The Bayesian Solution and Hierarchies of Beliefs

Tang, Qianfeng

The University of Texas at Austin

9 November 2010

The Bayesian solution and hierarchies of beliefs

Qianfeng Tang*

November 9, 2010

Abstract

The Bayesian solution is a notion of correlated equilibrium proposed by Forges (1993), and hierarchies of beliefs over conditional beliefs are introduced by Ely and Pęski (2006) in their study of interim rationalizability. We study the connection between the two concepts. We say that two type spaces are equivalent if they represent the same set of hierarchies of beliefs over conditional beliefs. We show that the correlation embedded in equivalent type spaces can be characterized by partially correlating devices, which send correlated signals to players in a belief invariant way. Since such correlating devices also implement the Bayesian solution, we establish that the Bayesian solution is invariant across equivalent type spaces.

*Department of Economics, The University of Texas at Austin. Email: tangqianfeng198@mail.utexas.edu.
The author thanks Marcin Pęski for his invaluable guidance and detailed comments. All errors are my own.

1 Introduction

Harsanyi (1967-1968) proposes type spaces to model players' beliefs and higher-order beliefs in games with incomplete information, and later Mertens and Zamir (1985) constructs a universal type space which incorporates all hierarchies of beliefs. These works provide the foundations for strategic analysis in games with incomplete information. One phenomenon that has recently attracts game theorists' attention is that, for a given solution concept, type spaces and hierarchies of beliefs are not always strategically equivalent. To be more precise, fix any hierarchy of beliefs, there are multiple type spaces that could represent it. These type spaces, although equivalent in the set of hierarchies of beliefs that they represent, may differ in the amounts of correlations incorporated in the types; and these correlations potentially matter for the behavioral prediction of various solution concepts.

The characterization of correlations embedded in type spaces with the same set of Mertens-Zamir (conventional, hereafter) hierarchies of beliefs has been done in Liu (2005). Liu shows that any redundant type spaces (ones in which multiple types of the same player have the same hierarchy of beliefs) can be generated by operating a state-dependent correlating mechanism on the non-redundant type space. The correlation provided by a state-dependent correlating mechanism can be viewed as *ex post*, because in the mechanism, correlated signals are sent to players depending on information in the ex post stage of the game–both states of nature and players' types.

We focus on hierarchies of beliefs over conditional beliefs, i.e., Δ-hierarchies of beliefs, which are introduced by Ely and pęski (2006); and we are interested in *interim-stage correlations* among players, i.e., the correlations that depend only on interim stage information–players' types. We define type spaces with the same set of Δ-hierarchies of beliefs to be equivalent, then show that correlations embedded in equivalent type spaces can be characterized by partially correlating devices. Depending on players' type profiles, partially correlating devices send correlated signals to players in a belief invariant way in the interim stage.

We use the following example to illustrate the difference between *interim* and *ex post* correlation.

Example 1. Consider a two-player game with payoff uncertainties parameterized by $\Theta = \{+1, -1\}$. The action sets of players' are $A_i = \{a_i, b_i\}$ for $i \in \{1, 2\}$, and the payoffs of players' are given by

	a_2	b_2
a_1	$1, 1$	$0, 0$
b_1	$0, 0$	$1, 1$

$\theta = +1$

	a_2	b_2
a_1	$0, 0$	$1, 1$
b_1	$1, 1$	$0, 0$

$\theta = -1$

Figure 1.

From the payoffs, players would like to match their actions in state $\theta = +1$ and to mismatch in state $\theta = -1$. Consider a type space T on Θ in which the sets of players' types are described by $T_1 = T_2 = \{+1, -1\}$, and the type profiles in $T_1 \times T_2$ are equally distributed. Suppose if $t_1 = t_2, \theta = +1$ and if $t_1 \neq t_2, \theta$ equals $+1$ or -1 each with probability $\frac{1}{2}$. With no correlation in actions, the ex ante payoff for each player from playing any strategy is $\frac{1}{2}$.

To implement interim stage correlation, assume there is a mediator who observes both players' types. When $t_1 = t_2$, the mediator tosses a coin; if the outcome is head (H), she tells player 1 to play a_1 and player 2 to play a_2, and if the outcome is tail (T), she tells player 1 to play b_1 and player 2 to play b_2. Recommendations are privately made to each player. When $t_1 \neq t_2$, the mediator's information on t does not provide her with any extra information on θ, and she does not make recommendations. By following the mediator's recommendations, players match their actions perfectly with probability $\frac{1}{2}$. The ex ante expected payoff for each player is $\frac{3}{4}$.

To implement ex post correlation, assume there is a mediator who observes both players' types and the true state of nature. At both $\theta = +1$ and $\theta = -1$ the mediator tosses a coin. When $\theta = +1$, the mediator recommends (a_1, a_2) at H and (b_1, b_2) at T; and when $\theta = -1$, the mediator recommends (a_1, b_2) at H and (b_1, a_2) at T. Recommendations are privately made to each player. By following the mediator's recommendations, players match their actions perfectly in both states. The expected payoff for each player is 1.

Here the mediator's role is exactly implementor of a partially correlating device and a state-dependent correlating mechanism. Moreover, it is not difficult to check that in the interim stage correlation, the signals (recommendations) from the mediator do not change players' Δ-hierarchies of beliefs[1]; and in the ex post correlation, the signals do not change players' conventional hierarchies of beliefs. Further more, we can also see from the example that signals from the ex post correlation change the set of conditional beliefs, and hence Δ-hierarchies of beliefs: prior to receiving signals, at $t_1 = +1$, player 1's belief over Θ conditional on player 2's type $t_2 = -1$ is $\frac{1}{2}\{\theta = +1\} + \frac{1}{2}\{\theta = -1\}$; however, after receiving signals, player 1's belief over Θ at type $(+1, a_1)$ conditional on player 2's type $(-1, a_2)$, for example, becomes certainty of $\{\theta = +1\}$.

For any type space and a partially correlating device, we can generate a larger type space when we incorporate signals into players' private information; and when signals are recommendations of actions, these newly generated type spaces are exactly the epistemic models used by Forges (1993) in her definition of the Bayesian solution. A partially correlating device is canonical if the set of signals a player could receive is exactly her action set. Forges (2006) uses canonical partially correlating devices to explicitly implement the Bayesian solution. Based on the characterization of correlations, we establish that the set of Bayesian solution payoffs on a type space is the union of Bayesian Nash equilibria payoffs in

[1]Please refer to Section 2.2 for explicit formulations of Δ-hierarchies of beliefs.

its equivalent type spaces; and in an immediate corollary, we show that the Bayesian solution is invariant across equivalent type spaces.

This paper is organized as follows. We present notations and formulations of hierarchies of beliefs in Section 2, and derive the characterization of correlations embedded in equivalent type spaces in Section 3. Section 4 presents that the Bayesian solution is invariant across equivalent type spaces. Section 5 discusses and concludes.

2 Model

2.1 Notations

We begin with some notations. For any metric space X, let ΔX denote the space of probability measures on the Borel σ-algebra of X endowed with the weak*-topology. Let the product of two metric spaces be endowed with the product Borel σ-algebra. For any measure $\mu \in \Delta(X \times Y)$, denote $\text{marg}_X \, \mu$ the marginal distribution of μ on X.

We study games with incomplete information with n players. The set of players is $N = \{1, 2, ..., n\}$. For each $i \in N$, let $-i$ denote the set of i's opponents. Players play a game in which the payoffs are uncertain and parameterized by a finite set Θ. Each element $\theta \in \Theta$ is called a state of nature. For each $i \in N$, denote A_i the set of actions for player i, and $A \equiv \times_{i\in N} A_i$ the set of action profiles. A *(strategic form) game* is a profile $G = (g_i, A_i)_{i\in N}$. For each $i \in N$, we assume the payoff function is bounded: $g_i : A \times \Theta \to [-M, M]$, for some positive real number M. The set of finite bounded games is denoted by $\mathcal{G}$.

A *type space* over Θ is defined as $T = (T_i, \pi_i)_{i\in N}$, where for each i, T_i is a finite set of types for player i and $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$ is a mapping such that $\pi_i(t_i)[(t_{-i}, \theta)]$ describes player i's belief on the others' types being t_{-i} and the state of nature being θ.

Throughout, given arbitrary $x \in X$ and $y \in Y$, we use the notation $\pi_i(x)[y]$ to denote player i's belief about y conditional on x. More precisely, the object in the round bracket always denotes the object player i conditions on, and the object in the square bracket always denotes the object player i assigns probability to.

2.2 Formulations of hierarchies of beliefs

We first present Mertens and Zamir's standard formulation of hierarchies of beliefs (see also Brandenburger and Dekel (1993)), and based on that present Ely and Pęski's construction of Δ-hierarchies of beliefs. For convenience, we call Mertens-Zamir hierarchy of beliefs the conventional hierarchy of beliefs.

Let $X_0 = \Theta$, and for $k \geq 1, X_k = X_{k-1} \times \times_{j \neq i} \Delta(X_{k-1})$. Let $h^1(t_i) = \text{marg}_\Theta \, \pi_i(t_i)$, which is player i's belief over Θ at type t_i. For each $k \geq 1$, let $h^k(t_i)[S] = \pi_i(t_i)[\{(\theta, t_{-i}) : (\theta, (h^l(t_{-i}))_{1 \leq l \leq k-1}) \in S\}]$, for any measurable subset $S \subseteq X_k$. In the construction, $h^k(t_i) \in \Delta(X_{k-1})$ represents player i's k-th order belief at t_i. The profile $h(t_i) = (h^1(t_i), ..., h^k(t_i), ...) \in \times_{k=0}^{\infty} \Delta X_k$ is called player i's conventional hierarchy of beliefs at type t_i.

A Δ-hierarchy of beliefs describes a player's belief and higher-order beliefs about conditional beliefs on states of nature. The concept was introduced by Ely and Pęski (2006) in their study of interim independent rationalizability. We begin with defining conditional beliefs. Given belief $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$, the conditional belief of type t_i over Θ, conditioning on the others' types being t_{-i}, is $\pi_i(t_i)(t_{-i}) \in \Delta\Theta$, also written as $\pi_i(t_i, t_{-i})$. For any type space T, denote the set of all possible conditional beliefs at t_i as $B_i(t_i) = \{\pi_i(t_i, t_{-i}) \in \Delta\Theta : t_{-i} \in T_{-i}\}$. Type t_i's belief over T_{-i} then induces a belief over $\Delta\Theta$: for any measurable subset $S \subseteq \Delta\Theta, \pi_i(t_i)[S] = \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) \in S\}]$.

Now we can define Δ-hierarchy of beliefs at t_i by treating the set of possible conditional beliefs, i.e., $\Delta\Theta$, as the set of basic uncertainty. Let the first-order belief be player i's belief over the set of conditional beliefs, second-order belief be player i's belief over the others' beliefs over the set of conditional beliefs, and so on.

Formally, for any type space $T = (T_i, \pi_i)_{i \in N}$ on Θ, we can transform it into a type space $T^\Delta = (T_i, \pi_i^\Delta)_{i \in N}$ on $\Delta\Theta$. In the new type space, players' types are unchanged, and $\pi_i^\Delta(t_i) \in \Delta(T_{-i} \times \Delta\Theta)$ is given by

$$\pi_i^\Delta(t_i)[S] = \pi_i(t_i)[\{t_{-i} : (t_{-i}, \pi_i(t_i, t_{-i})) \in S\}],$$

for any measurable subset $S \subseteq \Delta(T_{-i} \times \Delta\Theta)$.

Denote the conventional hierarchy of beliefs at t_i in the type space T^Δ as $h(t_i|T^\Delta)$.

Definition 1. *For any type space T, for any $k \geq 1$, let the k-th order Δ-hierarchy of beliefs at $t_i \in T_i$ be $h^k(t_i|T^\Delta)$ and denote it as $\delta^k(t_i)$. Also, denote the Δ-hierarchy of beliefs at t_i as $\delta(t_i) = (\delta^1(t_i), ..., \delta^k(t_i), ...)$.*

By definition, $\delta(t_i) = h(t_i|T^\Delta)$. For player i, we use δ_{-i} to denote the profile of the others' Δ-hierarchies of beliefs.

3 Characterization of correlations

3.1 Equivalence of type spaces

For any type space T, denote the set of all Δ-hierarchies of beliefs that T has as $\Lambda(T) = \{\delta(t_i) : t_i \in T_i, i \in N\}$. However, the set of Δ-hierarchies of beliefs does not uniquely pin down the type space, instead, there are multiple type spaces that induce the same set of Δ-hierarchies of beliefs.

Definition 2. *Two type spaces T and T' are equivalent, write as $T \sim T'$, if they have the* *same set of* Δ*-hierarchies of beliefs, that is, if*

$$\Lambda(T) = \Lambda(T').$$

A type space in which different types of a player always have different hierarchies of beliefs is called a *reduced* type space (Aumann, 1998), or a *non-redundant* type space (Liu, 2005). For any conventional hierarchy of beliefs, we are able to construct such a type space that generates it, but this is not true for Δ-hierarchies of beliefs. We illustrate this with a simple type space taken from Ely and Pęski (2006).

Example 2. Consider a type space T in which $\Theta = T_1 = T_2 = \{+1, -1\}$, and players' beliefs are updated from a common prior $\pi \in \Delta(\Theta \times T_1 \times T_2)$ such that

$$\pi(t_1, t_2, \theta) = \begin{cases} \frac{1}{4} \text{ if } t_i \cdot t_2 = \theta, \\ 0 \text{ otherwise.} \end{cases}$$

In this type space, the set of conditional beliefs for each type contain point mass on $\theta = +1$ and point mass on $\theta = -1$, and at each type of both players', the Δ-hierarchy of beliefs is common certainty of equal probability of the point masses. Moreover, type space T is the most compact one that supports this Δ-hierarchy beliefs.

Although we can alternatively define the most compact type space that generates a Δ-hierarchy of beliefs as non-redundant, we prefer not to do that here. Without distinguishing non-redundant and redundant type spaces, we can achieve a partial characterization of the correlation embedded in equivalent type spaces, which is sufficient for proving our result in the next section.

Definition 3. *For any type space* T*, a partially correlating device on* T *is a profile* $Q = (q_i, S_i)_{i \in N}$*, where for each* $i \in N, S_i$ *is a finite set of signals and* $q_i : T \to \Delta S, S = \times_{i \in N} S_i$*,* *such that*

1. *player i believes that when players' type profile is $t \in T$, the device selects a signal profile $s \in S$ according to the distribution $q_i(t) \in \Delta S$, and for each $j \in N$, s_j is reported by a mediator to player j.*

2. *belief invariance is satisfied. Formally, at different types t_{-i}, t'_{-i} of the others', player i receives s_i with the same probability, i.e.,*

$$\sum_{\{s' \in S : s'_i = s_i\}} q_i(t_i, t_{-i})[s'] = \sum_{\{s' \in S : s'_i = s_i\}} q_i(t_i, t'_{-i})[s'], \forall i, t_i, s_i.$$

If $S_i = A_i, \forall i \in N, Q$ is a canonical partially correlating device.

From the definition, partially correlating devices are subjective; for each $i \in N$, player i holds a subjective belief q_i over the signals. Belief invariance ensures that from the signals that the players receive, they cannot infer any extra information about the others' types. Also note that the correlated signals depend only on players' types, not on states of nature. There is a key distinction between the partially correlating device and Liu (2005)'s state-dependent correlating mechanism. The latter assumes that correlated signals depend on both players' types and states of nature, i.e., on states of the world. One can also view the distinction as that between interim stage correlation and ex post stage correlation. A canonical correlating device uses actions as signals, and thus the signals can be viewed as recommendations of play.

Denote $q_i(t_i, t_{-i})[s_{-i}|s_i]$ as player i's posterior about s_{-i}, conditional on receiving signal s_i.

Definition 4. *For any type space $T = (T_i, \pi_i)_{i \in N}$ and any partially correlating device $Q = (q_i, S_i)_{i \in N}$, denote T^Q as the type space generated from T through operating Q on T. More* *precisely,* $T^Q = (T_i^Q, \pi_i^Q)_{i \in N}$ *such that*

$$T_i^Q = \{(t_i, s_i) : q_i(t_i, t_{-i})[s_i] > 0, \text{ for some } t_{-i} \in T_{-i}\},$$

and for all $(t_{-i}, s_{-i}) \in T_{-i}^Q, \theta \in \Theta$ *and* $(t_i, s_i) \in T_i^Q$,

$$\pi_i^Q((t_i, s_i))[((t_{-i}, s_{-i}), \theta)] = \pi_i(t_i)[(t_{-i}, \theta)] \cdot q_i(t_i, t_{-i})[s_{-i}|s_i].$$

3.2 The characterization

The following theorem provides a partial characterization of the correlation embedded in equivalent type spaces.

Proposition 1. *We have*

1. *for any type space* T *and partially correlating device* Q, $T^Q \sim T$*; more specifically, for any* $(t_i, s_i) \in T_i^Q, \delta((t_i, s_i)) = \delta(t_i)$.

2. *for any pair of type spaces* T *and* $\hat{T}$ *with* $T \sim \hat{T}$*, there exist partially correlating devices* Q *and* $\hat{Q}$ *such that* $T^Q = \hat{T}^{\hat{Q}}$.

Proof. Part I. We use induction to show that for any $(t_i, s_i) \in T_i^Q, \delta((t_i, s_i)) = \delta(t_i)$. First note that for any $(t_i, s_i) \in T_i^Q, (t_{-i}, s_{-i}) \in T_{-i}^Q$, and $\theta \in \Theta$,

$$\begin{aligned}
\pi_i^Q((t_i, s_i), (t_{-i}, s_{-i}))[\theta] &= \frac{\pi_i^Q((t_i, s_i))[(t_{-i}, s_{-i}), \theta]}{\pi_i^Q((t_i, s_i))[(t_{-i}, s_{-i})]} \\
&= \frac{\pi_i(t_i)[(t_{-i}, \theta)] \cdot q_i(t_i, t_{-i})[(s_i, s_{-i})]}{\pi_i(t_i)[t_{-i}] \cdot q_i(t_i, t_{-i})[(s_i, s_{-i})]} \\
&= \pi_i(t_i, t_{-i})[\theta].
\end{aligned}$$

Therefore, for any $(t_i, s_i) \in T_i^Q$, the set of conditional beliefs at (t_i, s_i) is the same as that at t_i. Furthermore, for any conditional belief $\beta \in B_i(t_i)$,

$$\begin{aligned}
\pi_i^Q((t_i,s_i))[\beta] &= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i^Q((t_i,s_i),(t_{-i},s_{-i})) = \beta\}] \\
&= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i(t_i,t_{-i}) = \beta\}] \\
&= \pi_i^Q((t_i,s_i))[\{t_{-i} : \pi_i(t_i,t_{-i}) = \beta\}] \\
&= \pi_i(t_i)[\{t_{-i} : \pi_i(t_i,t_{-i}) = \beta\}] \\
&= \pi_i(t_i)[\beta].
\end{aligned}$$

The fourth equation above comes from belief invariance. We have proved that for all $(t_i,s_i) \in T_i^Q$, $\delta^1((t_i,s_i)) = \delta^1(t_i)$. For higher-order beliefs, we prove by induction. Now suppose for all $0 < l \leq k$ and $(t_i,s_i) \in T_i^Q, \delta^l((t_i,s_i)) = \delta^l(t_i)$, we show that for all $(t_i,s_i) \in T_i^Q$, $\delta^{k+1}((t_i,s_i)) = \delta^{k+1}(t_i)$. Denote the support of the l-th order belief at type t_i as $B_i^l(t_i)$. As a result, the set of conditional beliefs is relabeled as $B_i^1(t_i)$. By the premises of induction, for all $(t_i,s_i) \in T_i^Q$ and $0 < l \leq k, B_i^l((t_i,s_i)) = B_i^l(t_i)$. Indeed, for any $(\beta,\delta^1,...,\delta^k) \in \times_{0<l\leq k} B_i^l(t_i)$,

$$\begin{aligned}
&\delta^{k+1}((t_i,s_i))[(\beta,\delta^1,...,\delta^k)] \\
&= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i^Q((t_i,s_i),(t_{-i},s_{-i})) = \beta, \delta^1((t_{-i},s_{-i})) = \delta^1,...,\delta^k((t_{-i},s_{-i})) = \delta^k\}] \\
&= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i(t_i,t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1,...,\delta^k(t_{-i}) = \delta^k\}] \\
&= \pi_i(t_i)[\{t_{-i} : \pi_i(t_i,t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1,...,\delta^k(t_{-i}) = \delta^k\}] \\
&= \delta^{k+1}(t_i)[(\beta,\delta^1,...,\delta^k)].
\end{aligned}$$

By induction, for all $(t_i,s_i) \in T_i^Q, \delta((t_i,s_i)) = \delta(t_i)$. Naturally, T^Q and T have the same set of Δ-hierarchies of beliefs, $T^Q \sim T$.

Part II. Fix a pair of type spaces $T = (T_i,\pi_i)_{i\in N}$ and $\hat{T} = (\hat{T}_i,\hat{\pi}_i)_{i\in N}$. Suppose $T \sim \hat{T}$, we now construct Q and $\hat{Q}$ such that $T^Q = \hat{T}^{\hat{Q}}$. To do that, we manipulate the type space $\hat{T}$ into a partially correlating device Q and manipulate T into a partially correlating device $\hat{Q}$. We then show that the generated type spaces T^Q and $\hat{T}^{\hat{Q}}$ are the same.

Step 1. Before we start, we need a few intermediate results.

Lemma 1. *Fix a type space T. If $t_i, t'_i \in T_i$ and $\delta(t_i) = \delta(t'_i)$, then $\pi_i(t_i)[(\beta, \delta_{-i})] = \pi_i(t'_i)[(\beta, \delta_{-i})], \forall \beta, \delta_{-i}$.*

Proof. With the basic property of probability measures,

$$
\begin{aligned}
\pi_i(t_i)[(\beta, \delta_{-i})] &= \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1_{-i}, ..., \delta^n(t_{-i}) = \delta^n_{-i}, ...\}] \\
&= \pi_i(t_i)[\cap_n \{t_{-i} : \pi_i(t_i, t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1_{-i}, ..., \delta^n(t_{-i}) = \delta^n_{-i}\}] \\
&= \lim_n \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1_{-i}, ..., \delta^n(t_{-i}) = \delta^n_{-i}\}] \\
&= \lim_n \delta^{n+1}(t_i)[(\beta, \delta^1, ..., \delta^n)] \\
&= \lim_n \delta^{n+1}(t'_i)[(\beta, \delta^1, ..., \delta^n)] \\
&= \pi_i(t'_i)[(\beta, \delta_{-i})].
\end{aligned}
$$

□

Lemma 2. *Fix a type space T. Suppose $t_i, t'_i \in T_i$ and $\delta(t_i) = \delta(t'_i)$. Then for any t_{-i} that satisfies $\pi_i(t_i)[t_{-i}] > 0$, there exists t'_{-i} that satisfies $\delta(t'_{-i}) = \delta(t_{-i})$, such that $\pi_i(t_i, t_{-i}) = \pi_i(t'_i, t'_{-i})$.*

Proof. We prove by contradiction. Suppose it is not the case. Then there exists a t_{-i} that satisfies $\pi_i(t_i)[t_{-i}] > 0$ and $\pi_i(t_i, t_{-i}) = \beta$, such that for all t'_{-i} that satisfies $\pi_i(t'_i, t'_{-i}) = \beta, \delta(t'_{-i}) \neq \delta(t_{-i})$. As a result, $\pi_i(t'_i)[(\beta, \delta_{-i}(t_{-i}))] = 0$. However, $\pi_i(t_i)[(\beta, \delta_{-i}(t_{-i}))] \geq \pi_i(t_i)[t_{-i}] > 0$. Given Lemma 1, this is in contradiction with $\delta(t_i) = \delta(t'_i)$.

□

Step 2. Using information in type space $\hat{T}$, we now construct a partially correlating device $Q = (q_i, S_i)_{i \in N}$ which is to be operated on type space T. For each $i \in N$, let the set of signals for player i be $S_i = \hat{T}_i$, and define $S \equiv \times_{i \in N} S_i$. Define

$$S_i(t_i) \equiv \{\hat{t}_i \in \hat{T}_i : \delta(\hat{t}_i) = \delta(t_i)\}$$

and

$$S_{-i}(t_i, t_{-i}|\hat{t}_i)) \equiv \{\hat{t}_{-i} \in \hat{T}_{-i} : \delta(\hat{t}_{-i}) = \delta(t_{-i}) \text{ and } \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i}) = \pi_i(t_i, t_{-i})\}.$$

Intuitively, we are going to construct $q_i : T \to \Delta S$ in a way such that the set of signals that player i could possibly receive when her type is t_i is restricted to be $S_i(t_i)$, which is the set of t_i's equivalent types in $\hat{T}_i$. Similarly, $S_{-i}(t_i, t_{-i}|\hat{t}_i))$ will be the restricted set of signals that the others may receive at type profile t_{-i} from player i's view, when her own type is t_i and she receives signal $\hat{t}_i$.

We need the following corollary, which is immediate from Lemma 1 and Lemma 2, in the construction of $q_i : T \to \Delta S$.

Corollary 1. *If* $\hat{t}_i, \hat{u}_i \in S_i(t_i)$, *then* $\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i}|\hat{t}_i))] = \hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]$.

Define on the type space $\hat{T}$ a prior $\hat{p}_i \in \Delta(\hat{T}_i \times \hat{T}_{-i} \times \Theta)$ for player i as follows:

$$\hat{p}_i[(\hat{t}_i, \hat{t}_{-i}, \theta)] = \frac{1}{|\hat{T}_i|}\hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)], \forall (\hat{t}_i, \hat{t}_{-i}, \theta) \in \hat{T}_i \times \hat{T}_{-i} \times \Theta.$$

Then we can construct q_i in the partially correlating device Q as follows:

$$q_i(t_i, t_{-i})[(\hat{t}_i, \hat{t}_{-i})] = \begin{cases} \frac{\hat{p}_i[(\hat{t}_i, \hat{t}_{-i})]}{\hat{p}_i[S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)]}, & \text{if } (\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i); \\ 0, & \text{otherwise.} \end{cases}$$

With Corollary 1, for any $(\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)$,

$$
\begin{aligned}
q_i(t_i, t_{-i})[(\hat{t}_i, \hat{t}_{-i})] &= \frac{\hat{p}_i[\hat{t}_i]\hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)]}{\sum_{\hat{u}_i \in S_i(t_i)} \hat{p}_i[\hat{u}_i]\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i})|\hat{u}_i]} \\
&= \frac{1/|\hat{T}_i|}{1/|\hat{T}_i| \cdot |S_i(t_i)|} \cdot \frac{\hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)]}{\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i})|\hat{t}_i]}.
\end{aligned}
$$

The expression of q_i can be rewritten as

$$
q_i(t_i, t_{-i})[(\hat{t}_i, \hat{t}_{-i})] = \begin{cases} \frac{1}{|S_i(t_i)|} \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i}|\hat{t}_i)]}, & \text{if } (\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i); \\ 0, & \text{otherwise.} \end{cases}
$$

Now we prove that the Q defined above satisfies belief invariance, and thus is indeed a partially correlating device. For any $(t_i, t_{-i}) \in T_i$ and any $\hat{u}_i \in S_i(t_i)$, the probability that player i will receive signal $\hat{u}_i$ is

$$
\begin{aligned}
\sum_{\{\hat{t} \in \hat{T}: \hat{t}_i = \hat{u}_i\}} q_i(t_i, t_{-i})[(\hat{u}_i, \hat{t}_{-i})] &= \sum_{\{\hat{t}_{-i}: \hat{t}_{-i} \in S_{-i}(t_i, t_{-i}|\hat{u}_i)\}} \frac{1}{|S_i(t_i)|} \cdot \frac{\hat{\pi}_i(\hat{u}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]} \\
&= \frac{1}{|S_i(t_i)|} \frac{\sum_{\{\hat{t}_{-i}: \hat{t}_{-i} \in S_{-i}(t_i, t_{-i}|\hat{u}_i)\}} \hat{\pi}_i(\hat{u}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]} \\
&= \frac{1}{|S_i(t_i)|} \frac{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]}{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]} \\
&= \frac{1}{|S_i(t_i)|},
\end{aligned}
$$

which is independent of t_{-i}, and thus the signal does not provide extra information on the others' types.

Step 3. Given the partially correlating device Q constructed using information in $\hat{T}$, we can generate a new type space $T^Q = (T_i^Q, \pi_i^Q)_{i \in N}$ from the type space T. In $T^Q, T_i^Q =$

$\{(t_i, \hat{t}_i) : t_i \in T_i, \hat{t}_i \in S_i(t_i)\}$, and for any $(\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)$,

$$\pi_i^Q((t_i, \hat{t}_i))[((t_{-i}, \hat{t}_{-i}), \theta)] = \pi_i(t_i)[(t_{-i}, \theta)] \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i}|\hat{t}_i)]}.$$

Similarly, we can construct another partially correlating device $\hat{Q}$ using information in the type space T, and generate a new type space $\hat{T}^{\hat{Q}}$ from $\hat{T}$. In $\hat{T}^{\hat{Q}}, \hat{T}_i^{\hat{Q}} = \{(\hat{t}_i, t_i) : \hat{t}_i \in \hat{T}_i, t_i \in S_i(\hat{t}_i)\}$, and for any $(t_i, t_{-i}) \in S_i(\hat{t}_i) \times S_{-i}(\hat{t}_i, \hat{t}_{-i}|t_i)$,

$$\pi_i^{\hat{Q}}((\hat{t}_i, t_i))[((\hat{t}_{-i}, t_{-i}), \theta)] = \hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)] \cdot \frac{\pi_i(t_i)[t_{-i}]}{\pi_i(t_i)[S_{-i}(\hat{t}_i, \hat{t}_{-i}|t_i)]}.$$

It is straightforward that $T_i^Q = \hat{T}_i^{\hat{Q}}, \forall i \in N$. Now we show $\pi_i^Q((t_i, \hat{t}_i)) = \pi_i^{\hat{Q}}((\hat{t}_i, t_i))$. By the definition, for any (t_i, t_{-i}) and $(\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)$, we know that $\pi_i(t_i, t_{-i}) = \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i}) = \beta, \delta(t_{-i}) = \delta(\hat{t}_{-i}) = \delta_{-i}$, for some β and δ_{-i}. We can decompose the belief π_i^Q as follows:

$$\begin{aligned}
&\pi_i^Q((t_i, \hat{t}_i))[((t_{-i}, \hat{t}_{-i}), \theta)] \\
&= \pi_i(t_i, t_{-i})[\theta] \cdot \pi_i(t_i)[t_{-i}] \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{t}_i)[\{\hat{t}'_{-i} : \delta(\hat{t}'_{-i}) = \delta(t_{-i}), \hat{\pi}_i(\hat{t}_i, \hat{t}'_{-i}) = \pi_i(t_i, t_{-i})\}]} \\
&= \pi_i(t_i, t_{-i})[\theta] \cdot \frac{\pi_i(t_i)[t_{-i}] \cdot \hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\pi_i(t_i)[(\beta, \delta_{-i})]}.
\end{aligned}$$

Similarly, $\pi_i^{\hat{Q}}((\hat{t}_i, t_i))[((\hat{t}_{-i}, t_{-i}), \theta)]$ can also be decomposed:

$$\pi_i^{\hat{Q}}((\hat{t}_i, t_i))[((\hat{t}_{-i}, t_{-i}), \theta)] = \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i})[\theta] \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] \cdot \pi_i(t_i)[t_{-i}]}{\hat{\pi}_i(\hat{t}_i)[(\beta, \delta_{-i})]}.$$

We compare π_i^Q and $\pi_i^{\hat{Q}}$ term by term. First, $\pi_i(t_i, t_{-i})[\theta] = \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i})[\theta]$. Second, $\pi_i(t_i)[t_{-i}] \cdot \hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] = \hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] \cdot \pi_i(t_i)[t_{-i}]$. Third, from Lemma 1, $\pi_i(t_i)[(\beta, \delta_{-i})] = \hat{\pi}_i(\hat{t}_i)[(\beta, \delta_{-i})]$.

Since for any $i \in N, (t_i, \hat{t}_i) \in T_i^Q = \hat{T}_i^{\hat{Q}}, \pi_i^Q((t_i, \hat{t}_i)) = \pi_i^{\hat{Q}}((\hat{t}_i, t_i))$, we have $T^Q = \hat{T}^{\hat{Q}}$.

□

4 The Bayesian solution

4.1 Definition

The Bayesian solution is a notion of correlated equilibrium for games with incomplete information proposed by Forges (1993). Its definition is inspired by Aumann's Bayesian view and aims at capturing Bayesian rationality.

Following Forges (2006), the definition of *the Bayesian solution* involves the use of an epistemic model $Y = (Y, \vartheta, (\mathcal{S}_i, \tau_i, \alpha_i, p_i)_{i\in N})$ into which the type space $T = (T_i, \pi_i)_{i\in N}$ can be embedded[2]. In the epistemic model, Y is the set of states of the world which is large enough to characterize uncertainties in states of nature, agents' types, and agents' actions, $\mathcal{S}_i$ denotes player i's informational partition, and p_i is player i's subjective prior. The mapping $\vartheta : Y \to \Theta$ indicates the state of nature, $\tau_i : Y \to T_i$ indicates player i's type, and $\alpha_i : Y \to A_i$ indicates i's action. Both τ_i and α_i are assumed to be $\mathcal{S}_i$ measurable, thus given any state, player i knows both her type and action. The consistency in probabilities requires that for any measurable subset $S \subseteq T_{-i} \times \Theta$ and $S' \subseteq T_{-i}$,

$$p_i[(\tau_{-i}, \vartheta)^{-1}(S)|\mathcal{S}_i] = \pi_i[S|\tau_i], \tag{4.1}$$
$$p_i[{\tau_{-i}}^{-1}(S')|\mathcal{S}_i] = p_i[{\tau_{-i}}^{-1}(S')|\tau_i], \forall i \in N.$$

The first condition requires that the epistemic model does not give players more information on the joint distribution of the others' types and states of nature, and the second

[2] Forges's definition of the Bayesian solution is restricted to two-player games for type spaces with common priors; what we present here is the n-player non-common prior anaologue of her definition.

condition further requires it does not give more information on the others' types. The two conditions together, guarantees *belief invariance* (the invariance of conditional beliefs). Given the epistemic model, we can define Bayesian rationality for player i: player i is *Bayesian rational* if

$$E[g_i(\alpha_i, \alpha_{-i}, \vartheta)|\mathcal{S}_i] \geq E[g_i(a_i, \alpha_{-i}, \vartheta)|\mathcal{S}_i], \forall a_i \in A_i,$$

where the expectation is taken over T_{-i} and Θ.

Definition 5. *Given a game G and a type space T, a Bayesian solution for the game is an epistemic model $Y = (Y, \vartheta, (\mathcal{S}_i, \tau_i, \alpha_i, p_i)_{i\in N})$ constructed as above that satisfies the Bayesian rationality of every player.*

For any Bayesian solution Y, let $\mu_i(y) \in \Delta(\Theta \times A_{-i})$ be player i's belief over states of nature and the others' actions in the state of the world y, and $\mu(y) = (\mu_i(y))_{i\in N}$ be a profile of players' beliefs. Denote the set of payoffs of player i in a Bayesian solution Y as

$$B_i(Y) = \{g_i = \max_{a_i \in A_i} g_i(a_i, \mu_i(y)) : y \in Y\},$$

and let $B(Y) \equiv (B_i(Y))_{i\in N} \in R^N$. From a point of view analogous to the "revelation principle" in the mechanism literature, Forges (2006) characterizes Bayesian solutions with partially correlating devices.

Proposition 2. *For any game G and type space T, the set of payoffs $B(Y)$ from a Bayesian solution Y can be achieved by a canonical partially correlating device, $Q = (q_i, A_i)_{i\in N}$, that is incentive compatible, i.e., such that each player does not have incentive to deviate from the mediator's recommendation.*

We can also view $B(Y)$ as the set of players' payoffs from the set of Bayesian Nash equilibria in the game G with type space T^Q. Alternatively, any incentive compatible canonical partially correlating device Q is itself a Bayesian solution.

4.2 Invariance of the Bayesian solution

It is not a coincidence that both the characterization of correlations embedded in equivalent type spaces and the implementation of the Bayesian solution involve using partially correlating devices.

For any game G and any type space T, denote the set of players' all possible payoffs from Bayesian solutions as

$$B(G,T) = \{g = (g_i)_{i \in N} \in R^N : g \in B(Y) \text{ for some Bayesian solution } Y \text{ of } G\}.$$

Denote players' all possible interim payoffs from Bayesian Nash equilibria of the game G with type space T as $NE(G,T)$. The result below states that the set of players' payoffs from Bayesian solutions at a type space is exactly the union of Bayesian Nash equilibria payoffs in equivalent type spaces.

Proposition 3. $B(G,T) = \cup_{\{\hat{T}:\hat{T}\sim T\}} NE(G,\hat{T})$.

Proof. First, notice that each Bayesian solution Y corresponds to a partially correlating device and the payoffs from Y can be implemented by a canonical partially correlating device. Therefore, $B(G,T)$ is equivalent to the union of Bayesian Nash equilibria payoffs in type spaces generated from T by partially correlating devices. Denote the set of partially correlating devices on T as $\mathcal{Q}$, then

Lemma 3. $B(G,T) = \cup_{\{Q:Q\in\mathcal{Q}\}} NE(G,T^Q)$.

Now we only need to show that for any $\hat{T} \sim T$, there exists $Q \in \mathcal{Q}$, such that $NE(G,\hat{T}) \subseteq NE(G,T^Q)$. Suppose $\hat{T} \sim T$, Proposition 1 ensures that there exists partially correlating devices $\hat{Q}$ and Q such that $\hat{T}^{\hat{Q}} = T^Q$.

Lemma 4. *For any partially correlating devices $\hat{Q}$ on $\hat{T}$, $NE(G,\hat{T}) \subseteq NE(G,\hat{T}^{\hat{Q}})$.*

Proof of this lemma is straightforward in that any Bayesian Nash equilibrium in $(G, \hat{T})$ can be replicated in $(G, \hat{T}^{\hat{Q}})$, provided that when facing type space $\hat{T}^{\hat{Q}}$, all players choose to use only information in $\hat{T}$ and ignore the signals sending from $\hat{Q}$.

As a result, $\cup_{\{\hat{T}:\hat{T}\sim T\}} NE(G, \hat{T}) \subseteq \cup_{\{Q:Q\in\mathcal{Q}\}} NE(G, T^Q)$, and since $T^Q \sim T$ for each Q, they must be equal.

□

It is immediate from Proposition 3 that if two type spaces represent the same set of Δ-hierarchies of beliefs, they must induce the same set of Bayesian solution payoffs in any game. In other words, the Bayesian solution is invariant on the equivalent class of type spaces.

Corollary 2. *If two type spaces $\hat{T}$ and T are equivalent in Δ-hierarchies of beliefs, i.e., $\hat{T} \sim T$, then $B(G, T) = B(G, \hat{T})$.*

Proof. Notice that if $\hat{T} \sim T$, then the expressions in Proposition 3 for $B(G, \hat{T})$ and $B(G, \hat{T})$ are the same.

□

Remark 1. Both the characterization of interim-stage correlations and the invariance result above parallel with Liu (2005). Liu characterizes ex-post correlations with state-dependent correlating mechanisms and based on that he defines another notion of correlated equilibrium, which is equivalent with the universal Bayesian solution (Forges, 1993).

5 Conclusion

We study the correlations embedded in type spaces with the same set of hierarchies of beliefs over conditional beliefs, it turns out that such correlations can be expressed explicitly with partially correlating devices, which operate in the interim stage of the game.

With these results, we compare two closely related literatures side by side. Partially correlating devices characterize correlations embedded in type spaces with the same set of Δ-hierarchies of beliefs, and implement the Bayesian solution. Tang (2010) shows that Δ-hierarchies of beliefs fully identify interim partially correlated rationalizability and that interim partially correlated rationalizability and the Bayesian solution are payoff equivalent.

State-dependent correlating mechanisms characterize correlations embedded in type spaces with the same set of conventional hierarchies of beliefs, and implement the universal Bayesian solution (Liu, 2005). Dekel, Fudenberg and Morris (2007) show that conventional hierarchies of beliefs fully identify interim correlated rationalizability and also discuss that interim correlated rationalizability and the universal Bayesian solution are payoff equivalent.

As we have argued in the introduction of Tang (2010), the distinction between the two literatures is purely methodological, in that in modeling incomplete information, the former adopts Harsanyi's principle while the latter adopts Aumann's Bayesian view.

References

[1] Aumann, Robert (1998), "Common priors: A reply to Gul." *Econometrica*, 66, 929-938.

[2] Brandenburger, Adam and Eddie Dekel (1993), "Hierarchies of beliefs and common knowledge." *Journal of Economic Theory*, 59, 189-198.

[3] Dekel, Eddie, Drew Fudenberg and Stephen Morris (2007), "Interim correlated rationalizability." *Theoretical Economics,* 2, 15-40.

[4] Ely, Jeff and Marcin Pęski (2007), "Hierarchies of belief and interim rationalizability." *Theoretical Economics,* 1, 19-65.

[5] Forges, Françoise (1993), "Five legitimate definitions of correlated equilibrium in games with incomplete information." *Theory and decision*, 35, 277-310.

[6] Forges, Françoise (2006), "Correlated equilibrium in games with incomplete information revisited." *Theory and decision*, 61, 329-344.

[7] Harsanyi, John (1967-1968), "Games with incomplete information played by 'Bayesian' players, Parts I, II, III." *Management Science*, 14, 159-182, 320-334, 486-502.

[8] Liu, Qingmin (2009), "On redundant types and Bayesian formulation of incomplete information." *Journal of Economic Theory*, 144, 2115-2145.

[9] Liu, Qingmin (2005), "Representation of belief hierarchies in incomplete information games." *working paper*.

[10] Mertens, Jean-François and Shmuel Zamir (1985), "Formulation of Bayesian analysis for games with incomplete information." *International Journal of Game Theory*, 14, 1-29.

[11] Tang, Qianfeng (2010), "Interim partially correlated rationalizability." *working paper*.