Some solutions to the equity premium and volatility puzzles

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Some Solutions to the Equity Premium and Volatility Puzzles

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Abstract. In this paper, I adopt an economic equilibrium model utilizing the framework introduced by Mehra and Prescott (1985) when they presented the equity premium puzzle. This model, in the long run and with respect to stationary probabilities, produces results that match the sample values derived from the U.S. economy between 1889 and 1978 as illustrated by the studies performed by Grossman and Shiller (1981), which includes the expected average, standard deviation, and first-order serial correlation of the growth rate of per capita real consumption and the expected returns and standard deviation of equity, risk-free security, and risk premium for equity. Therefore, this model solves the equity premium and volatility puzzles. I also explore the reasons why the equity premium puzzle was caused.

Keywords. The constant relative risk aversion class utility function; risk premium; time discount factor; parameters defining preferences; parameters defining technologies; the equity premium and volatility puzzles.

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1. Introduction

In the last 30 years, the study of financial economics models has rapidly developed, playing an increasingly important role in the fields of finance, micro-investment theory, economics of uncertainty, and others. Before a financial economics model can be utilized to investigate, forecast, or predict future economics and finance trends, it must first be able to accurately describe the historical economics and finance behaviors. Therefore, for a given economics sample, it is necessary to build a financial economics model that provides values that exactly match the values from the sample.

In 1981, Grossman and Shiller studied the U.S. economy from the period 1889 through 1978, providing the average, standard deviation, and first-order serial correlation of growth rate of per capita in real consumption and the average returns and standard deviations of equity, risk-free securities, and risk premium in equity for this sample. Mehra and Prescott (1985) published a paper entitled *The Equity Premium, A Puzzle*, in which they formulized a very efficient economics equilibrium model by employing a variation of Lucas' pure exchange model under an assumption that the growth rate of the endowment follows a Markov process. In that paper, they selected a case using two states of growth rates with a special symmetrical transition matrix for the Markov process. From this special model, after matching the average, standard deviation, and first-order serial correlation of the growth rate of per capita consumption from their model to the sample, they discovered that the average returns on equity, risk-free security, and risk premium from the model did not match the respective actual values from the sample. The differences, which were significantly large, formed the equity premium puzzle. It is apparently impossible for their model to match the standard deviations of equity, risk-free security, and risk premium to the respective values from the sample; and therefore it is impossible to match the volatility, which is a financial instrument refers to the standard deviation of the returns of this financial instrument within a specific time horizon. More precisely, Mehra and Prescott (1985) described the puzzle as follows:

> 'The average real return on relatively riskless, short- term securities over the 1889-1978 period was 0.80 percent. The average real return on the Standard and Poor's 500 Composite Stock Index over the ninety years considered was 6.98 percent per annum. This leads to an average equity premium of 6.18 percent. Given the estimated process on consumption, fig. 4 depicts the set of values of the average risk-free rate and equity risk premium which are both consistent with the model and result in average real risk-free rates between zero and four percent. These are values that can be obtained by varying preference parameters α between zero and ten and β between zero and one. The observed real return of 0.80 percent and equity premium of 6 percent is clearly inconsistent with the predictions of the model. The largest premium obtainable with the model is 0.35 percent, which is not close to the observed value'.

Very recently, Guvenen (2009) studied this puzzle and summarized this problem in his paper as:

> 'Since the 1980s, a vast body of empirical research has documented some interesting and puzzling features of asset prices. For example, Mehra and Prescott (1985) have shown that the equity premium observed in the historical US data was hard to reconcile with a

> canonical consumption-based asset pricing model, and as it later turned out, with many of its extensions'.

Then they concluded (See Rietz (1988):

> 'most likely, an equilibrium model which is not an Arrow-Debrea economy will be one that simultaneously rationalizes both historically observed large average equity return and the small average risk-free return'.

I believe that the general model with *n* states for growth rate introduced in Mehra and Prescott's paper is a very efficient model to fit the purpose to match the sample data from this model in an economy, which includes U.S. economy from the period 1889 through 1978, if the states and their Markov processes transition probabilities are appropriately chosen. The reason why this puzzle was formed is that they considered a special model that has two symmetric states to the average gross growth rate that follows a Markov processes with a symmetric transition matrix. In this paper, I will examine their model and the techniques in order to gain a deeper understanding of the causes that formed the puzzle and will build a modified model by employing more states and creating more efficient techniques. So that this modified model and these more efficient technique perfectly reconcile the theory and observation to provide solutions for resolving the equity premium and volatility puzzle. Of course, as a result, the equity premium is resolved.

This puzzle can be solved by employing a more realistic model that has a certain number of states and more powerful, comprehensive techniques from the general economic model used by Mehra and Prescott in their study. In this paper, I choose a general three states model and a special four states model, which are different from the model used by Rietz (1988). Therefore, I will adopt all of the notation and terminology of Mehra and Prescott.

This paper refers to this incompatibility of the standard deviations of equity, risk-free security, and risk premium of equity between the model and the sample the equity premium and volatility puzzle. The volatility of a financial instrument refers to the standard deviation of the returns of this financial instrument within a specific time horizon. This equity premium and volatility puzzle must be distinguished from the well-known volatility puzzle, which relates to the volatility and average returns for some financial instruments in a given period of time (see Chabi-Yo, Merton). A solution of the equity premium and volatility puzzle is described by an economics model from that the first moments and the second moments of the growth rate of per capita consumption and the returns on equity, risk-free security, and risk premium from the model match the respective actual values from the sample.

Since the equity premium puzzle was presented in 1985, many papers have been published to explain or to resolve this puzzle (see references). To my knowledge, there is no published literature that attempts to solve the equity premium and volatility puzzle. In this paper, I apply the economics equilibrium model developed by Mehra and Prescott (1985) and the simulating techniques to construct two types of modified economics models: three state type and four state type. In each type, we will claim that there may be infinitely many different models to matching the average, standard deviation, and first-order serial correlation growth rate of per capita consumption and the expected returns and standard deviations on equity, risk-free security, and risk premium to the respective values of the sample. These matches are exactly matches instead of estimation. In each type, I provide one solution with all details to show the perfect matches and to demonstrate the satisfaction of all conditions stated by Mehra and Prescott. I also provide more solutions for each type without details. These solutions are perfect mathematical solutions of the equity premium and volatility puzzle under the sense of date matching. In each solution, the parameters for states and their transition probabilities may not satisfy some economists for explaining their economies. But this paper provides the techniques to solve the puzzle. I believe that if one uses a super computer and chooses more states, then one can get solutions to satisfy some economists' various desires.

To sum up, after we discover the reasons that caused this puzzle and after we get many models to match the sample data, we can say that the "equity Premium Puzzle" is not a puzzle any more. It is also important to point out that as what I mention in the previous paragraphs, I strongly believe that the Mehra and Prescott's model is a very efficient model to match the sample data in an economy. On the other hand, I believe that this model, which is based on the Lucas' pure exchange model, can not reasonably describe an economy with a certain long periods. It is impossible to describe a very complicated economy by using such a simple mathematics model. More precisely (see Section 2), the growth rate of consumption in real capita will never follow any given ergodic Markov chain.

This paper is organized in five sections: Section 2 summarizes the Mehra and Prescott model and is devoted to the exploration of Mehra and Prescott's model and the discovery of the causes that formed the puzzle; Section 3 presents the modified model with three states and the simulating techniques; Section 4 provides a solution with three states, in details, that solves the equity premium and volatility puzzle and a set of additional solutions to the model built in this paper without details; Section 5 presents the modified model with four states; Section 6, similarly to Section 4, provides a solution with four states, in details, that resolves the equity premium and volatility puzzle and a set of additional solutions to the model built in this paper without details; Section 7 concludes this paper. The appendix provides complicated mathematical simulating calculations and programming, which will be available on the author's webpage and will not be published because it is extremely long.

2. Reexamining the Case n = 2 and Finding the Causes that Formed the Equity Premium Puzzle.

We outline some notation and techniques used by Mehra and Prescott, which will be frequently used in the content of this paper. For details, the reader is referred to Mehra and Prescott's paper (1985). In Mehra and Prescott's paper (1985), they employed a variation of Lucas' pure exchange model under an assumption that the growth rate of the endowment follows a Markov process with a utility function of the constant relative risk aversion class:

$$U(c_t, \alpha) = \frac{c_t^{1-\alpha} - 1}{1-\alpha}, \alpha > 0. \tag{1}$$

We want to optimize

$$E_0\left(\sum_{i=0}^{\infty}\beta^i U(c_{t+i},\alpha)\right), \tag{2}$$

where c_t is the per capita consumption at time t and β is the subjective time discount factor. α and β are parameters defining preferences.

Suppose that the economy has one productive unit and one equity share. A firm's output is the firm's dividend payment in each given time period, t, denoted by y_t. The growth rate, which is denoted by x_{t+1} in y_t, is subject to an ergodic Markov chain; that is

$$y_{t+1} = x_{t+1}y_t, \tag{3}$$

where $x_{t+1} \in \{\lambda_{1,} \lambda_{2,} \ldots\ldots, \lambda_n\}$. The transition matrix of the ergodic Markov chain is denoted by $\Phi = \left(\phi_{ij}\right)_{1\le i,\, j\le n}$ satisfying

$$\phi_{ij} = P(x_{t+1} = \lambda_j \mid x_t = \lambda_i), \text{ for } i, j = 1, 2, \ldots, n. \tag{4}$$

Grossman and Shiller (1981) studied the U.S. economy in the period 1889 through 1978 and provided sample data for the average, standard deviation, and first-order serial correlation of the growth rate per capita in real consumption, which are denoted and given by, respectively, as follows:

$$\bar{\mu} = 1 + 0.0183, \tag{5}$$

$$\bar{\delta} = 0.0357, \tag{6}$$

$$\bar{\rho} = -\,0.14; \tag{7}$$

the average returns on equity, on risk-free security, and on risk premium for equity, denoted as follows:

$$\bar{R}^e = 0.0698, \tag{8}$$

$$\bar{R}^f = 0.008, \tag{9}$$

$$\bar{R}^p \equiv \bar{R}^e - \bar{R}^f = 0.0618; \tag{10}$$

and the standard deviations of equity, risk-free security, and risk premium for equity, denoted as follows:

$$\bar{\delta}^e = 0.1654, \tag{11}$$

$$\bar{\delta}^f = 0.0567, \tag{12}$$

$$\bar{\delta}^p = 0.1667. \tag{13}$$

Once the model is built, the values derived from the model corresponding to the actual values as defined in (5) – (13) will be conveniently denoted by the same notations without the top bar—that is, μ, δ, ρ, R^e, R^f, R^p, δ^e, δ^f, and δ^p, respectively.

The goal of this paper is to solve the equity premium and volatility puzzle by building a Mehra - Prescott economics equilibrium model that, with respect to the stationary probabilities, matches the expected average, standard deviation, and first-order serial correlation of the growth rate of per capita consumption and the expected returns and standard deviations of equity, risk-free security, and risk premium for equity with the respective values from the sample. Symbolically, a model is built such that equations (5) – (13) hold for the same notations without the top bars.

In rest of this section, we investigate the reasons that the equity premium puzzle was formed. In their paper, Mehra and Prescott used a case $n = 2$. They chose the states $\{\lambda_1, \lambda_2\}$ of the gross growth rate per capita consumption as follows

$$\lambda_1 = \bar{\mu} - \bar{\delta}, \quad \lambda_2 = \bar{\mu} + \bar{\delta},$$

with a symmetric transition matrix of the ergodic Markov chain as

$$\Phi = \begin{pmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{pmatrix},$$

for some $0 < \phi < 1$. From the symmetric property of this transition matrix, the fixed probability vector of this Markov chain must be equally likely. That is

$$(\pi_1, \pi_2) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Under conditions that all parameters determined by this model match the observed data given by (5) -- (10), with respect to the model's stationary probability distribution, Mehra and Prescott obtained

$$\lambda_1 = 0.982, \lambda_2 = 1.054,$$

and

$$\Phi = \begin{pmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{pmatrix}.$$

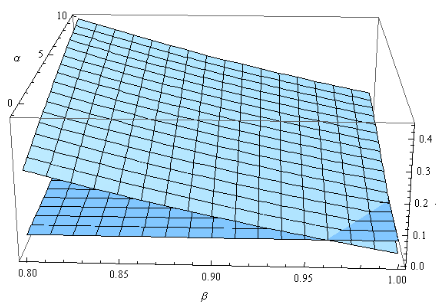
All technology parameters introduced for the case $n = 2$ have been automatically determined. Consequently, the expected return on equity, the expected return on risk-free security and therefore, the risk premium for equity from this model turn out to be independent of the technology parameters and only depend on the preferences parameters. The expected return on equity is

$$\frac{1}{2\beta}\left[\frac{0.43(0.982)\left[1+\beta(0.14)1.054^{1-\alpha}\right]+0.57(1.054)\left[1+\beta(0.14)0.982^{1-\alpha}\right]}{-0.57(0.982^{1-\alpha}-1.054^{1-\alpha})+\left[1+\beta(0.14)1.054^{1-\alpha}\right]0.982^{1-\alpha}}\right.$$
$$\left.+\frac{0.57(0.982)\left[1+\beta(0.14)1.054^{1-\alpha}\right]+0.43(1.054)\left[1+\beta(0.14)0.982^{1-\alpha}\right]}{0.57(0.982^{1-\alpha}-1.054^{1-\alpha})+\left[1+\beta(0.14)0.982^{1-\alpha}\right]1.054^{1-\alpha}}\right]-1.$$

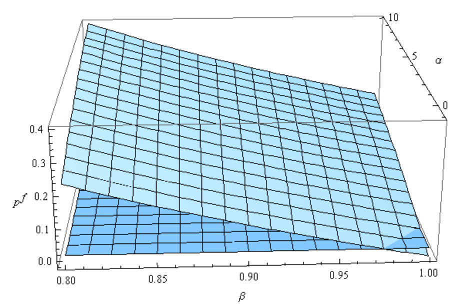
The expected return on risk-free security is

$$\frac{1}{2\beta}\left[\frac{0.982^{-\alpha}+1.054^{-\alpha}}{\left(0.43(0.982^{-\alpha})+0.57(1.054^{-\alpha})\right)\left(0.57(0.982^{-\alpha})+0.43(1.054^{-\alpha})\right)}\right]-1.$$

In Graph 1 below, the surface is the graph of the expected return on equity from the model as a function of α and β and the plane is the graph of observed average return on equity 0.0698. The α and β coordinators of every point on the space curve -- the intersection of the surface and the plane -- are determined values for the two reference parameters α and β, which satisfy that the expected return on equity from this model match the observed return on equity data 0.0698.



Graph 1: The expected return on equity from the model = 0.0698.



Graph 2: The expected return on risk-free security from the model = 0.008.

Similarly to Graph 1, in Graph 2, the surface is the graph of the expected return on risk-free security from the model as a function of α and β and the plane is the graph of observed average return on risk-free security 0.008. The α and β coordinators of every point on the space curve -- the intersection of the surface and the plane -- are determined values for the two reference parameters α and β, which satisfy that the expected return on risk-free security from this model, match the observed return on risk-free security data 0.008. One can see that if these two graphs are drawn in the same system, these two curves do not have any joint point. It implies that there does not exist risk aversion $\alpha \in (0, 10)$ (in fact, even though it is in (0, 30)) and a discount factor $\beta \in (0, 1)$ at which the expected return on equity and the expected return on risk-free security from this model simultaneously match the observed parameters 0.0698 and 0.008, respectively. Consequently, the equity premium puzzle is formed.

To summarize, if we select only two states with a symmetric transition matrix as in the Mehra and Prescott's paper, then all the technology parameters will be immediately and automatically determined. The states become symmetric from the sample gross grow rate and the fixed probability distribution of the Markov chain immediately becomes equally likely. As a result, all technology parameters introduced in the model will become constant and cannot be used as variables. This model will then become a simple model in which states are symmetric with equally likely stationary probability distribution. When we match the parameters from the model with the observed parameters, the model will lose its power to impact the estimated preferences parameters, which are the risk aversion and discount factor. To sum up, the puzzle is formed because, in Mehra and Prescott's model, there is no parameter to be chosen to justify the model to closely fit the economy described by (5) -- (10).

3. Models with Three States and More Powerful Simulating Techniques

As mentioned in the last section, if a model can provide a solution to the equity premium and volatility puzzle, then the parameters must be solutions of the system of eight equations given by (5) – (13). A two-state model generally has a total of six parameters: four technology parameters and two reference parameters. Normally, a system of eight equations with six variables has no solutions. It implies that a two-state model cannot solve the equity premium and volatility puzzle. As with my solution to the equity premium puzzle (2009), I use three states $\{\lambda_1, \lambda_2, \lambda_3\}$ of the growth rate per capita consumption:

$$\lambda_1 = \bar{\mu} + a\bar{\delta} = 1.0183 + 0.0357\,a,$$

$$\lambda_2 = \bar{\mu} + b\bar{\delta} = 1.0183 + 0.0357\,b,$$

$$\lambda_3 = \bar{\mu} + c\bar{\delta} = 1.0183 + 0.0357\,c, \tag{14}$$

where *a*, *b*, and *c* are parameters defining technology. The growth rates are assumed to follow an ergodic Markov chain with the following general non-symmetric transition matrix:

$$\Phi = \begin{pmatrix} p & q & 1-p-q \\ s & t & 1-s-t \\ u & v & 1-u-v \end{pmatrix}, \tag{15}$$

where p, q, s, t, u, and v are also technology parameters satisfying $0 < p, q, s, t, u, v < 1$. From the fundamental theorem of ergodic Markov chains, Φ has a unique fixed probability vector, which is the stationary probability distribution of the growth rate per capita in consumption. This fixed probability row vector is denoted by (π_1, π_2, π_3) and it is the solution of the following system of linear equations:

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3)\,\Phi,$$

with $\pi_1 + \pi_2 + \pi_3 = 1$. As a function of p, q, s, t, u and v, the solution is given by

$$\pi_1 = -\frac{-u + tu - sv}{1 - p - qs - t + pt + u + qu - tu + v - pv + sv},$$

$$\pi_2 = \frac{qu + v - pv}{1 - p - qs - t + pt + u + qu - tu + v - pv + sv},$$

$$\pi_3 = \frac{1 - p - qs - t + pt}{1 - p - qs - t + pt + u + qu - tu + v - pv + sv}. \tag{16}$$

All with respect to the model's stationary probability distribution as given in (16), the expected average, variance, and first-order serial correlation of the growth rate per capita consumption in this model are functions of the technology parameters *a, b, c, p, q, s, t, u* and *v*, which are defined below:

$$\mu = \mu(a, b, c, p, q, s, t, u, v) = \lambda_1 \pi_1 + \lambda_2 \pi_2 + \lambda_3 \pi_3,$$

$$\delta = \delta(a, b, c, p, q, s, t, u, v) = (\lambda_1 - \mu)^2 \pi_1 + (\lambda_2 - \mu)^2 \pi_2 + (\lambda_3 - \mu)^2 \pi_3,$$

$$\begin{aligned}
\rho = \rho(a, b, c, p, q, s, t, u, v) \\
= \Big(&\big((\lambda_1 - \mu)^2 p + (\lambda_1 - \mu)(\lambda_2 - \mu) q + (\lambda_1 - \mu)(\lambda_3 - \mu)(1 - p - q)\big)\pi_1 \\
&+ \big((\lambda_1 - \mu)(\lambda_2 - \mu) s + (\lambda_2 - \mu)^2 t + (\lambda_2 - \mu)(\lambda_3 - \mu)(1 - s - t)\big)\pi_2 \\
&+ \big((\lambda_1 - \mu)(\lambda_3 - \mu) u + (\lambda_2 - \mu)(\lambda_3 - \mu) v + (\lambda_3 - \mu)^2 (1 - u - v)\big)\pi_3 \Big) / \\
&\big((\lambda_1 - \mu)^2 \pi_1 + (\lambda_2 - \mu)^2 \pi_2 + (\lambda_3 - \mu)^2 \pi_3\big).
\end{aligned}$$

The expected returns on equity and on risk-free security, R^e, R^f (and therefore on risk premium for equity, $R^p \equiv R^e - R^f$), from the model are calculated by using formulas (11), (13) and (14) in Mehra and Prescott's paper as functions of α, β, *a, b, c, p, q, s, t, u*, and *v*, as given below:

$$\begin{aligned}
R^e = \; & R^e(\alpha, \beta, a, b, c, p, q, s, t, u, v) \\
= & \left(\left(\frac{\lambda_1(w_1+1)}{w_1} - 1\right)p + \left(\frac{\lambda_2(w_2+1)}{w_1} - 1\right)q + \left(\frac{\lambda_3(w_3+1)}{w_1} - 1\right)(1-p-q)\right)\pi_1 \\
+ & \left(\left(\frac{\lambda_1(w_1+1)}{w_2} - 1\right)s + \left(\frac{\lambda_2(w_2+1)}{w_2} - 1\right)t + \left(\frac{\lambda_3(w_3+1)}{w_2} - 1\right)(1-s-t)\right)\pi_2 \\
+ & \left(\left(\frac{\lambda_1(w_1+1)}{w_3} - 1\right)u + \left(\frac{\lambda_2(w_2+1)}{w_3} - 1\right)v + \left(\frac{\lambda_3(w_3+1)}{w_3} - 1\right)(1-u-v)\right)\pi_3,
\end{aligned}$$

$$\begin{aligned}
R^f = \; & R^f(\alpha, \beta, a, b, c, p, q, s, t, u, v) \\
= & \left(\frac{1}{\beta(\lambda_1^{-\alpha} p + \lambda_2^{-\alpha} q + \lambda_3^{-\alpha}(1-p-q))} - 1\right)\pi_1 + \left(\frac{1}{\beta(\lambda_1^{-\alpha} s + \lambda_2^{-\alpha} t + \lambda_3^{-\alpha}(1-s-t))} - 1\right)\pi_2 \\
+ & \left(\frac{1}{\beta(\lambda_1^{-\alpha} u + \lambda_2^{-\alpha} v + \lambda_3^{-\alpha}(1-u-v))} - 1\right)\pi_3.
\end{aligned}$$

Where, by following equation (9) from Mehra and Prescott's paper, w_1, w_2 and w_3 can be solved from the following system of linear equations:

$$w_1 = \beta p \lambda_1^{1-\alpha}(w_1+1) + \beta q \lambda_2^{1-\alpha}(w_2+1) + \beta(1-p-q)\lambda_3^{1-\alpha}(w_3+1),$$

$$w_2 = \beta s \lambda_1^{1-\alpha}(w_1+1) + \beta t \lambda_2^{1-\alpha}(w_2+1) + \beta(1-s-t)\lambda_3^{1-\alpha}(w_3+1),$$

$$w_3 = \beta u\lambda_1^{1-\alpha}(w_1+1)+\beta v\lambda_2^{1-\alpha}(w_2+1)+\beta(1-u-v)\lambda_3^{1-\alpha}(w_3+1).$$

By applying the expressions for R^e and R^f above, the variances of the equity, the risk-free security and the risk premium for equity are calculated as follows:

$$(\delta^e)^2 = V^e(\alpha,\beta,a,b,c,p,q,s,t,u,v)$$

$$= \left(\left(\frac{\lambda_1(w_1+1)}{w_1}-1\right)p+\left(\frac{\lambda_2(w_2+1)}{w_1}-1\right)q+\left(\frac{\lambda_3(w_3+1)}{w_1}-1\right)(1-p-q)\right)^2\pi_1$$

$$+\left(\left(\frac{\lambda_1(w_1+1)}{w_2}-1\right)s+\left(\frac{\lambda_2(w_2+1)}{w_2}-1\right)t+\left(\frac{\lambda_3(w_3+1)}{w_2}-1\right)(1-s-t)\right)^2\pi_2$$

$$+\left(\left(\frac{\lambda_1(w_1+1)}{w_3}-1\right)u+\left(\frac{\lambda_2(w_2+1)}{w_3}-1\right)v+\left(\frac{\lambda_3(w_3+1)}{w_3}-1\right)(1-u-v)\right)^2\pi_3 \quad -(R^e)^2,$$

$$(\delta^f)^2 = V^f(\alpha,\beta,a,b,c,p,q,s,t,u,v)$$

$$= \left(\frac{1}{\beta(\lambda_1^{-\alpha}p+\lambda_2^{-\alpha}q+\lambda_3^{-\alpha}(1-p-q))}-1\right)^2\pi_1+\left(\frac{1}{\beta(\lambda_1^{-\alpha}s+\lambda_2^{-\alpha}t+\lambda_3^{-\alpha}(1-s-t))}-1\right)^2\pi_2$$

$$+\left(\frac{1}{\beta(\lambda_1^{-\alpha}u+\lambda_2^{-\alpha}v+\lambda_3^{-\alpha}(1-u-v))}-1\right)^2\pi_3 \quad -(R^f)^2$$

$$(\delta^p)^2 = V^p(\alpha,\beta,a,b,c,p,q,s,t,u,v)$$

$$= \left(\left(\frac{\lambda_1(w_1+1)}{w_1}-1\right)p+\left(\frac{\lambda_2(w_2+1)}{w_1}-1\right)q+\left(\frac{\lambda_3(w_3+1)}{w_1}-1\right)(1-p-q)-\left(\frac{1}{\beta(\lambda_1^{-\alpha}p+\lambda_2^{-\alpha}q+\lambda_3^{-\alpha}(1-p-q))}-1\right)\right)^2\pi_1$$

$$+\left(\left(\frac{\lambda_1(w_1+1)}{w_2}-1\right)s+\left(\frac{\lambda_2(w_2+1)}{w_2}-1\right)t+\left(\frac{\lambda_3(w_3+1)}{w_2}-1\right)(1-s-t)-\left(\frac{1}{\beta(\lambda_1^{-\alpha}s+\lambda_2^{-\alpha}t+\lambda_3^{-\alpha}(1-s-t))}-1\right)\right)^2\pi_2$$

$$+\left(\left(\frac{\lambda_1(w_1+1)}{w_3}-1\right)u+\left(\frac{\lambda_2(w_2+1)}{w_3}-1\right)v+\left(\frac{\lambda_3(w_3+1)}{w_3}-1\right)(1-u-v)-\left(\frac{1}{\beta(\lambda_1^{-\alpha}u+\lambda_2^{-\alpha}v+\lambda_3^{-\alpha}(1-u-v))}-1\right)\right)^2\pi_3$$

$$-(R^e-R^f)^2.$$

The expressions of the above functions are extremely complicated. The details are reduced and given in the appendix.

Building the mathematical simulating model to solve the equity premium and volatility puzzle is equivalent to solving for the parameters α, β, a, b, c, p, q, s, t, u, and v from the following system of eight equation (17) – (24) while satisfying the four constraints (25) – (28):

$$\mu(a, b, c, p, q, s, t, u, v) = 1.0183, \tag{17}$$

$$\delta(a, b, c, p, q, s, t, u, v) = 0.0357^2, \tag{18}$$

$$\rho(a, b, c, p, q, s, t, u, v) = -0.14, \tag{19}$$

$$R^e(\alpha, \beta, a, b, c, p, q, s, t, u, v) = 0.0698, \tag{20}$$

$$R^f(\alpha, \beta, a, b, c, p, q, s, t, u, v) = 0.008, \tag{21}$$

$$V^e(\alpha, \beta, a, b, c, p, q, s, t, u, v) = 0.1654^2, \tag{22}$$

$$V^f(\alpha, \beta, a, b, c, p, q, s, t, u, v) = 0.0567^2, \tag{23}$$

$$V^p(\alpha, \beta, a, b, c, p, q, s, t, u, v) = 0.1667^2, \tag{24}$$

$$0 \le p, q, s, t, u, v \le 1, \tag{25}$$

$$p + q \le 1, s + t \le 1, u + v \le 1, \tag{26}$$

$$0 \le \alpha \le 10, \tag{27}$$

$$0 \le \beta \le 1. \tag{28}$$

Normally, the system of equations (17) – (24) should have infinitely many solutions. If there exists a solution satisfying the above constraints (25) – (28), by substituting the values of the technology parameters a, b, c, p, q, s, t, u and v into (14) and (15), taking the risk aversion α in (1) and discount factor β in (2), then a three-state model is obtained. From this model, with respect to the stationary probabilities (16), the expected growth rate, standard deviation, and first-order serial correlation of the growth rate of per capita in consumption and the expected returns and standard deviations of equity, risk-free security, and risk premium for equity exactly match the values from the sample listed in (5) – (13). Hence, such a model provides a solution to this puzzle.

4. A Solution to the Equity Premium and Volatility Puzzles

A solution to the system of equations (17) – (24) satisfying the constraints (25) – (28) is given below. It is obtained by using Mathematica. The programming and the procedure to obtain the solution are available in the Appendix 1. In this section, I describe the procedure with details to build a model to solve the Equity Premium and Volatility Puzzles. Additional solutions are provided late without these details.

$p = 0.00461561332332569$, $q = 0.5347057009293275$,

$s = 0.07212269178915119$, $t = 0.35828816104829075$,

$u = 0.00302051111953540$, $v = 0.00064334232930200$,

$a = -16.406881670510224$, $b = 5.551214709489761$, $c = 0.03339328376013917$,

$\alpha = 6.728012992773973, \quad \beta = 0.5734537514033831.$

Then a model is built by substituting a, b, and c from the above solution into (14), which results in the three states $\{\lambda_1, \lambda_2, \lambda_3\}$ of the growth rate as follows:

$$\lambda_1 = 1.0183 + 0.0357(-16.4069) = 0.432574,$$

$$\lambda_2 = 1.0183 + 0.0357(5.55121) = 1.21648,$$

$$\lambda_3 = 1.0183 + 0.0357(0.0333933) = 1.01949; \quad (29)$$

and simultaneously substituting the solutions of p, q, s, t, u and v into (15) gives the transition matrix of the ergodic Markov chain as follows:

$$\Phi = \begin{pmatrix} 0.00461561332332569 & 0.5347057009293275 & 0.4606786658 \\ 0.07212269178915119 & 0.35828816104829075 & 0.5695891472 \\ 0.00302051111953540 & 0.00064334232930200 & 0.9963361466 \end{pmatrix}, \quad (30)$$

with the following stationary probabilities:

$$(\pi_1, \pi_2, \pi_3) = (0.00328361, 0.00373157, 0.992985);$$

substituting the risk aversion as α = 6.728012992773973 into (1) results in the following utility function:

$$U(c_t, 6.728012992773973) = -\frac{c_t^{-5.728012992773973} - 1}{5.728012993},$$

and substituting the discount factor of β = 0.5734537514033831 in (2). Since the set of the technology parameters and reference parameters in this model is a solution of the system of equations (17) – (24), for the model built by the solution given in this section, we have the following endogenous results below (see the attached Appendix 1 for the details):

$\mu = 1.0183,$

$\delta = 0.0012744900000000005` = 0.00127449 = 0.0357^2,$

$\rho = -0.14000000000000007 = -0.14,$

$R^e = 0.06979999999999953` = 0.0698,$

$R^f = 0.007999999999999993` = 0.008,$

$R^p = 0.06979999999999953` - 0.007999999999999993` = 0.0618,$

$(\delta^e)^2 = 0.027357159999999995` = 0.02735716 = 0.1654^2,$

$(\delta^f)^2 = 0.0032148899999999993` = 0.00321489 = 0.0567^2,$

$$(\delta^p)^2 = 0.027788889999999997` = 0.02778889 = 0.1667^2.$$

These values from this model built in this example "exactly" match the sample data for the U.S. economy from 1889 through 1978.

Finally, I have to show the existence of the expected utility (2) in this model. Mehra and Prescott (1984) proved that if the matrix A, defined by (31) below, is stable, then the expected utility (2) exists. Where the matrix A of this model is given by

$$A = \begin{pmatrix} \beta \mathrm{p} \lambda_1^{1-\alpha} & \beta \mathrm{q} \lambda_2^{1-\alpha} & \beta(1-p-\mathrm{q})\lambda_3^{1-\alpha} \\ \beta \mathrm{s} \lambda_1^{1-\alpha} & \beta \mathrm{t} \lambda_2^{1-\alpha} & \beta(1-s-\mathrm{t})\lambda_3^{1-\alpha} \\ \beta \mathrm{u} \lambda_1^{1-\alpha} & \beta \mathrm{v} \lambda_2^{1-\alpha} & \beta(1-u-\mathrm{v})\lambda_3^{1-\alpha} \end{pmatrix}. \tag{31}$$

Substituting the parameters by the values in this solution results in the following:

$$A = \begin{pmatrix} 0.321645 & 0.0998008 & 0.236523 \\ 5.02596 & 0.0668732 & 0.29244 \\ 0.210488 & 0.000120077 & 0.511542 \end{pmatrix}.$$

By using Mathematica, we get

$$A^{20000} = \begin{pmatrix} 8.55189\times10^{-120} & 9.28794\times10^{-121} & 4.83146\times10^{-120} \\ 4.79516\times10^{-119} & 5.20787\times10^{-120} & 2.70907\times10^{-119} \\ 3.80276\times10^{-120} & 4.13005\times10^{-121} & 2.1484\times10^{-120} \end{pmatrix},$$

which clearly shows that

$$\lim_{n\to\infty} A^n = 0.$$

This implies the stability of A. Hence, the model defined by (29) and (30), with the risk aversion α = 6.728012992773973, and the discount factor β = 0.5734537514033831, provides a solution to the Equity Premium and Volatility Puzzles. Incidentally, the equity premium puzzle is also resolved.

The following we provide a list of additional solutions without giving the details:

1. p=0.00505384, q=0.399756, s=0.091704, t=0.0237336, u=0.00295961, v=0.00203708, a=−16.5675, b=5.4517, c=0.0358042, α=6.85414, β=0.524433;

2. p=0.00339233, q=0.0869019, s=0.420592, t=0.0148888, u=0.00201901, v=0.0027704, a=−16.5295, b=5.4787, c=0.0381379, α=6.80014, β=0.651005;

3. p=0.0034313, q=0.0729206, s=0.529812, t=0.150151, u=0.00226601, v=0.00240743,

$a=-15.2399$, $b=5.18714$, $c=0.0439007$, $\alpha=7.27847$, $\beta=0.71614$;

4. $p=0.00544038$, $q=0.463159$, $s=0.0894655$, $t=0.325345$, $u=0.00333755$, $v=0.000824639$, $a=-15.3491$, $b=6.01644$, $c=0.0340865$, $\alpha=7.0215$, $\beta=0.643564$;

5. $p=0.00388512$, $q=0.0861559$, $s=0.455512$, $t=0.0419301$, $u=0.00235272$, $v=0.00267497$, $a=-15.5144$, $b=5.41257$, $c=0.0419138$, $\alpha=7.14913$, $\beta=0.690432$;

6. $p=0.00393438$, $q=0.0783984$, $s=0.497521$, $t=0.0198361$, $u=0.00239422$, $v=0.00274313$, $a=-15.2748$, $b=5.10517$, $c=0.0445802$, $\alpha=7.33628$, $\beta=0.68479$;

7. $p=0.00382558$, $q=0.0815112$, $s=0.471688$, $t=0.0149579$, $u=0.0023056$, $v=0.0027571$, $a=-15.5847$, $b=5.22547$, $c=0.0427738$, $\alpha=7.19291$, $\beta=0.676497$;

8. $p=0.00326907$, $q=0.0737998$, $s=0.515796$, $t=0.162462$, $u=0.00216859$, $v=0.00237811$, $a=-15.4908$, $b=5.28546$, $c=0.0423763$, $\alpha=7.15875$, $\beta=0.714015$;

9. $p=0.00390263$, $q=0.0837348$, $s=0.470835$, $t=0.0490654$, $u=0.00238405$, $v=0.00265768$, $a=-15.3715$, $b=5.33926$, $c=0.0428318$, $\alpha=7.21717$, $\beta=0.694766$;

10. $p=0.00528123$, $q=0.42947$, $s=0.0869867$, $t=0.105418$, $u=0.0032116$, $v=0.00165855$, $a=-15.9923$, $b=5.31739$, $c=0.037709$, $\alpha=7.08275$, $\beta=0.541648$;

11. $p=0.00302414$, $q=0.0711496$, $s=0.528391$, $t=0.218185$, $u=0.00208709$, $v=0.00222775$, $a=-15.5383$, $b=5.34048$, $c=0.0417739$, $\alpha=7.1121$, $\beta=0.726259$;

12. $p=0.00409136$, $q=0.0903165$, $s=0.436325$, $t=0.00183161$, $u=0.00241186$, $v=0.00276837$, $a=-15.5251$, $b=5.44494$, $c=0.0418263$, $\alpha=7.15381$, $\beta=0.681104$;

13. $p=0.00522467$, $q=0.473036$, $s=0.0815291$, $t=0.246173$, $u=0.00330761$, $v=0.00107173$, $a=-15.7356$, $b=5.39372$, $c=0.0371423$, $\alpha=7.09014$, $\beta=0.572447$.

5. Models with Four States

In this section, we build a model with four states similarly to what we did in last two sections. I use four states $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of the growth rate per capita consumption as below:

$$\begin{aligned}
\lambda_1 &= \bar{\mu} + a\bar{\delta} = 1.0183 + 0.0357\, a, \\
\lambda_2 &= \bar{\mu} + b\bar{\delta} = 1.0183 + 0.0357\, b, \\
\lambda_3 &= \bar{\mu} + c\bar{\delta} = 1.0183 + 0.0357\, c, \\
\lambda_4 &= \bar{\mu} + d\bar{\delta} = 1.0183 + 0.0357\, d, \qquad (32)
\end{aligned}$$

where a, b, c, and d are parameters defining technology. We suppose that the transitions from time to time only take place among λ_1 and λ_2, and among λ_3 and λ_4 separately. Then the growth rates are assumed to follow a Markov chain that is not ergodic:

$$\Phi = \begin{pmatrix} p & 1-p & 0 & 0 \\ q & 1-q & 0 & 0 \\ 0 & 0 & s & 1-s \\ 0 & 0 & t & 1-t \end{pmatrix}, \tag{33}$$

where p, q, s, and t are also technology parameters satisfying $0 < p, q, s, t < 1$. It is clear that the two 2×2 sub matrices on the main diagonal of Φ are matrices ergodic Markov chains. It implies that Φ has a unique stationary probability distribution of the growth rate per capita in consumption, which is denoted by $(\pi_1, \pi_2, \pi_3, \pi_4)$ and it can be solved from

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (\pi_1, \pi_2, \pi_3, \pi_4)\ \Phi,$$

with $\pi_1 + \pi_2 = \gamma$, $\pi_3 + \pi_4 = 1 - \gamma$, for some $\gamma \in (0, 1)$, which is also a parameter needed to be determined in the late contents. As a function of p, q, s, and t, the solution is given by

$$\pi_1 = -\frac{q\gamma}{-1+p-q},$$

$$\pi_2 = \frac{(-1+p)\gamma}{-1+p-q},$$

$$\pi_3 = \frac{t(-1+\gamma)}{-1+s-t},$$

$$\pi_4 = 1 - \frac{t(-1+\gamma)}{-1+s-t} - \gamma = \frac{(-1+\gamma)(1-s)}{-1+s-t}. \tag{34}$$

Similarly to the three states case, we can calculate the expected average, variance, and first-order serial correlation of the growth rate per capita consumption in this model are functions of the technology parameters a, b, c, d, p, q, s, t and γ, all with respect to the model's stationary probability distribution as given in (34). By using (32)—(34), the expected returns on equity and on risk-free security, R^e, R^f, and on risk premium for equity, $R^p \equiv R^e - R^f$, from the model are calculated as given below:

$$R^e = R^e(\alpha, \beta, a, b, c, d, p, q, s, t, \gamma)$$

$$= \left(\left(\frac{\lambda_1(w_1+1)}{w_1} - 1 \right) p + \left(\frac{\lambda_2(w_2+1)}{w_1} - 1 \right) (1-p) \right) \pi_1$$

$$+\left(\left(\frac{\lambda_1(w_1+1)}{w_2}-1\right)q+\left(\frac{\lambda_2(w_2+1)}{w_2}-1\right)(1-q)\right)\pi_2$$

$$+\left(\left(\frac{\lambda_1(w_3+1)}{w_3}-1\right)s+\left(\frac{\lambda_2(w_4+1)}{w_3}-1\right)(1-s)\right)\pi_3$$

$$+\left(\left(\frac{\lambda_1(w_3+1)}{w_4}-1\right)t+\left(\frac{\lambda_2(w_4+1)}{w_4}-1\right)(1-t)\right)\pi_4;$$

$$R^f = R^f(\alpha,\beta,a,b,c,d,p,q,s,t,\gamma)$$

$$= \left(\frac{1}{\beta(\lambda_1^{-\alpha}p+\lambda_2^{-\alpha}(1-p)}-1\right)\pi_1+\left(\frac{1}{\beta(\lambda_1^{-\alpha}q+\lambda_2^{-\alpha}(1-q)}-1\right)\pi_2$$

$$+\left(\frac{1}{\beta(\lambda_3^{-\alpha}s+\lambda_4^{-\alpha}(1-s)}-1\right)\pi_3+\left(\frac{1}{\beta(\lambda_3^{-\alpha}t+\lambda_4^{-\alpha}(1-t)}-1\right)\pi_4$$

Where w_1, w_2, w_3, and w_4 can be solved from the following system of linear equations:

$$\begin{aligned}
w_1 &= \beta p\lambda_1^{1-\alpha}(w_1+1)+\beta(1-p)\lambda_2^{1-\alpha}(w_2+1),\\
w_2 &= \beta q\lambda_1^{1-\alpha}(w_1+1)+\beta(1-q)\lambda_2^{1-\alpha}(w_2+1),\\
w_3 &= \beta s\lambda_3^{1-\alpha}(w_3+1)+\beta(1-s)\lambda_4^{1-\alpha}(w_4+1),\\
w_4 &= \beta t\lambda_3^{1-\alpha}(w_3+1)+\beta(1-t)\lambda_4^{1-\alpha}(w_4+1).
\end{aligned}$$

By applying the expressions for R^e and R^f above, the variances of the equity, the risk-free security and the risk premium for equity are calculated as follows:

$$(\delta^e)^2 = V^e(\alpha,\beta,a,b,c,d,p,q,s,t,\gamma)$$

$$= \left(\left(\frac{\lambda_1(w_1+1)}{w_1}-1\right)p+\left(\frac{\lambda_2(w_2+1)}{w_1}-1\right)(1-p)\right)^2\pi_1$$

$$+\left(\left(\frac{\lambda_1(w_1+1)}{w_2}-1\right)q+\left(\frac{\lambda_2(w_2+1)}{w_2}-1\right)(1-q)\right)^2\pi_2$$

$$+\left(\left(\frac{\lambda_1(w_3+1)}{w_3}-1\right)s+\left(\frac{\lambda_2(w_4+1)}{w_3}-1\right)(1-s)\right)^2\pi_3$$

$$+\ \left(\left(\frac{\lambda_1(w_3+1)}{w_4}-1\right)t+\left(\frac{\lambda_2(w_4+1)}{w_4}-1\right)(1-t)\right)^2\pi_4\ \ -(R^e)^2;$$

$$(\delta^f)^2\ =\ V^f(\alpha,\beta,a,b,c,d,p,q,s,t,\gamma)$$

$$=\ \left(\frac{1}{\beta(\lambda_1^{-\alpha}p+\lambda_2^{-\alpha}q+\lambda_3^{-\alpha}(1-p-q))}-1\right)^2\pi_1+\left(\frac{1}{\beta(\lambda_1^{-\alpha}s+\lambda_2^{-\alpha}t+\lambda_3^{-\alpha}(1-s-t))}-1\right)^2\pi_2$$

$$+\left(\frac{1}{\beta(\lambda_1^{-\alpha}u+\lambda_2^{-\alpha}v+\lambda_3^{-\alpha}(1-u-v))}-1\right)^2\pi_3\ \ -(R^f)^2$$

$$(\delta^p)^2=\ V^p(\alpha,\beta,a,b,c,d,p,q,s,t,\gamma)$$

$$=\ \left(\left(\frac{\lambda_1(w_1+1)}{w_1}-1\right)p+\left(\frac{\lambda_2(w_2+1)}{w_1}-1\right)(1-p)-\left(\frac{1}{\beta(\lambda_1^{-\alpha}p+\lambda_2^{-\alpha}(1-p))}-1\right)\right)^2\pi_1$$

$$+\ \left(\left(\frac{\lambda_1(w_1+1)}{w_2}-1\right)q+\left(\frac{\lambda_2(w_2+1)}{w_2}-1\right)(1-q)-\left(\frac{1}{\beta(\lambda_1^{-\alpha}q+\lambda_2^{-\alpha}(1-q))}-1\right)\right)^2\pi_2$$

$$+\ \left(\left(\frac{\lambda_1(w_3+1)}{w_3}-1\right)s+\left(\frac{\lambda_2(w_4+1)}{w_3}-1\right)(1-s)-\left(\frac{1}{\beta(\lambda_3^{-\alpha}s+\lambda_4^{-\alpha}(1-s))}-1\right)\right)^2\pi_3$$

$$+\ \left(\left(\frac{\lambda_1(w_3+1)}{w_4}-1\right)t+\left(\frac{\lambda_2(w_4+1)}{w_4}-1\right)(1-t)-\left(\frac{1}{\beta(\lambda_3^{-\alpha}t+\lambda_4^{-\alpha}(1-t))}-1\right)\right)^2\pi_4$$

$$-(R^e-R^f)^2.$$

The details are reduced and given in the appendix; it is because that is extremely complicated. As in the three states case, we solve for the parameters α, β, a, b, c, d, p, q, s, t, and γ from the system of eight equation (17) – (24) while satisfying the four constraints (25) – (28).

6. A Solution to the Equity Premium and Volatility Puzzles with Four States

By using Mathematica, similarly to the three states case, we can get many solutions of the system of equations (17) – (24) satisfying the constraints (25) – (28). Following every one solution, we can build a model to solve the Equity Premium and Volatility Puzzles. We provide the following solution with the details how to build a model. Additional solutions are provided without these details.

p = 0.0029197806886129313\`, $q$ = 0.0004777799300437738\`,

$s = 0.1437706005043603`$, $t = 0.5464432221143865`$

$a = -16.007044654730475`$, $b = -0.03089344194734167`$, $c = -6.905222865935489`$,

$d = 10.880469586812413`$, $\gamma$=0.9903400088952371`

α =7.478241969826183`, $\beta$ = 0.916814051785879`. (36)

Then a model is built by substituting a, b, and c from the above solution into (32), which results in the three states $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of the growth rate as follows:

$$\begin{aligned}
\lambda_1 &= 1.0183+0.0357(-16.007044654730475`) = 0.446848505826122`,\\
\lambda_2 &= 1.0183+0.0357(-0.03089344194734167`) = 1.0171971041224799`,\\
\lambda_3 &= 1.0183+0.0357(-6.905222865935489`) = 0.771783543686103`,\\
\lambda_4 &= 1.0183+0.0357(10.880469586812413`) = 1.4067327642492031`;
\end{aligned} \tag{37}$$

and simultaneously substituting the solutions of p, q, s, t, and γ into (33) gives the transition matrix of the Markov chain as follows:

$$\Phi = \begin{pmatrix} 0.00291978 & 0.99708022 & 0 & 0 \\ 0.00047778 & 0.99952222 & 0 & 0 \\ 0 & 0 & 0.14377060 & 0.85622940 \\ 0 & 0 & 0.54644322 & 0.45355678 \end{pmatrix}, \tag{38}$$

which has the following stationary probabilities:

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.0004743228769949451`, 0.9898656860182421`,$$
$$0.003763270618930301`, 0.005896720485832563`). \tag{39}$$

Substituting the risk aversion as α = 7.478241969826183` into (1) results in the following utility function:

$$U(c_t, 7.478241969826183`) = -\frac{c_t^{-6.478241969826183`} - 1}{6.478241969826183}, \tag{40}$$

and by taking the discount factor of β = 0.916814051785879`, similarly to (31), the matrix A in this solution is given by

$$A = \begin{pmatrix} 0.494285 & 0.818539 & 0 & 0 \\ 0.0808826 & 0.820543 & 0 & 0 \\ 0 & 0 & 0.705964 & 0.086044 \\ 0 & 0 & 2.68323 & 0.0455787 \end{pmatrix}.$$

By using Mathematica, we get A^{200000} is almost 0, which clearly shows that

$$\lim_{n \to \infty} A^n = 0.$$

This implies the stability of A. If we take the solution given in (36) and build the model by (37)—(40), then we have the following endogenous results (see the attached Appendix 2 for the details):

$\mu = 1.0183$,

$\delta = 0.0012744899999999942` = 0.00127449 = 0.0357^2$,

$\rho = -0.14000000000000093` = -0.14$,

$R^e = 0.06980000000000058` = 0.0698$,

$R^f = 0.008000000000000227` = 0.008$,

$R^p = 0.06980000000000058` - 0.008000000000000227` = 0.0618$,

$(\delta^e)^2 = 0.027357159999999999` = 0.02735716 = 0.1654^2$,

$(\delta^f)^2 = 0.0032148899999999833` = 0.00321489 = 0.0567^2$,

$(\delta^p)^2 = 0.02778888999999998` = 0.02778889 = 0.1667^2$.

These values from this model built in this example also "exactly" match the sample data for the U.S. economy from 1889 through 1978. As the case of three states, we list more solutions below without providing the details for building the corresponding models of four states.

1. α=6.667493474976215`,$\beta$=0.9564540374558674`, p=0.009922870704505744`,$q$=0.0032091019,$s$=0.013444047076287892`,t=0.38980897944264, a=−12.886746228938259`, $b$=0.03156243645785`,c=−13.5018226196575`, d=8.46517042362, γ=0.99548614039545

2. α=6.698100388429119`,$\beta$=0.9856810780865292`, P=0.0049553471390962`,$q$=0.003503071970451`,s=0.0124466960040615`,$t$=0.378826894004,$a$ =−12.13606388612816`,b=0.03136209183887`,$c$=−13.509854662`,d=8.48298801728326`, $\gamma$=0.99527572797257`.

3. α=8.176146356980405\`,$\beta$=0.6742125190613665\`,
   p=0.0108068647137\`,$q$=0.0038723563469389\`,s=0.01380785199012\`,$t$=0.573215805932275\`,
   a=−13.469710035113\`, $b$=0.05962495594224\`, c=−12.7022962122497\`,$d$ =4.672627797503\`,
   γ=0.9960089107228071\`.

4. α=6.80062893110629\`,$\beta$=0.9833866566962598\`,
   p=0.005803560260318\`,$q$=0.0033338783275248\`,s=0.03327596381298\`,$t$=0.42703794110669\`,
   a=−12.16819908674\`,$b$=0.02554168242832\`,c=−11.1097392830174\`, $d$=8.892136647487748\`,
   $\gamma\rightarrow$0.9945261564968048\`.

5. α=6.667493474976215\`,$\beta$=0.9564540374558674\`,
   p=0.0099228707045\`,$q$=0.0032091019422424\`,s=0.013444047076287892\`,$t$=0.38980897944\`,
   a=−12.886746228938259\`,$b$=0.03156243645785\`,c=−13.5018226196575\`,$d$=8.465170423623\`,
   γ=0.9954861403954505\`.

6. α=7.66769421010036\`,$\beta$=0.8842873057891891\`,
   p=0.01344470575132\`,$q$=0.004580440678444535\`,s=0.024095293346705\`,$t$=0.476405757525\`,
   a=−12.00469618499\`,$b$=0.051633318276993\`,c=−11.350285564265\`,$d$=6.881265923336441\`,
   γ=0.9954864290121973\`.

7. α=7.098936113731737\`,$\beta$=0.9980507961729107\`,
   p=0.0009121989147902\`,$q$=0.004807470092594\`,s=0.007837113341223676\`,$t$=0.372554694\`,
   a=−10.839563453517\`, $b$=0.04360381959615\`,c=−13.937882068982436\`,$d$=7.832416892625\`,
   γ=0.995514043984058\`.

8. α=7.084988027176065\`,$\beta$=0.9989797998334727\`,
   p=0.0003871167987021\`,$q$=0.0048256882805975\`,s=0.007991483180187\`,$t$=0.373362505223\`,
   a=−10.825336191089\`,$b$=0.0436360193002\`,c=−13.916910574585136\`,$d$=7.87061214603984\`,
   γ=0.9955331421285455\`.

9. α=7.164206507376002\`,$\beta$=0.9974388915561673\`,
   p=0.0013900102978694\`,$q$=0.004950692184326\`,s=0.007765397708428\`,$t$=0.373884421091\`,
   a=−10.7332335223545\`,$b$=0.04497247919\`,c=−13.8740018623675\`, $d$=7.723518541515675\`,
   γ=0.9954977753073198\`.

10. α=6.667493474976215\`,$\beta$=0.9564540374558674\`,
    p=0.0099228707045057\`,$q$=0.0032091019422424\`,s=0.0134440470762879\`,$t$=0.38980897944\`,
    a=−12.886746228938\`,$b$=0.03156243645785\`,c=−13.501822619657538\`,$d$=8.4651704236234\`,
    γ=0.9954861403954505\`.

7. Conclusion

In this paper, I applied the Mehra and Prescott's economic model to solve the Equity Premium and Volatility Puzzles, which incidentally solves the equity premium puzzle that was highlighted by Mehra and Prescott's model. I find that, in general, the framework of the economic model formulated by Mehra and Prescott, as a variation of Lucas' pure exchange model, can accurately describe a historical economic period. The procedures and techniques of numerical simulation adopted in this paper provide a useful methodology to design a model that describes complex behaviors of an economy under the utility function given by (1), if three or more states of the growth rates of the endowment are chosen.

As mentioned in Section 3, the system of equations (17) – (24) may have infinitely many solutions satisfying the constraints (25) – (28). It implies that the Equity Premium and Volatility Puzzles have multiple, maybe infinitely many, solutions. Of course, this is similarly true for the equity premium puzzle. All the solutions listed in the appendix have some common properties:

1. In the long run, there exists a state of growth rate very close to the sample average rate 1.0183 with a high stationary probability.

2. There are some states that are very low. For example, in the example in Section 6, the first state $\lambda_1 = 0.446848505826122`$. It implies that there existed some factors with average drop rate almost 0.57% (=1.0183 −0.446848505826122) with a very small probability. It seems to be a disaster. It is because that the worst case in USA is that the real per capita in GDP falls 31% during 1929 to 1933 (over all sectors) (See Barro).

3. The risk aversions in all solutions listed in this paper are higher than 6. It may be considered too high for some economists' estimations.

I believe that if we use a supper computer and choose more states, then we can get some more desirable solutions. For example, if we take the following solution in the four state case:

α=7.43867332402421`, $\beta$=0.986020527346233`,
p=0.0058589365545052155`,$q$=0.0034641791865`,s=0.076456743729752`,t=0.495567839088
a=−11.41263879877478`,$b$=0.018551952092873`, c=−8.39103608667`,$d$=9.10405453304085`,
γ=0.9929891742075543`

We can get the four states below

$$\lambda_1 = 0.610869, \lambda_2 = 1.01896, \lambda_3 = 0.71874, \lambda_4 = 1.34331,$$

with the following stationary distribution:

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.00344815,\ 0.989541,\ 0.00244825,\ 0.00456258).$$

In this solution, the disaster is also the state λ_1 (= 0.610869), which indicates that some sectors had decreasing rate 0.4074 (=1.0183 −0.610869) with a very small probability 0.00344815, in the long run. It is very close to the lowest falling rate (over all sectors) 0.31.

The results obtained in this paper seem to be "mechanically" developed by following Mehra and Prescott's economic model. Meanwhile, I believe that this is also the strong point of the results. It is because that the puzzle was solved by using the exactly same model which the puzzle was raised.

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