-i})) \in S\}],$$

for any measurable subset $S \subseteq \Delta(T_{-i} \times \Delta\Theta)$.

Ely and Pęski show that if conditional beliefs are jointly measurable in t_i and t_{-i}, then $\pi_i^\Delta(t_i) \in \Delta(T_{-i} \times \Delta\Theta)$ is measurable and hierarchies of beliefs over conditional beliefs can be constructed[7].

Lemma 1 (Ely and Pęski, 2006). *If $\pi_i(\cdot, \cdot) : T_i \times T_{-i} \to \Delta\Theta$ is jointly measurable in t_i and t_{-i}, then $\pi_i^\Delta(\cdot) : T_i \to \Delta(T_{-i} \times \Delta\Theta)$ is measurable.*

Denote the conventional hierarchy of beliefs at t_i in the type space T^Δ as $h(t_i|T^\Delta)$.

Definition 3. *In any type space T, for any $k \geq 1$, let the k-th order Δ-hierarchy of beliefs at $t_i \in T_i$ be $h^k(t_i|T^\Delta)$ and denote it as $\delta^k(t_i)$. Also, denote the Δ-hierarchy of beliefs at t_i as $\delta(t_i) = (\delta^1(t_i), ..., \delta^k(t_i), ...)$.*

By definition, $\delta(t_i) = h(t_i|T^\Delta)$.[8]

[7] Shmaya (2007) shows the existence of a regular conditional probability that is jointly measurable in t_i and t_{-i}, given that $\Delta(T_{-i} \times A_{-i})$ is Polish.

[8] Although Ely and Pęski (2006) constructs Δ-hiearchies of beliefs only for two players, the construction and all relevant proofs extend in an obvious way for type spaces with more than two players.

4 Rationalizability and hierarchies of beliefs

Let us illustrate intuitively how conditional beliefs matter for players' rational behavior. At the interim stage of the game, player i knows her type t_i, but does not know the types of other players t_{-i} and the state of nature θ. We can view (t_i, t_{-i}, θ) as an ex post state of the world, and (t_i, t_{-i}) an interim scenario. At t_i, before making the decision on which action to play, player i will take the following thought process: first she assigns probability $\pi_i(t_i)[t_{-i}]$ to the interim scenario (t_i, t_{-i}), then conditional on the others' types being t_{-i}, she conjectures that they will play some correlated strategy $\sigma_{-i}(t_{-i})[\cdot] \in \Delta A_{-i}$, and at the same time, she updates her belief over Θ to be $\pi_i(t_i, t_{-i}) \in \Delta\Theta$. The thought process helps us to further decompose a conjecture v of player i such that its marginal on $\Theta \times A_{-i}$ can be written as

$$m^v = \int_{T_{-i}} \pi_i(t_i, t_{-i})[\theta] \cdot \sigma_{-i}(t_{-i})[a_{-i}] \cdot \pi_i(t_i)[dt_{-i}],$$

where $\pi_i(t_i, t_{-i}) \in \Delta\Theta$ is player i's conditional belief at t_i given t_{-i}, as previously defined. Since type-correlated strategies $\sigma_{-i}(\cdot)$ can be arbitrary, the set of conjectures is determined by a player's belief on conditional beliefs.

4.1 Main theorem

The following result shows that two types provide the same IPCR prediction if and only if they have the same Δ-hierarchy of beliefs.

Theorem 1. *If $t_i \in T, t'_i \in T'$, then $\delta(t_i) = \delta(t'_i)$ if and only if $R_i^T(t_i|G) = R_i^{T'}(t'_i|G), \forall G \in \mathcal{G}$.*

Proof. We present the proof for sufficiency here. The proof necessity, preceded with a sketch of its key idea, is presented in the appendix.

Fix a game $G \in \mathcal{G}$. We need to show that if $\delta(t_i) = \delta(t'_i)$, then $R_i^T(t_i) = R_i^{T'}(t'_i)$. Denote the set of all possible conjectures of player i in the k-th round of the elimination procedure by

$$V_i^k(t_i) = \begin{cases} v \in \Delta(T_{-i} \times \Theta \times A_{-i}) \text{ such that:} \\ (1) v[(t_{-i}, \theta, a_{-i})] > 0 \Rightarrow a_{-i} \in R^T_{-i,(k-1)}(t_{-i}); \\ (2) \int_{T_{-i}} v[(t_{-i}, \theta, a_{-i})] dt_{-i} = \int_{T_{-i}} \pi_i(t_i, t_{-i})[\theta] \sigma_{-i}(t_{-i})[a_{-i}] \pi_i(t_i)[dt_{-i}]. \end{cases}.$$

Denote the set of marginals of $V_i^k(t_i)$ on $\Theta \times A_{-i}$ by $\operatorname{marg}_{\Theta \times A_{-i}} V_i^k(t_i)$. From the definition of rationalizability, the set of marginals on $\Theta \times A_{-i}$ determines the set of justifiable expected payoffs, thus determines the set of rationalizable actions. That is, if $\operatorname{marg}_{\Theta \times A_{-i}} V_i^k(t_i) = \operatorname{marg}_{\Theta \times A_{-i}} V_i^k(t_i')$, then $R^T_{i,k}(t_i) = R^{T'}_{i,k}(t_i')$.

Step 1. We start with the case of $k = 1$ and then prove the rest inductively. Consider the probability space $(T_{-i}, \pi_i(t_i)[\cdot], \mathcal{T}_{-i})$, where $\pi_i(t_i)[\cdot] \in \Delta T_{-i}$ is the marginal of $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$ over T_{-i} and $\mathcal{T}_{-i}$ is the usual Borel σ-algebra. View $\pi_i(t_i, \cdot) : T_{-i} \to B_i(t_i) \subseteq \Delta\Theta$ as a random variable on T_{-i}, and denote the σ-algebra generated by it by $\sigma(\pi_i(t_i, \cdot))$. Since T_{-i} is a compact metric space, there exists a regular conditional probability that maps from $T_{-i} \times \mathcal{T}_{-i}$ to $[0, 1]$ given $\sigma(\pi_i(t_i, \cdot))$ (see, for example, Durrett (2004)). Since the conditional probability is $\sigma(\pi_i(t_i, \cdot))$ measurable, by a little abuse of notation, we can write it as $\pi_i(t_i, \cdot) : B_i(t_i) \to \Delta T_{-i}$. Now, the marginal distribution for a given conjecture $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ over $\Theta \times A_{-i}$ can be expressed as

$$\begin{aligned} m^v &= \int_{T_{-i}} \pi_i(t_i, t_{-i})[\theta] \sigma_{-i}(t_{-i})[a_{-i}] d\pi_i(t_i)[t_{-i}] \\ &= \int_{B_i(t_i)} \int_{\{t_{-i}: \pi_i(t_i, t_{-i}) = \beta\}} \pi_i(t_i, t_{-i})[\theta] \sigma_{-i}(t_{-i})[a_{-i}] \pi_i(t_i, \beta)[dt_{-i}] \delta^1(t_i)[d\beta] \\ &= \int_{B_i(t_i)} \beta[\theta] \pi_i(t_i, \beta)[\sigma_{-i}(t_{-i})[a_{-i}]] \delta^1(t_i)[d\beta] \end{aligned}$$

We are ready to construct a conjecture v' for type t_i' such that $v' = v$. Suppose t_i' believes that the others play the following type-correlated strategy: for any type t_{-i}' such that $\pi'_i(t'_i, t'_{-i}) = \beta$,

$$\sigma'_{-i}(t'_{-i})[a_{-i}] = \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta\}} \sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i,\beta)[dt_{-i}]$$
$$= \pi_i(t_i,\beta)[\sigma_{-i}(t_{-i})[a_{-i}]], \forall a_{-i} \in A_{-i}.$$

Intuitively, t'_i believes that at all types $t'_{-i}, \pi'_i(t'_i, t'_{-i}) = \beta$, action a_{-i} is played with the average of the probabilities it is played with at types $t_{-i}, \pi_i(t_i, t_{-i}) = \beta$. The marginal distribution over $\Theta \times A_{-i}$ of the conjecture v' is

$$\begin{aligned}
m^{v'} &= \int_{T'_{-i}} \pi'_i(t'_i, t'_{-i})[\theta]\sigma'_{-i}(t'_{-i})[a_{-i}]\pi'_i(t'_i)[dt'_{-i}] \\
&= \int_{B_i(t'_i)} \int_{\{t'_{-i}:\pi'_i(t'_i,t'_{-i})=\beta\}} \pi'_i(t'_i, t'_{-i})[\theta]\sigma'_{-i}(t'_{-i})[a_{-i}]\pi'_i(t'_i,\beta)[dt'_{-i}]\delta^1(t'_i)[d\beta] \\
&= \int_{B_i(t'_i)} \beta[\theta] \int_{\{t'_{-i}:\pi'_i(t'_i,t'_{-i})=\beta\}} \pi_i(t_i,\beta)[\sigma_{-i}(t_{-i})[a_{-i}]]\pi'_i(t'_i,\beta)[dt'_{-i}]\delta^1(t'_i)[d\beta] \\
&= \int_{B_i(t_i)} \beta[\theta]\pi_i(t_i,\beta)[\sigma_{-i}(t_{-i})[a_{-i}]]\delta^1(t_i)[d\beta] \\
&= m^v,
\end{aligned}$$

where the first and second equality are natural, the third equality comes the construction of $\sigma'_{-i}(t'_{-i})[a_{-i}]$, and the fourth equality due to $B_i(t_i) = B_i(t'_i), \delta^1(t_i) = \delta^1(t'_i)$ and that $\int_{\{t'_{-i}:\pi'_i(t'_i,t'_{-i})=\beta\}} \pi'_i(t'_i,\beta)[dt'_{-i}] = 1$.

We have shown that any marginal in $\operatorname{marg}_{\Theta\times A_{-i}} V_i^1(t_i)$ also belongs to $\operatorname{marg}_{\Theta\times A_{-i}} V_i^1(t'_i)$, i.e., $\operatorname{marg}_{\Theta\times A_{-i}} V_i^k(t_i) \subseteq \operatorname{marg}_{\Theta\times A_{-i}} V_i^k(t'_i)$. By symmetry, $\operatorname{marg}_{\Theta\times A_{-i}} V_i^1(t'_i) \subseteq \operatorname{marg}_{\Theta\times A_{-i}} V_i^1(t_i)$, and hence $\operatorname{marg}_{\Theta\times A_{-i}} V_i^1(t_i) = \operatorname{marg}_{\Theta\times A_{-i}} V_i^1(t'_i)$. By definition, $R^T_{i,1}(t_i) = R^{T'}_{i,1}(t'_i)$, for all $G \in \mathcal{G}$.

Step 2. We prove inductively for cases of $k > 1$. Suppose $R^T_{i,(k-1)}(t_i) = R^{T'}_{i,(k-1)}(t'_i)$ for all $G \in \mathcal{G}$, and $\delta^k(t_i) = \delta^k(t'_i)$. Denote the support of $\delta^k(t_i)$ and $\delta^k(t'_i)$ as $D^{k-1}(t_i)$ and $D^{k-1}(t'_i)$, respectively. We know instantly that $D^{k-1}(t_i) = D^{k-1}(t'_i)$. Denote a typical element in $D^{k-1}(t_i)$ as $(\beta, \delta_1^{k-1}) \equiv (\beta, (\delta^l)_{1\leq l\leq k-1})$. Similar to step 1, we can express the marginal of any conjecture $v \in \Delta(T_{-i} \times \Theta \times R^T_{-i,(k-1)})$ as

$$\begin{aligned}\underset{\Theta\times R^T_{-i,(k-1)}}{\operatorname{marg}} v &= \int_{D^{k-1}(t_i)} \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta,\delta_1^{k-1}(t_{-i})=\delta_1^{k-1}\}} \pi_i(t_i,t_{-i})[\theta] \\ &\qquad \sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i,(\beta,\delta_1^{k-1}))[dt_{-i}]\delta^k(t_i)[d(\beta,\delta_1^{k-1})] \\ &= \int_{D^{k-1}(t_i)} \beta[\theta] \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta,\delta_1^{k-1}(t_{-i})=\delta_1^{k-1}\}} \sigma_{-i}(t_{-i})[a_{-i}] \\ &\qquad \pi_i(t_i,(\beta,\delta_1^{k-1}))[dt_{-i}]\delta^k(t_i)[d(\beta,\delta_1^{k-1})],\end{aligned}$$

where $\pi_i(t_i,(\beta,\delta_1^{k-1}))$ is the conditional belief of t_i over t_{-i} at (β,δ_1^{k-1}). To construct the corresponding $v' \in \Delta(T'_{-i} \times \Theta \times A_{-i})$ for v, for any t'_{-i} such that $\pi'_i(t'_i,t'_{-i}) = \beta, \delta_1^{k-1}(t'_{-i}) = \delta_1^{k-1}(t_{-i})$, let

$$\sigma'_{-i}(t'_{-i})[a_{-i}] = \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta,\delta_1^{k-1}(t_{-i})=\delta_1^{k-1}\}} \sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i,(\beta,\delta_1^{k-1}))[dt_{-i}],$$

for all $a_{-i} \in R^T_{-i,(k-1)}$, and 0 otherwise. We can check that again the induced marginal on $\Theta \times A_{-i}$ from the conjecture v' coincides with that from v. Following the same argument as in step 1, $R^T_{i,k}(t_i) = R^T_{i,k}(t'_i)$, for all $G \in \mathcal{G}$. $\square$

The proof above also indicates that if $\delta^k(t_i) = \delta^k(t'_i)$, then $R^T_{i,k}(t_i|G) = R_{i,k}(t'_i|G), \forall G \in \mathcal{G}$. That is, k-th order of beliefs over conditional beliefs characterize level-k interim partially correlated rationalizable actions. To see the intuition, notice that whether an action is first-order rationalizable is determined by the set of conjectures that can be supported by type-correlated strategies, and this set is in turn characterized by players' beliefs over conditional beliefs. The k-order conjectures depend on both beliefs on conditional beliefs and beliefs on the others' level-$(k-1)$ rationalizable actions, thus are determined by the k-th order beliefs.

The sufficiency part of Theorem 1 parallels with Proposition 1, and the whole theorem is an extension of Ely and Pęski's main result (Ely and Pęski, 2006, section 4, theorem 2.) from two-player games to n-player games. Our proof of the sufficiency part differs from that of Ely and Pęski's; and the proof of necessity, which we present in the appendix, adapts Ely and Pęski's, but uses a different approach that is more direct and accessible. We refrain from working with abstract structures like conditional belief preserving mappings, the universal type space of Δ-hierarchies of beliefs, the universal type space for rationalizability, and so on.

4.2 Connections between IPCR and ICR

4.2.1 Harsanyi vs. Aumann

The definitions of IPCR and ICR adopt Harsanyi's principle and Aumann's Bayesian view, respectively. The two approaches differ mainly in whether actions are treated as type-contingent variables or not. In Harsanyi's principle, it is common knowledge among players that all players believe that the others' actions depend only on their types and nature affects actions only indirectly through types; that is, it is common knowledge that for all i, player i believes that conditional on t_{-i}, a_{-i} is independent of θ. However, common knowledge of such beliefs is not inherent in Aumann's Bayesian view; according to this viewpoint, player i forms a subjective belief $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$, and a_{-i} can correlate with t_{-i} and θ arbitrarily. The distinction is indicated more clearly in the following corollary:

Corollary 1 (Dekel, Fudenberg and Morris, 2007)**.** *The constraint on a conjecture v in the definition of ICR can be equivalently expressed as: there exists a state-and-type correlated strategy $\sigma_{-i}^{\Theta} : T_{-i} \times \Theta \to \Delta A_{-i}$ such that*

$$m^v = \int_{T_{-i}} \pi_i(t_i)[(t_{-i}, \theta)] \cdot \sigma_{-i}^{\Theta}(t_{-i}, \theta)[a_{-i}] \cdot \pi_i(t_i)[dt_{-i}].$$

An ICR conjecture needs to be supported by some strategy which depends also on states of nature. In other words, there is information about Θ that affects the others' decision but is not incorporated in the type profile t_{-i}. A "deep" Bayesian player[9] would be able to locate such information and incorporate it into the others' types such that conditional on the new types of the others', player i believes that a_{-i} is independent of θ. As a result, the new type space which is a (an) refinement (enlargement) of T satisfies Harsanyi's principle.

To define solution concepts based on different viewpoints, Harsanyi's and Aumann's, we need to construct type spaces that incorporate different amounts of information. Alternatively, fix any artificially constructed type space, the choice of the "right" solution concept should be determined by the information incorporated in the types. The distinction between IPCR and ICR is methodological.

The following proposition describes a consistency between the two solution concepts: the set of ICR actions at any type is exactly the union of the IPCR actions in its refinements.

Proposition 2. *Fix any game $G \in \mathcal{G}$. For any type t_i,*

$$\bigcup_{\{t_i':h(t_i')=h(t_i)\}} R_i(t_i') = ICR_i(t_i).$$

Proof. We first prove that $LHS \subseteq RHS$. Since ICR and IPCR can be identified by conventional hierarchies of beliefs and Δ-hierarchies of beliefs, respectively, and that two types have the same Δ-hierarchy of beliefs only if they have the same conventional hierarchy of beliefs, it is sufficient to show that for any t_i,

$$R_i(t_i) \subseteq ICR_i(t_i).$$

This is trivially true as the set of marginals of conjectures over $\Theta \times A_{-i}$ of IPCR in each

[9] Equivalently, we may view that a player modeled by the partial Bayesian approach reasons "deeper" than one modeled by the universal Bayesian approach.

round of elimination is a subset of that of ICR, which means fewer actions can be justified and more actions are to be eliminated.

Second, $RHS \subseteq LHS$. We need to show that for any $a_i \in ICR_i(t_i)$, there exists t_i' with $h(t_i') = h(t_i)$ such that $a_i \in R_i(t_i')$. We start with constructing a hierarchy of beliefs over conditional beliefs. Suppose t_i belongs to some type space $(T_i, \pi_i)_{i \in N}$ on Θ. Now consider a new type space $\tilde{T}$ defined on $\Delta\Theta$, with the same set of types for each player, and states of nature replaced with point masses, i.e., replace θ with $\mathbf{1}_{\{\theta\}}$. And for any measurable subset S of T_{-i}, $\tilde{\pi}_i(t_i)[(S, \mathbf{1}_{\{\theta\}})] = \pi_i(t_i)[(S, \theta)]$. Now let t_i' be some type such that $\delta(t_i')$ equals $h(t_i|\tilde{T})$, the conventional hierarchy of beliefs of t_i in $\tilde{T}$. Since $\delta(t_i')$ characterizes exactly the same information as $h(t_i)$, $R_i(t_i')$ necessarily equals $ICR_i(t_i)$. To see this, suppose t_i' is in some type space T'. If at t_i, $a_i \in ICR_i^1(t_i)$ is justified by some conjecture supported by a state-and-type correlated strategy σ_{-i}^{Θ}, we can construct σ_{-i}' for t_i' as follows: for any t_{-i}' such that $\pi_i(t_i', t_{-i}') = \mathbf{1}_{\{\theta\}}$, let $\sigma_{-i}'(t_{-i}')[a_{-i}] = \sigma_{-i}^{\Theta}(t_{-i}, \theta)[a_{-i}], \forall a_{-i} \in A_{-i}$. $\square$

4.2.2 Nature as another player

An example in DFM (2007, section 3.2) suggests that IPCR is potentially sensitive to the addition of an omniscient player (e.g., nature) and may not be a good solution concept. We argue that there is a very bright side behind that example, by showing that when nature is added as another player, IPCR coincides with ICR. Therefore, compared with ICR, for any fixed type space, adopting IPCR as the solution concept is more general.

Consider that we add nature as another player into a game G with type space T. Nature's type space is Θ. Since nature knows her own type, at each type θ she knows the true state is θ. Suppose that nature's action does not affect the payoff of the others', and that players' beliefs over nature's types are consistent with their beliefs on $T_{-i} \times \Theta$ in T. Denote the expanded game as G^N and the expanded type space as T^N.

It is obvious from Corollary 1 that the set of IPCR actions G^N is the same as the set of

ICR actions in G, at any type t_i. This is because for player i, a type-correlated strategy of the others' in G^N becomes $\sigma_{-i} : T_{-i} \times \Theta \to \Delta A_{-i}$, which is the same as a state-and-type correlated strategy in G. In accordance, the Δ-hierarchy of beliefs at any type t_i in T^N reduces into its conventional hierarchy of beliefs in T. Denote the Δ-hierarchy of type t_i in the expanded type space T^N as $\delta(t_i|T^N)$.

Proposition 3. *Fix a game G and type spaces T, T':*

1. $R_i^{T^N}(t_i|G^N) = ICR_i^T(t_i|G), \forall t_i \in T_i$.

2. *For any $t_i \in T_i, t_i' \in T_i'$, $h(t_i) = h(t_i')$ if and only if $\delta(t_i|T^N) = \delta(t_i'|T_i'^N)$.*

Proof. Part 1 is by definition. For part 2, observe that when nature is added as another player, the conditional belief at t_i conditioning on the others' types (t_{-i}, θ) reduces to point mass on θ. □

The proposition is directly implied by the fact that when nature is added in to the game, Harsanyi's principle and Aumann's Bayesian view are equivalent.

4.3 Relevant issues

4.3.1 Equivalent formulations of IPCR

Recall that in complete information games, correlated rationalizability can be defined in multiple equivalent ways. There are also multiple equivalent ways of defining ICR, as discussed and checked in DFM (2007). To show that IPCR is as legitimate as ICR as an extension of correlated rationalizability in complete information games, we present its iterative elimination of strictly dominated actions formulation and check its equivalence with the iterative elimination of never best response actions formulation. Its equivalence with other formulations can be routinely checked.

Definition 4. *Fix a game G and a type space T. For all $t_i \in T_i, U^T_{i,0}(t_i) = A_i$. An action is level-$(k+1)$ rationalizable at t_i, i.e., $a_i \in U^T_{i,k+1}(t_i)$, if there does not exist $\rho_i \in \Delta A_i$ such that*

$$\sum_{a_{-i},\theta} g_i(a_i, a_{-i}, \theta) m^v[(a_{-i}, \theta)] < \sum_{a_{-i},\theta} g_i(\rho_i, a_{-i}, \theta) m^v[(a_{-i}, \theta)],$$

for all $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ that satisfies $(t_{-i}, \theta, a_{-i}) \in \operatorname{supp} v \Rightarrow a_{-i} \in (U^T_{j,k}(t_j))_{j \neq i}$ and the constraint on conjectures (item 3). And $U^T_i(t_i) = \cap_{k=1}^{\infty} U^T_{i,k}(t_i)$.

Proposition 4. $U^T_i(t_i) = R^T_i(t_i)$.

Proof. If an action is strictly dominated, it is never a best response. Therefore, $U^k_i(t_i) \subseteq R^k_i(t_i), \forall k \geq 1$. We only need to show the other direction, that $\forall k \geq 1$, $R^k_i(t_i) \subseteq U^k_i(t_i)$. We prove by induction. First notice that $R^T_{i,0}(t_i) = U^T_{i,0}(t_i)$. Suppose for some $k \geq 1, R^T_{i,k}(t_i) = U^T_{i,k}(t_i)$, we show that $R^T_{i,k+1}(t_i) \subseteq U^T_{i,k+1}(t_i)$. If $a_i \notin R_{i,k+1}(t_i)$, given any ICR conjecture $v \in \Delta(T_{-i} \times \Theta \times R^T_{-i,k}(t_{-i}))$, there exists $\rho_i \in \Delta A_i$ such that

$$v[g_i(a_i, a_{-i}, \theta)] < v[g_i(\rho_i, a_{-i}, \theta)].$$

Since the inequality holds for all ICR conjectures v, and the set of ρ_i's is compact,

$$\inf_v \sup_{\rho_i} (v[g_i(\rho_i, a_{-i}, \theta)] - v[g_i(a_i, a_{-i}, \theta)]) > 0.$$

Observe that as a function of v and ρ_i, $(v[g_i(\rho_i, a_{-i}, \theta)] - v[g_i(a_i, a_{-i}, \theta)])$ is linear in both arguments, that the set of α_i's is convex compact, and that the set of IPCR conjectures is a convex subset of a vector space, we can apply the minimax theorem and obtain

$$\sup_{\rho_i} \inf_v (v[g_i(\rho_i, a_{-i}, \theta)] - v[g_i(a_i, a_{-i}, \theta)]) > 0.$$

That is, for all conjecture that satisfy the constraints, there exists ρ_i that strictly dominates a_i, $a_i \notin U^T_{i,k+1}(t_i)$. Therefore, $R^T_{i,k+1}(t_i) \subseteq U^T_{i,k+1}(t_i)$. □

4.3.2 Insufficiency of Δ-hierarchies of beliefs for IIR

We show by example that Δ-hierarchies of beliefs are not sufficient for IIR.

Example 4. *Given the game form and type space T in Example 3, we construct another type space T' as follows: $T'_1 = T'_2 = \{-1, +1\}, T'_3 = \{*\}$, and there is a common prior $\pi(t'_1, t'_2, *, *) \in \Delta(T'_1 \times T'_2 \times T'_3 \times \Theta)$ such that*

$$\pi(t'_1, t_2, *, *) = \begin{cases} \frac{1}{2} \text{ if } t'_1 = t'_2, \\ 0 \text{ otherwise.} \end{cases}$$

The types of player 3 in T and T' have the same Δ-hierarchy of beliefs, which is common knowledge on the point mass of θ. However, the sets of IIR actions at them are different. To see that, suppose player 3 believes that $\sigma_i(+1) = a_i, \sigma_i(-1) = b_i$ for $i = 1, 2$, she thinks the others play (a_1, a_2) and (b_1, b_2) each with probability $\frac{1}{2}$. Under this belief, c_3 is an IIR action for her. But in T it is not. This is because T' is redundant with respect to Δ-hierarchies of beliefs, and the redundancy enlarges player 1 and 2's action set and provides extra correlation.

The type space T' can be generated from T with a partially correlating device defined in the next section.

5 The Bayesian solution

5.1 Definition

The Bayesian solution is a notion of correlated equilibrium for games with incomplete information proposed by Forges (1993). Its definition is inspired by Aumann's Bayesian view and aims at capturing Bayesian rationality. In this section, we establish the equivalence between the Bayesian solution and IPCR.

Following Forges (2006), the definition of *the Bayesian solution* involves the use of an *epistemic* model $Y = (Y, \vartheta, (\mathcal{S}_i, \tau_i, \alpha_i, p_i)_{i \in N})$ into which the type space $T = (T_i, \pi_i)_{i \in N}$ can be embedded[10]. In the epistemic model, Y is the set of states of the world which is large enough to characterize uncertainties in states of nature, agents' types, and agents' actions; $\mathcal{S}_i$ denotes player i's informational partition, and p_i denotes player i's subjective prior. The mapping $\vartheta : Y \to \Theta$ indicates the state of nature, $\tau_i : Y \to T_i$ indicates player i's type, and $\alpha_i : Y \to A_i$ indicates i's action. Both τ_i and α_i are assumed to be $\mathcal{S}_i$ measurable; hence at any state, player i knows both her type and action. The consistency in beliefs requires that for any measurable subset $S \subseteq T_{-i} \times \Theta$ and $S' \subseteq T_{-i}$,

$$\begin{aligned} p_i[(\tau_{-i}, \vartheta)^{-1}(S)|\mathcal{S}_i] &= \pi_i[S|\tau_i], \\ p_i[{\tau_{-i}}^{-1}(S')|\mathcal{S}_i] &= p_i[{\tau_{-i}}^{-1}(S')|\tau_i], \forall i \in N. \end{aligned} \tag{5.1}$$

The first condition requires that the epistemic model does not give players extra information on the joint distribution of the others' types and states of nature, and the second condition further requires that it does not give extra information on the others' types. The two conditions together, guarantees *belief invariance* (the invariance of conditional beliefs). Given the epistemic model, we define Bayesian rationality for player i: player i is *Bayesian rational* if

$$E[g_i(\alpha_i, \alpha_{-i}, \vartheta)|\mathcal{S}_i] \geq E[g_i(a_i, \alpha_{-i}, \vartheta)|\mathcal{S}_i], \forall a_i \in A_i,$$

where the expectation is taken over T_{-i} and Θ.

Definition 5. *Given a game G and a type space T, a Bayesian solution for the game is an*

[10] Forges's definition of the Bayesian solution is restricted to two-player games with type spaces with a common prior; what we present here is the n-player non-common prior analogue of her definition.

epistemic model $Y = (Y, \vartheta, (\mathcal{S}_i, \tau_i, \alpha_i, p_i)_{i \in N})$ *constructed as above that satisfies the Bayesian rationality of every player.*

For any Bayesian solution Y, let $\mu_i(y) \in \Delta(\Theta \times A_{-i})$ be player i's belief over states of nature and the others' actions in the state of the world y, and $\mu(y) \equiv (\mu_i(y))_{i \in N}$ be a profile of players' beliefs. From a point of view analogous to the "revelation principle", the set of profiles of beliefs $\{\mu(y) : y \in Y\}$ can be implemented canonically from a partial correlating device $q = (q_i)_{i \in N}$, such that $q_i : T \to \Delta A$ satisfies:

1. player i believes that at any type profile t an action profile $a \in A$ is selected according to $q_i(t) \in \Delta A$, and then a_j is recommended to player $j, \forall j \in N$, by an omniscient mediator who observes all players' types.

2. belief invariance is satisfied, i.e., from the recommendations they receive, players cannot infer any information on the others' types. Formally, at different types t_{-i}, t'_{-i} of the others, type t_i of player i receive recommendation a_i with the same probability,

$$\sum_{\{a' \in A : a'_i = a_i\}} q_i(t_i, t_{-i})[a'] = \sum_{\{a' \in A : a'_i = a_i\}} q_i(t_i, t'_{-i})[a'], \forall i, t_i, a_i,$$

and that each player does not have incentive to deviate from the mediator's recommendation at any of her types.

Remark 1. *The definition of the Bayesian solution involves using epistemic models, this indirectly provides us with conditions on the epistemic foundation of IPCR. DFM (*2007, *section* 3.4*) show that ICR characterizes common certainty of rationality and of the correctness of the standard type space; by correctness of the standard type space, they require only that players have correct beliefs about* $T_{-i} \times \Theta$*. To justify IPCR with epistemic models, we also need the model to preserve conditional beliefs, which can be achieved by requiring belief* *invariance. Intuitively, IPCR characterizes common certainty of rationality, correct beliefs and invariance of conditional beliefs.*

5.2 Equivalence with IPCR

A Bayesian solution is equivalent to a partial correlating device q under which players are *incentive compatible*. Recall that in the definition of IPCR (Definition 1), a conjecture of player $i, v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ needs to be justified by a correlated strategy $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ of the others'. This correlated strategy, however, is not natural since it assumes that the strategy of each $j \neq i$ is not measurable with respect to j's own types, but with respect to the type profile of $-i$. The following lemma states that all conjectures in IPCR can be justified by an incentive compatible partial correlating device, and hence by a Bayesian solution[11].

Lemma 2. *Any correlated strategy $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ can be induced from a profile of strategies $(\tilde{\sigma}_j)_{j \neq i}, \tilde{\sigma}_j : T_j \times A_j \to A_j$, in which each player's action depends only on her own type and the action recommended from an incentive compatible partially correlating device q.*

Proof. Fix a correlated strategy $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$. Suppose that player i's type is t_i. Construct the partial correlating device such that $q_i(t_i, t_{-i})[a_{-i}] = \sigma_{-i}(t_{-i})[a_{-i}], \forall t_{-i} \in T_{-i}$, and let q_j be arbitrary, for all $j \neq i$. Given that player i believes at each $t_{-i} \in T_{-i}$, the others are recommended actions according to $q_i(t_i, t_{-i})$, if player i conjectures that the others follow

[11] The agent-normal-form correlated equilibrium proposed by Samuelson and Zhang (1989) is of similar form to the Bayesian solution. An agent-normal-form correlated equilibrium can be implemented by a correlating device $Q \in \Delta(\times_i A_i^{T_i})$ and a mediator. A profile of strategies $\sigma = (\sigma_i)_{i \in N}$ is chosen randomly according to Q and the mediator who observes t_i recommends the action $\sigma_i(t_i)$ to agent t_i. If no type has the incentive to deviate, Q implements equilibrium.

Such correlated equilibria also satisfy belief invariance and provide some correlations in the interim stage. However, a close look at it would reveal that the correlations are not interim types dependent; it operates in the ex ante stage, and the correlation happens only across ex ante strategies but not interim type dependent actions of the others. Consequently, many type-correlated strategies of IPCR cannot be justified by agent-normal-form correlated equilibrium. (See Forges (2006) for more discussion.)

independent strategies $\tilde{\sigma}_j(t_j, a_j) = a_j$, i.e., they always follow the recommendations, then i believes that the others' play at t_{-i} is exactly $q_i(t_i, t_{-i})[a_{-i}]$, which equals to $\sigma_{-i}(t_{-i})[a_{-i}]$. $\square$

Lemma 2 is directly implied by the fact that both IPCR and the Bayesian solution follow the same viewpoint, Harsanyi's principle, in characterizing correlation. In both concepts, the correlation can be achieved by sending to players type profile dependent signals (recommendations) in a belief invariant way.

We can further show the payoff equivalence between IPCR and the Bayesian solution, as an analogue of Brandenburger and Dekel (1993) which establishes the payoff equivalence between correlated rationalizability and *a posteriori* equilibrium in complete information games. For any game G and any type space T, an interim IPCR payoff of player i at type t_i is the maximal payoff i can possibly obtain given some IPCR conjecture $v \in (T_{-i} \times \Theta \times R^T_{-i}(t_{-i}|G))$. Let $W_i(t_i)$ be the set of interim payoffs of player i at type t_i.

Proposition 5. *Fix any game G and type space T. A vector $u = (u_i(t_i))_{i \in N, t_i \in T_i} \in \times_{i \in N, t_i \in T_i} W_i(t_i)$ is a profile of interim partially rationalizable payoffs if and only if there is a Bayesian solution in which it is a vector of interim payoffs.*

Proof. Necessity is straightforward due to Lemma 2. In any incentive compatible partial correlating device q, if an action a_i of player i is played in the Bayesian solution at type t_i, then a_i is a best response to $q_i(t_i, t_{-i})[a_{-i}]$. Let the support of q be $\operatorname{supp} q \subseteq A$, then $\operatorname{supp} q$ satisfies the best response property, i.e., any action profile $a \in \operatorname{supp} q$ is IPCR. Thus any vector of interim payoffs is interim partially correlated rationalizable.

Sufficiency. Suppose at type t_i, $u_i(t_i)$ is achieved by playing a_i against a correlated strategy $\sigma_{-i} : T_{-i} \to R^T_{-i}(t_{-i}|G)$. Construct q_i such that $q_i(t_i, t_{-i})[a_i, a_{-i}] = \sigma_{-i}(t_{-i})[a_{-i}], \forall t_{-i}, a_{-i}$, then when i is recommended to play a_i, she believes that the others are playing the correlated strategy σ_{-i}. The same construction of $q_i(t_i, \cdot)$ can be done for each $i \in N$ and $t_i \in T_i$. We also restrict q to be incentive compatible on other action profiles that do not support u.

The partial correlating device thus defined implements a Bayesian solution, and when a_i is recommended to type t_i, player i's expected payoff is $u_i(t_i)$. $\square$

6 Conclusion

For any fixed type space, we propose a notion of interim "correlated" rationalizability that respects the structure of the type space in the least sense, by assuming that the actions of the others' are dependent on their types. It then turns out that hierarchies of beliefs on conditional beliefs play a key role in the characterization of the solution. The characterization also implies that to construct type spaces that satisfy Harsanyi's principle, we need more information than just players' beliefs and higher-order beliefs about states of nature. This paper belongs to the literature that characterize implications of type spaces with respect to different solution concepts.

APPENDIX

In this appendix, we present the proof of necessity in the main theorem. We use an approach different from that used in Ely and Pęski's proof of their main theorem, but that uses their intermediate results. Our proof can be viewed as an adaptation and at the same time a simplification of Ely and Pęski's proof. The approach we use is very similar to that used by Gossner and Mertens (2001) in constructing zero-sum betting games to separate the behavior of types with different conventional hierarchies of beliefs.

Before moving on to the notationally involved proof, we summarize its key idea, which is simple. We construct inductively games that separate the behaviors of types that differ in each order of beliefs. More specifically, in the first step, for any pair of types that have different first-order beliefs, we construct a game in which the two types have different sets of rationalizable actions. Then, for any two types with different second-order beliefs, we let the player play against the other players who are playing games constructed in the first step. Since the types have different beliefs about the others' first-order beliefs, which determines the others' rationalizable actions, they will have different beliefs about the others' action sets. The difference in beliefs again allows us to construct a game in which the two types have different set of rationalizable actions. This procedure can be replicated in a way that for any two types that differ in the k-th order beliefs, we let the player play against the others who are playing games constructed in the $(k-1)$-th step. The separating games are very much like betting games in which players are asked to bet on the others' actions. This is because in each separating game a player's payoff depends on the others' actions, but the others are playing games constructed one step lower and their payoffs are not affected by this player's action in the current game. And we know that bets reveal beliefs.

Proof. Assume $\delta(t_i) \neq \delta(t_i')$. Due to the consistency of Δ-hierarchy of beliefs, we decompose the proof by discussing cases of $\delta^k(t_i) \neq \delta^k(t_i'), \delta^l(t_i) = \delta^l(t_i'), \forall 1 \leq l \leq k$, i.e., in the k-th case, the Δ-hierarchies of beliefs at t_i and t_i' differ starting from the k-th level belief. For each case, we construct a game that separates the types in their IPCR behavior. The construction of games is inductive.

Step 1 ($k = 1$). In the first step we consider the case of $\delta^1(t_i) \neq \delta^1(t_i')$, i.e., when two types have different beliefs over conditional beliefs. We first present an adapted version of lemma 5′ in Ely and Pęski (2006). Let $F = \{f : \Delta\Theta \to [0, \infty)$ such that $f(\beta) = \max_{\mathbf{k} \in \{1,...,m\}^{N-1}} \beta[\psi(\mathbf{k}, \theta)]$ for some natural number m and continuous bounded function $\psi : \{1, ..., m\}^{N-1} \times \Theta \to [0, \infty)\}$.

Lemma 3. *The collection of sets $\{\mu : \mu[f] < 0\} \subseteq \Delta(\Delta\Theta)$ for $f \in F$ generate the weak*-topology on $\Delta(\Delta\Theta)$. This topology is normal, and therefore any pair of disjoint closed subsets $S, S' \in \Delta(\Delta\Theta)$ can be separated by open sets, and there is a function $f \in F$ such that $\forall \mu \in S$*

and $\mu' \in S'$,

$$\mu[f] \neq \mu'[f].$$

Since the proof to Lemma 3 is a special case of lemma $5'$ in Ely and Pęski (2006), we only sketch the idea here. Let H denote the Hilbert cube $[0,1]^{\mathbb{N}}$, since $\Delta\Theta$ is a second countable Hausdorff space, there is a mapping $H : \Delta\Theta \to \mathbf{H}$ that embeds $\Delta\Theta$ into $\mathbf{H}$ (Urysohn metrization theorem, cf. Aliprantis and Border (2006), theorem 3.40). Since H is an embedding, the problem of showing $\{\mu : \mu[f] < 0\} \subseteq \Delta(\Delta\Theta)$ for $f \in F$ generates the weak*-topology on $\Delta(\Delta\Theta)$ transforms into showing that there is a family of continuous functions $f : \mathbf{H} \to R$ such that the collection of sets $\{\mu : \mu[f(h)] < 0\}$ generates the weak*-topology on $\Delta(\mathbf{H})$. Let $F'_n = \{f : [0,1]^n \to R$ such that $f(h_1, ..., h_n) = \max_{\eta \in \{\eta_1,...,\eta_m\}} \eta \cdot h\}$ for some natural number m and a profile of vectors $\eta_1, ..., \eta_m \in [0,1]^n$. We can prove that the set $L'_n = \{f - g : f, g \in F'_n\}$ is uniformly dense in the set $C([0,1]^n)$, and hence the family of functions $\cup_n L'_n$ generates the topology on $\Delta(\mathbf{H})$. Now define $F = \{f : f(\beta) = f'(H(\beta))$ for some $f' \in \cup_n L'_n\}$, we see that $\cup_n L'_n$ corresponds to the image of F from the embedding H. Since the topology is Hausdorff on a compact space, it is normal, therefore any pair of disjoint closed subsets can be separated by two open sets.

In order to construct a game in which t_i and t'_i have distinct sets of rationalizable actions, we need the following corollary which is immediate from Lemma 3.

Corollary 2. *If* $\delta^1(t_i), \delta^1(t'_i) \in \Delta(\Delta\Theta)$ *and* $\delta^1(t_i) \neq \delta^1(t'_i)$, *then there exists a natural number* m *and a continuous bounded function* $\psi : \{1, ..., m\}^{N-1} \times \Theta \to [0, \infty)$ *such that for* $f : \Delta\Theta \to R$ *defined by* $f(\beta) = \max_{\mathbf{k} \in \{1,...,m\}^{N-1}} \beta[\psi(\mathbf{k}, \theta)]$, *we have*

$$\delta^1(t_i)[f] \neq \delta^1(t'_i)[f].$$

Without loss of generality, suppose $\delta^1(t'_i)[f] < \delta^1(t_i)[f]$. By linearity of expectation, there is a $\lambda > 0$ such that $\delta^1(t_i')[\lambda f - 1] < 0 < \delta^1(t_i)[\lambda f - 1]$.

With Corollary 2 we construct a finite game $G_i(\delta^1(t_i), \delta^1(t_i')) = (u_i, A_i)_{i \in N}$ for player i to separate the behavior at types with first-order belief $\delta^1(t_i)$ and types with first-order belief $\delta^1(t_i')$. Let $A_i = \{0, 1\}$, and $A_j = \{1, ..., m\}, \forall j \neq i$. Let the payoffs to the others be constant, e.g., for all $a_j, a_{-j}, \theta, u_j(a_j, a_{-j}, \theta) = 0$, and let the payoff to player i be

$$u_i(a_i, a_{-i}, \theta) = a_i[\lambda\psi(a_{-i}, \theta) - 1].$$

With these payoffs, for any other player, all actions in $\{1, ...m\}$ are rationalizable. For player i, playing $a_i = 0$ gives her 0, while the payoff from playing $a_i = 1$ depends on the actions of the others and states of nature. Player i's payoff from playing $a_i = 1$ is maximized if the others play the following type-correlated strategy:

$$\sigma_{-i}(t_{-i}) = \arg\max_{\mathbf{k}} \beta[\psi(\mathbf{k}, \theta)], \forall t_i \text{ such that } \pi_i(t_i, t_{-i}) = \beta, \forall \beta \in \Delta\Theta.$$

The maximal payoff is $\delta^1(t_i)[\lambda \max_{\mathbf{k}} \beta[\psi(\mathbf{k}, \theta)] - 1] = \delta^1(t_i)[\lambda f - 1]$. Since player i's playoff from playing 1, $\delta^1(t_i)[\lambda f - 1]$, is greater than the payoff from playing $a_i = 0$, which is 0, $a_i = 1$ is rationalizable at t_i. However, at type t_i', the maximal payoff from playing $a_i = 1$ is $\delta^1(t_i')[\lambda f - 1] < 0$. Therefore playing $a_i = 1$ is strictly dominated by playing $a_i = 0$; $a_i = 1$ is not rationalizable at t_i'.

By applying Lemma 3, for any pair of disjoint closed subsets of first-order beliefs, we can construct a game that separates them in rationalizability. For any pair of disjoint closed subsets $S, S' \in \Delta(\Delta\Theta)$, there is a game $G(S, S')$ such that for all $\delta^1 \in S, 1 \in R_i(\delta^1|G(S, S'))$ and for all $\tilde{\delta}^1 \in S', 1 \notin R_i(\tilde{\delta}^1|G(S, S'))$.

Step 2 (Induction). To carry out induction, we first introduce an intermediate result in Ely and Pęski (2006). For any game $G = (u_i, A_i)_{i \in N}$, the mapping $t_{-i} \rightarrow R_{-i}(t_{-i}|G)$

defines the set of rationalizable actions for any profile of the others' types. For any set A, denote 2^A the set of subsets of A. For any measurable subset $S \subseteq \Delta\Theta \times 2^{A_{-i}}$, let

$$\omega(t_i|G)[S] = \pi_i(t_i)[\{t_{-i} : (\pi_i(t_i, t_{-i}), R_{-i}(t_{-i}|G)) \in S\}].$$

We call $\omega(t_i|G) \in \Delta(\Delta\Theta \times 2^{A_{-i}})$ player i's *rationalizable belief* at t_i. It is straightforward to see that rationalizable beliefs at types determine the sets of rationalizable conjectures and therefore the sets of best response actions.

If $\delta^2(t_i) \neq \delta^2(t_i')$, the two types must differ in their beliefs at some closed subset $S \subseteq \times_{i \neq j}\Delta(\Delta\Theta)$, thus there must be some pair of disjoint closed subsets $S, S' \subseteq \times_{j \neq i}\Delta(\Delta\Theta)$ and a game $G(S, S')$ that separates them such that $\omega(t_i|G(S, S')) \neq \omega(t_i'|G(S, S'))$. If player i believes the other players are playing $G(S, S')$, at t_i, t_i' she will have different sets of conjectures about the others' actions and states of nature; this suggests that she will have different sets of rationalizable actions at t_i and t_i' given that her payoff function is properly designed.

Theorem 2 (Ely and Pęski, 2006, theorem 3)**.** *If two types t_i and t_i' differ in terms of their rationalizable belief in game G, i.e., $\omega(t_i|G) \neq \omega(t_i'|G)$, then there is a finite game G' in which t_i and t_i' have distinct rationalizable sets, i.e., $R_i(t_i|G') \neq R_i(t_i'|G')$.*

As an immediate result, if $\delta^2(t_i) \neq \delta^2(t_i')$, then there is a finite game G' such that $R_i(t_i|G') = R_i(t_i'|G')$. The construction of G' is very similar to the construction of $G(\delta^1(t_i), \delta^1(t_i'))$ in step 1; it uses a lemma more general than Lemma 3.

Let F be the set of $f : \Delta\Theta \times 2^{A_{-i}} \to [0, \infty)$ such that for any $\beta \in \Delta\Theta, S_j \subseteq A_j, \forall j \neq i$,

$$f(\beta, S_{-i}) = \max_{\substack{\mathbf{k} \in \{1,\dots,m\}^{N-1} \\ a_{j1},\dots,a_{jm'} \in S_j, \forall j \neq i}} \beta[\psi(\mathbf{k}, (a_{j,1}, \dots, a_{j,m'})_{j \neq i}, \theta)]$$

for some natural numbers m and m', and continuous bounded function $\psi : \{1, \dots, m\}^{N-1} \times$

$A_{-i}^{m'} \times \Theta \to [0, \infty)$.

Lemma 4. *The collection of sets* $\{\mu : \mu[f] < 0\} \subseteq \Delta(\Delta\Theta \times 2^{A_{-i}})$ *for* $f \in F$ *generate the weak*-topology on* $\Delta(\Delta\Theta \times 2^{A_{-i}})$. *This topology is normal, and therefore any pair of disjoint closed subsets* $S, S' \in \Delta(\Delta\Theta \times 2^{A_{-i}})$ *can be separated by open sets, and there is a function* $f \in F$ *such that* $\forall \mu \in S$ *and* $\mu' \in S'$,

$$\mu[f] \neq \mu'[f].$$

As a result of this lemma, there is a game $G(S.S')$ that separates any pair of disjoint closed subsets S, S' of second-order beliefs.

The induction works as follows. If $\delta^3(t_i) \neq \delta^3(t_i')$, the two types must differ in their beliefs at some closed subset $S \in \times_{j \neq i} \Delta(\Delta(\Delta\Theta))$; hence there must be some pair of disjoint closed subsets $S, S' \in \times_{j \neq i} \Delta(\Delta(\Delta\Theta))$ and a game $G(S, S')$ that separate them such that $\omega(t_i | G(S, S')) \neq \omega(t_i' | G(S, S'))$. Applying Theorem 2 again, there must be a finite game G' such that $R_i(t_i | G') = R_i(t_i' | G')$.

For $\delta^k(t_i) \neq \delta^k(t_i'), k \geq 3$, respective separating games can be constructed inductively by applying Lemma 4 and Theorem 2. $\square$

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