Consecutive k-within-m-out-of-n:F system with exchangeable components

Eryilmaz, Serkan and Kan, Cihangir and Akici, Fatih

Izmir University of Economics

2009

Consecutive k-Within-m-Out-of-n:F System with Exchangeable Components

Serkan Eryılmaz, Cihangir Kan, Fatih Akıcı

Department of Mathematics, Statistical Reliability and Process Control Research Group, Izmir University of Economics, 35330, Balcova, Izmir, Turkey

Received 28 April 2008; revised 23 January 2009; accepted 4 February 2009
DOI 10.1002/nav.20354
Published online 9 June 2009 in Wiley InterScience (www.interscience.wiley.com).

Abstract: As a generalization of k-out-of-n:F and consecutive k-out-of-n:F systems, the consecutive k-within-m-out-of-n:F system consists of n linearly ordered components such that the system fails iff there are m consecutive components which include among them at least k failed components. In this article, the reliability properties of consecutive k-within-m-out-of-n:F systems with exchangeable components are studied. The bounds and approximations for the survival function are provided. A Monte Carlo estimator of system signature is obtained and used to approximate survival function. The results are illustrated and numerics are provided for an exchangeable multivariate Pareto distribution. © 2009 Wiley Periodicals, Inc. Naval Research Logistics 56: 503–510, 2009

Keywords: exchangeable lifetimes; meantime to failure; Monte Carlo simulation; moving order statistics; multivariate Pareto distribution; Samaniego's signature

1. INTRODUCTION

Over the past two decades there has been much significant progress in reliability studies of consecutive type systems, which have been used to model telecommunication and oil pipeline systems, and vacuum systems in accelerators. The reliability properties and characteristics of such systems have been widely studied in the literature under various assumptions. One of the most widely studied systems is called consecutive k-out-of-n:F system, which consists of n linearly ordered components such that the system fails if and only if at least k consecutive components fail. Recent discussions on consecutive k-out-of-n systems appear in the works of Yun et al. [27], Xiao et al. [26], Navarro and Eryilmaz [16], Eryilmaz [4–6]. See also Kuo and Zuo [14] for an extensive review of the topic.

A general consecutive system is known as consecutive k-within-m-out-of-n:F system, consisting of n linearly ordered components such that the system fails iff there are m consecutive components which include among them at least k failed components. This system was first introduced by Griffith [8] and several alternative names, such as k-within-consecutive-m-out-of-n:F and consecutive k-out-of-m-from-n:F have also been used for this system in the literature. This model involves consecutive k-out-of-n:F and k-out-of-n:F systems for $m = k$ and $m = n$, respectively, and has applications in quality control and radar detection. Bounds and approximations for the reliability of consecutive k-within-m-out-of-n:F system consisting of independent components have been proposed in the literature. For example, Sfakianakis et al. [23] provided lower and upper bounds for the reliability of such systems which consist of independent identical components. Iyer [12] studied the lifetime distribution of this system with independent exponentially distributed component lifetimes. Papastavridis and Koutras [21] presented upper and lower bounds for the reliability of linear and circular systems consisting of independent nonidentical components. Habib and Szantai [10] improved the bounds obtained by Sfakianakis et al. [23] by applying higher order Boole-Bonferroni bounds. Recently, Habib et al. [9] presented an algorithm to compute the reliability of multi-state consecutive k-within-m-out-of-n:G system, which is the generalization of consecutive k-within-m-out-of-n:G system to the multi-state case.

Dependence among component lifetimes emerges from the common random production and operating environments. Analysis of systems that consist of dependent components might be difficult, especially whenever the system has a complex structure. In this article, we study the reliability properties of a consecutive k-within-m-out-of-n:F system which consists of exchangeable components. Systems with exchangeable components have been widely studied in the literature. See e.g. [1, 15, 17, 18, 20, 24].

In the second section, we provide the definitions and notations that will be used throughout the article. In Section

Correspondence to: S. Eryılmaz (serkan.eryilmaz@ieu.edu.tr)

3, we provide bounds and approximations for the survival function of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. In the fourth section, we develop a method based on Samaniego's signature for simulating the reliability characteristics of the corresponding system. In Section 5, we provide numerical illustrations whenever the lifetimes of components have exchangeable Pareto distribution.

2. DEFINITIONS AND NOTATIONS

Below we provide the notations and definitions that will be used throughout the article.

n, the number of components;
T_i, the lifetime of component i;
$X_i(t)$, the state of component i at time t: $X_i(t) = 1(0)$ if $T_i \le t (T_i > t)$;
$T_{k:m}^{(j)}$, kth smallest among $T_j, T_{j+1}, \dots, T_{j+m-1}, k \le m, 1 \le j \le n-m+1$;
A_j, the event of $\{T_{k:m}^{(j)} > t\}$;
$T_{k,m:n}$, the lifetime of consecutive k-within-m-out-of-n:F system, $1 \le k \le m \le n$;
$R_{k,m:n}(t) = P\{T_{k,m:n} > t\}$, the survival function of consecutive k-within-m-out-of-n:F system;
$R_X(t) = P\{X > t\}$, the survival function of X;
$E(X)$, the mean time to failure (MTTF) for the system with lifetime X.

The main goal of this article is to study the reliability properties of consecutive k-within-m-out-of-n:F system with exchangeable lifetimes. A sequence of lifetimes $T_1, T_2, \dots, T_n$ is exchangeable if for each n,

$$P\{T_1 \le t_1, \dots, T_n \le t_n\} = P\{T_{\pi(1)} \le t_1, \dots, T_{\pi(n)} \le t_n\},$$

for any permutation $\pi = (\pi(1), \dots, \pi(n))$ of $\{1, 2, \dots, n\}$, i.e. the joint distribution (survival function) of $T_1, T_2, \dots, T_n$ is symmetric in $t_1, t_2, \dots, t_n$. The exchangeability means that the components have identical distributions, but they affect one another within the system. It is obvious that a sequence of independent, identically distributed (i.i.d.) lifetimes is exchangeable. Therefore, the results obtained in this article readily hold for a system with i.i.d. lifetimes.

Consecutive k-within-m-out-of-n:F system can be represented as a series system of $n-m+1$ dependent k-out-of-m:F systems. That is,

$$T_{k,m:n} = \min\left(T_{k:m}^{(1)}, T_{k:m}^{(2)}, \dots, T_{k:m}^{(n-m+1)}\right), \tag{1}$$

where $T_{k:m}^{(j)}$ shows the lifetime of k-out-of-m:F system of components with the lifetimes $T_j, T_{j+1}, \dots, T_{j+m-1}$, $1 \le j \le n-m+1$. It is clear that the random variables $(T_{k:m}^{(1)}, T_{k:m}^{(2)}, \dots, T_{k:m}^{(n-m+1)})$ have the common terms and this makes the problem of finding the exact reliability difficult, especially whenever $T_1, T_2, \dots, T_n$ are dependent, which is the case in this article. The random variables $T_{k:m}^{(j)}$, $1 \le j \le n-m+1$ are known as moving order statistics in the literature. Although the theory of usual order statistics has been well developed in the literature, less work has been done for moving order statistics. We may refer to David and Nagaraja [3, p. 140] for limited results on moving order statistics.

Using (1) we can represent the survival function of consecutive k-within-m-out-of-n:F system as

$$\begin{aligned} R_{k,m:n}(t) &= P\{T_{k,m:n} > t\} \\ &= P\left\{T_{k:m}^{(1)} > t, T_{k:m}^{(2)} > t, \dots, T_{k:m}^{(n-m+1)} > t\right\}. \end{aligned}$$

Consider the random variable $S_m^{(j)}(t) = \sum_{i=j}^{j+m-1} X_i(t)$ which denotes the total number of failed components among $T_j, T_{j+1}, \dots, T_{j+m-1}$ at time t. By the exchangeability we have

$$\begin{aligned} &P\left\{S_m^{(j)}(t) = s\right\} \\ &\quad = P\left\{S_m^{(1)}(t) = s\right\} \\ &\quad = \binom{m}{s} \sum_{i=0}^{m-s} (-1)^i \binom{m-s}{i} P\{T_1 \le t, \dots, T_{s+i} \le t\} \\ &\quad = \binom{m}{s} \sum_{i=0}^{s} (-1)^i \binom{s}{i} P\{T_1 > t, \dots, T_{m-s+i} > t\}. \end{aligned} \tag{2}$$

The latter equations can be obtained using Theorem 2.1 of George and Bowman [7]. For simplicity hereafter let

$$f(a, b) = \sum_{i=0}^{a} (-1)^i \binom{a}{i} P\{T_1 \le t, \dots, T_{b+i} \le t\},$$

and

$$g(a, b) = \sum_{i=0}^{a} (-1)^i \binom{a}{i} P\{T_1 > t, \dots, T_{b+i} > t\}.$$

With the notation given above, Eq. (2) can be rewritten as

$$P\left\{S_m^{(j)}(t) = s\right\} = \binom{m}{s} f(m-s, s) = \binom{m}{s} g(s, m-s).$$

3. BOUNDS AND APPROXIMATIONS FOR THE SURVIVAL FUNCTION

In this section, we evaluate the probability

$$R_{k,m:n}(t) = P\left\{\bigcap_{i=1}^{n-m+1} A_i\right\}, \tag{3}$$

using various inequalities. We first obtain a lower bound using the second order Bonferroni inequality, also known as Hunter-Worsley inequality [11, 25]. This variant of Bonferroni inequality has been found to be very quick and useful for the reliability evaluation of consecutive k-within-m-out-of-n:F system consisting of i.i.d. components [10]. The proofs of the following Theorems are presented in Appendix.

THEOREM 1: Let $(T_1, T_2, \dots, T_n)$ be an exchangeable random vector representing the lifetimes. Then for $1 \le k \le m \le n$,

$$R_{k,m:n}(t) \ge 1 - (n-m+1)P\{T_{k:m}^{(1)} \le t\} + (n-m)P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\},$$

where

$$P\{T_{k:m}^{(1)} \le t\} = \sum_{s=k}^{m} \binom{m}{s} f(m-s, s) = \sum_{s=k}^{m} \binom{m}{s} g(s, m-s),$$

and

$$P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\} = \binom{m-1}{k-1} f(m-k, k+1) + \sum_{l=k}^{m-1} \binom{m-1}{l} f(m-l-1, l), \quad (4)$$

or in terms of the joint survival function

$$P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\} = \binom{m-1}{m-k} [g(k-1, m-k) - 2g(k-1, m-k+1) + g(k-1, m-k+2)] + \sum_{l=k}^{m-1} \binom{m-1}{l} g(l, m-l-1). \quad \square \quad (5)$$

The probabilities given in (4) and (5) can be easily calculated if the joint distribution (survival) function of lifetimes of the components is given.

An approximation formula for the survival function can also be obtained using the following product-type approximation formula [see, e.g. [2]].

$$R_{k,m:n}(t) = P\left\{\bigcap_{i=1}^{n-m+1} A_i\right\} \simeq \frac{\prod_{i=2}^{n-m+1} P\{A_{i-1}A_i\}}{\prod_{i=2}^{n-m} P\{A_i\}} = \frac{[P\{A_1A_2\}]^{n-m}}{[P\{A_1\}]^{n-m-1}}, \quad (6)$$

where the last equation follows from exchangeability. The probabilities in (6) can be easily evaluated using the equations given in Theorem 1. For example,

$$\begin{aligned} P\{A_1A_2\} &= P\{T_{k:m}^{(1)} > t, T_{k:m}^{(2)} > t\} \\ &= 1 - P\{T_{k:m}^{(1)} \le t\} - P\{T_{k:m}^{(2)} \le t\} \\ &\quad + P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\}. \end{aligned} \quad (7)$$

It should be noted that the probability given by (7) with $m = n-1$ is actually the exact survival function of consecutive k-within-$(n-1)$-out-of-n:F system.

THEOREM 2: Let $(T_1, T_2, \dots, T_n)$ be an exchangeable random vector. Then for $1 \le k \le m \le n$,

$$\begin{aligned} R_{k,m:n}(t) &\le \sum_{j_1, j_2, \dots, j_r = 0}^{k-1} \binom{m}{j_1} \cdots \binom{m}{j_r} f\left(r \cdot m - \sum_{i=1}^{r} j_i, \sum_{i=1}^{r} j_i\right) \\ &= \sum_{j_1, j_2, \dots, j_r = 0}^{k-1} \binom{m}{j_1} \cdots \binom{m}{j_r} g\left(\sum_{i=1}^{r} j_i, r \cdot m - \sum_{i=1}^{r} j_i\right), \end{aligned}$$

where $r = [\frac{n}{m}]$.

4. SIMULATION BASED ON SAMANIEGO'S SIGNATURE

Samaniego [22] [see also Kochar et al. [13]] proved that any coherent system with lifetime T and i.i.d. component lifetimes $T_1, T_2, \dots, T_n$ having absolutely continuous c.d.f.s, satisfies

$$P\{T > t\} = \sum_{i=1}^{n} p_i P\{T_{i:n} > t\}, \quad (8)$$

where p_i is the probability that the system fails upon the occurrence of the ith component failure, i.e. $p_i = P\{T = T_{i:n}\}$. More explicitly,

$$p_i = \frac{\text{number of orderings for which the } i\text{th failure causes system failure}}{n!},$$

$i = 1, 2, \dots, n$. The vector $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is called the system signature. Navarro and Rychlik [18] proved that the representation (8) also holds in the case whenever the lifetimes $T_1, T_2, \dots, T_n$ have an absolutely continuous exchangeable distribution. Signatures of consecutive k-out-of-n systems with several components are listed in Table 1 of Navarro and Eryılmaz [16]. The determination of the signature of a coherent system might be difficult except for some special

Table 1. Order statistic representation of consecutive 2-within-3-out-of-4:F system.

Ordering	$T_{2,3:4}$	Ordering	$T_{2,3:4}$
$T_1 < T_2 < T_3 < T_4$	$T_{2:4}$	$T_3 < T_1 < T_4 < T_2$	$T_{2:4}$
$T_1 < T_2 < T_4 < T_3$	$T_{2:4}$	$T_3 < T_1 < T_2 < T_4$	$T_{2:4}$
$T_1 < T_3 < T_4 < T_2$	$T_{2:4}$	$T_3 < T_2 < T_1 < T_4$	$T_{2:4}$
$T_1 < T_3 < T_2 < T_4$	$T_{2:4}$	$T_3 < T_2 < T_4 < T_1$	$T_{2:4}$
$T_1 < T_4 < T_3 < T_2$	$T_{3:4}$	$T_3 < T_4 < T_2 < T_1$	$T_{2:4}$
$T_1 < T_4 < T_2 < T_3$	$T_{3:4}$	$T_3 < T_4 < T_1 < T_2$	$T_{2:4}$
$T_2 < T_1 < T_3 < T_4$	$T_{2:4}$	$T_4 < T_1 < T_3 < T_2$	$T_{3:4}$
$T_2 < T_1 < T_4 < T_3$	$T_{2:4}$	$T_4 < T_1 < T_2 < T_3$	$T_{3:4}$
$T_2 < T_3 < T_4 < T_1$	$T_{2:4}$	$T_4 < T_2 < T_1 < T_3$	$T_{2:4}$
$T_2 < T_3 < T_1 < T_4$	$T_{2:4}$	$T_4 < T_2 < T_3 < T_1$	$T_{2:4}$
$T_2 < T_4 < T_1 < T_3$	$T_{2:4}$	$T_4 < T_3 < T_1 < T_2$	$T_{2:4}$
$T_2 < T_4 < T_3 < T_1$	$T_{2:4}$	$T_4 < T_3 < T_2 < T_1$	$T_{2:4}$

cases. In Table 1, we present the order statistic representation of the lifetime of consecutive 2-within-3-out-of-4:F system by writing out all possible permutations of T_1, T_2, T_3, T_4. From Table 1, we compute

$$\begin{aligned} p_1 &= P\{T_{2,3:4} = T_{1:4}\} = 0, \\ p_2 &= P\{T_{2,3:4} = T_{2:4}\} = 20/24, \\ p_3 &= P\{T_{2,3:4} = T_{3:4}\} = 4/24, \\ p_4 &= P\{T_{2,3:4} = T_{4:4}\} = 0. \end{aligned}$$

The signature of a system does not depend on the distribution of $T_1, T_2, \ldots, T_n$ because

$$P\{T_1 < T_2 < \cdots < T_n\} = P\{T_{\pi(1)} < T_{\pi(2)} < \cdots < T_{\pi(n)}\}$$

holds for any permutation $\pi = (\pi(1), \ldots, \pi(n))$ [see also Theorem 3.2 of Navarro et al. [19]]. Thus the system with exchangeable components has the same signature vector with the system with i.i.d. components. This is crucial for the development of our simulation. The simulation of the lifetime of consecutive k-within-m-out-of-n:F system without using this fact needs to generate random vectors from the distribution $F(t_1, t_2, \ldots, t_n) = P\{T_1 \le t_1, \ldots, T_n \le t_n\}$. Because of the difficulty of this task, we first obtain the Monte Carlo estimates of the signature of consecutive k-within-m-out-of-n:F system consisting of i.i.d. components and then use these estimates to estimate the survival function of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. That is, the estimator of survival function is given by

$$\hat{R}_{k,m:n}(t) = \sum_{i=1}^{n} \hat{p}_i P\{T_{i:n} > t\}, \tag{9}$$

Table 2. Monte Carlo estimates of system signature.

n	m	k	$\hat{\mathrm{p}}$
4	3	2	(0,0.8320,0.1700,0)
10	3	2	(0, 0.3855, 0.4611, 0.1646, 0.0049, 0, 0, 0, 0, 0)
10	7	2	(0, 0.8683, 0.1323, 0, 0, 0, 0, 0, 0, 0)
10	7	5	(0, 0, 0, 0, 0.2594, 0.4464, 0.2523, 0.0350, 0, 0)
15	7	5	(0, 0, 0, 0, 0.0481, 0.1447, 0.2498, 0.2901, 0.2264, 0.0345, 0, 0, 0, 0, 0)
15	10	4	(0, 0, 0, 0.4610, 0.4055, 0.1206, 0.0102, 0, 0, 0, 0, 0, 0, 0, 0)
20	10	7	(0, 0, 0, 0, 0, 0, 0.0133, 0.0547, 0.1336, 0.2139, 0.2612, 0.2189, 0.1068, 0, 0, 0, 0, 0, 0, 0)
20	10	9	(0, 0, 0, 0, 0, 0, 0, 0, 0.0011, 0.0043, 0.0150, 0.0456, 0.0965, 0.1670, 0.2359, 0.2488, 0.1720, 0, 0, 0)

where $\hat{p}_i$ is the Monte Carlo estimate of the ith element of the signature vector and

$$\begin{aligned} P\{T_{i:n} > t\} &= 1 - \sum_{j=i}^{n} (-1)^{j-i} \binom{j-1}{i-1} \binom{n}{j} P\{T_{j:j} \le t\} \\ &= 1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} P\{T_{1:j} \le t\}, \end{aligned}$$

where $T_{1:j} = \min(T_1, \ldots, T_j)$ and $T_{j:j} = \max(T_1, \ldots, T_j)$. We readily have $P\{T_{1:j} \le t\} = 1 - P\{T_1 > t, \ldots, T_j > t\}$, $P\{T_{j:j} \le t\} = P\{T_1 \le t, \ldots, T_j \le t\}$.

In Table 2, we present the Monte Carlo estimate $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ for various values of n, m, and k. All simulation results are based on 50,000 repetitions.

Via the same simulation method we can also approximate the other reliability characteristics of consecutive k-within-m-out-of-n:F system. For example, mean time to failure (MTTF) of the system can be estimated from

$$\hat{E}(T_{k,m:n}) = \sum_{i=1}^{n} \hat{p}_i E(T_{i:n}).$$

Table 3. Bounds, approximations and exact value for the survival function when $n = 5, m = 3, k = 2$.

a	t	$R_{k,m:n}(t)$	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{\text{LB+UB}}{2}$
1.5	1.1	0.8760	0.8735	0.8771	0.8725	0.9329	0.9027
	1.3	0.5861	0.5786	0.5871	0.5711	0.7187	0.6449
	1.5	0.4138	0.4061	0.4132	0.3945	0.5547	0.4746
	1.7	0.3099	0.3031	0.3102	0.2903	0.4404	0.3653
	1.9	0.2426	0.2369	0.2424	0.2242	0.3593	0.2917
2.0	1.1	0.8205	0.8170	0.8192	0.8150	0.8999	0.8574
	1.3	0.4644	0.4571	0.4638	0.4450	0.6179	0.5314
	1.5	0.2873	0.2813	0.2874	0.2656	0.4300	0.3478
	1.7	0.1935	0.1892	0.1930	0.1737	0.3127	0.2432
	1.9	0.1388	0.1356	0.1389	0.1217	0.2366	0.1791

Table 4. Bounds, approximations, and exact value for the survival function when $n = 8$, $m = 3$, $k = 2$.

a	t	$R_{k,m:n}(t)$	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{\text{LB+UB}}{2}$
1.5	1.1	0.8044	0.7914	0.8051	0.7820	0.8776	0.8298
	1.3	0.4581	0.4180	0.4566	0.3496	0.5760	0.4628
	1.5	0.2961	0.2544	0.2977	0.1543	0.3994	0.2768
	1.7	0.2101	0.1731	0.2108	0.0652	0.2952	0.1802
	1.9	0.1587	0.1268	0.1570	0.0215	0.2290	0.1252
2.0	1.1	0.7240	0.7068	0.7246	0.6877	0.8210	0.7543
	1.3	0.3281	0.2908	0.3270	0.1857	0.4497	0.3177
	1.5	0.1802	0.1488	0.1807	0.0189	0.2704	0.1446
	1.7	0.1130	0.0890	0.1117	0.0000	0.1786	0.0893
	1.9	0.0773	0.0589	0.0775	0.0000	0.1263	0.0631

Table 6. Bounds and approximations for the survival function when $n = 15$, $m = 10$, $k = 8$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{\text{LB+UB}}{2}$
1.5	1.1	0.9973	0.9869	0.9973	0.9990	0.9982
	1.3	0.8964	0.9055	0.8955	0.9430	0.9192
	1.5	0.7366	0.7386	0.7326	0.8247	0.7787
	1.7	0.6000	0.6200	0.5932	0.7043	0.6487
	1.9	0.4951	0.5120	0.4867	0.6015	0.5441
2.0	1.1	0.9942	0.9841	0.9942	0.9979	0.9960
	1.3	0.8260	0.8371	0.8235	0.8989	0.8612
	1.5	0.6090	0.6374	0.6009	0.7237	0.6623
	1.7	0.4501	0.4698	0.4387	0.5690	0.5038
	1.9	0.3416	0.3571	0.3291	0.4508	0.3900

5. NUMERICAL RESULTS

In this section, we present some computational results when $(T_1, T_2, \ldots, T_n)$ has a multivariate Pareto distribution whose survival function is

$$\bar{F}_a(t_1, \ldots, t_n) = \left(\sum_{i=1}^{n} t_i - n + 1\right)^{-a}, \quad a > 0,$$
$$t_i > 1, \quad i = 1, \ldots, n.$$

It is easy to see that $(T_1, \ldots, T_n)$ is exchangeable, and

$$\begin{aligned} P\{T_{1:j} \le t\} &= 1 - P\{T_1 > t, \ldots, T_j > t\} \\ &= 1 - \bar{F}_a(t, \ldots, t) = 1 - (j(t-1)+1)^{-a}. \end{aligned}$$

Thus

$$P\{T_{i:n} > t\} = 1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \times (1 - (j(t-1)+1)^{-a}),$$

On the other hand, if $a > 1$, then $E(T_{1:j}) = \frac{1}{j(a-1)}$, and hence

$$E(T_{i:n}) = \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \frac{1}{j(a-1)}.$$

We were able to compute the precise value of $\mathbf{p}$ for small values of n generating all the permutations of numbers from 1 up to n (MATLAB code is available on request). The precise values of $\mathbf{p}$ for $n = 5$, $m = 3$, $k = 2$, and $n = 8$, $m = 3$, $k = 2$ are found to be $\mathbf{p} = (0, 84/120, 36/120, 0, 0)$ and $\mathbf{p} = (0, 0.4643, 0.4643, 0.0714, 0, 0, 0, 0)$, respectively. These allow computation of the exact value of the survival function for $n = 5$, and $n = 8$ as provided in Tables 3 and 4. These tables also include the bounds and approximations for the survival function. From Tables 3 and 4 it can be observed that the approximation based on simulation is rather effective, which suggests (9) could be used as a reference value for larger n where the computation of the exact value is not possible.

The simulation results along with the bounds and approximations for the survival function are presented in Tables 5–8 for $n = 15, m = 12, k = 8$; $n = 15, m = 10, k = 8$; $n = 30$, $m = 10$, $k = 8$, and $n = 30$, $m = 10$, $k = 6$, respectively. In

Table 5. Bounds and approximations for the survival function when $n = 15$, $m = 12$, $k = 8$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{\text{LB+UB}}{2}$
1.5	1.1	0.9927	0.9858	0.9927	0.9960	0.9944
	1.3	0.8340	0.8473	0.8332	0.8772	0.8552
	1.5	0.6441	0.6524	0.6422	0.7066	0.6744
	1.7	0.5032	0.5205	0.5007	0.5675	0.5341
	1.9	0.4034	0.4122	0.4006	0.4635	0.4321
2.0	1.1	0.9852	0.9900	0.9852	0.9917	0.9885
	1.3	0.7358	0.7493	0.7342	0.7973	0.7657
	1.5	0.4979	0.5052	0.4946	0.5709	0.5327
	1.7	0.3481	0.3562	0.3444	0.4134	0.3789
	1.9	0.2544	0.2592	0.2508	0.3091	0.2800

Table 7. Bounds and approximations for the survival function when $n = 30$, $m = 10$, $k = 8$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{\text{LB+UB}}{2}$
1.5	1.1	0.9920	0.9925	0.9920	0.9973	0.9947
	1.3	0.7700	0.8169	0.7528	0.8841	0.8184
	1.5	0.5249	0.6394	0.4563	0.7095	0.5829
	1.7	0.3710	0.4873	0.2599	0.5664	0.4131
	1.9	0.2760	0.3967	0.1420	0.4601	0.3010
2.0	1.1	0.9832	0.9898	0.9831	0.9941	0.9886
	1.3	0.6409	0.7281	0.5974	0.8045	0.7009
	1.5	0.3629	0.4782	0.2328	0.5702	0.4015
	1.7	0.2228	0.3334	0.0478	0.4083	0.2280
	1.9	0.1485	0.2434	0.0000	0.3024	0.1512

Table 8. Bounds and approximations for the survival function when $n = 30$, $m = 10$, $k = 6$.

a	t	$\tilde{R}_{k,m:n}(t)$	$\hat{R}_{k,m:n}(t)$	LB	UB	$\frac{\text{LB+UB}}{2}$
1.5	1.1	0.9214	0.9619	0.9192	0.9677	0.9434
	1.3	0.4587	0.5997	0.3555	0.6793	0.5174
	1.5	0.2463	0.3965	0.0596	0.4655	0.2666
	1.7	0.1540	0.2789	0.0000	0.3384	0.1692
	1.9	0.1066	0.2107	0.0000	0.2589	0.1295
2.0	1.1	0.8609	0.9084	0.8539	0.9402	0.8970
	1.3	0.2981	0.4506	0.1088	0.5374	0.3231
	1.5	0.1257	0.2422	0.0000	0.3100	0.1550
	1.7	0.0666	0.1513	0.0000	0.1976	0.0988
	1.9	0.0407	0.1035	0.0000	0.1360	0.0680

these Tables, $\tilde{R}_{k,m:n}(t)$ denotes the approximation computed from (6) and $\hat{R}_{k,m:n}(t)$ shows the simulated reliability given in (9). LB and UB denote the lower and upper bounds given in Theorem 1 and Theorem 2, respectively. We also compute (LB + UB)/2 as an alternative approximation. The performance of the approximation computed from (6) is relatively effective if m is close enough to n and/or k is close enough to m. That is, the closer m to n and/or the closer k to m, the better approximation. The approximation computed from (LB + UB)/2 seems stronger for larger n when m and k are fixed. However, since the lower bounds are much better approximations than the upper ones for small n (this can be observed comparing the rows of Table 3 with Table 4, and Table 6 with Table 7) it might be more appropriate to use weighted average of bounds, e.g. $(3 \cdot \text{LB} + \text{UB})/4$ for small n. We also observe that for fixed a, the bounds and approximations perform better for smaller values of t (or equivalently for high reliability structures).

6. SUMMARY AND CONCLUSIONS

In this article, we studied the reliability of consecutive k-within-m-out-of-n:F system consisting of exchangeable components. The bounds and approximations based on the probabilities associated with moving order statistics were provided for the survival function of this system. The formulas have been represented both in terms of the joint c.d.f. and the joint survival function of $T_1, T_2, \ldots, T_n$ so that the computations can be easily performed if either the joint c.d.f. or joint survival is known.

A simulation study based on Samaniego's signature was also performed to estimate the system reliability. The proposed method does not need to generate random vectors from the joint distribution of $T_1, T_2, \ldots, T_n$, which is a difficult task in Monte Carlo simulation. This method can be also used to estimate the other reliability characteristics of systems consisting of exchangeable components.

The performance of the approximations is satisfactory under particular selections of k, m, and n. The results obtained in the article are readily applicable for consecutive k-out-of-n:F ($m = k$) and k-out-of-n:F ($m = n$) systems which consist of exchangeable components.

APPENDIX

PROOF OF THEOREM 1: According to the Hunter-Worsley variant of Bonferroni inequality we have

$$P\left\{\bigcup_{i=1}^{n} C_i\right\} \le \sum_{i=1}^{n} P\{C_i\} - \sum_{i=1}^{n-1} P\{C_i C_{i+1}\}.$$

Using this inequality for (3) one obtains

$$R_{k,m:n}(t) \ge 1 - \sum_{i=1}^{n-m+1} p\{T_{k:m}^{(i)} \le t\} + \sum_{i=1}^{n-m} p\{T_{k:m}^{(i)} \le t, T_{k:m}^{(i+1)} \le t\}.$$

By the exchangeability we have

$$R_{k,m:n}(t) \ge 1 - (n-m+1)P\{T_{k,m}^{(1)} \le t\} + (n-m)P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\}. \quad (10)$$

The probabilities in (10) can be computed using the following equations.

$$P\{T_{k:m}^{(1)} \le t\} = P\left\{\sum_{i=1}^{m} X_i(t) \ge k\right\} = \sum_{s=k}^{m} \binom{m}{s} f(m-s, s) = \sum_{s=k}^{m} \binom{m}{s} g(s, m-s), \quad (11)$$

and

$$\begin{aligned} P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\} &= P\left\{\sum_{i=1}^{m} X_i(t) \ge k, \sum_{i=2}^{m+1} X_i(t) \ge k\right\} \\ &= P\left\{X_1(t) + \sum_{i=2}^{m} X_i(t) \ge k, \sum_{i=2}^{m} X_i(t) + X_{m+1}(t) \ge k\right\} \\ &= \sum_{l} P\left\{X_1(t) \ge k-l, X_{m+1}(t) \ge k-l, \sum_{i=2}^{m} X_i(t) = l\right\}. \quad (12) \end{aligned}$$

Consider the probability in (12). It is clear that

$$P\left\{X_1(t) \ge k-l, X_{m+1}(t) \ge k-l, \sum_{i=2}^{m} X_i(t) = l\right\} = \begin{cases} P\left\{\sum_{i=2}^{m} X_i(t) = l\right\} & \text{if } k \le l \\ P\left\{X_1(t) = 1, X_{m+1}(t) = 1, \sum_{i=2}^{m} X_i(t) = l\right\} & \text{if } k = l+1 \\ 0 & \text{if } k > l+1. \end{cases}$$

Thus

$$P\{T_{k:m}^{(1)} \le t, T_{k:m}^{(2)} \le t\} = P\{X_1(t)=1, X_{m+1}(t)=1, S_{m-1}^{(2)}(t)=k-1\} + \sum_{l=k}^{m-1} P\{S_{m-1}^{(2)}(t)=l\} \tag{13}$$

$$= \binom{m-1}{k-1}\sum_{i=0}^{m-k}(-1)^i\binom{m-k}{i}P\{T_1 \le t,\dots,T_{k+i+1}\le t\} + \sum_{l=k}^{m-1}\binom{m-1}{l}\sum_{i=0}^{m-l-1}(-1)^i\binom{m-l-1}{i}P\{T_1\le t,\dots,T_{1+i}\le t\}. \tag{14}$$

Therefore the proof of (4) is completed.

For the proof of (5) we need to write (13) in terms of joint survival function (or $g(a,b)$). It is clear that

$$P\{X_1(t)=1, X_{m+1}(t)=1, S_{m-1}^{(2)}(t)=k-1\} = P\{E_{k,m}\} - P\{E_{k,m}\cap\{T_1>t\}\} - P\{E_{k,m}\cap\{T_{m+1}>t\}\} + P\{E_{k,m}\cap\{T_1>t\}\cap\{T_{m+1}>t\}\}, \tag{15}$$

where $E_{k,m}$ denotes the event of $\{m-k$ of $T_2, T_3, \dots, T_m$ are greater than $t\}$. Thus we have

$$p\{E_{k,m}\} = \binom{m-1}{m-k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}P\{T_1>t,\dots,T_{m-k+i}>t\} = \binom{m-1}{m-k}g(k-1,m-k), \tag{16}$$

$$P\{E_{k,m}\cap\{T_1>t\}\} = P\{E_{k,m}\cap\{T_{m+1}>t\}\} = \binom{m-1}{m-k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}P\{T_1>t,\dots,T_{m-k+i+1}>t\} = \binom{m-1}{m-k}g(k-1,m-k+1), \tag{17}$$

and

$$P\{E_{k,m}\cap\{T_1>t\}\cap\{T_{m+1}>t\}\} = \binom{m-1}{m-k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}P\{T_1>t,\dots,T_{m-k+i+2}>t\} = \binom{m-1}{m-k}g(k-1,m-k+2). \tag{18}$$

Using (16)–(18) in (15) and considering (15) in (13), we complete the proof of (5). □

PROOF OF THEOREM 2: It is clear that

$$R_{k,m:n}(t) = P\{T_{k:m}^{(1)}>t, T_{k:m}^{(2)}>t,\dots,T_{k:m}^{(n-m+1)}>t\} \le P\{T_{k:m}^{(1)}>t, T_{k:m}^{(m+1)}>t,\dots,T_{k:m}^{(s)}>t\},$$

where $s = ([\frac{n}{m}]-1)\cdot m+1$. As the order statistics $T_{k:m}^{(1)}, T_{k:m}^{(m+1)}, \dots, T_{k:m}^{(s)}$ are nonoverlapping (they do not have the common terms) we have

$$P\{T_{k:m}^{(1)}>t, T_{k:m}^{(m+1)}>t,\dots,T_{k:m}^{(s)}>t\} = P\left\{\sum_{i=1}^{m}X_i(t)<k, \sum_{i=m+1}^{2m}X_i(t)<k,\dots,\sum_{i=s}^{s+m-1}X_i(t)<k\right\}$$

$$= \sum_{j_1,j_2,\dots,j_r=0}^{k-1} P\left\{\sum_{i=1}^{m}X_i(t)=j_1, \sum_{i=m+1}^{2m}X_i(t)=j_2,\dots,\sum_{i=s}^{s+m-1}X_i(t)=j_r\right\}$$

$$= \sum_{j_1,j_2,\dots,j_r=0}^{k-1}\binom{m}{j_1}\dots\binom{m}{j_r}P\{T_1\le t,\dots,T_{j_1+\dots+j_r}\le t, T_{j_1+\dots+j_r+1}>t,\dots,T_{s+m-1}>t\}.$$

The proof is completed by noting that

$$P\{T_1\le t,\dots,T_{j_1+\dots+j_r}\le t, T_{j_1+\dots+j_r+1}>t,\dots,T_{s+m-1}>t\} = \sum_{i=0}^{s+m-1-(j_1+\dots+j_r)}(-1)^i\binom{s+m-1-(j_1+\dots+j_r)}{i} \times P\{T_1\le t,\dots,T_{j_1+\dots+j_r+i}\le t\}$$

$$= \sum_{i=0}^{j_1+\dots+j_r}(-1)^i\binom{j_1+\dots+j_r}{i}P\{T_1>t,\dots,T_{s+m-1-(j_1+\dots+j_r)+i}>t\}.$$

□

ACKNOWLEDGMENTS

The authors would like to express gratitude to the referees for a thorough review and valuable comments that led to improvements in this article.

REFERENCES

[1] B. Bassan and F. Spizzichino, Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, J Multivariate Anal 93 (2005), 313–339.
[2] T.M. Costigan, Combination setwise-Bonferroni-type bounds, Naval Res Logistics 43 (1996), 59–77.
[3] H.A. David and H.N. Nagaraja, Order Statistics, Wiley Series in Probability and Statistics, Wiley, New Jersey, 2003.
[4] S. Eryılmaz, On the lifetime distribution of consecutive k-out-of-n:F system, IEEE Trans Reliabil 56 (2007), 35–39.
[5] S. Eryılmaz, Lifetime of combined k-out-of-n, and consecutive k_c-out-of-n systems, IEEE Trans Reliabil 57 (2008), 331–335.
[6] S. Eryılmaz, Reliability properties of consecutive k-out-of-n systems of arbitrarily dependent components, Reliabil Eng System Safety 94 (2009), 350–356.
[7] E.O. George and D. Bowman, A full likelihood procedure for analysing exchangeable binary data, Biometrics 51 (1995), 512–523.
[8] W.S. Griffith, On consecutive k-out-of-n failure systems and their generalizations, Reliabil Quality Control (1986), 157–165.
[9] A. Habib, R.O. Al-Seedy, and T. Radwan, Reliability evaluation of multi-state consecutive k-out-of-r-from-n:G system, Appl Math Model 31 (2007), 2412–2423.

[10] A. Habib and T. Szantai, New bounds on the reliability of the consecutive k-out-of-r-from-n:F system, Reliabil Eng System Safety 68 (2000), 97–104.

[11] D. Hunter, An upper bound for the probability of a union, J Appl Probabil 13 (1976), 597–603.

[12] S. Iyer, Distribution of the lifetime of consecutive k-within-m-out-of-n:F systems, IEEE Trans Reliabil 41 (1992), 448–450.

[13] S. Kochar, H. Mukerjee, and F. Samaniego, The "signature" of a coherent system and its application to comparison among systems, Naval Res Logist 46 (1999), 507–523.

[14] W. Kuo and M.J. Zuo, Optimal reliability modeling, principles and applications, Wiley, New York, 2003.

[15] J. Navarro, Likelihood ratio ordering of order statistics, mixtures, and systems, J Stat Plann Inference 138 (2008), 1242–1257.

[16] J. Navarro and S. Eryılmaz, Mean residual lifetimes of consecutive k-out-of-n systems, J Appl Probabil 44 (2007), 82–98.

[17] J. Navarro, J.M. Ruiz, and C.J. Sandoval, A note on comparisons among coherent systems with dependent components using signatures, Stat Probabil Lett 72 (2005), 179–185.

[18] J. Navarro and T. Rychlik, Reliability and expectation bounds for coherent systems with exchangeable components, J Multivariate Anal 98 (2007), 102–113.

[19] J. Navarro, F.J. Samaniego, N. Balakrishnan, and D. Bhattacharya, On the application and extension of system signatures in engineering reliability, Naval Res Log 55 (2008), 313–327.

[20] S.G. Papastavridis, Lifetime distribution of circular consecutive k-out-of-n:F systems with exchangeable lifetimes, IEEE Trans Reliabil 38 (1989), 460–461.

[21] S.G. Papastavridis and M.V. Koutras, Bounds for reliability of consecutive k-within-m-out-of-n:F systems, IEEE Trans Reliabil 42 (1993), 156–160.

[22] F. Samaniego, On closure of the IFR class under formation of coherent systems, IEEE Trans Reliabil R-34 (1985), 69–72.

[23] M. Sfakianakis, S. Kounias, and A. Hillaris, Reliability of a consecutive k-out-of-r-from-n:F system, IEEE Trans Reliabil 41 (1992), 442–447.

[24] J.G. Shanthikumar, Lifetime distribution of consecutive k-out-of-n:F systems with exchangeable lifetimes, IEEE Trans Reliabil R-34 (1985), 480–483.

[25] K.J. Worsley, An improved Bonferroni inequality and applications, Biometrika 69 (1982), 297–302.

[26] G. Xiao, Z. Li, and T. Li, Dependability estimation for non-Markov consecutive k-out-of-n:F repairable systems by fast simulation, Reliabil Eng System Safety 92 (2007), 293–299.

[27] W.Y. Yun, G.R. Kim, and H. Yamamoto, Economic design of a circular consecutive k-out-of-n:F system with $(k-1)$ step Markov dependence, Reliabil Eng System Safety 92 (2007), 464–478.